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Néron models of Jacobians over bases of arbitrary dimension

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Abstract. We work with a smooth relative curve X_U/U with nodal reduction over an excellent and locally factorial scheme S. We show that blowing up a nodal model of X_U in the ideal sheaf of a section yields a new nodal model and describe how these models relate to each other. We construct a Néron model for the Jacobian of X_U and describe it locally on S as a quotient of the Picard space of a well-chosen nodal model. We provide a combinatorial criterion for the Néron model to be separated.

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1. Introduction

1.1. Néron models

Given a scheme S and a dense open $U \subset S$, proper and smooth schemes over U often have no proper and smooth model over S. Even so, they may still have a canonical smooth S-model, the *Néron model*, first introduced in [Nér64]. The Néron model of X_U/U is defined as a smooth S-model satisfying the *Néron mapping property*: for every smooth S-scheme T and every U-morphism $\phi_U : T_U \to X_U$, there exists a unique morphism $\phi : T \to N$ extending ϕ_U. Néron models are unique up to a unique isomorphism and inherit a group structure from X_U when it has one.

Néron proved in the original article [Nér64] that abelian varieties over a dense open of a Dedekind scheme always have Néron models. Recently, people have taken interest in constructing Néron models in different settings. It was proved by Qing Liu and Jilong Tong in [LT13] that smooth and proper curves of positive genus over a dense open of a Dedekind scheme always have Néron models. This does not apply to genus 0: if S is the spectrum of a discrete valuation ring with fraction field K, then $\mathbb{P}^1_K$ does not have a Néron model. Indeed, the Néron model, if it existed, would be the smooth and proper model $\mathbb{P}^1_S$. But $\mathbb{P}^1_S$ does not have the Néron mapping property since many automorphisms of $\mathbb{P}^1_K$ do not extend to automorphisms of $\mathbb{P}^1_S$ (*e.g.* multiplication by a uniformizer).

Among the concrete applications of the theory of Néron models, we can cite the semi-stable reduction theorem (an abelian variety over the fraction field of a discrete valuation ring acquires semi-abelian reduction after a finite extension of the base field), the Néron–Ogg–Shafarevich criterion for good reduction of abelian varieties, the computation of canonical heights on Jacobians, as well as the linear and quadratic Chabauty methods to determine whether or not a list of rational points on a curve is exhaustive. For a geometric description of the quadratic Chabauty method, see [EL19]. Parallels can also be drawn to some problems in which Néron models do not explicitly intervene, such as extending the double ramification cycle on the moduli stack of smooth curves to the whole moduli stack of stable curves as in [Hol21]. Here, one is interested in models in which one given section extends, instead of all sections simultaneously, but the two problems are closely related.

1.2. Models of Jacobians

Some constructions have already been made relating to Néron models of Jacobians over higher-dimensional bases. When S is a regular scheme of arbitrary dimension and X/S is a nodal curve[(1)] smooth over U, David Holmes exhibited in [Hol19] a combinatorial criterion on X/S called ***alignment***, necessary for the Jacobian of X_U to have a separated Néron model and sufficient when X is regular. In [Ore18], Giulio Orecchia introduces the ***toric-additivity*** criterion. Consider an abelian scheme A/U with semi-abelian reduction $\mathcal{A}/S$, where S is a regular base and U the complement in S of a strict normal crossings divisor. Toric-additivity is a condition on the Tate module of A. When A is the generic Jacobian of an S-curve with a nodal model, toric-additivity is sufficient for a separated Néron model of A to exist. It is also necessary up to some restrictions on the base characteristic. For general abelian varieties, it is proven in [Ore19] that toric-additivity is still sufficient when S has equicharacteristic 0, and a partial converse holds; ***i.e.*** the existence of a separated Néron model implies a weaker version of toric-additivity. When S is a toroidal variety and X/S a nodal curve, smooth over the complement U of the boundary divisor, a construction of the Néron model of the Jacobian of X_U is given in [HMO+20], together with a moduli interpretation for it.

Let $g \geq 3$ be an integer. In [Cap08], Lucia Caporaso constructs a "balanced Picard stack" $\mathcal{P}_g^d$, naturally mapping to the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus g. This stack acts as a universal Néron model of the degree d Jacobian for test curves corresponding to regular stable curves, ***i.e.*** if $T \to \overline{\mathcal{M}}_g$ is a morphism from a trait[(2)] such that the corresponding stable curve X/T is regular, then $\mathcal{P}_g^d \times_{\overline{\mathcal{M}}_g} T$ is canonically isomorphic to the Néron model of the degree d Jacobian of the generic fibre of X. The balanced Picard stack does not admit a group structure compatible with that of the Jacobian. In [Hol15], Holmes exhibits an algebraic space $\widetilde{\mathcal{M}}_g$ over $\overline{\mathcal{M}}_g$ which is regular, in which $\mathcal{M}_g$ is dense, over which the universal Jacobian has a separated Néron model, and which is universal with respect to these properties.

1.3. Notation

We will adopt the following conventions:

- If $f : X \to S$ is a morphism of algebraic spaces locally of finite type, we call ***smooth locus*** of f, and denote by $(X/S)^{\text{smooth}}$ (or X^{smooth} if there is no ambiguity), the open subspace of X at which f is smooth. Likewise, the ***étale locus*** $(X/S)^{\text{étale}}$ (or just $X^{\text{étale}}$) of f is the open subspace of X at which f is étale.
- If $f : X \to S$ is a morphism of schemes which is locally of finite presentation and has geometric fibres of pure dimension 1, we call ***singular locus*** of f, and denote by $\mathrm{Sing}(X/S)$, the closed subscheme of X cut out by the first Fitting ideal of the sheaf of relative 1-forms of X/S. The set-theoretical complement of $\mathrm{Sing}(X/S)$ in X is precisely $(X/S)^{\text{smooth}}$.
- Unless specified otherwise, if A is a local ring, we write $\mathfrak{m}_A$ for its maximal ideal, k_A for the its residue field and $\widehat{A}$ for its $\mathfrak{m}_A$-adic completion.
- When M is a monoid, or sheaf of monoids, we write $\overline{M}$ for the quotient of M by its units.

1.4. Structure of the paper, main results

In this article, we work with a nodal curve X/S, smooth over a dense open $U \subset S$, where S is an excellent scheme satisfying certain conditions of local factoriality. We are interested in constructing a Néron model for the Jacobian of X_U.

In Section 2, we start with a general discussion about the base change properties of Néron models, and we show the following result.

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(1) A nodal curve is a proper, flat, finitely presented morphism with geometric fibres of pure dimension 1 and at worst ordinary double point singularities.

(2) A trait is the spectrum of a discrete valuation ring.

Corollary 1.1 (*cf.* Corollary 2.10). *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open subscheme, $N_U \to U$ a smooth, separated U-group algebraic space and $f: N \to S$ a smooth S-group model of N_U. Denote by E the scheme-theoretic closure of the unit section in N and by $E^{\text{étale}}$ the étale locus of E/S. Then, for any smooth S-scheme Y, the sequence of abelian groups*

$$0 \to \mathrm{Hom}(Y, E^{\text{étale}}) \to \mathrm{Hom}(Y, N) \to \mathrm{Hom}(Y_U, N_U)$$

is exact. In particular, the quotient space $N/E^{\text{étale}}$ is a smooth S-group model of N_U with uniqueness in the Néron mapping property.

In Section 3, we present some generalities about nodal curves, their local structure and their dual graphs.

In Section 4, we are interested in *smooth-factorial* schemes, *i.e.* those schemes S such that any smooth S-scheme is locally factorial. We give conditions under which a prime divisor in a smooth-factorial scheme S remains prime in Y for various kinds of morphisms $Y \to S$.

In Section 5, we work with a section $\sigma: S \to X$. We introduce a combinatorial invariant, the *type* of σ at a point $s \in S$. We discuss some properties of this invariant, and we show that there are étale quasisections of every possible type through any singular point of X/S.

In Section 6, we study blow-ups $X' \to X$ in the ideal sheaves of S-sections. We show that X'/S is a nodal curve and that, locally on S, it is characterized by the type of the section. We compute the smoothing parameters of the nodes of X'/S in terms of those of X/S. We show that, étale-locally on S, we can always obtain a model of X_U satisfying strong conditions of local factoriality by repeatedly blowing up X in S-sections. This can be seen as a higher-dimensional variant of the smoothening process of [BLR90]. The reader familiar with logarithmic geometry can establish a parallel between these blow-ups and logarithmic modifications of a log curve inducing a given subdivision of its tropicalization (although one should be careful with this analogy; see Remark 7.8).

In Section 7, we construct the Néron model of the Jacobian. We describe how blowing up a nodal curve in a S-section affects its relative Picard scheme, giving us a "bigger" model of the Jacobian (*cf.* Lemma 7.2). Then, we show that one obtains a Néron model by appropriately quotienting a union of such models. The main result is as follows.

Theorem 1.2 (*cf.* Theorem 7.6). *Let S be a smooth-factorial (e.g. regular) and excellent scheme, and let $U \subset S$ be a dense open subscheme. Let X_U/U be a smooth curve that admits a nodal model over S. Then:*

(1) The Jacobian $J = \mathrm{Pic}^0_{X_U/U}$ of X_U/U admits a Néron model N over S.
(2) For any nodal model X/S of X_U/U, the map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}/E^{\text{étale}} \to N$ extending the identity over U is an open immersion, where E is the scheme-theoretic closure of the unit section in $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$.
(3) For any étale morphism $V \to S$ and nodal V-model X of $X_{U\times_S V}$, if $\bar{s} \to V$ is a geometric point such that the singularities of $X_{\bar{s}}$ have prime thicknesses, then the canonical map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/V} \to N$ is surjective on $\mathrm{Spec}(\mathcal{O}^{\text{ét}}_{S,\bar{s}})$-points.

When X/S comes from a vertical logarithmic curve over a logarithmically regular base (*e.g.* S is regular and X smooth over the complement of a normal crossings divisor, or S is a toroidal variety and X smooth over the complement of the boundary divisor), the Néron model exists and has a moduli interpretation by [HMO+20, Corollary 6.13]. The difference is that we require S to be smooth-factorial but allow the discriminant locus to be arbitrary.

We give a local description of the Néron model in terms of Picard spaces in Remark 7.7. Finally, we give a simple way to determine whether or not the Néron model is separated. Following the ideas of [Hol19], we say X/S is *strictly aligned* when the smoothing parameters of its singularities satisfy a certain combinatorial condition (*cf.* Definition 7.9), and we prove the following result.

Theorem 1.3 (*cf.* Theorem 7.13). *Let S be a regular and excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Denote by J the Jacobian of X_U/U. Then, the S-Néron model N of J exhibited in Theorem 7.6 is separated if and only if X/S is strictly aligned.*

With the notation of Theorem 1.3, when X/S is strictly aligned, the Néron model was already constructed in [Hol19] (see Proposition 3.6 of *loc. cit.*) under the additional assumption that U is the complement of a normal crossings divisor in S. This additional assumption guarantees the existence of a global nodal model X'/S whose total space is regular, in which case the Néron model of the Jacobian is the quotient of $\mathrm{Pic}^{\mathrm{tot}\,0}_{X'/S}$ by the closure of its unit section. In our setting, the phenomenon illustrated by Example 5.14 prevents the existence of such an X' in general, but a separated Néron model of the Jacobian still exists.

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2. Generalities about Néron models

2.1. Definitions

Definition 2.1. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a U-algebraic space. An *S-model* of Z (or just *model* if there is no ambiguity) is an S-algebraic space X together with an isomorphism $X_U = Z$. A *morphism of S-models* between two models X and Y of Z is an S-morphism $X \to Y$ that commutes over U with the given isomorphisms $X_U = Z$ and $Y_U = Z$.

Definition 2.2. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth U-scheme and N an S-model of Z. We say that S has the *Néron mapping property* (resp. *existence in the Néron mapping property*, *uniqueness in the Néron mapping property*) if for each smooth S-algebraic space Y, the restriction map

$$\mathrm{Hom}_S(Y,N) \to \mathrm{Hom}_U(Y_U,Z)$$

is bijective (resp. surjective, injective). If N is S-smooth and has the Néron mapping property, we say that it is an *S-Néron model* of Z (or just *Néron model* if there is no ambiguity).

Remark 2.3. Various authors require Néron models to be separated and of finite type over the base. Néron models without a quasicompactness condition are sometimes referred to as Néron-lft models, where "lft" stands for "locally of finite type". We justify the definition above by observing that our Néron models are still unique up to a unique isomorphism by virtue of the universal property: we can always discuss their separatedness or quasicompactness *a posteriori*.

Remark 2.4. Let S, U, Z be as in Definition 2.2, and let N be a smooth, separated S-model of Z. Consider a smooth S-algebraic space Y/S and two morphisms $f_1, f_2 \colon Y \to N$ that coincide over U. The separatedness of N/S implies that the equalizer of f_1 and f_2 is a closed subspace of Y containing Y_U, and the latter is scheme-theoretically dense in Y by [GD66, théorème 11.10.5]. Thus, N automatically has uniqueness in the Néron mapping property.

Remark 2.5. By descent, the definition of Néron models is unchanged if we only require the Néron mapping property to hold when Y/S is a scheme. Therefore, when we ask if a Néron model exists for X_U/U, we are asking if the functor $Y \mapsto X_U(Y_U)$ on the small smooth site of S is representable by a smooth algebraic space. Similarly, we would get a looser (but still universal) definition of Néron models by asking for them to be smooth algebraic stacks.

2.2. Base change and descent properties

Proposition 2.6 (The formation of Néron models is compatible with smooth base change). *Consider a smooth morphism $S' \to S$, a scheme-theoretically dense open $U \subset S$ and a smooth S-model X of X_U with uniqueness in the Néron mapping property. Then, the base change $X_{S'}$ is a smooth S'-model of $X_{U'}$ with uniqueness in the Néron mapping property. If X is the Néron model of X_U, then X' is the Néron model of $X_{U'}$.*

Proof. First, note that $X_{S'}/S'$ is smooth since X/S is and that U' is scheme-theoretically dense in S' by [GD66, théorème 11.10.5]. Thus, we only need to check that X'/S' has uniqueness in the Néron mapping property and that it has existence if X/S does.

Let Y' be a smooth S'-scheme. A morphism $Y' \to X'$ is uniquely determined by the two projections $Y' \to S'$ and $Y' \to X$. Since $Y' \to S$ is smooth, it follows that X' has uniqueness in the Néron mapping property. Now, suppose that X is the Néron model of X_U, and consider a U'-morphism $u' : Y'_{U'} \to X_{U'}$. Composing with the projection: $X_{U'} \to X_U$, we get a U-morphism $Y'_{U'} \to X_U$, which extends to a unique S-morphism $Y' \to X$ by the Néron mapping property since Y'/S is smooth. Then the induced morphism $Y' \to X'$ extends u'. □

Corollary 2.7. *If S'/S is a cofiltered limit of smooth S-schemes (indexed by a cofiltered partially ordered set, e.g. a localization, a henselization when S is local,...) and X is the S-Néron model of X_U, then $X_{S'}$ is the S'-Néron model of $X_{U'}$.*

Proposition 2.8 (Néron models descend along smooth covers). *Let S be a scheme and U a scheme-theoretically dense open of S. Let $S' \to S$ be a smooth surjective morphism and $U' = U \times_S S'$. Let X_U be a smooth U-algebraic space, and suppose $X_{U'}$ has a S'-Néron model X'. Then X' comes via base change from an S-space X, which is the Néron model of X_U.*

Proof. Denote by p_1, p_2 the two projections $S'' := S' \times_S S' \to S'$. They are smooth morphisms, so by Proposition 2.6, there is a canonical isomorphism $p_1^* X' = p_2^* X'$ satisfying the cocycle condition. By the effectiveness of fppf descent for algebraic spaces (*cf.* [Stacks, Tag 0ADV]), X' comes via base change from an S-algebraic space X/S. The morphism $X \to S$ is smooth since X'/S' is (smoothness is even fpqc local on the base; see [Stacks, Tag 02VL]). Therefore, we only need to prove that X/S has the Néron mapping property. Let Y be a smooth S-algebraic space and f_U be in $\mathrm{Hom}(Y_U, X_U)$. The corresponding map $Y'_U \to X'_U$ extends uniquely to some $f' \colon Y' \to X'$. The two pull-backs of f' to $S'' := S' \times_S S'$ coincide over U; hence they coincide (since $X_{S''}$ has the Néron mapping property by Proposition 2.6). Hence f' comes from a morphism $f : Y \to X$, which is the only one extending f_U. □

2.3. Group models with injective restriction map

In this subsection, we show that any smooth group model of a group algebraic space has a biggest quotient with uniqueness in the Néron mapping property, namely the quotient by the étale locus over the base of its unit section (*cf.* Corollary 2.10). Given a scheme S, we will write S_{sm} (resp. $(\mathrm{Sch}/S)_{\mathrm{sm}}$) for the small smooth site (resp. big smooth site) of S.

We write Ab for the category of abelian groups. We will use without further mention the fact that a smooth S-group algebraic space is determined up to a unique isomorphism by the corresponding functor $(S_{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{Ab}$ (by combining the Yoneda lemma with a descent argument). Recall from Subsection 1.3 that if $f : E \to S$ is a morphism, then $(E/S)^{\text{étale}}$ (or just $E^{\text{étale}}$) denotes the étale locus of f.

Lemma 2.9 (*cf.* [Hol19, Lemma 5.18]). *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open and $f: E \to S$ an S-algebraic space. Suppose that f restricts to an isomorphism over U and that $U \times_S E$ is scheme-theoretically dense in E. Then any section of f factors through $E^{\text{étale}}$.*

Proof. The claim is étale-local on S and E, so we may assume that f is a morphism of affine schemes. In particular, f is separated. Then $S \to E$ is a closed immersion through which U factors, hence an isomorphism. *A fortiori*, f is étale. □

Corollary 2.10. *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open subscheme, $N_U \to U$ a smooth, separated U-group algebraic space and $f: N \to S$ a smooth S-group model of N_U. Denote by E the scheme-theoretic closure of the unit section in N. Then, for any smooth S-scheme Y, the sequence of abelian groups*

$$0 \to \mathrm{Hom}(Y, E^{\text{étale}}) \to \mathrm{Hom}(Y, N) \to \mathrm{Hom}(Y_U, N_U)$$

is exact. In particular, the quotient space $N/E^{\text{étale}}$ is a smooth S-group model of N_U with uniqueness in the Néron mapping property.

Remark 2.11. In the setting of Corollary 2.10, if N has existence in the Néron mapping property, it follows that $N/E^{\text{étale}}$ is the Néron model of N_U.

3. Nodal curves and dual graphs

3.1. First definitions

The results of this subsection mostly either are well-known facts about nodal curves or come from [Hol19]. When the proofs are short enough, we reproduce them for convenience.

Definition 3.1. A *graph* G is an ordered pair of finite sets (V, E), together with a map $f: E \to (V \times V)/\mathcal{S}_2$. We denote by V the set of vertices of G and by E its set of edges. We think of f as the map sending an edge to its endpoints. We call *loop* any edge in the preimage of the diagonal of $(V \times V)/\mathcal{S}_2$. We will often omit f in the notation and write $G = (V, E)$.

Let v, v' be two vertices of G. A *path between v and v' in G* is a finite sequence $(e_1, \dots, e_n)$ of edges such that there are vertices $v_0 = v, v_1, \dots, v_n = v'$ satisfying $f(e_i) = (v_{i-1}, v_i)$ for all $1 \le i \le n$. We denote by n the *length* of the path. A *chain* is a path as above, with positive n, where the only repetition allowed in the vertices $(v_i)_{0 \le i \le n}$ is $v_0 = v_n$. A *cycle* is a chain from a vertex to itself. The cycles of length 1 of G are its loops.

Let M be a monoid. A *labelled graph over M* (or *labelled graph* if there is no ambiguity) is the data of a graph $G = (V, E)$ and a map $l: E \to M \backslash \{0\}$, called *edge-labelling*. The image of an edge by this map is called the *label* of that edge.

Definition 3.2. Let X be an algebraic space. We call *geometric point* of X a morphism $\mathrm{Spec}\, \bar{k} \to X$ where the image of $\mathrm{Spec}\, \bar{k}$ is a point with residue field k, and $\bar{k}$ is a separable closure of k. Given two geometric points s, t of X, we say that t is an *étale generization* of s (or that s is an *étale specialization* of t) when the image of $t \to X$ is a generization of the image of $s \to X$. We will often omit the word "étale" and just call them specializations and generizations.

Definition 3.3. A *curve* over a separably closed field k is a proper, finitely presented morphism $X \to \mathrm{Spec}\, k$ with X of pure dimension 1. It is called *nodal* if it is connected and for every point x of X, either X/k is smooth at x, or x is an ordinary double point (*i.e.* the completed local ring of X at x is isomorphic to $k[[u, v]]/(uv)$).

A *curve* (resp. *nodal curve*) over a scheme S is a proper, flat, finitely presented morphism $X \to S$ whose geometric fibres are curves (resp. nodal curves).

Remark 3.4. By [Liu02, Proposition 10.3.7], our definition of nodal curves is unchanged if one defines geometric points with algebraic closures instead of separable closures.

Definition 3.5. Let S be a scheme, s a point of S and $\bar{s}$ a geometric point mapping to s. We will call *étale neighbourhood of $\bar{s}$ in S* the data of an étale morphism of schemes $V \to S$, a point v of V and a factorization $\bar{s} \to v \to s$ of $\bar{s} \to s$. Étale neighbourhoods naturally form a codirected system, and we call *étale stalk of S at s* the limit of this system. The étale stalk of S at s is an affine scheme, and we call *étale local ring at s*, and denote by $\mathcal{O}^{\text{ét}}_{S,s}$, its ring of global sections. We will sometimes keep the choice of geometric point $\bar{s}$ implicit and abusively write (V,v), or even V, for an étale neighbourhood of s in S.

Remark 3.6. The étale local ring of S at $\bar{s}$ is the strict henselization of the local ring (in the Zariski topology) $\mathcal{O}_{S,s}$ determined by the separable closure $k(s) \to k(\bar{s})$.

3.2. The local structure

Proposition 3.7. *Let S be a locally Noetherian scheme and X/S be a nodal curve. Let s be a geometric point of S and x be a non-smooth point of X_s. There exists a unique principal ideal $T = (\Delta)$ of the étale local ring $\mathcal{O}^{\text{ét}}_{S,s}$ such that*

$$\widehat{\mathcal{O}^{\text{ét}}_{X,x}} \simeq \widehat{\mathcal{O}^{\text{ét}}_{S,s}}[[u,v]]/(uv - \Delta).$$

We call T the thickness *of x. It can be seen as an element of the (multiplicative) monoid $\overline{\mathcal{O}^{\text{ét}}_{S,s}} = \mathcal{O}^{\text{ét}}_{S,s}/(\mathcal{O}^{\text{ét}}_{S,s})^{\times}$.*

Proof. This is [Hol19, Proposition 2.5]. □

Remark 3.8. The element Δ of $\mathcal{O}^{\text{ét}}_{S,s}$ is a nonzerodivisor if and only if X/S is generically smooth in a neighbourhood of x.

3.3. The dual graph at a geometric point

We discuss an important combinatorial object, the *dual graph* of a nodal curve at a geometric point. Throughout the literature, one can find many definitions of dual graphs (or tropicalizations of logarithmic curves), depending on how much information the authors need this object to carry. With a definition slightly heavier than ours, one can construct them functorially in families (see [HMO+20, Section 3]).

Definition 3.9. Let X/S be a nodal curve with S locally Noetherian, and let s be a geometric point of S. We define the *dual graph* of X at s as follows:

- Its vertices are the irreducible components of X_s.
- It has an edge for every singular point x of X_s, whose endpoints are the two (possibly equal) irreducible components containing the two preimages of x in the normalization of X_s.
- It has an edge-labelling by the multiplicative monoid $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$, mapping a singular point to its thickness.

When S is strictly local, we will sometimes refer to the dual graph of X at the closed point as simply "the dual graph of X".

Proposition 3.10. *Let $S' \to S$ be a morphism between locally Noetherian schemes, X/S a nodal curve, s a geometric point of S and s' a geometric point of S' mapping to a generization of s. Let X' be the base change of X to S'. Let Γ (resp. Γ') be the dual graph of X at s (resp. of X' at s').*

Then, Γ' is obtained from Γ by contracting the edges whose labels map to 1 in $M := \overline{\mathcal{O}}^{\text{ét}}_{S',s'}$ and replacing the labels of the other edges by their images in M.

In particular, if s' has image s, then Γ and Γ' are isomorphic as non-labelled graphs, and the labels of Γ' are the images in M of those of Γ.

Proof. This is [Hol19, Remark 2.12]. We re-prove it here.

We can reduce to $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ affine and strictly local,(3) with respective closed points s and s'.

Let x be a singular point of X with image s and $\Delta \in R$ be a lift of its thickness. Then we can choose an isomorphism $\widehat{\mathcal{O}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta)$.

This yields $\widehat{\mathcal{O}_{X,x}} \otimes_R R' = \widehat{R} \otimes_R R'[[u,v]]/(uv-\Delta)$. The ring $\widehat{R} \otimes_R R'$ is local, with completion $\widehat{R'}$ with respect to the maximal ideal: as desired, if Δ is invertible in R', then X' is smooth above a neighbourhood of x, and otherwise, X' has exactly one singular closed point of image x, with thickness $\Delta R'$. □

Example 3.11. With the same notation as above, in the case $S = S'$, we have defined the ***specialization maps*** of dual graphs: if s, ξ are geometric points of S with s specializing ξ, we have a canonical map from the dual graph at s to the dual graph at ξ, contracting an edge if and only if its label becomes the trivial ideal in $\mathcal{O}^{\text{ét}}_{S,\xi}$.

It can be somewhat inconvenient to always have to look at geometric points. We can often avoid it as in [Hol15], by reducing to a case in which the dual graphs already make sense without working étale-locally on the base.

3.4. Quasisplitness, dual graphs at non-geometric points

Definition 3.12 (see [Hol15, Definition 4.1]). We say that a nodal curve $X \to S$ is ***quasisplit*** if the two following conditions are met:

(1) For any point $s \in S$ and any irreducible component E of X_s, there is a smooth section $S \to (X/S)^{\text{smooth}}$ intersecting E.

(2) The singular locus $\operatorname{Sing}(X/S) \to S$ is of the form
$$\coprod_{i \in I} F_i \to S,$$
where the $F_i \to S$ are closed immersions.

Example 3.13. Consider the real conic
$$X = \operatorname{Proj}(\mathbb{R}[x,y,z]/(x^2+y^2)).$$
It is an irreducible nodal curve over $\operatorname{Spec} \mathbb{R}$, but the base change $X_{\mathbb{C}}$ has two irreducible components: X is not quasisplit over $\operatorname{Spec} \mathbb{R}$.

On the other hand, consider the real projective curve
$$Y = \operatorname{Proj}(\mathbb{R}[x,y,z]/(x^3+xy^2+xz^2)).$$
It has two irreducible components (respectively cut out by x and by $x^2+y^2+z^2$), both geometrically irreducible. The singular locus of $Y/\mathbb{R}$ consists of two $\mathbb{C}$-rational points, with projective coordinates $(0:i:1)$ and $(0:-i:1)$, at which $Y_{\mathbb{C}}$ is nodal. Since $\operatorname{Sing}(Y/\mathbb{R})$ is not a disjoint union of $\mathbb{R}$-rational points, Y is not quasisplit over $\operatorname{Spec} \mathbb{R}$. However, both X and Y become quasisplit after base change to $\operatorname{Spec} \mathbb{C}$.

Remark 3.14. Our definition of quasisplitness is slightly more restrictive than that of [Hol15].

Remark 3.15. Let X/S be a quasisplit nodal curve, s a point of S and $\bar{s} \to S$ a geometric point above s. The irreducible components of $X_{\bar{s}}$ are in canonical bijection with those of X_s by the first condition defining quasisplitness, and the thicknesses of X at $\bar{s}$ come from principal ideals of the local ring (in the Zariski topology) $\mathcal{O}_{S,s}$ by the second condition. Therefore, we can define without ambiguity the dual graph of X at s: its vertices are the irreducible components of X_s, its edges are the non-smooth points $x \in X_s$, with

(3) In other words, S and S' are isomorphic to spectra of strictly henselian local rings.

endpoints the two components meeting at x, and the label of x is the preimage in $\overline{\mathcal{O}_{S,s}}$ of the thickness of some (equivalently, any) point above x in a geometric fibre of X/S.

From now on, we will call the label of x defined as above the thickness of X at x and talk freely about the dual graphs of quasisplit curves at (not necessarily geometric) field-valued points of S. This can clash with Definition 3.9 when x is a singular point of a geometric fibre of X/S. Unless specified otherwise, when there is an ambiguity, we will always privilege Definition 3.9.

Lemma 3.16. *Quasisplit curves are stable under arbitrary base change.*

Proof. The two conditions forming quasisplitness are preserved by base change. □

Lemma 3.17. *Let S be a Noetherian strictly local scheme and X/S a nodal curve. Then X/S is quasisplit.*

Proof. There is a section through every closed point in the smooth locus of X/S, so in particular there is a smooth section through every irreducible component of every fibre. Proposition 3.7 implies that the map $\mathrm{Sing}(X/S) \to S$ is a disjoint union of closed immersions. □

Corollary 3.18. *Let S be a locally Noetherian scheme and X/S a nodal curve. Then there is an étale cover $V \to S$ such that X_V/V is quasisplit.*

Lemma 3.19. *Let S be a locally Noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Quasisplitness of X/S gives a factorization*

$$x \to F \to \mathrm{Sing}(X/S) \to X \to S,$$

where $F \to S$ is a closed immersion and $F \to \mathrm{Sing}(X/S)$ is the connected component containing x. Then, there exist an étale neighbourhood (V, y) of x in X, two effective Cartier divisors C, D on V and an isomorphism $V \times_X F = C \times_V D$ such that $V \times_S F$ is the union of C and D.

Proof. Let $\bar{s}$ be a geometric point of S mapping to s, and $\bar{x} = x \times_s \bar{s}$. By Proposition 3.7, we have an isomorphism $\widehat{\mathcal{O}^{\text{ét}}_{X,\bar{x}}} = \widehat{\mathcal{O}^{\text{ét}}_{S,\bar{s}}}[[u,v]]/(uv - \Delta)$, where Δ is a lift to $\mathcal{O}^{\text{ét}}_{S,\bar{s}}$ of the thickness of x. The base change of F/S to $\mathrm{Spec}\,\mathcal{O}^{\text{ét}}_{S,\bar{s}}$ is cut out by Δ, and the zero loci C_u of u and C_v of v are effective Cartier divisors on $\widehat{\mathcal{O}^{\text{ét}}_{X,\bar{x}}}$, intersecting in $\widehat{\mathcal{O}^{\text{ét}}_{X,\bar{x}}}/(u,v) = F \times_X \mathrm{Spec}\,\widehat{\mathcal{O}^{\text{ét}}_{X,\bar{x}}}$. The union of C_u and C_v is $\widehat{\mathcal{O}^{\text{ét}}_{S,\bar{s}}}[[u,v]]/(\Delta, uv)$, so the proposition follows by a limit argument. □

4. Primality, local factoriality and base change

4.1. Smooth-factorial schemes

The main results of this article hold when the base scheme S is quasiexcellent, locally Noetherian and smooth-factorial (*cf.* Definition 4.2). In this subsection, we discuss smooth-factoriality and try to give some intuition for it.

Lemma 4.1 (Popescu's theorem). *Let R be a Noetherian and excellent local ring; then $\widehat{R}$ is a directed colimit of smooth R-algebras.*

Proof. This is a special case of [Stacks, Tag 07GC]. □

Definition 4.2. Let S be a scheme. We say that S is *smooth-factorial* (resp. *étale-factorial*) if any smooth (resp. étale) S-scheme is locally factorial.

Remark 4.3. Any regular scheme S is smooth-factorial.

Lemma 4.4 (*cf.* [Dan70, Proposition 1]). *Let $R \to R'$ be a faithfully flat morphism of Noetherian, integrally closed local rings. Then any ideal $p \subset R$ is principal if $p \otimes_R R'$ is. In particular, if R' is a unique factorization domain, then R is a unique factorization domain.*

Proof. By faithfully flat descent, p is a finitely generated projective R-module of rank 1, so it is principal. □

Corollary 4.5. *Let S be a normal and locally Noetherian scheme. Then S is étale-factorial if and only if all of its étale local rings are unique factorization domains.*

Remark 4.6. In view of Corollary 4.5, étale-factoriality is a relatively easy condition to understand and verify, but smooth-factoriality is *a priori* harder to grasp. It seems reasonable to hope that they are equivalent under mild assumptions (*e.g.* local Noetherianity). By Lemma 4.7, proving this reduces to showing that given a Noetherian, strictly henselian, local unique factorization domain R, the strict localizations of $R[X]$ at $\mathfrak{m}_R$ and at $\mathfrak{q} := (\mathfrak{m}_R, T)$ are unique factorization domains. In [Dan70], Vladimir Danilov states the related conjecture that $R[[X]]$ must be a unique factorization domain. When R is excellent, the equivalent claims in Lemma 4.7 imply Danilov's conjecture by Lemma 4.1. Conversely, Danilov's conjecture implies that $(R[X]_{\mathfrak{q}})^{\mathrm{sh}} = (R[X]_{\mathfrak{q}})^{\mathrm{h}}$ is a unique factorization domain by Lemma 4.4.

Lemma 4.7. *The following three claims are equivalent:*

(1) A locally Noetherian scheme S is smooth-factorial if and only if it is étale-factorial.
(2) If S is a locally Noetherian étale-factorial scheme, then $\mathbb{A}^1_S$ is étale-factorial.
(3) If R is a strictly henselian, Noetherian local ring with $\operatorname{Spec} R$ étale-factorial, then the strict localizations of $R[X]$ at the prime ideals $\mathfrak{m}_R$ and at $(\mathfrak{m}_R, T)$ are unique factorization domains.

Proof. We clearly have (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3). Any smooth morphism of schemes $Y \to S$ factors locally as $Y \to \mathbb{A}^n_S \to S$, where $Y \to \mathbb{A}^n_S$ is étale, so by induction we have (2) $\Longrightarrow$ (1). For (3) $\Longrightarrow$ (2), suppose that (3) holds, and consider a locally Noetherian, étale-factorial scheme S. It suffices to show that the étale local ring of $\mathbb{A}^1_S$ at an arbitrary point y is a unique factorization domain. Denote by s the image of y in S. By Lemma 4.4, we may assume that $S = \operatorname{Spec} R$ is strictly local, with closed point s. Translating by a S-section of $\mathbb{A}^1_S$ if necessary, we may assume that the image of y in $\mathbb{A}^1_s$ is either the generic point or the origin. Therefore, y corresponds to one of the ideals $\mathfrak{m}_R$ and $(\mathfrak{m}_R, T)$ of $R[T]$, and we are done. □

Remark 4.8. The claims of Lemma 4.7 remain equivalent if one removes the Noetherianity assumptions in all three of them.

4.2. Permanence of primality under certain morphisms of local rings

Let R be a strictly henselian local ring such that $\widehat{R}$ is a unique factorization domain, let $X/\operatorname{Spec} R$ be a nodal curve, and let x be a singular closed point of X whose thickness is prime in $\widehat{R}$. We will see in Lemma 6.2 that X is locally factorial at x. Since a generic line bundle on a locally factorial scheme always extends to a line bundle, it follows that, in order to construct Néron models of Jacobians, we are interested in questions of permanence of primality (of an element of a smooth-factorial ring) under smooth maps, étale maps and completions. We discuss these matters in this section.

An element Δ of an integral local ring R is prime in R^{sh} when the quotient $R/(\Delta)$ is *geometrically unibranch* (*i.e.* its strict henselization is integral, or, equivalently, its normalization is local with purely inseparable residue extension), so we are interested in questions of permanence of geometrically unibranch rings under tensor product. For a more detailed discussion on unibranch rings or counting branches in general, see [Ray70, chapitre IX] or [GD64, section 23.2].

In [Swe75], Moss Eisenberg Sweedler gives a necessary and sufficient condition for the tensor product of two local algebras over a field to be local. We are interested in a sufficient condition for algebras over a strictly local ring. The proof of [Swe75] carries over without much change: we reproduce it here.

Lemma 4.9. *Let R be a strictly henselian local ring, $R \to A$ an integral morphism of local rings with purely inseparable residue extension and $R \to B$ any morphism of local rings. Then $A \otimes_R B$ is local, and its residue field is purely inseparable over that of B.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $A \otimes_R B$. The map $B \to A \otimes_R B$ is integral, so it has the going-up property (*cf.* [Stacks, Tag 00GU]); therefore, the inverse image of $\mathfrak{m}$ in B is a maximal ideal: it must be $\mathfrak{m}_B$. Thus $\mathfrak{m}$ contains $A \otimes_R \mathfrak{m}_B$.

In particular, $\mathfrak{m}$ also contains the image of $\mathfrak{m}_R$ in $A \otimes_R B$: it corresponds to a maximal ideal of $(A \otimes_R B)/(\mathfrak{m}_R A \otimes_R B)$, which we will still call $\mathfrak{m}$. We have a commutative diagram

$$\begin{tikzcd} k_R \arrow[r] \arrow[d] & B/\mathfrak{m}_R B \arrow[d] \\ A/\mathfrak{m}_R A \arrow[r] & (A \otimes_R B)/(\mathfrak{m}_R A \otimes_R B). \end{tikzcd}$$

Since $A/\mathfrak{m}_R A$ is local and integral over the field k_R, its maximal ideal m_A is nilpotent and is its only prime ideal. The inverse image of $\mathfrak{m}$ in $A/\mathfrak{m}_R A$ is a prime ideal, so it can only be $\mathfrak{m}_A$. This shows that, as an ideal of $A \otimes_R B$, $\mathfrak{m}$ also contains $\mathfrak{m}_A \otimes_R B$.

Every maximal ideal of $A \otimes_R B$ contains both $\mathfrak{m}_A \otimes_R B$ and $A \otimes_R \mathfrak{m}_B$, so the maximal ideals of $A \otimes_R B$ are in bijective correspondence with those of $k_A \otimes_{k_R} k_B = A \otimes_R B/(\mathfrak{m}_A \otimes_R B + A \otimes_R \mathfrak{m}_B)$. We will now show that the latter is local, with purely inseparable residue extension over k_B.

By hypothesis, the extension k_A/k_R is purely inseparable. If k_R has characteristic 0, then $k_A = k_R$, and we are done. Suppose k_R has characteristic $p > 0$. For any $x \in k_A \otimes_{k_R} k_B$, we can write x as a finite sum $\sum_{i \in I} \lambda_i \otimes \mu_i$ with the λ_i, μ_i in k_A, k_B, respectively. There is an integer $N > 0$ such that for all i, $\lambda_i^{p^N}$ is in k_R. Therefore, $x^{p^N} = \sum_{i \in I} \lambda_i^{p^N} \mu_i^{p^N}$ is in k_B, so x is either nilpotent or invertible. It follows that $k_A \otimes_{k_R} k_B$ is local, with maximal ideal its nilradical, and that its residue field is purely inseparable over k_B, as claimed. □

Lemma 4.10. *Let $(R, \mathfrak{m})$ be an integral and strictly local Noetherian ring. Let $R \to R'$ be a smooth ring map, let $\mathfrak{p}$ be a prime ideal of R' containing $\mathfrak{m}R'$, and let $(R'_p)^{\mathrm{sh}}$ be a strict henselization of $R'_\mathfrak{p}$. Then $R'_\mathfrak{p}$ is geometrically unibranch; i.e. $(R'_p)^{\mathrm{sh}}$ is an integral domain.*

Proof. We know $(R'_p)^{\mathrm{sh}}$ is reduced since it is a filtered colimit of smooth R-algebras. Let B, B' be the integral closures of $R, R'_\mathfrak{p}$ in their respective fraction fields. The ring $R'_\mathfrak{p}$ is integral, so by [Ray70, corollaire IX.1], $(R'_p)^{\mathrm{sh}}$ is an integral domain if and only if B' is local and the extension of residue fields of $R'_\mathfrak{p} \to B'$ is purely inseparable. But any smooth base change of B/R remains normal (see [Liu02, Corollary 8.2.25]), so $B \otimes_R R'_\mathfrak{p}$ is normal as a filtered colimit of normal B-algebras. Moreover, any normal algebra over $R'_\mathfrak{p}$ must factor through $B \otimes_R R'_\mathfrak{p}$, so we have $B' = B \otimes_R R'_\mathfrak{p}$. Applying Lemma 4.9, we find that B' is local and the extension of residue fields of $R'_\mathfrak{p} \to B'$ is purely inseparable, which concludes the proof. □

Corollary 4.11. *Let S be a smooth-factorial scheme and $Y \to S$ a smooth morphism, and consider a commutative square*

$$\begin{tikzcd} y \arrow[r] \arrow[d] & s \arrow[d] \\ Y \arrow[r] & S, \end{tikzcd}$$

where $s \to S$ and $y \to Y$ are geometric points. Then for any prime element Δ of $\mathcal{O}^{\text{ét}}_{S,s}$, the image of Δ in $\mathcal{O}^{\text{ét}}_{Y,y}$ is prime.

Proof. Base change to $\operatorname{Spec} \mathcal{O}^{\text{ét}}_{S,s}/(\Delta)$, replace Y by an affine neighbourhood of y in Y, and apply Lemma 4.10. □

Lemma 4.12. *Let R be a strictly henselian and excellent local ring. Then an element Δ of R is prime in R if and only if it is prime in $\widehat{R}$.*

Proof. The nontrivial implication is the "only if" part. Suppose Δ is prime in R. By Lemma 4.1, $\widehat{R}$ is a directed colimit of smooth R-algebras. Therefore, $R/(\Delta) \to \widehat{R}/(\Delta)$ is a colimit of smooth $R/(\Delta)$-algebras, and we conclude by Lemma 4.10. □

Lemma 4.13. *Let S be an excellent and smooth-factorial scheme, X/S a S-scheme of finite presentation, $\bar{s}$ a geometric point of S and x a closed point of $X_{\bar{s}}$ with an isomorphism $\widehat{\mathcal{O}^{\text{ét}}_{X,x}} = \widehat{\mathcal{O}^{\text{ét}}_{S,\bar{s}}}[[u,v]]/(uv - \Delta)$ for some $\Delta \in \mathfrak{m}_{\bar{s}} \subset \mathcal{O}^{\text{ét}}_{S,\bar{s}}$. For every $t_1, t_2 \in \mathfrak{m}_{\bar{s}}$ such that $t_1 t_2 = \Delta$, there exist an étale neighbourhood $S' \to S$ of $\bar{s}$ and a section $S' \to X$ through x such that the induced map $\widehat{\mathcal{O}^{\text{ét}}_{X,x}} \to \widehat{\mathcal{O}^{\text{ét}}_{S',\bar{s}}}$ sends u, v to generators of (t_1) and (t_2), respectively.*

Proof. Put $R = \mathcal{O}^{\text{ét}}_{S,\bar{s}}$. Then $\widehat{R}$ is a unique factorization domain by Lemma 4.1. Consider the map $\widehat{\mathcal{O}^{\text{ét}}_{X,x}} \to \widehat{R}$ that sends u, v to t_1, t_2, respectively. Compose it with $\mathcal{O}^{\text{ét}}_{X,x} \to \widehat{\mathcal{O}^{\text{ét}}_{X,x}}$ to get a map $f_0 : \mathcal{O}^{\text{ét}}_{X,x} \to \widehat{R}$.

For Noetherian local rings, quotients commute with completion with respect to the maximal ideal, so two distinct ideals are already distinct modulo some power of the maximal ideal. Let $\prod_{i=1}^n \Delta_i^{\nu_i}$ be the prime factor decomposition of Δ in R. Principal ideals of R of the form $(\Delta_i^{\mu_i})$ with $0 \le \mu_i \le \nu_i$ are pairwise distinct and in finite number, so there exists some $N \in \mathbb{N}$ such that their images in $R/\mathfrak{m}_R^N$ are pairwise distinct. Since R is henselian and excellent, it has the Artin approximation property (*cf.* [Stacks, Tag 07QY]), so there exists a map $f : \mathcal{O}^{\text{ét}}_{X,x} \to R$ which coincides with f_0 modulo m_R^N. This f induces a map $\widehat{f} : \widehat{\mathcal{O}^{\text{ét}}_{X,x}} \to \widehat{R}$. Denote by a, b the respective images of u, v by $\widehat{f}$; we have $a = t_1$ and $b = t_2$ in $R/\mathfrak{m}_R^N$. But $ab = \Delta$ in $\widehat{R}$ and, by Lemma 4.12, Δ has the same prime factor decomposition in R and $\widehat{R}$, so the only principal ideals of $\widehat{R}$ containing Δ are of the form $(\Delta_i^{\mu_i})$ with $0 \le \mu_i \le \nu_i$. By the definition of N, we get $a\widehat{R} = t_1\widehat{R}$ and $b\widehat{R} = t_2\widehat{R}$. Since X/S is finitely presented, f comes from an S-morphism $S' \to X$, where S' is an étale neighbourhood of $\bar{s}$ in S. □

5. Sections of nodal curves

We present some technicalities regarding sections of nodal curves, with a view towards studying those morphisms that are locally the blow-up in the ideal sheaf of a section. The basis for this formalism was thought of together with Giulio Orecchia.

5.1. Type of a section

We will define a combinatorial invariant, the *type* of a section, summarizing information about the behaviour of the said section around the singular locus of a nodal curve X/S. Later on, we will show that sections of all types exist étale-locally on the base (*cf.* Proposition 5.12) and that the type of a section locally characterizes the blow-up of X in the ideal sheaf of that section (Corollary 6.6).

Definition 5.1. Let S be a locally Noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let F be the connected component of $\mathrm{Sing}(X/S)$ containing x. Then the set of connected components of $(X \backslash F) \times_X \operatorname{Spec} \mathcal{O}^{\text{ét}}_{X,x} \times_S F$ is a pair $\{C, D\}$ (see Proposition 3.7 and Lemma 3.19), on which the Galois group $\mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}^{\text{ét}}_{S,s}) = \mathrm{Gal}(k(s)^{\text{sep}}/k(s))$ acts. If this action is trivial, we say X/S is *orientable at* x, and we call *orientations of* X/S *at* x the ordered pairs (C, D) and (D, C). The scheme-theoretic closures of C and D in $\operatorname{Spec} \mathcal{O}^{\text{ét}}_{X,x}$ are effective Cartier divisors, and we will often also call them C and D.

If X/S is orientable at a singular point x, and if x' is a singular point specializing to x, then X/S is orientable at x' and there is a canonical bijection between orientations at x and at x'. Given an orientation (C_1, C_2) at x, we will often still write (C_1, C_2) for the induced orientation at x'.

We say X/S is *orientable* if it admits orientations at all points, compatible with the generization isomorphisms between orientations. In that case, we call *global orientation* (or just *orientation*) of X/S a compatible family $(C_{1,x}, C_{2,x})_{x\in \mathrm{Sing}(X/S)}$, where $(C_{1,x}, C_{2,x})$ is an orientation at x. We will often abusively write global orientations as ordered pairs (C_1, C_2) and confuse them with the induced orientation at any given point $x \in \mathrm{Sing}(X/S)$.

Remark 5.2. If X/S is orientable, then it is orientable at every point, but the converse is not true in general.

Remark 5.3. The curve X/S is orientable at x if and only if the preimage of x in the normalization of X_s consists of two $k(s)$-rational points, in which case an orientation is the choice of one of these points. Roughly speaking, this also corresponds to picking an orientation of the edge corresponding to x in the dual graph of X at s. The "roughly speaking" is due to the case of loops: there is an ambiguity on how to orient them. We could get rid of this ambiguity by using a heavier notion of dual graphs (such as the tropical curves often used in log geometry), but this work does not require it.

Lemma 5.4. *Let S be a locally Noetherian scheme and X/S a quasisplit nodal curve. Then, there exists an étale cover $V \to S$ such that X_V/V is orientable.*

Proof. It suffices to show that any $s \in S$ has an étale neighbourhood over which X is orientable, which follows from observing that X/S is finitely presented and that a nodal curve over a strictly local scheme is orientable. □

Lemma 5.5. *Let $S' \to S$ be a morphism between locally Noetherian schemes. Let X/S be a quasisplit nodal curve. If X/S is orientable at a point $x \in X$, then $X' := X \times_S S'$ is orientable at any singular point x' above x, and orientations of X at x naturally pull back to orientations of X' at x'. In particular, if X/S is orientable, then X'/S' is orientable.*

Proof. This follows from Remark 5.3. □

Definition 5.6 (Type of a section). Let X/S be a quasisplit nodal curve with S smooth-factorial. Suppose X is smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s. We call *type at* x any element of the monoid $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$ strictly comprised between 1 and the thickness of x (for the order induced by divisibility). There are only finitely many types at x.

Suppose that X/S admits a global orientation (C_1, C_2). Pick an isomorphism

$$\widehat{\mathcal{O}^{\text{ét}}_{X,x}} = \widehat{\mathcal{O}^{\text{ét}}_{S,s}}[[u,v]]/(uv - \Delta_x),$$

where C_1 corresponds to $u = 0$ and $\Delta_x \in \mathcal{O}^{\text{ét}}_{S,s}$ maps to the thickness of x in $\overline{\mathcal{O}}^{\text{ét}}_{S,s}$. Let σ be a section of X/S through x. It induces a morphism

$$\widehat{\sigma^{\#}} : \widehat{\mathcal{O}^{\text{ét}}_{S,s}}[[u,v]]/(uv - \Delta_x) \to \widehat{\mathcal{O}^{\text{ét}}_{S,s}}.$$

By Lemma 4.12, Δ_x has the same prime factor decomposition in $\mathcal{O}^{\text{ét}}_{S,s}$ and in $\widehat{\mathcal{O}^{\text{ét}}_{S,s}}$, so there is a canonical embedding of the submonoid of $\widehat{\mathcal{O}^{\text{ét}}_{S,s}}/(\widehat{\mathcal{O}^{\text{ét}}_{S,s}})^*$ generated by the factors of Δ_x into $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$. We call *type of* σ *at* x *relative to* (C_1, C_2) the image of u in $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$. It is a type at x and does not depend on our choice of isomorphism $\widehat{\mathcal{O}^{\text{ét}}_{X,x}} = \widehat{\mathcal{O}^{\text{ét}}_{S,s}}[[u,v]]/(uv - \Delta_x)$ as long as C_1 is given by $u = 0$. When they are clear from context, we will omit x and (C_1, C_2) from the notation and just call it the *type of* σ. In general, given a type T at x, there need not exist a section of type T.

Lemma 5.7. *Let X/S, s, x, (C_1, C_2) and U be as in Definition 5.6 and σ be a section $S \to X$ of type T at x. Let s' be a generization of s. Then there is a singular point of $X_{s'}$ specializing to x if and only if the thickness of x does not map to 1 in $\overline{\mathcal{O}^{\text{ét}}_{S,s'}}$. Suppose it is the case, and write x' for this singular point. Then:*

- *If the image of T in $\overline{\mathcal{O}^{\text{ét}}_{S,s'}}$ is either 1 or the thickness of x', then $\sigma(s')$ is a smooth point of $X_{s'}$.*

- *Otherwise, the image of T is a type at s', which we still denote by T, and σ is of type T at x' relative to (C_1, C_2).*

Proof. By Proposition 3.7, the set of non-smooth points of $X_{s'}/s'$ specializing to x is empty if the thickness of x maps to 1 in $\overline{\mathcal{O}^{\text{ét}}_{S,s'}}$, and it is a singleton $\{x'\}$ otherwise. Suppose the latter holds; then we conclude using Proposition 3.7 and the definition of the type of a section. □

Remark 5.8. One can think of the thickness of x as the relative version of a length, and of the type of a section σ relative to an orientation (C_1, C_2) as a measure of the intersection of σ with C_1, seen as an effective Cartier divisor locally around x as in Lemma 3.19. In other words, the type is a measure of "how close to C_1" the section is.

Proposition 5.9. *Let S be a smooth-factorial and excellent scheme, and let X/S be a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let σ and σ' be two S-sections of X. Then the union of $(X/S)^{\text{smooth}}$ with the set of singular points x of X/S at which σ and σ' have the same type is an open subscheme of X, which we call the* same type locus *of σ and σ'.*

Proof. Since the smooth locus of X/S is open in X, the proposition reduces to showing that if σ and σ' have the same type at a singular point x of X, then they have the same type at every singular point in an open neighbourhood of x. Let s be the image of x in S. Pick an isomorphism

$$\widehat{\mathcal{O}^{\text{ét}}_{X,x}} = \widehat{\mathcal{O}^{\text{ét}}_{S,s}}[[u,v]]/(uv - \Delta_x),$$

where $\Delta_x \in \mathcal{O}_{S,s}$ is a lift of the thickness of x. By hypothesis, the images Δ, Δ' of u under the two morphisms $\widehat{\mathcal{O}^{\text{ét}}_{X,x}} \to \widehat{\mathcal{O}^{\text{ét}}_{S,s}}$ given by σ and σ' are the same up to a unit $\lambda \in \widehat{\mathcal{O}^{\text{ét}}_{S,s}}^{\times}$. By Lemma 4.1, base changing to a smooth neighbourhood of s in S, we may assume that $\Delta, \Delta', \Delta_x$ come from global sections of $\mathcal{O}_S$ and λ from a global section of $\mathcal{O}_S^{\times}$. Pick a smooth neighbourhood W of x in X such that u, v come from global sections of W. Shrinking S, we may assume that σ, σ' factor through W and that their comorphisms map u to Δ and Δ', respectively. Shrinking further, we may assume that the non-smooth locus of W/S is cut out by (u, v, Δ_x). Then, the image of W in X is a Zariski open neighbourhood of x contained in the same type locus of σ and σ'. □

5.2. Admissible neighbourhoods

Here we show that when one works étale-locally on the base (in a sense that we will make precise), one can always assume that sections of all types exist.

Definition 5.10. Let S be a smooth-factorial scheme and X/S a nodal curve, smooth over a dense open U of S. Let s be a point of S and (V, v) an étale neighbourhood of s in S. We say that (V, v) is an *admissible neighbourhood of s* (relative to X/S) when the following conditions are met:

(1) The curve X_V/V is quasisplit and orientable.
(2) For any singular point x of X_v, every prime factor in $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$ of the thickness of x lifts to a global section of $\mathcal{O}_V$.
(3) For every singular point x of X_v (however oriented), there are sections $V \to X_V$ of all types at x.

When $\bar{s} \to S$ is a geometric point with image s and (V, v) an admissible neighbourhood of s with a factorization $\bar{s} \to v$, we will also sometimes call V an *admissible neighbourhood of $\bar{s}$.*

Remark 5.11. In the situation of Definition 5.10, if S is strictly local, then it is an admissible neighbourhood of its closed point.

Proposition 5.12. *Let X/S be a nodal curve, where S is a smooth-factorial and excellent scheme. Then any point $s \in S$ has an admissible neighbourhood.*

Proof. All three conditions in the definition of admissibility are stable under base change to an étale neighbourhood of s. We can assume that X/S is quasisplit by Corollary 3.18 and that it is orientable by Lemma 5.4. Let $x_1, \dots, x_n$ be the singular points of X_s. The thickness of each x_i has only finitely many prime factors in $\overline{\mathcal{O}^{\text{ét}}_{S,s}}$, so we can shrink S again into a neighbourhood satisfying condition (2) of the definition of admissibility. The fact that this V can be shrunk again until it meets all three conditions follows from Lemma 4.13. □

Remark 5.13. If (V, v) is an admissible neighbourhood of a point s of S, then V is not necessarily admissible even at generizations of v (see Example 5.14). Thus, it is not easy ***a priori*** to find a good global notion of admissible cover.

Example 5.14. Let $R = \operatorname{Spec} \mathbb{C}[[u, v, w]]$. Then R is regular (hence smooth-factorial), local and strictly henselian. The element $\Delta := u^2(v - w) - v^2(u + w)$ is prime in R. Let X_K be the elliptic curve over $K := \operatorname{Frac} R$ cut out in $\mathbb{P}^2_K$ by the equation $y^2 = (x-1)(x^2 - \Delta)$ (in affine coordinates x, y). The minimal Weierstrass model of X_K is a nodal curve X over $\operatorname{Spec} R$, whose closed fibre has exactly one singular point with label Δ. Let t be the point of $S := \operatorname{Spec} R$ corresponding to the prime ideal (u, v) of R. Then Δ has two prime factors in the étale local ring of S at t (because the invertible elements $v - w$ and $u + w$ of this étale local ring admit square roots). Hence, S is an admissible neighbourhood of its closed point but not of t (relative to X/S).

The next proposition states that admissible neighbourhoods behave well with respect to the ***smooth*** topology.

Proposition 5.15. *Let S be a smooth-factorial scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let $\pi \colon Y \to S$ be a smooth morphism, y a point of Y and $s = \pi(y)$. Let V be an admissible neighbourhood of s in S; then $V \times_S Y$ is an admissible neighbourhood of y in Y.*

Proof. This follows from Corollary 4.11 and the definition of admissible neighbourhoods. □

6. Relating different nodal models

This section is dedicated to constructing inductively nodal models of a smooth curve with prime thicknesses, starting from any nodal model.

6.1. Arithmetic complexity and motivation for refinements

Definition 6.1. Let M be the free commutative monoid over a set of generators G. We call ***arithmetic complexity*** of $m \in M \backslash \{0\}$ the integer $n-1$, where n is the (unique) $n \in \mathbb{N}^*$ such that we can write $m = \prod_{i=1}^n g_i$ with all the g_i in G. Given a graph Γ labelled by M, we define the arithmetic complexity of an edge to be that of its label, and the arithmetic complexity of Γ to be the sum of the arithmetic complexities of its edges. Given a nodal curve X/S, where $S = \operatorname{Spec} R$ is a local unique factorization domain, the monoid $\overline{R}$ of principal ideals of R is freely generated by the prime principal ideals. From now on, when we talk about arithmetic complexities of edges of dual graphs, we will always be referring to this set of generators. We define the arithmetic complexity of X at a geometric point $s \to S$ as that of its dual graph at s and the arithmetic complexity of a singular point $x \in X_s$ as that of the corresponding edge. We give similar definitions when X/S is quasisplit and $s \in S$ is a point.

Note that X has arithmetic complexity 0 if and only if every point has prime thickness: it is an integer measuring "how far away from being prime" the thicknesses are.

The following lemma essentially shows that nodal curves are locally factorial around their singular points that have prime thicknesses. In particular, any generic line bundle extends locally around such a point.

Lemma 6.2. *Let R be a complete and local unique factorization domain and Δ be an element of $\mathfrak{m}_R$. Then $\widehat{A} := R[[u,v]]/(uv-\Delta)$ is a unique factorization domain if and only if Δ is prime in R.*

Proof. Suppose that $\widehat{A}$ is a unique factorization domain, and let d be a prime factor of Δ in R. Denote by S the complement of the prime ideal (u,d) in $\widehat{A}$. Let $\mathfrak{p}$ be a nonzero prime ideal of $S^{-1}\widehat{A}$. Then, p contains a nonzero element $x = ux_u + x_v$, with x_u and x_v in $R[[u]]$ and $R[[v]]$, respectively. Since $\mathfrak{p} \neq S^{-1}\widehat{A}$, we have $d|x_v$. Let n and m be the maximal elements of $\mathbb{N}^* \cup \{+\infty\}$ such that $u^n|ux_u$ and $d^m|x_v$. Since x is nonzero, we know that either n or m is finite. If $n \leq m$, then $v^n x = \Delta^n \frac{x_u}{u^{n-1}} + d^n \frac{v^n x_v}{d^n}$ is in $\mathfrak{p}$ and is associated to d^n in $S^{-1}\widehat{A}$, so we obtain $d \in \mathfrak{p}$, from which it follows that $\mathfrak{p} = (u,d)$. If $m < n$, a similar argument shows that $\mathfrak{p}$ contains u^m and thus equals (u,d). Therefore, $S^{-1}\widehat{A}$ has Krull dimension 1, *i.e.* (u,d) has height 1 in $\widehat{A}$. Since $\widehat{A}$ is a unique factorization domain, it follows that (u,d) is principal in it, from which we deduce that Δ and d are associated in $\widehat{A}$. In particular, Δ is prime in R.

The interesting part is the converse: let us assume that Δ is prime in R. We want to show that $\widehat{A}$ is a unique factorization domain. We first prove that $A := R[u,v]/(uv-\Delta)$ is a unique factorization domain: let p be a prime ideal of A of height 1; we have to show that p is principal in A. We observe that u is a prime element of A since the quotient $A/(u) = R/(\Delta)[v]$ is an integral domain. Therefore, if p contains u, then $p = (u)$ is principal. Otherwise, p gives rise to a prime ideal of height 1 in $A_u := A[u^{-1}]$, which is principal since $A[u^{-1}] = R[u,u^{-1}]$ is a unique factorization domain. In that case, write $pA_u = fA_u$ for some $f \in A_u$. Multiplying by a power of the invertible element u of A_u, we can choose the generator f to be in $A \backslash uA$. Since p is a prime ideal of A not containing u, we know p contains f and thus fA. We will now prove the reverse inclusion. Let x be an element of p. The localization $pA_u = fA_u$ contains x, so x satisfies a relation of the form $u^n x = fy$ for some $n \in \mathbb{N}$ and some $y \in A$. But since u is prime in A, we know that u^n divides y and x is in fA.

Now, we will deduce the factoriality of $\widehat{A}$ from that of A. The author would like to thank Ofer Gabber for providing the following proof. Let q be a prime ideal of $\widehat{A}$ of height 1; we will show that q is principal. We put $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $\widehat{X} = \operatorname{Spec} \widehat{A}$ and $Z = \operatorname{Spec}(\widehat{A}/q)$, so that Z is a prime Weil divisor on $\widehat{X}$. Let η, η' be the generic points of the respective zero loci of u,v in the closed fibre $\operatorname{Spec} k_R[[u,v]]/(uv)$ of $\widehat{X} \to S$. Since u and v are prime elements of $\widehat{A}$, we can once again assume that Z contains neither η nor η'. It follows that the closed fibre of $Z \to S$ is of dimension 0: the morphism $Z \to S$ is quasifinite, hence finite by [GD60, section 0.7.4]. A fortiori, $\widehat{A}/q$ is finite over A, so by Nakayama's lemma, the morphism $A \to \widehat{A}/q$ is surjective. Denote by p its kernel. Then $A/p = \widehat{A}/q$ is $\mathfrak{m}_A$-adically complete and separated, so it maps isomorphically to its completion $\widehat{A}/p\widehat{A}$. The prime ideal p of A is of height 1 since $\widehat{X} \to X$ is a flat map of normal Noetherian schemes. Therefore, p is principal in A, and $q = p\widehat{A}$ is principal in $\widehat{A}$. □

6.2. Refinements of graphs

In order to reap the benefits from the properties of nodal models with prime labels, all we need is an algorithm that takes an arbitrary nodal model as an input and returns another one with strictly lower arithmetic complexity.

Definition 6.3. As in [Hol19, Definition 3.2], for a graph $\Gamma = (V,E,l)$ with edges labelled by elements of a monoid M, we call *refinement* of Γ the data of another labelled graph $\Gamma' = (V',E',l')$ labelled by M and two maps

$$E' \to E,$$
$$V' \to E \coprod V$$

such that:

- Every vertex v in V has a unique preimage v' in V';

- For every edge $e \in E$ with endpoints $v_1, v_2 \in V$, there is a chain $C(e)$ from v_1', v_2' in Γ' such that the preimage of $\{e\}$ in $V' \coprod E'$ consists of all edges and intermediate vertices of $C(e)$;
- For all $e \in E$, the length of e is the sum of the lengths of all edges of $C(e)$.

We will often keep the maps implicit in the notation, in which case we call Γ' a *refinement* of Γ and write $\Gamma' \leq \Gamma$. We say Γ' is a *strict refinement* of Γ and write $\Gamma' < \Gamma$ if, in addition, the map $E' \to E$ is not bijective.

Remark 6.4. Informally, a refinement of a graph is obtained by "replacing every edge by a chain of edges of the same total length". Suppose $\Gamma' \leq \Gamma$; then $\Gamma' < \Gamma$ if and only if at least one of the chains $C(e)$ is of length at least 2, *i.e.* if and only if Γ' has strictly more edges than Γ.

Now we want to blow up X in a way that does not affect X_U but refines the dual graph. We will define *refinements* of curves (*cf.* Definition 6.7). We will show that, étale-locally on the base, any refinement of a dual graph of X comes from a refinement of curves.

6.3. Refinements of curves

Lemma 6.5. *Let $f: X \to S$ be a quasisplit nodal curve with S smooth-factorial and excellent. Suppose X is smooth over some dense open $U \subset S$. Let $\sigma: S \to X$ be a section and $\phi: X' \to X$ be the blow-up in the ideal sheaf of σ.*

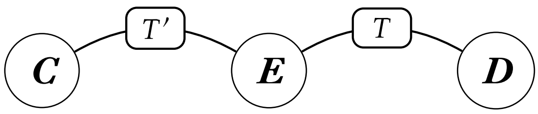
Then ϕ is an isomorphism above the complement in X of $\mathrm{Sing}(X/S) \cap \sigma(S)$. In particular, it is an isomorphism above the smooth locus of X/S, which contains X_U, so X' is a model of X_U.

Moreover, X'/S is a nodal curve, and its dual graphs are refinements of those of X. More precisely, let s be a point of S, and suppose $\sigma(s)$ is a singular point x of X_s. Choose an orientation (C,D) of $X_{\mathcal{O}^{\text{ét}}_{S,s}}$ at x. Let T_x be the thickness of x. In $\widehat{\mathcal{O}^{\text{ét}}_{S,s}}$, write

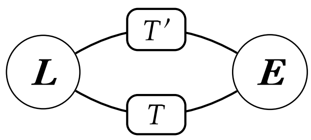
$$T_x = TT',$$

where T is the type of σ at x relative to (C,D). Let Γ, Γ' be the respective dual graphs of X and X' at s, and let e be the edge of Γ corresponding to x. Then e has label T_x, and one obtains Γ' from Γ as follows:

- *If e is not a loop, then C and D come from two distinct irreducible components of X_s (that we still call C and D). In that case, Γ' is obtained from Γ by replacing e by a chain*



  where the strict transforms of C and D are still called C and D, and where E is the inverse image of x.
- *If e is a loop, i.e. x belongs to only one irreducible component L of X_s, then Γ' is obtained from Γ by replacing e by a cycle*



  where the strict transform of L is still called L and E is the inverse image of x.

Proof. The ideal sheaf of σ is already Cartier above the smooth locus of X/S and outside the image of σ, so by the universal property of blow-ups (*cf.* [Stacks, Tag 0806]), we only need to describe ϕ above the étale localizations $\mathrm{Spec}\, \mathcal{O}^{\text{ét}}_{X,x}$, where $x, s, (C,D)$ are as in the statement of the lemma. We can assume that $S = \mathrm{Spec}\, R$ is strictly local, with closed point s. Lift T and T' to global sections Δ, Δ' of S, and pick an isomorphism

$$\widehat{\mathcal{O}^{\text{ét}}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta\Delta')$$

such that C is locally given by $u = 0$. The map

$$\widehat{\sigma}\colon \widehat{\mathcal{O}^{\text{ét}}_{X,x}} \to \widehat{R}$$

induced by σ sends u to a generator of $\Delta\widehat{R}$ and v to a generator of $\Delta'\widehat{R}$. Scaling u and v by a unit of $\widehat{R}$ if necessary, we can assume $\widehat{\sigma}(u) = \Delta$ and $\widehat{\sigma}(v) = \Delta'$.

The completed local rings of $\operatorname{Spec}\mathcal{O}^{\text{ét}}_{X,x} \times_X X'$ can be computed using the blow-up of the algebra $B := R[u,v]/(uv - \Delta\Delta')$ in the ideal $(u - \Delta, v - \Delta')$ (since the completion of B at $(u, v, \mathfrak{m}_R)$ is $\widehat{\mathcal{O}_{X,x}}$).

The ideal $(u - \Delta, v - \Delta')$ is covered by two affine patches (with the obvious gluing maps):

- The patch where $u - \Delta$ is a generator, given by the spectrum of

$$R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta), u\alpha + \Delta') \simeq R[u,\alpha]/(u\alpha + \Delta')$$

  since, in the ring $R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta))$, the element $uv - \Delta\Delta'$ is equal to $(u - \Delta)(u\alpha + \Delta')$;
- The patch where $v - \Delta'$ is a generator, where we obtain analogously the spectrum of

$$R[v,\beta]/(v\beta + \Delta).$$

Thus we see that X' remains nodal and that the edge e of Γ (of label $(\Delta\Delta')$) is replaced in Γ' by a chain of two edges, one labelled (Δ) and one labelled (Δ'). It also follows from this description that the strict transform of C (resp. D) in $X' \times_X \operatorname{Spec}\widehat{\mathcal{O}^{\text{ét}}_{X,x}}$ contains the singularity of label (Δ') (resp. (Δ)). □

Corollary 6.6. *With the same hypotheses and notation as in Lemma 6.5, for any two sections σ, σ' of X/S, denote by $Y \to X$ and $Y' \to X$ the blow-ups in the respective ideal sheaves of σ and σ'. Then, Y and Y' are canonically isomorphic above the same type locus $X_{\sigma,\sigma'}$ of σ and σ' in X. Conversely, if x is in $X \backslash X_{\sigma,\sigma'}$, then Y and Y' are not isomorphic above $\mathcal{O}_{X,x}$.*

Proof. The "conversely" part is immediate from Lemma 6.5. Pick a point $s \to S$ and a singular point x of X_s such that $\sigma(s) = \sigma'(s) = x$ and σ, σ' have the same type T at x. It suffices to exhibit a Zariski neighbourhood V of x in X and an isomorphism $Y \times_X V \to Y' \times_X V$ compatible with the canonical isomorphisms $Y \times_X X^{\text{smooth}} = X^{\text{smooth}} = Y' \times_X X^{\text{smooth}}$. Since X, Y, Y' are finitely presented over S, this can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. Using the universal property of blow-ups (*cf.* [Stacks, Tag 0806]), we reduce to proving that the pull-back of the ideal sheaf of σ' (resp. σ) to Y (resp. Y') is Cartier. The proofs are analogous, so we will only show that the pull-back to Y of the ideal sheaf of σ' is Cartier. This, in turn, reduces to proving that the ideal sheaf of σ' in $\operatorname{Spec}\widehat{\mathcal{O}^{\text{ét}}_{X,x}}$ becomes Cartier in $Y \times_X \operatorname{Spec}\widehat{\mathcal{O}^{\text{ét}}_{X,x}}$. Pick an isomorphism

$$\widehat{A} := \widehat{R}[[u,v]]/(uv - \Delta_x) \simeq \widehat{\mathcal{O}^{\text{ét}}_{X,x}},$$

where $\Delta_x \in R$ is a lift of the thickness of x. The map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ sends u, v to elements Δ, Δ' of $\widehat{R}$ with $\Delta\Delta' = \Delta_x$. Since σ and σ' have the same type at x, there is a unit $\lambda \in \widehat{R}^\times$ such that the map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ' sends u and v to $\lambda\Delta$ and $\lambda^{-1}\Delta'$, respectively. We have reduced to proving that the sheaf given by the ideal $(u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of $\widehat{A}$ becomes Cartier in the blow-up of $\widehat{A}$ in $(u - \Delta, v - \Delta')$. Put

$$A = \widehat{R}[u,v]/(uv - \Delta\Delta');$$

then it is enough to prove that the ideal $I = (u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of A becomes invertible in the two affine patches (as described in the proof of Lemma 6.5) forming the blow-up of A in $(u - \Delta, v - \Delta')$. By analogy, we only check it in the patch where $u - \Delta$ is a generator, which is the spectrum of

$$A_1 = \widehat{R}[u, \alpha]/(u\alpha + \Delta'),$$

where v maps to $\Delta' + \alpha(u - \Delta)$. We have $I = (u - \lambda\Delta, \lambda v - \Delta')$, and in A_1 we can write

$$\begin{aligned} \lambda v - \Delta' &= \lambda(\Delta' + \alpha(u - \Delta)) + u\alpha \\ &= -\lambda\alpha\Delta + u\alpha \\ &= \alpha(u - \lambda\Delta). \end{aligned}$$

Thus, the preimage of I in A_1 is the invertible ideal $(u - \lambda\Delta)$, and we are done. ☐

Definition 6.7. Let S be a smooth-factorial scheme and X/S a quasisplit nodal curve, smooth over a dense open subscheme U of S. We call *basic refinement* of X/S any morphism $f : X' \to X$ isomorphic to the blow-up of X in the ideal sheaf of a section $\sigma : S \to X$. If X/S is orientable at a point x above which f is not an isomorphism, it follows from Corollary 6.6 that the type T of σ at x relative to an orientation (C, D) is independent of the choice of σ: we say that T is the *type of* $X' \to X$ *at* x, or that $X' \to X$ is a *basic* T*-refinement* (at x, relative to (C, D)).

We call *refinement* of X/S any morphism $f : X' \to X$ which, Zariski locally on S, is a composition of basic refinements.

Remark 6.8.

- If S is excellent, then any geometric point $s \in S$ has an admissible neighbourhood V by Proposition 5.12, so X_V/V has a basic T-refinement for any type T at any singular point of X_s.
- Consider any morphism $S' \to S$, where S' is still smooth-factorial (*e.g.* any smooth map $S' \to S$). Let x be a singular point of X and x' a singular point of X' of image x. Then any type T at x pulls back to a type T' at x', and the base change to S' of a basic refinement of type T at x is a basic refinement of type T' at x'.
- Let $f : X' \to X$ be a basic refinement, let $x \in X$ be a singular point at which X/S is orientable and above which f is not an isomorphism, and let y be a generization of x. Let T be the type of f at x. By Lemma 5.7, either T corresponds to a type (still denoted by T) at y, in which case $X' \to X$ has type T at y, or T becomes trivial at y, in which case f restricts to an isomorphism above a Zariski neighbourhood of y.

Proposition 6.9. *Let S be a smooth-factorial and excellent scheme and $U \subset S$ a dense open subscheme. Let X/S be a nodal curve, smooth over U. Suppose that S is an admissible neighbourhood of some point $s \in S$. Then, there exists a refinement $X' \to X$ such that all the singularities of X'_s have prime thicknesses.*

Proof. By the definition of admissibility, S remains an admissible neighbourhood of s if we replace X with a basic refinement. If $X' \to X$ is a basic refinement which is not an isomorphism above s, then by Lemma 6.5, the arithmetic complexity of X' at s is strictly lower than that of X, so we obtain the proposition by induction. ☐

Lemma 6.10. *Let S be a smooth-factorial and excellent scheme and $U \subset S$ a dense open subscheme. Let X/S be a quasisplit nodal curve, smooth over U. Let x be a singular point of x with prime thickness. Then X is locally factorial at x. In particular, if $s \in S$ is such that all the singular points of X_s have prime thicknesses, then $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is locally factorial.*

Proof. Let t be the image of x in S. Then $\widehat{\mathcal{O}_{S,t}}$ is a unique factorization domain by Lemma 4.1. Thus, $\widehat{\mathcal{O}^{\text{ét}}_{X,x}}$ is a unique factorization domain by Lemmas 4.12 and 6.2. Therefore, $\mathcal{O}_{X,x}$ itself is a unique factorization domain by Lemma 4.4. ☐

7. Néron models of Jacobians

If $X \to S$ is a morphism of schemes, its relative Picard functor is the fppf sheafification of the functor sending an S-scheme T to the group of isomorphism classes of line bundles on X_T. When X/S is a nodal curve, by [BLR90, Theorems 8.3.1 and 9.4.1], the Picard functor is representable by a smooth, quasiseparated S-group algebraic space $\mathrm{Pic}_{X/S}$, the ***Picard space***. We write $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ for the kernel of the degree map from $\mathrm{Pic}_{X/S}$ to the constant sheaf $\mathbb{Z}$ on S and $\mathrm{Pic}^0_{X/S}$ for the fibrewise-connected component of identity of $\mathrm{Pic}_{X/S}$, parametrizing line bundles of degree 0 on every irreducible component of every fibre.

A classical way of obtaining a Néron model for the Jacobian J of a proper smooth curve X_U/U with a nodal model X/S, when X is "nice enough", is to consider the quotient P/E, where $P = \mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ and E is the closure of the unit section in P, so that P/E is the biggest separated quotient of P (see, for example, [BLR90, Section 9.5]). This works well when three conditions are met: P is representable by an S-algebraic space, E is flat over S (so that the quotient is also representable), and $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ satisfies existence in the Néron mapping property (*e.g.* X is regular). However, this approach fails most of the time when S is of arbitrary dimension since E is rarely S-flat (*cf.* [Hol19, Theorem 5.17]). The reason is that this method is designed to produce *separated* Néron models, and most Néron models over higher-dimensional bases turn out to be non-separated.

In this section, we will work assuming S is a smooth-factorial scheme and $U \subset S$ a dense open subscheme. In view of Corollary 2.10, it is tempting to try to construct the Néron model as the quotient of P by the étale locus of E/S. This works when P has existence in the Néron mapping property, *i.e.* when X is parafactorial along X_U after any smooth base change (*e.g.* regular). However, even if X_U has nodal models, it may be that none of them remains parafactorial after every smooth base change. We will construct a Néron model N for J when X/S is arbitrary and give a local description of N in terms of Picard spaces of local nodal models of X_U. Then, we will give a simple combinatorial criterion for N to be separated, related to the alignment condition of [Hol19].

7.1. Construction of the Néron model

Remark 7.1. Let S be a smooth-factorial and excellent scheme, $U \subset S$ a dense open subscheme and X/S a quasisplit nodal curve, smooth over U. Suppose that every singular point of X/S has prime thickness. Then X/S is locally factorial by Lemma 6.10, so any U-point of $P := \mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ extends to an S-section. By Corollary 4.11, this remains true after base change to any smooth S-scheme, so P satisfies existence in the Néron mapping property. Thus, by Corollary 2.10, the quotient of P by the étale locus of the closure of its unit section is the Néron model of the Jacobian of X_U. However, we cannot always use Lemma 5.12 and Proposition 6.9 to reduce locally to this situation since some singular points of X/S with prime thickness may have singular generizations whose thickness is not prime (see Example 5.14).

Lemma 7.2. *Let S be a smooth-factorial excellent scheme and $U \subset S$ a dense open subscheme. Let X/S be a nodal curve smooth over U, and let f be a refinement $X' \to X$. Write $P = \mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ and $P' = \mathrm{Pic}^{\mathrm{tot}\,0}_{X'/S}$. Denote by E (*resp.* E') the scheme-theoretic closure of the unit section in P (*resp.* P'). Then, the canonical morphisms $P \to P'$, $P/E \to P'/E'$ and $P/E^{\text{étale}} \to P'/E'^{\text{étale}}$ are open immersions. In addition, if X/S is quasisplit and f is an isomorphism above every singular point of X/S which is not disconnecting in its fibre, then $P/E \to P'/E'$ is an isomorphism.*

Proof. When X/S is quasisplit, $\mathrm{Sing}(X/S)$ is the disjoint union of its open and closed subschemes consisting of points that are, respectively, disconnecting in their fibre and non-disconnecting in their fibre. This partition is compatible with base change and refinements, by Lemma 6.5 and Proposition 3.10. By Corollary 3.18, we may assume that X/S and X'/S are quasisplit. By induction, we may assume that f is a basic refinement. By the proof of Lemma 6.5, we may therefore assume that there exists a closed subscheme F of $\mathrm{Sing}(X/S)$

such that f is an isomorphism above $X\backslash F$ and such that $X'\times_X F \simeq \mathbb{P}^1_F$. Hence, the pull-back along f induces an equivalence of categories between line bundles on X and line bundles on X' having degree 0 on every irreducible component of every fibre of $X'\times_X F \to F$. In particular, we have a canonical isomorphism $\mathrm{Pic}^0_{X/S} = \mathrm{Pic}^0_{X'/S}$. As $\mathrm{Pic}^0_{X/S}$ is an open neighbourhood of the unit section in P (and similarly for P'), it follows that $P \to P'$ is a local isomorphism. Since it is also injective, it is an open immersion. It follows that all squares are cartesian in the commutative diagram

$$\begin{tikzcd}
E^{\text{étale}} \arrow[r] \arrow[d] & E'^{\text{étale}} \arrow[d] \\
E \arrow[r] \arrow[d] & E' \arrow[d] \\
P \arrow[r] & P'.
\end{tikzcd}$$

Therefore, $P/E \to P'/E'$ and $P/E^{\text{étale}} \to P'/E'^{\text{étale}}$ are open immersions as well.

We now prove that $P/E \to P'/E'$ is surjective, assuming that every point of F is disconnecting in its fibre over S. This may be checked at the level of étale stalks over S: it suffices to show that $P(S)$ surjects onto $P'/E'(S)$, assuming that $S = \operatorname{Spec} R$ is strictly local with closed point s. If F is empty, we are done. Otherwise, F_s is a disconnecting singular point x of X_s, and a line bundle on X' is in the image of $P(S)$ if and only if its restriction to $X'\times_X F$ is trivial, *i.e.* if and only if it has degree 0 on $X'\times_X x \simeq \mathbb{P}^1_x$. Therefore, it suffices to show that E' contains a line bundle $\mathcal{L}$ of degree 1 on $X'\times_X x \simeq \mathbb{P}^1_x$. Let x' be a singular point of X'_x, and let $\Delta \in R$ be a lift of its thickness. Let E be the connected component of $\mathrm{Sing}(X'/S)$ containing x'. The map $E \to S$ is a closed immersion cut out by Δ, and $(X'\times_S E))\backslash E$ has two connected components. Denote by C the one whose fibre over x is nonempty and D the other one. The scheme-theoretic closures of C and D in X' are effective Cartier divisors, which we still call C and D (to see that they are Cartier at x', notice that they coincide locally with an orientation at x' as in Definition 5.1). Let $\mathcal{L}$ be the line bundle corresponding to D. Then $\mathcal{L}$ has degree 1 on $X'\times_X x$ since C and D meet transversally at x', and the S-point of P' corresponding to $\mathcal{L}$ is in E' since $\mathcal{L}$ is trivial over U, so we are done. □

Lemma 7.3. *Let $X \to S$ be a proper and finitely presented morphism of schemes, s be a point of S and $\mathcal{L}$ be a line bundle on $X^s := X\times_S \mathrm{Spec}(\mathcal{O}_{S,s})$. Then there exists a Zariski open neighbourhood S' of s in S such that $\mathcal{L}$ extends to a line bundle on $X\times_S S'$.*

Proof. Pick a Cartier divisor $D = \{(U_i, f_i)\}_{i\in I}$ representing $\mathcal{L}^t$. Since X^s is quasicompact, we may assume that the index set I is finite. For each $i \in I$, pick an open subscheme V^i of X containing U_i and a V_i-section g_i of $\mathcal{K}_X := \operatorname{Frac}\mathcal{O}_X$ restricting to f_i. Shrinking V_i if necessary, we may assume that g_i/g_j is in $\mathcal{O}_X^\times(V_i \cap V_j)$. The union of the V_i is an open subset V of X containing X_t. Therefore, since X/S is proper, the image of $X\backslash V$ in S is a closed subset not containing t, and its complement S' is an open neighbourhood of t in S. The V_i cover $X_{S'}$, so $\{(V_i, g_i)\}$ is a Cartier divisor on $X_{S'}$ which restricts to D, and the corresponding line bundle extends $\mathcal{L}$. □

Lemma 7.4. *Let S be a smooth-factorial and excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. For any geometric point $\bar{s} \to S$, write $J_{\bar{s}}$ for the set of prime factors of thicknesses of singular points of $X_{\bar{s}}$ and $\overline{M}_{\bar{s}}$ for the submonoid of $\overline{\mathcal{O}^{\text{ét}}_{S,\bar{s}}}$ spanned by $J_{\bar{s}}$. Consider the relation R on S given by sRt whenever t specializes s and for some (equivalently, any) étale specialization $\bar{t}$ of $\bar{s}$, where $\bar{t}, \bar{s}$ are geometric points above s and t, the restriction map of étale stalks induces a canonical isomorphism between $J_{\bar{t}}$ and $J_{\bar{s}}$. Denote by $\sim_S$ the transitive closure of R. Then, the equivalence classes for $\sim_S$ are locally constructible subsets of S.*

Proof. We immediately reduce to the following claim: given a point $s \in S$, if we denote by C_s its equivalence class for $\sim_S$, then the intersection of C_s with a small enough Zariski neighbourhood of s in S is locally constructible in S. We will now prove the claim. For any étale neighbourhood (V,v) of s, the preimage of C_s in X_V has a locally finite number of connected components, all of which are classes for $\sim_V$ (where $\sim_V$ is defined as $\sim_S$, but after replacing X/S by X_V/V). Therefore, there exists a Zariski neighbourhood (W,w) of v in V such that the preimage of C_s in W is the class C_w of w for $\sim_W$. It follows that the image of C_w in S is the intersection of C_s with an open of S (namely, the image of W). Thus, the claim may be proved after replacing $(X/S,s)$ by $(X_W/W,w)$. In particular, we may assume that S is an admissible neighbourhood of s (since admissibility is preserved by étale localization).

Pick a geometric point $\bar{s}$ above s. The singular locus of X/S has finitely many connected components $(F_1,\dots,F_r)$, and each $F_i \to S$ is cut out by a global section a_i of $\mathcal{O}_S$. Since S is an admissible neighbourhood of s, the elements of $J_{\bar{s}}$ lift to global sections $\Delta_1,\dots,\Delta_n$ of $\mathcal{O}_S$. Shrinking S further if necessary, we may assume that every Δ_i divides some a_j in $\mathcal{O}_S(S)$ (and not just in $\mathcal{O}^{\text{ét}}_{S,s}$). Recall that an integral scheme Y is called *geometrically unibranch* at a point $y \in Y$ if the strict henselization of $\mathcal{O}_{Y,y}$ is integral or, equivalently by [Ray70, corollaire IX.1], if the integral closure of $\mathcal{O}_{Y,y}$ is local with purely inseparable residue extension over $\mathcal{O}_{Y,y}$. Denote by Z_i the closed subscheme of S cut out by Δ_i. The set of points Z_i^{uni} at which Z_i is geometrically unibranch is locally constructible in Z_i by [GD66, corollaire 9.7.10]. Therefore, the intersection in S of the images of the Z_i^{uni} for all i is locally constructible in S. This intersection is precisely C_s, so we are done. □

Remark 7.5. With the hypotheses and notation of Lemma 7.4, the equivalence classes for $\sim_S$ form a partition of S into locally constructible subsets. In particular, locally on S, there are only finitely many such classes. Since $\sim_S$ only depends on X via the sets $J_{\bar{s}}$, it remains unchanged if we replace X with a refinement.

Theorem 7.6. *Let S be a smooth-factorial and excellent scheme, and let $U \subset S$ be a dense open subscheme. Let X_U/U be a smooth curve that admits a nodal model over S. Then:*

(1) The Jacobian $J = \mathrm{Pic}^0_{X_U/U}$ of X_U/U admits a Néron model N over S.
(2) For any nodal model X/S of X_U/U, the map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}/E^{\text{étale}} \to N$ extending the identity over U is an open immersion, where E is the scheme-theoretic closure of the unit section in $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$.
(3) For any étale morphism $V \to S$ and nodal V-model X of $X_{U\times_S V}$, if $\bar{s} \to V$ is a geometric point such that the singularities of $X_{\bar{s}}$ have prime thicknesses, then the canonical map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/V} \to N$ is surjective on $\mathrm{Spec}(\mathcal{O}^{\text{ét}}_{S,\bar{s}})$-points.

Remark 7.7. In the setting of Theorem 7.6, if X^0 is any nodal model of X_U, by Propositions 5.12 and 6.9, there exist an étale cover $V \to S$ and a refinement $X \to X^0_V$ with the following property: for any $s \in S$, there is some geometric point $v \to V$ above s such that the singularities of X_v have prime thicknesses. In particular, it follows from Theorem 7.6 that the canonical map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/V} \to N$ is an étale cover.

Proof of Theorem 7.6. Recall that the formation of Néron models is smooth local on the base (*cf.* Propositions 2.6 and 2.8), that the properties of morphisms "being étale" and "being an open immersion" are fpqc local on the target (*cf.* [Stacks, Tags 02L3 and 02VN]) and that for a nodal curve X/S, the formation of $\mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$ and of the closure of its unit section commute with flat base change. In particular, claims (1), (2) and (3) of the theorem hold if and only if they hold étale-locally on S.

First, let us assume (1) and (2) and prove (3). Let $X, V, \bar{s}$ be as in (3), and put $P = \mathrm{Pic}^{\mathrm{tot}\,0}_{X/V}$ and $T = \mathrm{Spec}\,\mathcal{O}^{\text{ét}}_{S,\bar{s}}$. It follows from (2) that $P \to N$ is étale, and we only need to show that it is surjective on T-points. Pick $f_U \in N(T)$. Then, f_U corresponds to a line bundle $\mathcal{L}_U$ on X_{T_U}. The curve X_T is locally factorial by Lemma 6.10. Pick a Weil divisor D on X_U representing $\mathcal{L}_U$. Its closure $\overline{D}$ in X_T is Cartier by local factoriality, hence defines a line bundle $\mathcal{L}$ extending $\mathcal{L}_U$, *i.e.* a T-point of P mapping to f_U.

Now, it suffices to prove that (1) and (2) hold. Fix a nodal S-model X of X_U. We say that a smooth S-scheme V is *good* if the following two conditions are met:

- There exists a V-Néron model N_V for $J_{U\times_S V}$. (The notation is unambiguous since if X_U has a Néron model N, then $N\times_S V$ is the V-Néron model of $X_{U\times_S V}$.)
- For any étale map $V'\to V$ and any refinement $X'\to X_{V'}$, the canonical map $\mathrm{Pic}^{\mathrm{tot}\,0}_{X'/V'}/E'^{\text{étale}}\to N_{V'}$ is an open immersion, where E' is the closure of the unit section in $\mathrm{Pic}^{\mathrm{tot}\,0}_{X'/V'}$.

We say that V has the property $\mathcal{P}$ if there exists a good open subscheme W of V such that V is an admissible neighbourhood of every point of $V\backslash W$. If s is a point of S, we say that s is good (resp. has $\mathcal{P}$) if some étale neighbourhood of s is good (resp. has $\mathcal{P}$). Goodness and $\mathcal{P}$ can both be checked locally on S for the étale topology. Therefore, the theorem reduces to the following two claims: goodness and $\mathcal{P}$ are equivalent, and any point of S has $\mathcal{P}$. We will now prove these claims, in order. Throughout the rest of the proof, when V is a smooth S-scheme, we will write $\sim_V$ for the equivalence relation on V defined as in Lemma 7.4 (relative to X_V/V or, equivalently, to any refinement of it).

Clearly goodness implies $\mathcal{P}$. We will show that S is good, assuming it has $\mathcal{P}$. Goodness and $\mathcal{P}$ are local, so it suffices to pick a point $s\in S$ and show that, after shrinking S at will to an arbitrarily small étale neighbourhood of s, J has a Néron model N and for any refinement $X'\to X$, the canonical map $P/E^{\text{étale}}\to N$ is an open immersion, where $P=\mathrm{Pic}^{\mathrm{tot}\,0}_{X'/S}$ and E is the closure of the unit section in P.

Let W be a good open subscheme of S such that S is an admissible neighbourhood of every point of $F:=S\backslash W$. Shrinking S, we may assume that there are only finitely many equivalence classes $(C_i)_{i\in I}$ for $\sim_S$. By Proposition 7.2, we may replace X' with a refinement. Pick an index $j\in I$ such that C_j meets F, and let s' be a point of $F\cap C_j$. By Proposition 6.9, we may assume that the singular points of X' above s' have prime thicknesses, which implies that the singular points of X' mapping to C_j have prime thicknesses by the definition of $\sim_S$. Iterating the process, we may assume that the thicknesses of all the singularities of X' above F are prime. Denote by N_W the W-Néron model of $J_{U\times_S W}$. The canonical map

$$(P/E^{\text{étale}})\times_S W\to N_W$$

is an open immersion since W is good. Denote by N the gluing of $P/E^{\text{étale}}$ and N_W along $(P/E^{\text{étale}})\times_S W$ (the notation N_W is unambiguous since $N\times_S W=N_W$). Then N is a smooth S-model of J with uniqueness in the Néron mapping property and with an open immersion $P/E^{\text{étale}}\to N$ restricting to the identity over U. Therefore, in order to prove that S is good, it suffices to show that for any smooth S-scheme Y, the restriction map

$$\mathrm{Hom}_S(Y,N)\to\mathrm{Hom}_U(Y_U,N_U)$$

is surjective. Pick some $f_U\in\mathrm{Hom}_U(Y_U,N_U)$. By uniqueness in the mapping property, it suffices to show that f_U extends to an S-map $Y'\to N$ for a Zariski neighbourhood Y' of a given point $y\in Y$. If y is in Y_W, this is clear. Otherwise, y lands in F so the singularities of X_Y above y have prime thicknesses by Corollary 4.11. In particular, the base change X^y of X_Y to $\mathrm{Spec}(\mathcal{O}_{Y,y})$ is locally factorial by Lemma 6.10, so the line bundle on $X^y\times_S U$ corresponding to f_U extends to a line bundle $\mathcal{L}$ on X^y. Then, Lemma 7.3 provides an open neighbourhood Y' of y in Y and a line bundle on $X_{Y'}$ extending $\mathcal{L}$, *i.e.* a morphism $Y'\to P$ extending f_U. We conclude by composing with $P\to N$.

We have shown that goodness and $\mathcal{P}$ are equivalent. Now, let s be a point of S; we will prove that s has $\mathcal{P}$. For any étale morphism $V\to S$, the locally closed pieces of the equivalence classes of $\sim_V$ form a partition of V into locally closed subsets. We write $n(V)$ for the number of pieces of this partition and $n(s)$ for the minimum of the $n(V)$ where V ranges through the étale neighbourhoods of s in S. By the local constructibility of the classes for $\sim_S$, we know that $n:=n(s)$ is finite. We will show that s has $\mathcal{P}$ by induction on n. Shrinking S, we may assume that $n(S)=n$. If $n=1$, then X/S is smooth, so S is good and we are done. Otherwise, denote by F_0 the equivalence class of s for $\sim_S$, and let F be the locally closed piece of F_0 containing s. By the minimality of $n(S)$, F is closed in S. Therefore, the open subscheme $W=S\backslash F$ of S is such that $n(W)=n(S)-1$, and by induction W has $\mathcal{P}$; *i.e.* W is good. Let (V,v) be an admissible neighbourhood of s in S. Shrinking V, we may assume that all the points of $V\backslash W_V$ are $\sim_V$-equivalent, from which it follows that V is an admissible neighbourhood of all of them. Then V has $\mathcal{P}$, which concludes the proof. $\square$

Remark 7.8. In [HMO+20], the authors describe the ***strict logarithmic Jacobian*** of a logarithmic curve. They show that when X/S is a nodal curve over a toroidal variety, smooth over the complement U of the boundary divisor, there are canonical log structures on X and S such that the strict logarithmic Jacobian is the Néron model of X_U. This gives a moduli interpretation in logarithmic geometry for the Néron model constructed in Theorem 7.6 when U is the complement in S of a divisor with normal crossings. A similar interpretation can be given when the discriminant locus of X/S is arbitrary. Indeed, let M_S be the étale subsheaf of monoids of $\mathcal{O}_S$ whose étale stalks are generated by the units and by the prime factors of the singular ideals of X. Let M_X be the submonoid of $\mathcal{O}_X$ whose étale stalk at a geometric point $x \to X$ above a geometric point $s \to S$ is:

- The submonoid of $\mathcal{O}^{\text{ét}}_{X,x}$ spanned by $(\mathcal{O}^{\text{ét}}_{X,x})^\times$ and $M_{S,s}$ if x is smooth over S;
- The submonoid of $\mathcal{O}^{\text{ét}}_{X,x}$ spanned by $(\mathcal{O}^{\text{ét}}_{X,x})^\times$, $M_{S,s}$ and local parameters for the two branches of X/S at x if x is singular.

Then, $M_S \to \mathcal{O}_S$ and $M_X \to \mathcal{O}_X$ are logarithmic structures in the sense of [Kat89], but they do not necessarily admit étale-local charts (*cf.* Example 5.14). Therefore, (X, M_X) and (S, M_S) are not logarithmic schemes in the usual sense, and we cannot apply directly the results of [HMO+20] (although many of the arguments remain valid in our context). However, replacing U with the maximal open subscheme of S over which X is smooth, *i.e.* on which $M_S = \mathcal{O}_S^\times$, we find that the groupification M_X^{gp} coincides with the direct image of $\mathcal{O}^\times_{X_U}$ in X. Hence, two isomorphism classes of M_X^{gp}-torsors which coincide over X_U are equal. Write $H^1(X, M_X^{\text{gp}})^\dagger$ for the subgroup of $H^1(X, M_X^{\text{gp}})$ consisting of torsors which, locally on S, come from a line bundle on a refinement of X. Then it follows that $H^1(X, M_X^{\text{gp}})^\dagger = \operatorname{Hom}(U, \operatorname{Pic}^0_{X_U/U})$. Combining this with the fact that the formation of M_S and M_X commutes with smooth base change by Corollary 4.11, we find that

$$\begin{aligned}(\mathrm{Sch}/S)^{\mathrm{op}} &\to \mathrm{Set}\\ T &\mapsto H^1(X_T, M^{\mathrm{gp}}_{X_T})^\dagger\end{aligned}$$

is the Hom functor of the Néron model of $\operatorname{Pic}^0_{X_U/U}$. As in [MW18] or [HMO+20], we can describe explicitly $H^1(X, M_X^{\text{gp}})^\dagger$ as the subgroup of $H^1(X, M_X^{\text{gp}})$ consisting of torsors satisfying a certain condition that can be expressed in terms of dual graphs, the condition of ***bounded monodromy***.

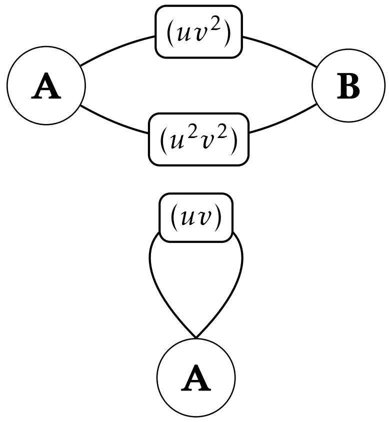
7.2. A criterion for separatedness

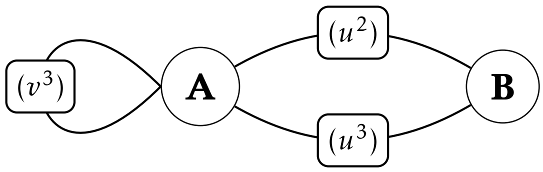
In this subsection, we exhibit a necessary and sufficient combinatorial condition for the Néron model of Theorem 7.6 to be separated, closely related to the alignment condition of [Hol19].

Definition 7.9. Let Γ be a graph labelled by a monoid M, written multiplicatively. Following [Hol19, Definition 2.11], we say that Γ is ***aligned*** when for every cycle Γ^0 in Γ, all the labels figuring in Γ^0 are positive powers of the same element l of M. When S is a smooth-factorial scheme and $s \to S$ a geometric point, we say that a nodal curve X/S is ***aligned at*** s when its dual graph Γ_s at s is aligned. We say X/S is ***aligned*** if it is aligned at every geometric point of S.

If M is the free commutative monoid over a set of generators G, we say that Γ is ***strictly aligned*** if l can be chosen in G. We say that X/S is ***strictly aligned at*** s if Γ_s is strictly aligned (here G is the set of principal prime ideals of $\mathcal{O}^{\text{ét}}_{S,s}$). We say that X/S is ***strictly aligned*** if it is strictly aligned at every geometric point of S.

Example 7.10. Over $S = \operatorname{Spec} \mathbb{C}[[u,v]]$, at the closed point, among the following three dual graphs, the first is not aligned, the second is aligned but not strictly, and the third is strictly aligned.





Example 7.11. In the setting of Example 5.14, the curve X/S is strictly aligned at the closed point s of S (since S is strictly local and the dual graph at s is a loop with prime label), but X/S is not aligned.

Proposition 7.12. *Let S be a regular scheme, $U \subset S$ a dense open and X/S a nodal curve, smooth over U. Let $P = \mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$, and let E be the scheme-theoretic closure in P of its unit section. Then the following conditions are equivalent:*

(1) E/S is flat.
(2) E/S is étale.
(3) X/S is aligned.

Proof. This is [Hol19, Theorem 5.17]. □

Theorem 7.13. *Let S be a regular and excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Denote by J the Jacobian of X_U/U. Then, the S-Néron model N of J exhibited in Theorem 7.6 is separated if and only if X/S is strictly aligned.*

Proof. First, suppose that N is separated. Let $s \to S$ be a geometric point; we will show that X/S is aligned at s. By Corollary 2.7, this may be checked assuming that S is strictly local with closed point s. Proposition 6.9 provides a refinement $X' \to X$ such that every singularity in the closed fibre of X'/S has prime thickness. Let Γ, Γ' be the dual graphs at s of X and X', respectively. The closure of the unit section in $\mathrm{Pic}^{\mathrm{tot}\,0}_{X'/S}$ is étale over S by [Hol19, Theorem 6.2]. Hence, Γ' is aligned at s by Proposition 7.12, and even strictly aligned since its labels are prime. Since Γ' refines Γ (*cf.* Lemma 6.5), it follows that Γ is strictly aligned as well.

Conversely, suppose that X is strictly aligned. We will show that $N \to S$ is separated. This may be done assuming that $S = \mathrm{Spec}\, R$ is strictly local with closed point s. Replacing X with a refinement, we may assume that every singular point of X_s has prime thickness by Proposition 6.9. Put $P = \mathrm{Pic}^{\mathrm{tot}\,0}_{X/S}$, and let E be the scheme-theoretic closure of the unit section in P. Then E is étale over S by Proposition 7.12, and there is a canonical open immersion $P/E \to N$ by Theorem 7.6. Since P/E is separated, it suffices to show that this open immersion is surjective. This can be checked on étale stalks over S: let $t \to S$ be a geometric point and $T = \operatorname{Spec} R'$, where $R' = \mathcal{O}_{S,t}^{\text{ét}}$; it suffices to show that $P/E \to N$ is surjective on T-points. Proposition 6.9 provides a refinement $X' \to X_T$ such that the singular points of X' above t have prime thicknesses. Put $P' = \operatorname{Pic}^{\text{tot}\,0}_{X'/T}$. By part (3) of Theorem 7.6, the map $P'(T) \to N(T)$ is surjective. Therefore, by the "in addition" part of Proposition 7.2, it suffices to show that $X' \to X_T$ is an isomorphism above every singular point of X_t which is non-disconnecting in X_t. Let x be such a point. By Lemma 6.5, it is enough to prove that the thickness $T \in \overline{R'}$ of x is prime. Let y be the singular point of X_s specializing x. Let $\Delta \in R$ be a lift of the thickness of y. Then T is the principal ideal of R' spanned by Δ. The ring $R'/(\Delta)$ is reduced as a filtered colimit of étale $R/(\Delta)$-algebras, so Δ is square-free in R'. But Δ is a prime power in R' since X is strictly aligned at t, so it is prime and we are done. □

Remark 7.14. As mentioned in the proof of [Hol19, Theorem 5.17], Proposition 7.12 should still hold if we only require S to be smooth-factorial instead of regular. If so, our proof of Theorem 7.13 remains valid when S is just an excellent and smooth-factorial scheme.

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