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An atlas of K3 surfaces with finite automorphism group

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Abstract. We study the geometry of K3 surfaces with finite automorphism group and Picard number at least 3. We describe these surfaces classified by Nikulin and Vinberg as double covers of simpler surfaces or as embedded in a projective space. We study, moreover, the configurations of their finite set of (-2)-curves.

Keywords. K3 surfaces, finite automorphism groups, finite number of (-2)-curves, Vinberg algorithms

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1. Introduction

Algebraic K3 surfaces X over $\mathbb{C}$ with finite automorphism group were classified by their Picard lattices $\mathrm{NS}(X)$ and Picard number $\rho = \mathrm{rk}(\mathrm{NS}(X)) \geq 3$ by Nikulin for $\rho = 5, 6, \dots$ in 1981, *cf.* [Nik83], and for $\rho = 3$ in 1985, *cf.* [Nik85], and by Vinberg for $\rho = 4$ (in 1981, published in 2007 in [Vin07]). The result of their classification is a list of 118 Néron–Severi lattices.

In the present paper, we exhibit for each of these surfaces a geometric construction, as a double plane, a double cover of the Hirzebruch surface $\mathbf{F}_n$, $n \in \{2, 3, 4\}$ (which part has also been more or less explicitly done in [AN06] and [Zha98]) or a complete intersection surface of degree $k \in \{4, 6, 8\}$ in $\mathbb{P}^{\frac{1}{2}k+1}$. Let us recall that a K3 surface with Picard number $\rho > 2$ has finite automorphism group if and only if it contains only a non-zero finite number of (-2)-curves. The (-2)-curves play a key role for K3 surfaces; for example, one can describe their ample cone using these curves. We are thus especially interested in the configuration of these curves on the K3 surfaces. In [Nik83] and especially in [Nik85] for $\rho = 3$, Nikulin described the number and some configurations of these curves. Our contribution to the subject comes for higher-rank Picard lattices, when there are more (-2)-curves than expected. Among the K3 surfaces we describe, there are two series which we find most remarkable.

The first series are the surfaces with Néron–Severi lattice of type $U(2)\oplus \mathbf{A}_1^{\oplus n}$, for $n\in\{2,\dots,7\}$. Such a surface X is the desingularization of the double cover of the plane branched over a sextic curve with $n+1$ nodes in general position. The surface X is also naturally the double cover $X\to Z$ of a degree $8-n$ del Pezzo surface Z branched over a curve in the linear system $|-2K_Z|$. The pull-back on X of the (-1)-curves on Z are (-2)-curves, and one has a description of the configuration of the (-2)-curves on X from the well-known configuration of (-1)-curves on the del Pezzo surface. In particular, for $n=2,\dots,7$, the number of (-2)-curves on the K3 surface is $6,10,16,27,56,240$, respectively. Let us describe the case with 240 (-2)-curves.

Theorem 1.1. *Let X be a general K3 surface with Néron–Severi lattice isometric to $U(2)\oplus\mathbf{A}_1^{\oplus 7}$. There exists a double cover $f_1\colon X\to\mathbb{P}^2$ branched over a sextic curve with 8 nodes $p_1,\dots,p_8$ such that the images of the 240 (-2)-curves on X are*

- *the 8 points $p_1,\dots,p_8$,*
- *the 28 lines through p_i,p_j with $i\neq j$,*
- *the 56 conics that go through 5 points in $\{p_1,\dots,p_8\}$,*
- *the 56 cubics that go through 7 points p_j with a double point at 1 of these points,*
- *the 56 quartics through the 8 points p_j with double points at 3 of them,*
- *the 28 quintics through the 8 points p_j with double points at 2 of them,*
- *the 8 sextics through the 8 points with double points at all except a single point with multiplicity 3.*

Let $Z\to\mathbb{P}^2$ be the blow-up at the points p_k, $k\in\{1,\dots,8\}$; the surface X is also a double cover of Z. The pull-back on X of the pencil $|-K_Z|$ gives another double cover $f_2\colon X\to\mathbb{P}^2$. Its branch locus is a smooth sextic curve to which 120 conics are tangent at every intersection point. These 120 conics are the images of the 240 (-2)-curves on X.

The two involutions corresponding to the covers f_1,f_2 generate the automorphism group $(\mathbb{Z}/2\mathbb{Z})^2$ of X.

The second series are the surfaces with Néron–Severi lattice of type $U\oplus\mathbf{E}_8\oplus\mathbf{A}_1^{\oplus 4}$, $U\oplus\mathbf{D}_8\oplus\mathbf{A}_1^{\oplus 3}$, $U\oplus\mathbf{D}_4^{\oplus 2}\oplus\mathbf{A}_1^{\oplus 2}$, $U\oplus\mathbf{D}_4\oplus\mathbf{A}_1^{\oplus 5}$ or $U\oplus\mathbf{A}_1^{\oplus 8}$. These surfaces contain, respectively, $27,39,59,90,145$ (-2)-curves. Let us give an example of the results obtained for the case of $U\oplus\mathbf{A}_1^{\oplus 8}$. Let $\mathbf{F}_4$ be the Hirzebruch surface with a section s such that $s^2=-4$. Let f be a fiber of the unique fibration, and let $L=4f+s$, $L'=5f+s$.

Theorem 1.2. *Let X be a general K3 surface with Néron–Severi lattice isometric to $U\oplus\mathbf{A}_1^{\oplus 8}$. There exists a double cover $f_1\colon X\to\mathbf{F}_4$ branched over a curve $B=s+b$, where b is a curve in $|3L|$ with 8 nodes $p_1,\dots,p_8$ and $s\cap b=\emptyset$; that double cover is such that the images of the 145 (-2)-curves on X are*

- *the 8 points $p_1,\dots,p_8$,*
- *the section s,*
- *the 8 fibers through the 8 points $p_1,\dots,p_8$,*
- *the 8 curves in $|L'|$ going through 7 of the 8 points $p_1,\dots,p_8$,*
- *the 56 sections in the linear system $|L|$ that pass through 5 of the 8 points $p_1,\dots,p_8$,*
- *the 56 curves in the linear system $|2L|$ that pass through 5 of the 8 points with multiplicity 1 and through the 3 remaining with multiplicity 2,*
- *the 8 curves in the linear system $|3L|$ that pass through the 8 points with multiplicity 2, except at 1 point where the multiplicity is 3.*

There exists, moreover, a double cover of $\mathbb{P}^2$ branched over a sextic curve with one node. Through that node, there are 8 lines that are tangent to the sextic at another intersection point. There are 64 conics that are 6-tangent to the sextic. The $72=8+64$ curves and the node are the images of the 145 (-2)-curves on X.

The two involutions corresponding to the two covers we described generate the automorphism group $(\mathbb{Z}/2\mathbb{Z})^2$ *of* X.

The other members of the family have a similar description as double covers of $\mathbf{F}_2$ or $\mathbf{F}_4$, and one can describe their (-2)-curves similarly. As above, these surfaces have another model as a double cover of $\mathbb{P}^2$ branched over a sextic curve. In most cases, the set of (-2)-curves can be naturally decomposed as the union of two sets of (-2)-curves having a different behavior: one set generates the Néron–Severi lattice; it has a dual graph which is directly related to the name of the lattice and can be easily represented. The second set is a set of extra (-2)-curves which, when they exist, form an interesting configuration, sometimes very symmetric. Often the curves of the second set occur as pull-backs under a double cover of conics or lines in $\mathbb{P}^2$ that are tangent to the sextic branch curve at an intersection point.

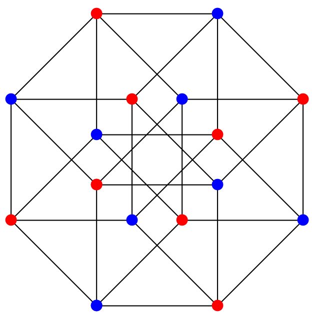
In this paper, we also describe the configurations of the (-2)-curves contained in the K3 surfaces. The perhaps nicest configuration is as follows.

Theorem 1.3. *Let* X *be a general K3 surface with Néron–Severi lattice* $U(4)\oplus A_1^{\oplus 3}$. *The surface* X *is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve* C_6 *such that there exist* 12 *conics that are tangent to the sextic at an intersection point; the* 24 (-2)*-curves on* X *are mapped in pairs to the* 12 *conics.*

There exists a partition of the 24 (-2)*-curves into three sets* S_1, S_2, S_3 *of* 8 *curves each, such that for curves* B, B' *in two different sets* S, S', *one has* $BB' \in \{0,4\}$ *and for any* $B \in S$, *there are exactly* 4 *curves* B' *in* S' *such that* $BB' = 4$, *and symmetrically for* B'. *The sets* S *and* S' *form an* 8_4 *configuration called the Möbius configuration.*

The moduli space of K3 surfaces polarized by $U(4)\oplus A_1^{\oplus 3}$ *is unirational.*

The following graph is the Levi graph of the Möbius configuration, where vertices in red are curves in S, vertices in blue are curves in S', and an edge links a red curve to a blue curve if and only if their intersection number is 4:



From that graph, we can moreover read off the intersection numbers of the curves in S (and S') as follows:

For any red curve B, there are four blue curves linked to it by an edge. Consider the complementary set of blue curves; this is another set of four blue curves, all linked through an edge to the same red curve B'. Then we have $BB' = 6$, and for any other red curve $B'' \neq B'$, we have $BB'' = 2$. Symmetrically, the intersection numbers between the blue curves follow the same rule.

Nikulin also studied that configuration of 24 curves, which he obtained by lattice considerations (see [Nik83, Section 8.3]), showing a very nice relation with the 24 roots of $\mathbf{D}_4$.

Another result is about the "famous" 95 families of K3 hypersurfaces in weighted projective threefolds. In Section 2.6, we remark that among these K3 surfaces, many are surfaces with finite automorphism group, and their moduli space is unirational. We also study the unirationality of many other moduli spaces of K3 surfaces with finite automorphism group.

At the end of the paper, in Section 16, we give the list of the K3 surfaces with finite automorphism group and their number of (-2)-curves, for Picard number at least 3. In this paper is missing the classification for Picard number 4: that is the subject of another paper [ACR20] with Artebani and Correa Diesler.

In [Kon89], Kondo classifies the automorphism group of general K3 surfaces in each of the 118 families. For 105 of these families, Kondo proves that the automorphism group of a general K3 surface is either trivial or the group $\mathbb{Z}/2\mathbb{Z}$, leaving indeterminate which cases actually happen. Our work clarifies the situation for each of these families; the automorphism group of a general K3 surface of each family can be found in Section 16.

It is not possible for the author to cite every contribution on the subject of K3 surfaces with finite automorphism group. However, let us give some details on [AN06], where Alexeev and Nikulin classify log del Pezzo surfaces of index at most 2 (*i.e.*, normal surfaces X with quotient singularities such that the anti-canonical divisor $-K_X$ is ample and $2K_X$ is a Cartier divisor). They also describe possible configurations of exceptional curves on a minimal resolution of X, *i.e.*, smooth rational curves with negative self-intersection.

The paper of Alexeev and Nikulin gives the description of the (-2)-curves in Theorems 1.1 and 1.2. Table 3 in [AN06] gives these configurations for the "special" resolutions of the most degenerate del Pezzo surfaces, those with the largest configuration of (-2)-curves. But the procedure for the general case is also provided in [AN06]: if a certain ADE configuration disappears on a deformation, then one should add the corresponding Weyl group orbit of the (-1)-curves. The (-2)-curves on the double cover K3 surface are the preimages of the (-1)-, (-2)- and (-4)-curves on these resolutions.

For example, for the double covers of the degree 1 del Pezzo surfaces: On the most degenerate "almost" del Pezzo surface, there is an $\mathbf{E}_8$ configuration of (-2)-curves and a single (-1)-curve. The corresponding K3 surface has $16+1=17$ (-2)-curves. On the most general del Pezzo surfaces, there are no (-2)-curves, and there is a single $W(\mathbf{E}_8)$-orbit of (-1)-curves, which has cardinality $|W(\mathbf{E}_8)/W(\mathbf{E}_7)| = 240$. Indeed, this description of the (-1)-curves on a del Pezzo surface is well known. The corresponding K3 surface has 240 (-2)-curves.

Table 3 in [AN06] contains this information for all 2-elementary hyperbolic lattices where the K3 surfaces have finite automorphism group, excluding the $(19,1,1)$ case which has $g=1$ (see the notation in [AN06]). The lattices in Theorems 1.1 and 1.2 are 2-elementary.

To be more precise, the paper [AN06] computes the fundamental chambers $P^{2,4}$ for all the 2-elementary lattices in question, for the full reflection group. That group is generated by the reflections in the (-2)-vectors and in the 2-divisible (-4)-vectors. And by [AN06, Proposition 2.4.1], the walls of the fundamental chamber P^2 for the Weyl group W^2 are the orbits of the (-2)-walls of $P^{2,4}$ under the group generated by the (-4)-walls of $P^{2,4}$. In terms of graphs, the set of walls of P^2 (that is, the (-2)-curves on X) is the union of the orbits of the white vertices in the Coxeter graph of $P^{2,4}$ (see [AN06, Table 3, p. 93]) under the Weyl group corresponding to the black subgraph.

On the subject of K3 surfaces with finite automorphism group, let us also mention the paper [Zha98], where Zhang classifies the quotients of K3 surfaces by an involution.

At last, let us also cite [AHL10] by Artebani, Hausen and Laface, and [McK10] by McKernan, where the Cox ring of some K3 surfaces is studied. It turns out that a K3 surface has a finitely generated Cox ring if and only if its cone of effective divisor classes is polyhedral or, equivalently, if its automorphism group is finite. K3 surfaces with finite automorphism group and Picard number $\rho=2$ have been described by Piatetskii-Shapiro and Shafarevich [PS71] and also studied in [GLP10] (see also [AHL10, Ott13] for their Cox rings). The Cox rings of K3 surfaces with a non-symplectic involution and $2 \le \rho \le 5$ have been described in [AHL10]. Also in [AHL10] is studied the Cox ring for all K3 surfaces which are general double covers of del Pezzo surfaces. Finally, let us mention (see [PS71]) that the general K3 surface with Picard number 1 has trivial automorphism group, unless its Néron–Severi lattice is generated by D with $D^2=2$, in which case the automorphism group is generated by the non-symplectic involution.

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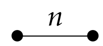
2. Notation, Preliminaries

2.1. Notation and conventions

We work over the complex numbers. Linear equivalence between divisors is denoted by $\equiv$. On a K3 surface, linear and numerical equivalences coincide. A (-2)-curve is unique in its numerical equivalence class, so we will often not distinguish between a (-2)-curve and its equivalence class.

The configuration of a set of (-2)-curves $C, C', \dots$ is given by its dual graph, where a vertex represents a (-2)-curve and two vertices are linked by an edge if the intersection $C \cdot C'$ (sometimes also denoted by CC') of the corresponding curves C, C' satisfies $C \cdot C' > 0$. Unless explicitly stated, when $n = C \cdot C'$, we label the edge n:



Moreover, a thick edge

between the two vertices C, C' means that $C \cdot C' = 2$, and no label means that $C \cdot C' = 1$.

We denote by $\mathbf{a}_n$, $\mathbf{d}_n$, $\mathbf{e}_n$ the Du Val curve singularities, also called simple curve singularities (see [BHP+04, Chapter II, Section 8]). Let $C_6 \hookrightarrow \mathbb{P}^2$ be a reduced plane sextic curve with at most $\mathbf{ade}$ singularities. Let X be the K3 surface which is the minimal desingularization of the double cover branched over C_6. We denote by $\eta \colon X \to \mathbb{P}^2$ the natural map. We say that a line is tritangent to C_6 if the multiplicities at the intersection points of the line and C_6 are even. Similarly, we say that a conic is 6-tangent to C_6 if the multiplicities of the intersection points at the conic and C_6 are even. The following result is well known.

Lemma 2.1. *Let R_d be a tritangent line ($d = 1$) or a 6-tangent conic ($d = 2$). The pull-back on X of R_d splits: $\eta^* R_d = A + B$, where A, B are two smooth rational curves with intersection number $d^2 + 2$. One has $D_2 \cdot A = D_2 \cdot B = d$ and $A + B \equiv dD_2$, where D_2 is the pull-back by η of a line.*

Conversely, if two (-2)-curves A, B are such that there exist a nef, base-point free divisor D_2 of square 2 and a $d \in \{1, 2\}$ such that $dD_2 = A + B$, then the image of A and B by the natural map obtained from $|D_2|$ is a rational curve of degree d.

For a symmetric matrix Q with integral entries, we denote by $[Q]$ the lattice with Gram matrix Q. We denote by U the lattice $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. If L is a lattice and $m \in \mathbb{Z}$, then $L(m)$ is the lattice with the same group as L but with Gram matrix multiplied by m.

We denote by $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_n$ the negative-definite lattices that correspond to the root systems denoted with the same letters. A double cover branched over a curve with a singularity $\mathbf{b}_n$ ($\mathbf{b}$ in $\mathbf{a}, \mathbf{d}, \mathbf{e}$) is a singular surface with B_n singularity (B in A, D, E), see [BHP+04, Chapter III, Section 6]. Its minimal resolution is by a union of (-2)-curves, which curves generate a lattice $\mathbf{B}_n$.

In this paper, by an elliptic fibration of a K3 surface X, we mean a morphism $X \to \mathbb{P}^1$ with connected fibers. We will frequently use implicitly the linear equivalence relations obtained from (singular) fibers of an elliptic fibration; such information can be read off the dual graph of the (-2)-curves. The Kodaira classification of singular fibers of elliptic fibrations and their dual graph with their weight can be found, *e.g.*, in [BHP+04, Section V.7]. We denote by $\tilde{\mathbf{A}}_n$, $\tilde{\mathbf{D}}_n$, $\tilde{\mathbf{E}}_n$ the types of singular fibers of an elliptic fibration in Kodaira's classification.

Let N be a lattice of signature $(1, \rho - 1)$. We denote by $\mathcal{M}_N$ the moduli of K3 surfaces X polarized by a primitive embedding $N \hookrightarrow \mathrm{NS}(X)$.

2.2. Some results of Saint-Donat on linear systems of K3 surfaces

To be self-contained, let us recall the following results of Saint-Donat [Sai74].

Theorem 2.2. *Let D be a divisor on a K3 surface X.*

a) *(*[Sai74, Proposition 2.6]*). If D is effective and non-zero and $D^2 = 0$, then $D = aE$, where $|E|$ is a free pencil and $a \in \mathbb{N}$.*

b) *(*[Sai74, (2.7.3)]*). If D is big and nef, then $h^0(D) = 2 + \frac{1}{2}D^2$ and either $|D|$ has no fixed part, or $D = aE + \Gamma$, where $|E|$ is a free pencil and Γ is an effective (-2)-class such that $\Gamma \cdot E = 1$.*

c) *(*[Sai74, Section 4.1]*). If D is big and nef and $|D|$ has no fixed part, then $|D|$ is base-point free and either $\varphi_{|D|}$ is 2-to-1 onto its image (hyperelliptic case), or it maps X birationally onto its image, contracting the (-2)-curves Γ such that $D \cdot \Gamma = 0$ to singularities of type ADE.*

For a divisor D such that the linear system $|D|$ has no base points, we denote by φ_D the morphism associated to $|D|$. In case the linear system $|D|$ is hyperelliptic, we have the following.

Theorem 2.3 (See [Sai74, Proposition 5.7] and its proof). *Let $|D|$ be a complete linear system on the K3 surface X. Suppose that $|D|$ has no fixed components. Then $|D|$ is hyperelliptic if and only if either*

a) *$D^2 \geq 4$, and there is a fiber of an elliptic fibration F such that $F \cdot D = 2$; or*

b) *$D^2 \geq 4$, and there is an irreducible curve D' with $D'^2 = 2$ and $D \equiv 2D'$ (thus in that case $D^2 = 8$);*
then the image of the map φ_D is the Veronese surface; or

c) *$D^2 = 2$;*
then φ_D is a double cover of $\mathbb{P}^2$.

In case a), the image of the associated map $\varphi_D \colon X \to \mathbb{P}^{\frac{1}{2}D^2+1}$ is a rational normal scroll of degree $\frac{1}{2}D^2$, except in the following three cases:

i) *$D \equiv 4F + 2\Gamma$, where F is a fiber and Γ is a (-2)-curve such that $F\Gamma = 1$.*
In that case, $D^2 = 8$, and $\varphi_D(X)$ is a cone in $\mathbb{P}^5$ over a rational normal quartic curve in a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$. The map φ_D factors through $X \xrightarrow{\varphi} \mathbf{F}_4 \to \mathbb{P}^5$, where φ is a morphism onto the Hirzebruch surface $\mathbf{F}_4$ and $\mathbf{F}_4 \to \mathbb{P}^5$ is the contraction map of the unique section s such that $s^2 = -4$. The branch locus of φ is the union of s and a reduced curve B' in the linear system $|3s + 12f|$ such that $s \cap B' = \emptyset$ (here f is a fiber of the unique fibration $\mathbf{F}_4 \to \mathbb{P}^1$).

ii) *$D \equiv 3F + 2\Gamma_0 + \Gamma_1$, where Γ_0, Γ_1 are (-2)-curves such that $\Gamma_0 \cdot F = 1$, $\Gamma_1 \cdot F = 0$ and $\Gamma_0 \cdot \Gamma_1 = 1$.*
In that case, $D^2 = 6$, and $\varphi_D(X)$ is a cone in $\mathbb{P}^4$ over a rational normal cubic curve in a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$. There is a factorization of φ_D through $X \xrightarrow{\varphi} \mathbf{F}_3 \to \mathbb{P}^4$. The branch locus of φ is the union of the unique section s such that $s^2 = -3$ and a reduced curve B' in the linear system $|3s + 10f|$ such that $sB' = 1$.

iii) *u) $D \equiv 2F + \Gamma_0 + \Gamma_1$, where Γ_0, Γ_1 are (-2)-curves such that $\Gamma_0 \cdot F = \Gamma_1 \cdot F = 1$ and $\Gamma_0 \cdot \Gamma_1 = 0$ (then $D^2 = 4$).*
v) $D \equiv 2F + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_n + \Gamma_{n+1} + \Gamma_{n+2}$, where $n \geq 0$ and the Γ_i are (-2)-curves, such that $D^2 = 4$.

In cases iii) u) and iii) v), $\varphi_D(X)$ is a quadric cone in $\mathbb{P}^3$, and there is a factorization $X \xrightarrow{\varphi} \mathbf{F}_2 \to \varphi_D(X)$. In these two cases, the branch locus B of φ is in $|4s + 8f|$. In case u), B does not contain the section s such that $s^2 = -2$; in case v), one has $B = s + B'$, with B' reduced and $sB' = 2$.

Remark 2.4. For brevity, we will say that a divisor D is base-point free or is hyperelliptic if the associated linear system $|D|$ is; we hope this will not induce any confusion to the reader.

2.3. About the computations

Let X be a K3 surface with finite automorphism group and Néron–Severi lattice L of rank $\rho > 2$. We did our search of (-2)-curves on X using an algorithm of Shimada [Shi14, Section 3] as described in [Rou19].

The inputs of that algorithm are the Gram matrix of a basis of the Néron–Severi lattice and an ample class. For each Néron–Severi lattice L involved, we computed such an ample class D as follows. The lattices L in this paper are mainly of the form

$$L = U(k) \oplus K \quad \text{or} \quad L = [m] \oplus K \quad (k, m \in \mathbb{N}^*),$$

where K is a direct sum of **ADE** lattices. To obtain an ample divisor $D \in L$, we construct a divisor $D' \in L$ such that on a set of roots of K which is also a basis of K, the divisor D' has positive intersection. Then we add a suitable multiple a of an element u in the $U[k]$- or $[m]$-part with positive square so that one gets a divisor $D'' = au + D' \in L$ with $(D'')^2 > 0$. We then check that the divisor D'' is ample by verifying that the negative-definite orthogonal complement $D''^{\perp}$ does not contain roots, *i.e.*, elements v such that $v^2 = -2$ (if this is not the case, then we increase the parameter a). Then, the transitivity of the action of the Weyl group

$$W = \langle s_\delta \colon x \to x + (x \cdot \delta)\delta \mid \delta \in \Delta \rangle$$

(with $\Delta = \{\delta \in L \mid \delta^2 = -2\}$) on the chambers of the positive cone (see [Huy16, Proposition 8.2.6]) allows us to choose $D = D''$ as an ample class for the K3 surfaces with lattice L.

In many cases, the first-found ample divisor D is such that D^2 is large. By computing the (-2)-curves on the K3 surface and by using Shimada's algorithm in [Rou19], we are able to find ample classes with smaller self-intersection. For each lattice, we give the ample class D with the smallest D^2 we found.

With that knowledge of an ample class D, one can run Shimada's algorithm for the computation of the (classes of) the (-2)-curves C on X which have degree $C \cdot D$ less than or equal to a fixed bound. We then use the test function in [ACL21] to check that we obtain the complete list of (-2)-curves, which worked well (and confirmed in another way the already known cases by Nikulin), except for the cases

$$U(2) \oplus \mathbf{A}_1^{\oplus 7}, \quad U \oplus \mathbf{A}_1^{\oplus 8}, \quad U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 5}, \quad U \oplus \mathbf{D}_4^{\oplus 2} \oplus \mathbf{A}_1^{\oplus 2}, \quad U \oplus \mathbf{D}_8 \oplus \mathbf{A}_1^{\oplus 3},$$

where the computation were too heavy to finish (there are too many facets on the effective cone), and thus we have only a lower bound. For these cases, one can obtain the exact number of (-2)-curves using the following approach communicated to us by one of the referees.

One starts by embedding the lattice $\mathrm{NS}(X)$ into another Néron–Severi lattice NS' of larger rank for which one has already determined the set of classes of (-2)-curves. For example, let us take

$$II_{1,17} = U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8$$

for NS', as in Section 15.4. The set of (-2)-curves $A_1, \dots, A_{19}$ and their configuration had been determined in the classical work of Vinberg [Vin75]. Let $\mathcal{P}_{18}$ be the positive cone of $II_{1,17} \otimes \mathbb{R}$ containing an ample class and $\mathcal{N}_{18}$ the set of the closures of connected components of the complement of the union of all hyperplanes $(r)^{\perp}$ of $\mathcal{P}_{18}$ defined by (-2)-vectors r. Suppose that we have a primitive embedding

$$\mathrm{NS}(X) \hookrightarrow \mathrm{NS}' = II_{1,17},$$

and let $\mathcal{P}_X$ be the positive cone of $\mathrm{NS}(X) \otimes S$ containing an ample class. We assume that $\mathcal{P}_X$ is embedded into $\mathcal{P}_{18}$ and regard $\mathcal{P}_X$ as a subspace of $\mathcal{P}_{18}$. We consider the closed subsets

$$\mathcal{P}_X \cap N' \quad (N' \in \mathcal{N}_{18})$$

of $\mathcal{P}_X$ that contain a non-empty open subset of $\mathcal{P}_X$. In [Shi15], these closed subsets $\mathcal{P}_X \cap N'$ are called induced chambers. Let N_X be the closure of the ample cone of $\mathrm{NS}(X)$. Since N_X has only finitely many walls, the cone N_X is tessellated by a finite number of induced chambers. By the method in [Shi15], we can determine the set of induced chambers contained in N_X, and hence the set of walls of N_X, that is, the classes of (-2)-curves on X. More precisely, let G be the subgroup of $O(\mathrm{NS}(X))$ consisting of the isometries $g \in O(\mathrm{NS}(X))$ that preserve N_X and extend to an isometry of $\mathrm{NS}' = II_{1,17}$. We can calculate a complete set of representatives of orbits of the action of G on the set of induced chambers contained in N_X.

For each NS(X) in the above five lattices, we embed NS(X) into $II_{1,17}$ primitively as follows. Let ρ be the rank of NS(X), and let $A_{i_1},\dots,A_{i_{18-\rho}}$ be the first $18-\rho$ elements of the sequence

$$A_3, A_5, A_7, A_9, A_1, A_{17}, A_{15}, A_{13}, A_{11}$$

of (-2)-curves $A_1,\dots,A_{19}$ in Section 15.4. Then NS(X) is isometric to the orthogonal complement of these classes $A_{i_1},\dots,A_{i_{18-\rho}}$ in $II_{1,17}$. The order of G and the orbit decomposition of induced chambers in N_X under the action of G are given in the following table, where, for example, 42456960 and $[1,55]$, $[2,7]$ in the first line mean that there exist 55 orbits of size $|G|$ and 7 orbits of size $|G|/2$ (with stabilizer group of order 2), and hence there exist

$$\left(55+\frac{7}{2}\right)|G| = 42456960$$

induced chambers contained in N_X.

ρ	NS(X)	#(-2)	$\lvert G\rvert$	Chambers	Orbits
9	$U(2)\oplus \mathbf{A}_1^{\oplus 7}$	240	725760	42456960	$[1,55],[2,7]$
10	$U\oplus \mathbf{A}_1^{\oplus 8}$	145	80640	30683520	$[1,372],[2,17]$
11	$U\oplus \mathbf{D}_4\oplus \mathbf{A}_1^{\oplus 5}$	90	10080	10800720	$[1,1061],[2,21]$
12	$U\oplus \mathbf{D}_4^{\oplus 2}\oplus \mathbf{A}_1^{\oplus 2}$	59	1440	2400480	$[1,1649],[2,36]$
13	$U\oplus \mathbf{D}_8\oplus \mathbf{A}_1^{\oplus 3}$	39	240	376200	$[1,1557],[2,21]$

We give at the end of this atlas a table with the number of (-2)-curves in the 118 different cases.

One can also compute the finite set of all fibrations on X as follows. By [Kov94, Proposition 2.4], if the class of a fiber F in NS(X) is indecomposable (*i.e.*, is not the sum of two effective divisors), then it is an extremal class in the closure of the effective cone. But by [Kov94, Theorem 6.1], since there are only a finite number of (-2)-curves in our surface X, the (-2)-curves are the extremal classes. Therefore, there is a singular fiber of the elliptic fibration which is the sum of (-2)-curves on X.

Now Kodaira's classification gives a finite number of possibilities for the types $\tilde{\mathbf{A}}_n$, $\tilde{\mathbf{D}}_n$, $\tilde{\mathbf{E}}_n$ of the reducible fibers F on X. Since a fiber F in NS(X) is a sum of (-2)-curves (which are finitely many), there is a finite number of possible degrees FD of fibers F with respect to D. Thus there is an upper bound for FD, so that one can compute all fibrations on X using Shimada's algorithm (see [Rou19]).

For the computations, we used Magma [BCP97]. Our algorithms are available as ancillary files of the submission of this paper on arXiv.

2.4. Algorithms for double planes

In this paper, many double plane models of K3 surfaces are studied. Algorithms to obtain geometric properties of the double covering (singularities of the branch curve B, configuration of irreducible components of B, enumeration of splitting curves, ...) from the lattice data are developed in the paper [Shi10]. They can be modified to the case where the linear system has a fixed component. Many arguments in the present paper (expressing a class D_2 such that $(D_2)^2 = 2$ as various sums of classes of (-2)-curves with non-negative coefficients) can be automated by these algorithms.

2.5. Nikulin star-shaped dual graphs and lattices of type $U\oplus K$

In [Nik83], Nikulin obtained the following result.

Proposition 2.5 ([Nik83, Corollary 1.6.5)]; see also [Kon89, Lemma 3.1]). *Let X be a K3 surface. Assume that* NS$(X)\simeq U\oplus K$.

- *i) There is an elliptic pencil $\pi\colon X\to\mathbb{P}^1$ with a section. A fiber and the section generate the lattice isometric to U.*

ii) If $K = \mathbf{G} \oplus K'$ with the lattice $\mathbf{G}$ generated by irreducible elements of square -2, then π has a singular fiber of type $\tilde{\mathbf{G}}$, where $\mathbf{G}$ is among the lattices

$$\mathbf{A}_n,\ n \geq 1, \quad \mathbf{D}_n,\ n \geq 4, \quad \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8.$$

Assume $\mathrm{NS}(X) \simeq U \oplus \bigoplus_{i \in I} \mathbf{G}^{(i)}$, where the $\mathbf{G}^{(i)}$ are lattices generated by irreducible elements of square -2. The dual graph of the (-2)-curves contained in the singular fibers of type $\tilde{\mathbf{G}}^{(i)}$ and the section form a (so-called) star-shaped graph (see [Nik83]), which we denote by $\mathrm{St}(\mathrm{NS}(X))$.

Theorem 2.6 (See [Nik83, Theorem 0.2.2 and Corollary 3.9.1]). *Let X be a K3 surface with Néron–Severi lattice isometric to $U \oplus K$, where K is in the following list:*

$$\mathbf{A}_2; \quad \mathbf{A}_1 \oplus \mathbf{A}_2, \mathbf{A}_3; \quad \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_2, \mathbf{A}_2^{\oplus 2}, \mathbf{A}_1 \oplus \mathbf{A}_3, \mathbf{A}_4;$$
$$\mathbf{A}_1 \oplus \mathbf{A}_2^{\oplus 2}, \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_3, \mathbf{A}_2 \oplus \mathbf{A}_3, \mathbf{A}_1 \oplus \mathbf{A}_4, \mathbf{A}_5, \mathbf{D}_5;$$
$$\mathbf{A}_2^{\oplus 3}, \mathbf{A}_3^{\oplus 2}, \mathbf{A}_2 \oplus \mathbf{A}_4, \mathbf{A}_1 \oplus \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_2 \oplus \mathbf{D}_4, \mathbf{A}_1 \oplus \mathbf{D}_5, \mathbf{E}_6; \quad \mathbf{A}_7,$$
$$\mathbf{A}_3 \oplus \mathbf{D}_4, \mathbf{A}_2 \oplus \mathbf{D}_5, \mathbf{D}_7, \mathbf{A}_1 \oplus \mathbf{E}_6; \quad \mathbf{A}_2 \oplus \mathbf{E}_6; \quad \mathbf{A}_2 \oplus \mathbf{E}_8; \quad \mathbf{A}_3 \oplus \mathbf{E}_8.$$

The K3 surface has finite automorphism group, and $\mathrm{St}(\mathrm{NS}(X))$ *is the dual graph of all* (-2)*-curves on X.*

Let X be a K3 surface such that $\mathrm{NS}(X) \simeq U \oplus \bigoplus_{i \in I} \mathbf{G}^{(i)}$, where the $\mathbf{G}^{(i)}$ are lattices generated by irreducible elements of square -2. Let F be a fiber of the natural fibration $\pi\colon X \to \mathbb{P}^1$ and E be the section as in Proposition 2.5. The divisor $D_2 = 2F + E$ is nef of square 2, with base points since $D_2 F = 1$. By Theorem 2.3, the divisor $D_8 = 2D_2$ is base-point free and hyperelliptic, and it defines a morphism $X \to \mathbb{P}^5$ which factors through the Hirzebruch surface $\mathbf{F}_4$ so that the branch locus of $\eta\colon X \to \mathbf{F}_4$ is the disjoint union of the unique section s such that $s^2 = -4$ and B, a curve in the linear system $|3s + 12f|$, where f is a fiber of the unique fibration of $\mathbf{F}_4$. We immediately have the folowing.

Proposition 2.7. *The image by η of the section E is the section s; the pull-back on X of the pencil $|f|$ is the pencil of elliptic curves in the elliptic fibration $\pi\colon X \to \mathbb{P}^1$. A singular fiber of π of type $\tilde{\mathbf{A}}_n$, $\tilde{\mathbf{D}}_n$, $\tilde{\mathbf{E}}_n$ is mapped onto a fiber of $\mathbf{F}_4 \to \mathbb{P}^1$ that cuts B at, respectively, an $\mathbf{a}_n$, $\mathbf{d}_n$, $\mathbf{e}_n$ singularity of B.*

2.6. About the famous 95, their moduli spaces and K3 surfaces with finite automorphism group

In [Bel02] are studied the Néron–Severi lattices of the so-called “famous 95” families of K3 surfaces. These “famous 95” have been constructed by Reid (unpublished); the list of these families appeared in Yomemura [Yon90] from the point of view of singularity theory. The K3 surfaces involved are (singular) anti-canonical divisors in weighted projective threefolds $\mathbb{WP}^3 = \mathbb{WP}^3(\bar{a})$ (here $\bar{a}$ is the weight $\bar{a} = (a_1, \dots, a_4)$). As we will see, many of these K3 surfaces have finite automorphism group.

Let $d_{\bar{a}}$ be the degree of an anti-canonical divisor in $\mathbb{WP}^3$. It turns out that each general degree $d_{\bar{a}}$ surface $\bar{X}$ in $\mathbb{WP}^3$ has the same singularities, and its minimal desingularization is a K3 surface X. The main result of [Bel02] is the computation of the Néron–Severi lattice $\mathrm{NS}(X)$ for a general member of each of the 95 families.

For $\bar{a}$ among the 95 possible weights, let $L_{\bar{a}} \simeq \mathrm{NS}(X)$ be the lattice of the Néron–Severi lattice of a general K3 surface X with singular model $\bar{X} \subset \mathbb{WP}^3(\bar{a})$, and let $\mathcal{M}_{\bar{a}}$ be the moduli space of $L_{\bar{a}}$-polarized K3 surfaces.

Proposition 2.8. *There is a birational map between $\mathcal{M}_{\bar{a}}$ and the moduli space of degree $d_{\bar{a}}$ surfaces in $\mathbb{WP}^3(\bar{a})$ modulo automorphisms.*

Proof. For each of the 95 cases of $\bar{a}$, we compute the dimension of the quotient space $\mathcal{Q} = \mathbb{P}(H^0(\mathbb{WP}^3(\bar{a}), \mathcal{O}(d_{\bar{a}}))^*)/\mathrm{Aut}(\mathbb{WP}^3(\bar{a}))$, using the formula

$$\dim \mathrm{Aut}(\mathbb{WP}^3(\bar{a})) = -1 + \sum h^0(\mathbb{WP}^3, \mathcal{O}(a_k)).$$

The 95 weights $\bar{a}$ are given in [Bel02, Table 3]; the degrees $d_{\bar{a}}$ can be found in [Yon90, Table 4.6]. It turns out that this quotient has dimension $20-\rho$, where $\rho = \mathrm{rank}(L_{\bar{a}})$ is the Picard number. There is a natural injective map from an open set in $\mathcal{Q}$ to the moduli space $\mathcal{M}_{\bar{a}}$; since $\mathcal{M}_{\bar{a}}$ is also $(20-\rho)$-dimensional, both spaces are birational. □

A direct consequence of Proposition 2.8 is the following.

Corollary 2.9. *The moduli spaces of the famous* 95 *polarized K3 surfaces are unirational.*

Comparing with Nikulin and Vinberg's lists, among the lattices $L_{\bar{a}}$ associated to the 95 weights $\bar{a}$, at least 43 lattices are such that the general K3 surface X with $\mathrm{NS}(X) \simeq L_{\bar{a}}$ (and a model $\bar{X} \subset \mathbb{WP}^3(\bar{a})$) has only a finite number of automorphisms. However, there are some weights $\bar{a}$, $\bar{b}$ among the 95 such that $L_{\bar{a}} \simeq L_{\bar{b}}$ (then there are two singular models of the same K3 surface; see [KM12]). Without counting the repetitions, one get (at least) 29 lattices which are lattices among the famous 95 and are also lattices of K3 surfaces with finite automorphism group. These lattices are

$$[2],\ [4],\ U,\ U(2),\ U\oplus\mathbf{A}_1,\ U(2)\oplus\mathbf{A}_1,\ U\oplus\mathbf{A}_2,\ U(2)\oplus\mathbf{D}_4,\ U\oplus\mathbf{D}_4,$$
$$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1,\ U\oplus\mathbf{D}_5,\ U\oplus\mathbf{A}_2\oplus\mathbf{D}_4,\ U\oplus\mathbf{E}_6,\ U\oplus\mathbf{E}_7,\ U\oplus\mathbf{A}_1\oplus\mathbf{E}_6,$$
$$U\oplus\mathbf{E}_8,\ U\oplus\mathbf{E}_7\oplus\mathbf{A}_1,\ U\oplus\mathbf{D}_4^{\oplus 2},\ U\oplus\mathbf{A}_2\oplus\mathbf{E}_6,\ U\oplus\mathbf{E}_8\oplus\mathbf{A}_1,$$
$$U(2)\oplus\mathbf{D}_4^{\oplus 2},\ U\oplus\mathbf{A}_2\oplus\mathbf{E}_8,\ U\oplus\mathbf{E}_8\oplus\mathbf{A}_1^{\oplus 3},\ U\oplus\mathbf{D}_8\oplus\mathbf{A}_1^{\oplus 3},\ U\oplus\mathbf{D}_8\oplus\mathbf{D}_4,$$
$$U\oplus\mathbf{E}_8\oplus\mathbf{D}_6,\ U\oplus\mathbf{E}_8\oplus\mathbf{E}_7,\ U\oplus\mathbf{E}_8\oplus\mathbf{E}_8,\ U\oplus\mathbf{E}_8\oplus\mathbf{E}_8\oplus\mathbf{A}_1,$$

plus the lattice $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$. For these lattices $L_{\bar{a}}$, we thus have a complete description of the K3 surfaces X with finite-order automorphism group with $\mathrm{NS}(X) \simeq L_{\bar{a}}$ as singular model(s) in weighted projective space(s).

It is worth mentioning that the 95 families have mirrors (see [Bel02]); not all mirrors are among the 95 families, but many mirrors are K3 surfaces with finite automorphism group. The families that do not already appear in the above list are

$$U\oplus\mathbf{A}_1^{\oplus 2},\ U(3)\oplus\mathbf{A}_2,\ U\oplus\mathbf{A}_1\oplus\mathbf{A}_2,\ U\oplus\mathbf{A}_2^{\oplus 2},$$
$$U\oplus\mathbf{A}_2\oplus\mathbf{A}_3,\ U\oplus\mathbf{A}_5,\ U\oplus\mathbf{E}_8\oplus\mathbf{A}_3.$$

2.7. On the automorphism groups

In [Kon89], Kondo studies the automorphism group of a general K3 surface with a finite number of (-2)-curves and Picard number at least 3. The main result of Kondo's paper [Kon89, Table 1] is that for a certain list of 12 lattices among the 118 lattices, the automorphism group is $(\mathbb{Z}/2\mathbb{Z})^2$, that for $U\oplus\mathbf{E}_8^{\oplus 2}\oplus A_1$ it is $\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z}$ and that otherwise, for the remaining 105 families, it is either the trivial group or $\mathbb{Z}/2\mathbb{Z}$.

Our study enables us to construct a hyperelliptic involution for surfaces in 90 out of 105 families, and we prove that the surfaces in the remaining 15 families have trivial automorphism group, thus completing in that way the results of Kondo. These results are summarized in the table in Section 16.

Remark 2.10. For the "general" assumption on the surface, which can be made more precise, we refer to the introduction of Kondo's paper [Kon89]. That hypothesis is important since Kondo constructed special K3 surfaces with finite automorphism group isomorphic to $\mathbb{Z}/42\mathbb{Z}$ or $\mathbb{Z}/66\mathbb{Z}$ (see [Kon86]). When computing the automorphism group, the K3 surfaces we consider are always supposed general.

When the K3 surface X has automorphism group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and Picard number at most 14, we describe the two hyperelliptic involutions generating the automorphism group $\mathrm{Aut}(X)$; when the Picard number is larger, we refer to the description in [Kon89].

For computing the automorphism group of a K3 surface X, it is important to know the image of the natural map

$$\varphi\colon\ \mathrm{Aut}(X)\to O(\mathrm{NS}(X)).$$

It turns out that when the automorphism group is not trivial, there is always a hyperelliptic involution σ. Using our description of the set of (-2)-curves, one can understand when an involution σ is in the kernel of φ. One may also use Proposition 2.7 in the case when the K3 lattice is of type $U \oplus \bigoplus \tilde{\mathbf{G}}_j$ as follows. When a K3 surface is the double cover branched over a curve with only singularities of type

$$\mathbf{a}_1, \mathbf{d}_4, \mathbf{d}_6, \mathbf{d}_8, \mathbf{e}_7, \mathbf{e}_8,$$

the action on (-2)-curves in the star of $U \oplus \bigoplus \tilde{\mathbf{G}}_j$ is trivial, and if there are any singularities of type

$$\mathbf{a}_2, \dots, \mathbf{a}_5, \mathbf{d}_5, \mathbf{d}_7, \mathbf{e}_6,$$

then the involution acts non-trivially on the set of (-2)-curves in the star of $U \oplus \bigoplus \tilde{\mathbf{G}}_j$. Since the Néron–Severi lattice is always generated by these (-2)-curves, one can then understand when an involution is in the kernel of φ. In the table in Section 16, we indicate when the action of a hyperelliptic involution on the Néron–Severi lattice is not trivial.

Let X be a K3 surface for which Kondo proved that $\mathrm{Aut}(X)$ is trivial or $\mathbb{Z}/2\mathbb{Z}$.

Proposition 2.11. *Suppose that X is general and the rank of $\mathrm{NS}(X)$ is less than or equal to 8 and*

i) there is no (-2)-curve A and fiber F such that $A \cdot F = 1$;
ii) there is no big and nef divisor D such that $D \cdot F = 2$ for a fiber F;
iii) there is no big and nef divisor D such that $D^2 = 2$.

Then the automorphism group of X is trivial.

Here, a fiber F means an irreducible curve with $F^2 = 0$.

Proof. By Theorems 2.2 and 2.3, the hypotheses i), ii) and iii) imply that there is no hyperelliptic involution acting on X. By [Kon89, Table 1 and Lemma 2.3], since the rank of $\mathrm{NS}(X)$ is less than or equal to 8, the automorphism group is either trivial or generated by a hyperelliptic involution; thus it must be trivial. □

2.8. On the irreducibility of the moduli spaces

The aim of this section is to prove the following result.

Proposition 2.12. *The 118 moduli spaces of K3 surfaces with Picard number at least 3 and finite automorphism group are irreducible.*

Let us recall that if L is an even lattice of rank ρ and signature $(1, \rho - 1)$, we denote by $\mathcal{M}_L$ the moduli space of K3 surfaces X polarized by a primitive embedding $j_X : L \hookrightarrow \mathrm{NS}(X)$. The moduli space $\mathcal{M}_L$ may depend upon the choice of the embedding of L in the K3 lattice $\Lambda_{K3} = U^{\oplus 3} \oplus \mathbf{E}_8^{\oplus 2}$: two non-isometric embeddings will give two different moduli spaces (see [Nik80] or [Dol96]). For that question, one can use the following result.

Theorem 2.13 (See [Huy16, Theorem 14.1.12]). *Let Λ be an even unimodular lattice of signature (n_+, n_-) and M be an even lattice of signature (m_+, m_-). If $m_+ < n_+$, $m_- < n_-$ and*

$$\ell(M) + 2 \leq \mathrm{rk}(\Lambda) - \mathrm{rk}(M), \tag{2.1}$$

then there exists a primitive embedding $M \hookrightarrow \Lambda$, which is unique up to automorphisms of Λ.

Here $\ell(M)$ is the minimal number of generators of the discriminant group of M and $\mathrm{rk}(M)$ is its rank.

Remark 2.14. Since $\ell(M) \leq \mathrm{rk}(M)$, if $\mathrm{rk}(M) \leq \frac{1}{2}(\mathrm{rk}(\Lambda) - 2)$, then condition (2.1) is verified.

In our situation, when $\Lambda = \Lambda_{K3}$ and $M = L$ is an even lattice of signature $(1, \mathrm{rk}(L) - 1)$, one has the following.

Corollary 2.15. *Suppose $\ell(L) \leq 20 - \mathrm{rk}(L)$. Then the primitive embedding $L \hookrightarrow \Lambda_{K3}$ is unique up to automorphisms.*

Suppose that the embedding $j\colon L \to \Lambda_{K3}$ is unique up to automorphisms. The following criterion of Dolgachev may be used to check if the moduli space $\mathcal{M}_L$ is irreducible.

Theorem 2.16 (See [Dol96, Proposition 5.6]). *Suppose that $L^\perp \subset \Lambda_{K3}$ contains a sublattice isometric to U or $U(2)$. Then the moduli $\mathcal{M}_L$ is irreducible.*

Let L be an even lattice of signature $(1, \mathrm{rk}(L)-1)$ such that $\ell(L) \leq 18 - \mathrm{rk}(L)$.

Corollary 2.17. *The moduli space $\mathcal{M}_L$ is irreducible.*

Proof. By Theorem 2.13, one can find a primitive embedding of L into the sublattice $U^{\oplus 2} \oplus \mathbf{E}_8^{\oplus 2}$ of Λ_{K3}. Then the orthogonal complement of L in Λ_{K3} contains a copy of U, and therefore by Theorem 2.16, the moduli space $\mathcal{M}_L$ is irreducible. □

Using that $\ell(L) \leq \mathrm{rk}(L)$, one obtains the following.

Corollary 2.18. *Suppose moreover that L has rank at most 9. Then the moduli space $\mathcal{M}_L$ is irreducible.*

The following table gives the discriminant groups of ADE lattices:

L	$\mathbf{A}_n\,(n \geq 1)$	$\mathbf{D}_{2n}\,(n \geq 2)$	$\mathbf{D}_{2n+1}\,(n \geq 2)$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
L^*/L	$\mathbb{Z}/(n+1)\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\{0\}$

There are 28 lattices L of rank at least 10 such that a K3 surface X with $\mathrm{NS}(X) \simeq L$ has finite automorphism group. For L among these lattices with $L \neq U \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_1$, using the above table and Corollary 2.17, one obtains that the moduli $\mathcal{M}_L$ is irreducible. For $L = U \oplus \mathbf{E}_8^{\oplus 2} \oplus \mathbf{A}_1$, the embedding in $U^{\oplus 3} \oplus \mathbf{E}_8^{\oplus 2}$ is unique up to automorphisms, and one sees that its orthogonal complement contains a copy of U; therefore, $\mathcal{M}_L$ is also irreducible. We have therefore proved Proposition 2.12.

3. Rank 3 lattices

3.1. Rank 3 cases

In [Nik85], Nikulin classifies rank 3 lattices that are Néron–Severi lattices of K3 surfaces with a finite number of automorphisms. Let us describe the classification when the fundamental domain of the associated Weyl group is not compact (see [Nik85, Section 2]).

Let $S_{1,1,1}$ be the rank 3 lattice generated by vectors a, b, c with intersection matrix

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{pmatrix}.$$

For r, s, $t \in \mathbb{Z}$, let $S_{r,s,t}$ denote the sublattice of $S_{1,1,1}$ generated by ra, sb, tc. Let also define $S'_{4,1,2}$, the lattice generated by $2a+c$, b, $2c$, and $S'_{6,1,2}$, the lattice generated by $6a+c$, b, $2c$. In [Nik85, Theorem 2.5], Nikulin gives a list of lattices which are Néron–Severi lattices of K3 surfaces with finite automorphism group. These lattices are of the form $S_{r,s,t}$ or $S'_{r,s,t}$. In [Nik14], Nikulin observes that some lattices in the list are isometric, thus giving the same families of K3 surfaces. One has

$$S_{2,1,2} \simeq S_{4,1,1}, \quad S_{4,1,2} \simeq S_{8,1,1}, \quad S_{6,1,2} \simeq S_{12,1,1}, \quad S'_{6,1,2} \simeq S_{6,1,1},$$

and when the fundamental domain is not compact, there are only 20 distinct cases. In the compact case, there are 6 lattices $S_1, \dots, S_6$, see [Nik85], and their geometric description is given in [Rou19] (see also [ACL21] for their Cox ring). So the total number of (isomorphism classes of) lattices of rank 3 which are Néron–Severi lattices of K3 surfaces with finite automorphism group is 26. For each case, Nikulin gives the number and sometimes the configurations of the (-2)-curves.

3.2. The rank 3 and compact cases

In [Rou19], we studied the six lattices $S_1, \dots, S_6$ of rank 3 such that the fundamental domain associated to the Weyl group is compact. These lattices are

$$S_1 = [6] \oplus \mathbf{A}_1^{\oplus 2}, \quad S_2 = [36] \oplus \mathbf{A}_2, \quad S_3 = [12] \oplus \mathbf{A}_2,$$
$$S_4 \subset [60] \oplus \mathbf{A}_2, \quad S_5 = [4] \oplus \mathbf{A}_2, \quad S_6 \subset [132] \oplus \mathbf{A}_2,$$

where the two inclusions have index 3. For completeness, let us recall the obtained results.

Theorem 3.1. *The K3 surfaces of type S_1, S_4, S_5, S_6 are double covers of the plane branched over a smooth sextic curve C_6 and such that*

- *if the Néron–Severi lattice is isometric to S_1, the six (-2)-curves on X are pull-backs of three conics that are 6-tangent to the sextic C_6;*
- *if the Néron–Severi lattice is isometric to S_4, the four (-2)-curves on X are pull-backs of a line tritangent to C_6 and a conic 6-tangent to C_6;*
- *if the Néron–Severi lattice is isometric to S_5, the four (-2)-curves on X are pull-backs of two lines tritangent to C_6;*
- *if the Néron–Severi lattice is isometric to S_6, the six (-2)-curves on X are pull-backs of one 6-tangent conic and two cuspidal cubics that cut C_6 tangentially and at their cusps.*

Let X be a K3 surface that has a Néron–Severi lattice isometric to S_2. There are three quadrics in $\mathbb{P}^3$ such that each intersection with $X \hookrightarrow \mathbb{P}^3$ is the union of two smooth degree 4 rational curves. These six rational curves are the only (-2)-curves on X.

Let X be a K3 surface that has a Néron–Severi lattice isometric to S_3. There exist two hyperplanes sections such that each hyperplane section is a union of two smooth conics. These four conics are the only (-2)-curves on X.

The cases S_3 and S_4 are linked to the surfaces $S_{1,1,4}$ and $S_{1,1,3}$; see Sections 3.6 and 3.5 below, respectively. We will only add the following result.

Proposition 3.2. *A general K3 surface X with a Néron–Severi lattice isometric to S_2 or S_3 has trivial automorphism group.*

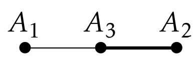
Proof. The Hilbert bases of the nef cones of these K3 surfaces are described in [ACL21]. Then as in Proposition 3.11 below, one can check that there is no hyperelliptic involution and conclude that the automorphism group is trivial. □

3.3. The lattice $S_{1,1,1}$

Let X be a K3 surface with rank 3 Néron–Severi lattice and intersection form

$$\begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -2 \end{pmatrix}.$$

The surface X contains three (-2)-curves A_1, A_2, A_3, with intersection matrix as above; their dual graph is



The divisor

$$D_{22} = 3A_1 + 6A_2 + 7A_3$$

is ample, of square 22, with $D_{22} \cdot A_1 = D_{22} \cdot A_3 = 1$, $D_{22} \cdot A_2 = 2$. The divisor

$$D_2 = A_1 + 2A_2 + 2A_3 \tag{3.1}$$

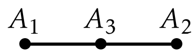
is nef, of square 2, with base points since $F = A_2 + A_3$ is a fiber of an elliptic fibration and $D_2F = 1$. It satisfies $D_2 \cdot A_1 = D_2 \cdot A_2 = 0$, $D_2 \cdot A_3 = 1$. The divisor $D_8 = 2D_2$ is base-point free and hyperelliptic. By Theorem 2.3, case i), we have the following.

Proposition 3.3. *The linear system $|D_8|$ defines a map $\varphi\colon X \to \mathbf{F}_4$ onto the Hirzebruch surface $\mathbf{F}_4$ such that the branch locus of φ is the disjoint union of the unique section s such that $s^2 = -4$ and a reduced curve B' in the linear system $|3s + 12f|$.*

By equation (3.1), the image of A_1 is the curve s, and the image of A_3 is the fiber through the point q onto which A_2 is contracted. The curve B' has a unique singularity, which is a node at q; its geometric genus is therefore 9.

3.4. The lattice $S_{1,1,2}$

Let X be a K3 surface with rank 3 Néron–Severi lattice of type $S_{1,1,2}$. The surface X contains three (-2)-curves A_1, A_2 , A_3, with dual graph



The divisor

$$D_{14} = 2A_1 + 2A_2 + 3A_3$$

is ample, of square 14, with $D_{14} \cdot A_1 = D_{14} \cdot A_2 = 2$ and $D_{14} \cdot A_3 = 4$. The divisor

$$D_2 = A_1 + A_2 + A_3$$

is nef, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_2 = 0$ and $D_2 \cdot A_3 = 2$. Thus, the following holds.

Proposition 3.4. *The K3 surface is the double cover of $\mathbb{P}^2$ branched over a sextic curve with two nodes p, q. The curves A_1, A_2 are contracted to p, q, and the image of A_3 is the line through p, q. That line cuts the sextic curve transversally in two other points.*

The Severi variety of plane curves of degree 6 with two nodes is rational, so the moduli space $\mathcal{M}_{S_{1,1,2}}$ of K3 surfaces with Néron–Severi lattice isometric to $S_{1,1,2}$ is unirational.

3.5. The lattice $S_{1,1,3}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,1,3}$. The surface X contains four (-2)-curves A_1, A_2, A_3, A_4, with intersection matrix

$$\begin{pmatrix} -2 & 3 & 2 & 0 \\ 3 & -2 & 0 & 2 \\ 2 & 0 & -2 & 6 \\ 0 & 2 & 6 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_2 = A_1 + A_2$$

is ample, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_2 = 1$ and $D_2 \cdot A_3 = D_2 \cdot A_4 = 2$. We have $2D_2 \equiv A_3 + A_4$; thus we have obtained the first part of the following proposition.

Proposition 3.5. *The surface X is the double cover of $\mathbb{P}^2$ branched over a smooth sextic C_6 which has a tritangent line and a 6-tangent conic. For general X, the sextic has equation*

$$C_6\colon \ell q g - f^2 = 0,$$

where ℓ, q, g and f are forms of degree 1, 2, 3 *and* 3, *respectively. The line $\ell = 0$ is the tritangent to the sextic, and the curve $q = 0$ is the* 6*-tangent conic. The moduli space $\mathcal{M}_{S_{1,1,3}}$ of K3 surfaces X with $\mathrm{NS}(X) \simeq S_{1,1,3}$ is unirational.*

Proof. Let $C_6 \colon \ell qg - f^2 = 0$ be a sextic curve as above, and let $Y \to \mathbb{P}^2$ be the double cover branched over C_6. Above the line $\ell = 0$ are two (-2)-curves A_1, A_2 such that $A_1 \cdot A_2 = 3$, and above the conic $q = 0$ are two (-2)-curves A_3, A_4 such that $A_3 \cdot A_4 = 6$. It remains to understand the intersections $A_j \cdot A_k$ for $j \in \{1, 2\}$ and $k \in \{3, 4\}$. This is done by computing Example 3.6 below, for which the intersection is (up to permutation of A_1, A_2 and A_3, A_4) the above intersection matrix of the four (-2)-curves on a K3 surface X with $\mathrm{NS}(X) \simeq S_{1,1,3}$. The intersection numbers remain the same for the (-2)-curves in that flat family of smooth surfaces.

If the equation of C_6 is general , the Picard number of Y is 3. The lattice generated by the (-2)-curves is $S_{1,1,3}$, of discriminant 18. The unique over-lattice containing $S_{1,1,3}$ is $S_{1,1,1}$, but a surface with Néron–Severi lattice isometric to $S_{1,1,1}$ contains only three (-2)-curves; thus $\mathrm{NS}(Y) \simeq S_{1,1,3}$.

In order to prove that a general K3 surface X with $\mathrm{NS}(X) \simeq S_{1,1,3}$ is branched over a sextic curve with an equation of the form $\ell qg - f^2 = 0$, let us consider the map

$$\Phi \colon H^0(\mathbb{P}^2, \mathcal{O}(1)) \oplus H^0(\mathbb{P}^2, \mathcal{O}(2)) \oplus H^0(\mathbb{P}^2, \mathcal{O}(3))^{\oplus 2} \to H^0(\mathbb{P}^2, \mathcal{O}(6))$$

defined by

$$w := (\ell, q, g, f) \mapsto f_{6,w} := \ell qg - f^2.$$

It is invariant under the action of the transformations $\Gamma \colon (\ell, q, g, f) \mapsto (\alpha\ell, \beta q, \gamma g, f)$ for $\alpha\beta\gamma = 1$. Suppose

$$\ell qg - f^2 = \ell' q' g' - f'^2$$

for $f' \neq \pm f$ and that the forms are chosen general so that the double cover Y branched over the sextic $f_{6,w} = 0$ has $\mathrm{NS}(Y) \simeq S_{1,1,3}$. Since the curves $\ell = 0$, $q = 0$ are the images of the four (-2)-curves on Y, up to rescaling by using a transformation Γ, one can suppose $\ell' = \ell$, $q' = q$, and then one obtains the relation

$$\ell q(g - g') = (f - f')(f + f').$$

Up to changing the sign of f', we can suppose that ℓ divides $f + f'$. If q does not divide $f + f'$, then one gets a codimension 1 family of such sextic curves; thus we can suppose that there exists a scalar α such that $\ell q = \alpha(f + f')$, and by solving the equations, we obtain that

$$f' = \frac{1}{\alpha}\ell q - f, \quad g' = g + \frac{1}{\alpha^2} lq - \frac{2}{\alpha} f.$$

The dimension of the (unirational) moduli space of sextic curves with an equation of the form $\ell qg - f^2 = 0$ is

$$(3 + 6 + 2 \cdot 10) - (9 + 2 + 1) = 17;$$

since $\mathcal{M}_{S_{1,1,3}}$ also has dimension 17, both spaces are birational. □

Example 3.6. Let us take

$$\begin{aligned} &\ell = 13x + 10y + z, \quad q = 4x^2 + 6xy + 26xz + 6yz + 23z^2, \\ &g = 9x^3 + 16x^2y + 5xy^2 + 24y^3 + 22x^2z + 26xyz + 4y^2z + 23xz^2 + 20yz^2 + 9z^3, \\ &f = 28x^3 + 8x^2y + 2xy^2 + 23y^3 + 2x^2z + 23xyz + 19y^2z + 24xz^2 + 20yz^2 + 17z^3. \end{aligned}$$

Let X be the associated K3 surface, and let X_p be its reduction modulo a prime p. Using the Tate conjectures, one finds that the K3 surfaces X_{23} and X_{29} have Picard number 4. By the Artin–Tate conjectures, one computes that

$$\begin{aligned} |\mathrm{Br}(X_{23})| \cdot |\mathrm{disc}(\mathrm{NS}(X_{23}))| = 3^2 \cdot 491, \\ |\mathrm{Br}(X_{29})| \cdot |\mathrm{disc}(\mathrm{NS}(X_{29}))| = 3^3 \cdot 5^2 \cdot 53, \end{aligned}$$

and using Van Luijk's trick (see [vLu07]), we conclude that the Picard number of X is 3 since the ratio of the two integers above is not a square (here Br is the Brauer group, and disc denotes the discriminant group).

Remark 3.7. In [Rou19], we study K3 surfaces with finite automorphism group and compact fundamental domain. The surface with Néron–Severi lattice of type S_4 (Nikulin's notation) is also a double cover of $\mathbb{P}^2$ branched over a smooth sextic which has one tritangent line and one 6-tangent conic. But the intersection matrix of the four (-2)-curves above the line and the conic (the only (-2) curves on that surface) is

$$\begin{pmatrix} -2 & 3 & 1 & 1 \\ 3 & -2 & 1 & 1 \\ 1 & 1 & -2 & 6 \\ 1 & 1 & 6 & -2 \end{pmatrix}.$$

Remark 3.8. By Section 2.8, we know that the unique moduli space $\mathcal{M}_{S_{1,1,3}}$ of $S_{1,1,3}$-polarized K3 surfaces is irreducible. By Proposition 3.5, there is a copy of $\mathcal{M}_{S_{1,1,3}}$ in the moduli space $\mathcal{M}_{[2]}$ of K3 surfaces with an ample divisor of square 2. Using the Hilbert basis of the nef cone, one can check that D_2 is the unique nef divisor of square 2 in the Néron–Severi group; therefore, the copy of $\mathcal{M}_{S_{1,1,3}}$ in $\mathcal{M}_{[2]}$ is unique.

3.6. The lattice $S_{1,1,4}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,1,4}$. The surface X contains four (-2)-curves A_1, A_2, A_3, A_4, with intersection matrix

$$M_1 = \begin{pmatrix} -2 & 4 & 0 & 2 \\ 4 & -2 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 2 & 0 & 4 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisors $F_1 = A_1 + A_4$ and $F_2 = A_2 + A_3$ are fibers of two distinct elliptic fibrations. The divisor

$$D_4 = A_1 + A_2 \equiv A_3 + A_4$$

is ample, of square 4, with $D_2 \cdot A_j = 2$ for $j \in \{1, \dots, 4\}$ and is non-hyperelliptic.

Proposition 3.9. *The surface X is a quartic in $\mathbb{P}^3$ which has two hyperplane sections which are unions of two conics. The general surface has a projective model of the form*

$$X : \ell_1\ell_2 q_3 - q_1 q_2 = 0 \hookrightarrow \mathbb{P}^3,$$

where ℓ_1, ℓ_2 are linear forms and q_1, q_2, q_3 are quadrics. The moduli space $\mathcal{M}_{S_{1,1,4}}$ of K3 surfaces X with $\mathrm{NS}(X) \simeq S_{1,1,4}$ *is unirational.*

Proof. Consider the map

$$\Phi : H^0(\mathbb{P}^3, \mathcal{O}(1))^{\oplus 2} \oplus H^0(\mathbb{P}^3, \mathcal{O}(2))^{\oplus 3} \to H^0(\mathbb{P}^3, \mathcal{O}(4))$$

defined by

$$w := (\ell_1, \ell_2, q_1, q_2, q_3) \mapsto Q_{4,w} := \ell_1\ell_2 q_1 - q_2 q_3.$$

By computing the differential $d\Phi_w$ at a randomly chosen point, we find that it has rank 33; thus the image of Φ is 33-dimensional, and the quotient W of that image by $GL_4(\mathbb{C})$ is at least 17-dimensional. The space W is unirational. For a general w, the quartic $Y : Q_{4,w} = 0$ is non-singular. The curves $A_1 : \ell_1 = q_2 = 0$, $A_2 : \ell_1 = q_3 = 0$, $A_3 : \ell_2 = q_2 = 0$, $A_4 : \ell_2 = q_3 = 0$ are (-2)-curves on Y with (up to permutation of A_3 and A_4) intersection matrix M_1. We thus obtain an injective map from W to the moduli space $\mathcal{M}_{S_{1,1,4}}$. Since $\mathcal{M}_{S_{1,1,4}}$ has dimension 17, this moduli space is unirational. □

Remark 3.10. a) One can construct, more geometrically, a member of $\mathcal{M}_{S_{1,1,4}}$ by considering two disjoint conics C_1 and C_3 in $\mathbb{P}^3$ and taking a general quartic among the 16-dimensional linear system of quartics containing them. We also obtain in that way a dominant map from a rational space to $\mathcal{M}_{S_{1,1,4}}$.

b) In [Rou19], we study K3 surfaces with finite automorphism group and compact fundamental domain. One of these surfaces, namely the surface with Néron–Severi lattice of type S_3 (Nikulin's notation) is also a quartic surface in $\mathbb{P}^3$ with two hyperplane sections which are the union of two conics. But the intersection matrix of the four (-2)-curves (which are the only (-2) curves on that surface) is

$$\begin{pmatrix} -2 & 4 & 1 & 1 \\ 4 & -2 & 1 & 1 \\ 1 & 1 & -2 & 4 \\ 1 & 1 & 4 & -2 \end{pmatrix}.$$

By using a construction as in part a), one finds that the moduli space of these surfaces is also unirational.

Proposition 3.11. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{1,1,4}$ is trivial.*

Proof. The walls of the nef cone are $A_k^\perp$, $k \in \{1, \dots, 4\}$. A Hilbert basis (see Definition 3.12) of the nef cone is given in [ACL21]; it is as follows. The divisors $H_1 = A_1 + A_4$ and $H_2 = A_2 + A_3$ are fibers of the two distinct elliptic fibrations on the K3 surface X; the seven remaining classes $H_3, \dots, H_9$ in the Hilbert basis are (in the basis A_1, A_2, A_3)

$$(1,1,0),(1,1,1),(1,2,0),(2,1,0),(2,1,1),(2,2,-1),(2,3,-1).$$

The intersection matrix $M_H = (H_i \cdot H_j)_{1 \le i,j \le 9}$ is

$$M_H = \begin{pmatrix} 0 & 8 & 4 & 8 & 8 & 4 & 8 & 4 & 8 \\ 8 & 0 & 4 & 4 & 4 & 8 & 8 & 8 & 8 \\ 4 & 4 & 4 & 6 & 6 & 6 & 8 & 6 & 8 \\ 8 & 4 & 6 & 6 & 10 & 8 & 8 & 12 & 16 \\ 8 & 4 & 6 & 10 & 6 & 12 & 16 & 8 & 8 \\ 4 & 8 & 6 & 8 & 12 & 6 & 8 & 10 & 16 \\ 8 & 8 & 8 & 8 & 16 & 8 & 8 & 16 & 24 \\ 4 & 8 & 6 & 12 & 8 & 10 & 16 & 6 & 8 \\ 8 & 8 & 8 & 16 & 8 & 16 & 24 & 8 & 8 \end{pmatrix}.$$

Let us check the condition of Proposition 2.11:

(i) Since the intersection matrix of the fibers H_1, H_2 with $A_1, \dots, A_4$ is

$$\begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix},$$

there are no (-2)-curves A and fibers F such that $AF = 1$.

(ii) A nef divisor D is a positive linear combination of the elements $H_1, \dots, H_9$ of the Hilbert basis of the nef cone. From the two first lines of the Gram matrix M_H, we see that there is no big and nef divisor D such that $D \cdot F = 2$ for any fiber $F = H_1$ or H_2.

(iii) From the matrix M_H, we see that there is no big and nef divisor D such that $D^2 = 2$.

Therefore, one can apply Proposition 2.11 and conclude that the automorphism group of X is trivial. □

Definition 3.12. Let us recall that if $C \subset \mathbb{R}^d$ is a polyhedral cone generated by integral vectors, a Hilbert basis $H(C)$ of C is a subset of integral vectors such that

a) each element of $C \cap \mathbb{Z}^d$ can be written as a non-negative integer combination of elements of $H(C)$, and

b) $H(C)$ has minimal cardinality with respect to all subsets of $C \cap \mathbb{Z}^d$ for which part a) holds.

3.7. The lattice $S_{1,1,6}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,1,6}$. The surface X contains six (-2)-curves $A_1,\dots,A_6$, with intersection matrix

$$\begin{pmatrix} -2 & 6 & 2 & 6 & 6 & 2 \\ 6 & -2 & 6 & 2 & 2 & 6 \\ 2 & 6 & -2 & 18 & 0 & 16 \\ 6 & 2 & 18 & -2 & 16 & 0 \\ 6 & 2 & 0 & 16 & -2 & 18 \\ 2 & 6 & 16 & 0 & 18 & -2 \end{pmatrix}.$$

The curves A_1, A_3, A_5 generate the Néron–Severi lattice. We have

$$A_2 = A_1 - 2A_3 + 2A_5, \quad A_4 = 4A_1 - 5A_3 + 4A_5, \quad A_6 = 4A_1 - 4A_3 + 3A_5.$$

The divisor

$$D_2 = A_1 - A_3 + A_5$$

is ample, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_2 = 2$ and $D_2 \cdot A_j = 4$ for $j \in \{3,\dots,6\}$. We have

$$2D_2 \equiv A_1 + A_2, \quad 4D_2 \equiv A_3 + A_4 \equiv A_5 + A_6.$$

Therefore, the following holds.

Proposition 3.13. *The K3 surface X is the double cover $\eta\colon X \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched over a smooth sextic curve which has a 6-tangent conic C, and there are two rational cuspidal quartics Q_1, Q_2 such that their three cusps are on the sextic curve and their remaining intersection points are tangent to the sextic (there are nine such points). The image by η of $A_1 + A_2$ is C, the image of $A_3 + A_4$ is Q_1, and the image of $A_5 + A_6$ is Q_2.*

3.8. The lattice $S_{1,1,8}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,1,8}$. The surface X contains eight (-2)-curves $A_1,\dots,A_8$, with intersection matrix

$$\begin{pmatrix} -2 & 2 & 0 & 8 & 16 & 8 & 18 & 14 \\ 2 & -2 & 8 & 0 & 8 & 16 & 14 & 18 \\ 0 & 8 & -2 & 14 & 18 & 2 & 16 & 8 \\ 8 & 0 & 14 & -2 & 2 & 18 & 8 & 16 \\ 16 & 8 & 18 & 2 & -2 & 14 & 0 & 8 \\ 8 & 16 & 2 & 18 & 14 & -2 & 8 & 0 \\ 18 & 14 & 16 & 8 & 0 & 8 & -2 & 2 \\ 14 & 18 & 8 & 16 & 8 & 0 & 2 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. We have

$$\begin{aligned} &A_4 = -2A_1 + 2A_2 + A_3, \quad && A_5 = -5A_1 + 3A_2 + 3A_3, \quad A_6 = -3A_1 + A_2 + 3A_3, \\ &A_7 = -6A_1 + 3A_2 + 4A_3, \quad && A_8 = -5A_1 + 2A_2 + 4A_3. \end{aligned}$$

The divisor

$$D_6 = -A_1 + A_2 + A_3$$

is ample, of square 6, base-point free and non-hyperelliptic, with $D_6 \cdot A_1 = D_6 \cdot A_2 = 4$, $D_6 \cdot A_3 = D_6 \cdot A_4 = 6$, $D_6 \cdot A_5 = D_6 \cdot A_6 = 10$, $D_6 \cdot A_7 = D_6 \cdot A_8 = 12$. The K3 surface X is a smooth complete intersection in $\mathbb{P}^4$. We remark that

$$A_3 + A_4 = 2D_6.$$

Proposition 3.14. *The automorphism group of a general K3 surface X with* $\mathrm{NS}(X) \simeq S_{1,1,4}$ *is trivial.*

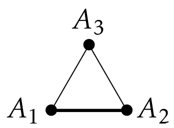
Proof. We proceed as in the proof of Proposition 3.11. The Hilbert basis of the nef cone of the K3 surface X is a set of 60 classes (see [ACL21, Table 1]), among which are the fibers

$$A_2 + A_3, \quad 4A_1 + 5A_2 - 7A_3, \quad 4A_1 + A_2 - 3A_3, \quad 8A_1 + 5A_2 - 11A_3.$$

Then one can check that any big and nef divisor D on X is base-point free, with $D^2 \geq 6$ or $D^2 = 0$. Moreover, there is no fiber F of an elliptic fibration such that $FD = 2$; thus by Theorems 2.2 and 2.3, there are no hyperelliptic involutions acting on X. From [Kon89, Table 1 and Lemma 2.3], the automorphism group of X is therefore trivial. □

3.9. The lattice $S_{1,2,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,2,1}$. The surface X contains three (-2)-curves $A_1, \dots, A_3$, with dual graph



These curves generate the Néron–Severi lattice. The divisor

$$D_6 = 2A_1 + 2A_2 + A_3$$

is ample, of square 6, with $D_6 \cdot A_1 = D_6 \cdot A_2 = 1$ and $D_6 \cdot A_3 = 2$. Since $D_6 \cdot (A_1 + A_2) = 2$ and $A_1 + A_2$ is a fiber, D_6 is hyperelliptic. The divisor

$$D_2 = A_1 + A_2 + A_3$$

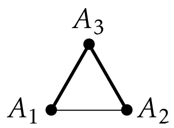
is nef, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_2 = 1$ and $D_2 \cdot A_3 = 0$.

Proposition 3.15. *The K3 surface X is a double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 which has a node at a point q and a line L containing the node, which is bitangent to C_6 at two points p_1, p_2. The moduli space $\mathcal{M}_{1,2,1}$ is unirational.*

Proof. It remains to prove the assertion on the moduli space $\mathcal{M}_{1,2,1}$. We identify this space with the moduli space of sextic curves with a node and a line through the node which is bitangent to the sextic. The imposition of a node and the tangency conditions are linear conditions on the space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ of sextic curves; thus the moduli space of such curves is unirational. □

3.10. The lattice $S_{1,3,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,3,1}$. The surface X contains three (-2)-curves A_1, A_2, A_3, with dual graph



These curves generate the Néron–Severi lattice. The divisor

$$D_4 = A_1 + A_2 + A_3$$

is very ample of square 4, with $D_4 \cdot A_1 = D_4 \cdot A_2 = 1$ and $D_4 \cdot A_3 = 2$.

Proposition 3.16. *The K3 surface X is a quartic surface in $\mathbb{P}^3$ with a hyperplane section which is the union of two lines and a conic. The general surface X with $\mathrm{NS}(X) \simeq S_{1,3,1}$ has an equation of the form*

$$\ell_1 f + q_1 \ell_2 \ell_3 = 0.$$

The moduli space $\mathcal{M}_{S_{1,3,1}}$ of such surfaces is unirational.

Proof. Let $Y\colon \ell_1 f + q_1\ell_2\ell_3 = 0$ be a quartic surface as in the proposition. The hyperplane section ℓ_1 cuts Y into a union of two lines $A_1\colon \ell_1 = \ell_2 = 0$, $A_2\colon \ell_1 = \ell_3 = 0$ and a conic $A_3\colon \ell_1 = q_1 = 0$. Their intersection graph is as above.

If the equation of Y is general , the Picard number of Y is 3; the lattice generated by the (-2)-curves is $S_{1,3,1}$, of discriminant 18. The unique over-lattice containing $S_{1,3,1}$ is $S_{1,1,1}$, but the configuration of the (-2)-curves in a surface with Néron–Severi lattice isometric to $S_{1,1,1}$ is different; thus $\mathrm{NS}(Y) \simeq S_{1,3,1}$.

Let us consider the map

$$\Phi\colon H^0(\mathbb{P}^3,\mathcal{O}(1))^{\oplus 3} \oplus H^0(\mathbb{P}^3,\mathcal{O}(2)) \oplus H^0(\mathbb{P}^3,\mathcal{O}(3)) \to H^0(\mathbb{P}^3,\mathcal{O}(4))$$

defined by

$$w := (\ell_1,\ell_2,\ell_3,q_1,f) \mapsto Q_{4,w} := \ell_1 f + q_1\ell_2\ell_3.$$

It is invariant under the action of the transformations $\Gamma\colon (\ell_1,\ell_2,\ell_3,q_1,f) \mapsto (\nu\ell_1,\alpha\ell_2,\beta\ell_3,\gamma q_1,\mu f)$ for $\nu\mu = 1$ and $\alpha\beta\gamma = 1$. Suppose

$$\ell_1 f + q_1\ell_2\ell_3 = \ell_1' f' + q_1'\ell_2'\ell_3' \tag{3.2}$$

and that the forms are chosen general so that the associated quartic satisfies $\mathrm{NS}(Y) \simeq S_{1,1,3}$. Since the lines $A_1\colon \ell_1 = \ell_2 = 0$, $A_2\colon \ell_1 = \ell_3 = 0$ and the conic $A_3\colon \ell_1 = q_1 = 0$ are the only (-2)-curves on X, we have (up to exchanging ℓ_2', ℓ_3')

$$A_1\colon \ell_1' = \ell_2' = 0, \quad A_2\colon \ell_1' = \ell_3' = 0, \quad A_3\colon \ell_1' = q_1' = 0.$$

The consequence is that—up to rescaling using a transformation Γ—one can suppose that there exist scalars α_1, α_2, α_3 and a linear form ℓ such that

$$\ell_1' = \ell_1, \quad \ell_2' = \alpha_2\ell_1 + \ell_2, \quad \ell_3' = \alpha_3\ell_1 + \ell_2, \quad q_1' = \alpha_1 q_1 + \ell_1\ell. \tag{3.3}$$

Substituting this into the formula (3.2) and taking the equation modulo ℓ_1, one finds that necessarily $\alpha_1 = 1$. Conversely, given forms as in equation (3.3), with $\alpha_1 = 1$, one can find a cubic form f' such that the linear forms ℓ_1', ℓ_2', ℓ_3', the quadric q_1' and the cubic f' satisfy equation (3.2). Therefore the dimension of the (unirational) moduli space $\mathcal{M}_4$ of quartic surfaces with an equation of type $\ell_1 f + q_1\ell_2\ell_3 = 0$ is

$$(3\cdot 4 + 10 + 20) - (16 + 1 + 2 + 6) = 17.$$

An open set of $\mathcal{M}_4$ is a subspace of the moduli space $\mathcal{M}_{S_{1,3,1}}$, which also has dimension 17; thus both spaces are birational. ☐

Proposition 3.17. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{1,3,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. The Hilbert basis of the nef cone of the K3 surface X is a set of four classes:

$$A_2 + A_3, \quad A_1 + A_3, \quad A_1 + A_2 + A_3, \quad 2A_1 + 2A_2 + A_1;$$

the first two classes are fibers of elliptic fibrations. The intersection matrix of these four classes is

$$\begin{pmatrix} 0 & 3 & 3 & 6 \\ 3 & 0 & 3 & 6 \\ 3 & 3 & 4 & 6 \\ 6 & 6 & 6 & 4 \end{pmatrix}.$$

From that one can check that any big and nef divisor D on X is base-point free, with $D^2 \geq 4$ or $D^2 = 0$, and there is no fiber F of an elliptic fibration such that $FD = 2$. Thus we conclude as in Proposition 3.11 that the automorphism group of X is trivial. ☐

3.11. The lattice $S_{1,4,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,4,1}$. The surface X contains four (-2)-curves A_1, A_2, A_3, A_4, with intersection matrix

$$\begin{pmatrix} -2 & 3 & 2 & 1 \\ 3 & -2 & 1 & 2 \\ 2 & 1 & -2 & 11 \\ 1 & 2 & 11 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_2 = A_1 + A_2$$

is ample, of square 2. We have

$$A_3 + A_4 \equiv 3D_2;$$

therefore, using the linear system $|D_2|$, we obtain the following.

Proposition 3.18. *The K3 surface X is the double cover $\eta\colon X \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched over a smooth sextic curve C_6 which has a tritangent line and a cuspidal cubic with a cusp on C_6 and such that the cuspidal cubic and C_6 are tangent at every other intersection points.*

The divisor $D_6 = A_1 + A_2 + A_3$ is very ample of square 6, with $D_6 \cdot A_j = 3, 2, 1, 14$ for $j = 1, \dots, 4$, so that A_1, A_2, A_3 are, respectively, a rational cubic, a conic and a line in a hyperplane of $\mathbb{P}^4$. That leads us to the following proposition.

Proposition 3.19. *The moduli space $\mathcal{M}_{S_{1,4,1}}$ of K3 surfaces with* $\mathrm{NS}(X) \simeq S_{1,4,1}$ *is unirational.*

Proof. We can construct a member of $\mathcal{M}_{S_{1,4,1}}$ as follows: Let $H_3 \subset \mathbb{P}^4$ be a hyperplane, let $B_3 \subset H_3$ be a line, let p_0, p_1, p_3 be three point on B_3, let $H_2 \subset H_3$ be a plane intersecting B_3 at p_0, let $B_2 \subset H_2$ be a smooth conic containing p_0, and let q_1, q_2, q_3 be three general points on B_2 (thus different from p_0). The moduli space of normal cubic curves passing through the points p_1, p_2, q_1, q_2, q_3 is unirational. Let $B_1 \subset H_3$ be such a rational normal cubic curve. By construction, for $j \neq k$ in $\{1, 2, 3\}$, the degree of the intersection scheme of B_j and B_k equals $A_j \cdot A_k$.

There is a unique quadric in H_3 that contains the curves B_1, B_2, B_3. The linear system of cubics in H_3 that contain these curves is 3-dimensional. Therefore, the linear systems L_2, L_3 of quadrics and cubics in $\mathbb{P}^4$ containing the three curves B_1, B_2, B_3 are, respectively, 4- and 18-dimensional.

Let $X \hookrightarrow \mathbb{P}^4$ be the degree 6 K3 surface which is the complete intersection of a general quadric in L_2 and a general cubic in L_3. It has Picard number at least 3, and there is an open subspace for which the Picard number is 3 since K3 surfaces with Néron–Severi lattice $S_{1,4,1}$ belong to that space. Thus for a general choice, the Néron–Severi lattice of X has rank 3 and contains the lattice $S_{1,4,1}$ generated by the (-2)-curves B_1, B_2, B_3. The only over-lattices of $S_{1,4,1}$ are $S_{1,1,1}$ and $S_{1,2,1}$. The K3 surfaces with such a Néron–Severi lattice contain only three (-2)-curves which do not have the same intersection matrix as the curves B_k, $k = 1, \dots, 3$. Thus $\mathrm{NS}(X) \simeq S_{1,4,1}$, and by the above construction, the moduli space $\mathcal{M}_{S_{1,4,1}}$ is unirational. □

Remark 3.20. a) We recall that a rational normal curve is a smooth, rational curve of degree n in projective n-space $\mathbb{P}^n$. Given $n+3$ points in $\mathbb{P}^n$ in linear general position (that is, with no $n+1$ lying in a hyperplane), there is a unique rational normal curve C passing through them. From, *e.g.*, [Har92, Theorem 1.18], the coefficients of the equations of that curve are rational functions in the coordinates of the $n+3$ points. In the above proof, for the unirationality of $\mathcal{M}_{S_{1,4,1}}$, we implicitly used the fact that the construction of a rational cubic curve passing through the four points in general position is rational in the coordinates of the points.

b) There are $\binom{n+2}{2} - 2n - 1$ independent quadrics that generate the ideal of a degree n rational normal curve in $\mathbb{P}^n$.

3.12. The lattice $S_{1,5,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,5,1}$. The surface X contains six (-2)-curves $A_1,\dots,A_6$, with intersection matrix

$$\begin{pmatrix} -2 & 6 & 4 & 2 & 2 & 14 \\ 6 & -2 & 2 & 4 & 14 & 2 \\ 4 & 2 & -2 & 11 & 1 & 23 \\ 2 & 4 & 11 & -2 & 23 & 1 \\ 2 & 14 & 1 & 23 & -2 & 66 \\ 14 & 2 & 23 & 1 & 66 & -2 \end{pmatrix}.$$

The curves A_1, A_3, A_5 generate the Néron–Severi lattice. The divisor

$$D_2 = 2A_1 + 2A_3 - A_5$$

is ample, of square 2, with

$$2D_2 \equiv A_1 + A_2, \quad 3D_2 \equiv A_3 + A_4, \quad 8D_2 \equiv A_5 + A_6.$$

Thus, by using the linear system $|D_2|$, we obtain the following.

Proposition 3.21. *The surface X is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve C_6 which has a 6-tangent conic, a tangent cuspidal cubic and a tangent rational cuspidal octic such that the cusps are on C_6, and at each intersection point of the octic or the cubic with C_6, the multiplicity is even.*

The divisor $D_4 = A_1 + A_3$ is very ample of square 4, with $D_4 \cdot A_j = 2, 8, 2, 13, 3, 37$ for $j = 1,\dots,6$. The divisor $D_8 = A_1 + A_3 + A_5$ is very ample of square 8, with $D_8 \cdot A_j = 4, 22, 3, 36, 1, 103$ for $j = 1,\dots,6$. This last model enables us to construct the surfaces with $\mathrm{NS}(X) \simeq S_{1,5,1}$ and to obtain the following proposition.

Proposition 3.22. *The moduli space $\mathcal{M}_{S_{1,5,1}}$ is unirational.*

Proof. Let us fix a hyperplane $H_4 \subset \mathbb{P}^5$, and let $B_3 \hookrightarrow H_4$ be a normal quartic curve. Let H_3 be a general hyperplane of H_4, let $q_1,\dots,q_4$ be the intersection points of B_3 with H_3. Let $B_5 \hookrightarrow H_4$ be a line passing through two general points p_1, p_2 of B_3, and let q_0 the intersection point of B_5 with H_3. Let $B_1 \hookrightarrow H_3$ be a rational normal cubic curve containing the points $q_0, q_1, \dots, q_4$.

By construction, the curves B_1, B_3, B_5 are such that the degree of the 0-cycle $B_j \cdot B_k$ equals $A_j \cdot A_k$ for $j \neq k$ in $\{1, 3, 5\}$. The linear system of quadrics in H_4 containing the curves B_1, B_2, B_5 is a net (a 2-dimensional linear space); thus the linear system $\mathcal{L}$ of quadrics in $\mathbb{P}^5$ containing these curves is 8-dimensional. A general net of quadrics in $\mathcal{L}$ defines a smooth K3 surface X such that $X \cdot H_4 = B_1 + B_3 + B_5$, and the curves B_k are (-2)-curves on X. The curves B_1, B_3, B_5 generate a lattice isometric to $S_{1,5,1}$, and for a general choice, $\mathrm{NS}(X) \simeq S_{1,5,1}$. That construction and Remark 3.20 on the parametrization of rational normal curves show that the moduli space $\mathcal{M}_{S_{1,5,1}}$ is unirational. □

3.13. The lattice $S_{1,6,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,6,1}$. The surface X contains four (-2)-curves $A_1,\dots,A_4$, with intersection matrix

$$\begin{pmatrix} -2 & 5 & 2 & 1 \\ 5 & -2 & 1 & 2 \\ 2 & 1 & -2 & 5 \\ 1 & 2 & 5 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_6 = A_1 + A_2 \equiv A_3 + A_4$$

is ample, of square 6, with $D_6 \cdot A_j = 3$ for $j \in \{1,\ldots,4\}$. It is base-point free and non-hyperelliptic, and therefore the surface X is a degree 6 surface in $\mathbb{P}^4$ with two hyperplane sections that split as the union of two rational normal cubic curves.

Proposition 3.23. *The linear system $|D_6|$ gives an embedding of X as a complete intersection in $\mathbb{P}^4$ with two hyperplanes sections which split as the unions of two rational cubic curves. The moduli space $\mathcal{M}_{S_{1,6,1}}$ of K3 surfaces X with $\mathrm{NS}(X) \simeq S_{1,6,1}$ is unirational.*

Proof. One can construct these surfaces by taking two degree 3 rational normal curves C_1, C_4 in two different hyperplanes H_1, H_2 such that the curves C_1, C_4 meet transversely in one point. The linear system $\mathcal{Q}$ of quadrics containing C_1 and C_4 is a pencil, and the linear system $\mathcal{C}$ of cubics containing C_1 and C_4 has dimension 15. Let X be the intersection of a general element in $\mathcal{Q}$ and a general element in $\mathcal{C}$. The intersections of X with H_1, H_2 break down into $C_1 + C_2$ and $C_3 + C_4$, where C_2, C_3 are two rational cubic normal curves. Using that $C_1 + C_2$, $C_3 + C_4$ are hyperplane sections and that we know that $C_1C_4 = 1$, one obtains that the curves $C_1,\ldots,C_4$ have the above intersection matrix and therefore generate a lattice isometric to $S_{1,6,1}$, which is equal to the Néron–Severi lattice for a general choice of X. The construction shows that the moduli space $\mathcal{M}_{S_{1,6,1}}$ is unirational. □

Proposition 3.24. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{1,6,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. The Hilbert basis of the nef cone of the K3 surface X contains 17 classes (see [ACL21, Table 1]). One can check that any big and nef divisor D on X is base-point free, with $D^2 \geq 6$ or $D^2 = 0$; moreover, there is no fiber F of an elliptic fibration such that $FD = 2$. Thus by Theorems 2.2 and 2.3, there are no hyperelliptic involutions acting on X. From [Kon89, Table 1 and Lemma 2.3], the automorphism group of X is therefore trivial. □

3.14. The lattice $S_{1,9,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{1,9,1}$. The surface X contains nine (-2)-curves $A_1,\ldots,A_9$, with intersection matrix

$$\begin{pmatrix} -2 & 10 & 2 & 8 & 10 & 26 & 2 & 26 & 8 \\ 10 & -2 & 8 & 2 & 10 & 2 & 26 & 8 & 26 \\ 2 & 8 & -2 & 1 & 26 & 37 & 25 & 46 & 37 \\ 8 & 2 & 1 & -2 & 26 & 25 & 37 & 37 & 46 \\ 10 & 10 & 26 & 26 & -2 & 8 & 8 & 2 & 2 \\ 26 & 2 & 37 & 25 & 8 & -2 & 46 & 1 & 37 \\ 2 & 26 & 25 & 37 & 8 & 46 & -2 & 37 & 1 \\ 26 & 8 & 46 & 37 & 2 & 1 & 37 & -2 & 25 \\ 8 & 26 & 37 & 46 & 2 & 37 & 1 & 25 & -2 \end{pmatrix}.$$

The family (A_1, A_3, A_4) is a basis of the Néron–Severi lattice. In that basis, the coordinates of the other (-2)-curves are

$$A_2 = (1,-2,2), \quad A_5 = (5,-6,4), \quad A_6 = (6,-9,7),$$
$$A_7 = (6,-5,3), \quad A_8 = (8,-11,8), \quad A_9 = (8,-8,5).$$

The divisor

$$D_4 = A_1 - A_3 + A_4$$

is ample, of square 4, and the degrees of the curves $A_1,\ldots,A_9$ are

$$4, 4, 5, 5, 10, 14, 14, 17, 17.$$

The divisor D_4 is very ample, and one has

$$2D_4 \equiv A_1 + A_2.$$

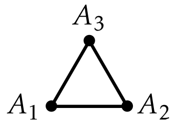
The divisor $D_{10} \equiv 2A_1 - A_3 + A_4$ is very ample of square 10.

Proposition 3.25. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{1,9,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. The Hilbert basis of the nef cone of the K3 surface X is a set of 72 classes (see [ACL21, Table 1]). □

3.15. The lattice $S_{4,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{4,1,1}$. The surface X contains three (-2)-curves A_1, A_2, A_3, with dual graph



These curves generate the Néron–Severi lattice. The divisor

$$D_6 = A_1 + A_2 + A_3$$

is ample, base-point free, non-hyperelliptic, of square 6, with $D_6 \cdot A_j = 2$ for $j \in \{1,2,3\}$, so that the surface X is a degree 6 surface in $\mathbb{P}^4$ with a hyperplane section containing three conics. That leads to the following.

Proposition 3.26. *The K3 surface X is degree 6 complete intersection in $\mathbb{P}^4$ with a hyperplane section which is the union of three conics. The moduli space $\mathcal{M}_{S_{4,1,1}}$ is unirational.*

Proof. Let P^3 be a hyperplane in $\mathbb{P}^4$, and let P_1, P_2, P_3 be three planes in P^3. For $\{i,j,k\} = \{1,2,3\}$, let us denote by L_k the line $P_i \cap P_j$. For each line L_k, let us choose two general points $p_k \neq q_k$ on L_k. For $\{i,j,k\} = \{1,2,3\}$, let C_k be a smooth conic in P_k passing through p_i, p_j, q_i, q_j. The linear system of quadrics in P^3 containing the conics C_1, C_2, C_3 is 0-dimensional, and the linear system of cubics containing C_1, C_2, C_3 is 4-dimensional. Thus the linear system of quadrics (resp. cubics) in $\mathbb{P}^4$ containing the three conics has dimension 5 (resp. 19). A general complete intersection of such a quadric and cubic is a K3 surface with $\mathrm{NS}(X) \simeq S_{4,1,1}$. By that construction, we see that the moduli space $\mathcal{M}_{S_{4,1,1}}$ is unirational. □

By [ACL21, Theorem 3.9], the surface X has equations of the form

$$X\colon\, q_2 = \ell_1\ell_2\ell_3 + \ell_4 g_2 = 0,$$

where q_2, g_2 are quadrics and ℓ_1, ℓ_2, ℓ_3, ℓ_4 are independent linear forms.

Proposition 3.27. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{4,1,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. The Hilbert basis of the nef cone of the K3 surface X is (see [ACL21, Table 1])

$$A_2 + A_3,\, A_1 + A_3,\, A_1 + A_2,\, A_1 + A_2 + A_3.$$

Their intersection matrix is

$$\begin{pmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 6 \end{pmatrix};$$

the result follows. □

3.16. The lattice $S_{5,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{5,1,1}$. The surface X contains four (-2)-curves $A_1,\dots,A_4$, with intersection matrix

$$\begin{pmatrix} -2 & 3 & 2 & 2 \\ 3 & -2 & 2 & 2 \\ 2 & 2 & -2 & 18 \\ 2 & 2 & 18 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_2 = A_1 + A_2$$

is ample, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_2 = 1$ and $D_2 \cdot A_3 = D_2 \cdot A_4 = 4$. We have

$$4D_2 \equiv A_3 + A_4;$$

thus, by using the linear system $|D_2|$, we obtain the following.

Proposition 3.28. *The surface X is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve C_6 which has a tritangent line and such that there is a rational cuspidal quartic curve Q_4 such that its three cusps are on C_6 and the intersection points of Q_4 and C_6 have even multiplicities. The moduli space $\mathcal{M}_{S_{5,1,1}}$ of K3 surfaces X with* $\mathrm{NS}(X) \simeq S_{5,1,1}$ *is unirational.*

Proof. The divisor $D_8 = A_1 + A_2 + A_3$ is very ample of square 8, with $D_8 \cdot A_j = 3, 3, 2, 22$. Thus the curves A_1, A_2, A_3 are, respectively, two rational normal cubics and a conic. The rational normal cubics cut each others in three points and cut the conic in two points.

Let $P_4 \subset \mathbb{P}^5$ be a hyperplane, and let P_3, P_3' be 3-dimensional projective subspaces of H_4. Let P_0 be the plane $P_0 = P_3 \cap P_3'$. Let $P_2 \subset P_4$ be a plane such that P_0 and P_2 meet at one point only. Let us define the lines $L_1 = P_2 \cap P_3$, $L_1' = P_2 \cap P_3'$, and let us fix two points p_1, p_2 (resp. p_1', p_2') on L_1 (resp. L_1') and three points q_1, q_2, q_3 in P_0. We now fix a smooth rational normal cubic curve B_1 (resp. B_2) on P_3 (resp. P_3') passing through p_1, p_2 (resp. p_1', p_2') and q_1, q_2, q_3. We also fix an irreducible conic B_3 in P_2 passing through p_1, p_2, p_1', p_2', so that

$$\mathrm{Degree}(B_j \cap B_k) = A_j \cdot A_k$$

for $j \neq k$ in $\{1, 2, 3\}$. The linear system $\mathcal{Q}$ of quadrics containing the curve $B_1 + B_2 + B_3$ is 11-dimensional; the complete intersection surface obtained by the intersection of the quadrics in a general net of $\mathcal{Q}$ is a K3 surface X containing the (-2)-curves B_1, B_2, B_3 which generate the lattice $S_{5,1,1}$. In fact, one has $\mathrm{NS}(X) \simeq S_{5,1,1}$ since the only over-lattice containing $S_{5,1,1}$ is $S_{1,1,1}$. From that construction and Remark 3.20 on the construction of rational normal curves, we obtain that the moduli space $\mathcal{M}_{S_{5,1,1}}$ is unirational. □

3.17. The lattice $S_{6,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{6,1,1}$. The surface X contains four (-2)-curves $A_1,\dots,A_4$, with intersection matrix

$$\begin{pmatrix} -2 & 4 & 2 & 2 \\ 4 & -2 & 2 & 2 \\ 2 & 2 & -2 & 10 \\ 2 & 2 & 10 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_4 = A_1 + A_2$$

is ample, of square 4, with $D_4 \cdot A_1 = D_4 \cdot A_2 = 2$ and $D_4 \cdot A_3 = D_4 \cdot A_4 = 4$. We have $2D_4 = A_3 + A_4$. The linear system $|D_4|$ is base-point free, non-hyperelliptic. This leads to the following.

Proposition 3.29. *The surface X is a quartic in $\mathbb{P}^3$ with a hyperplane section which is the union of two conics and a quadric section which is the union of two degree 4 smooth rational curves. The moduli space $\mathcal{M}_{S_{6,1,1}}$ of K3 surfaces X with $\mathrm{NS}(X) \simeq S_{6,1,1}$ is unirational.*

Proof. Let us construct these K3 surfaces. Let A_3 be a smooth degree 4 rational curve in $\mathbb{P}^3$, let p_1, p_2 be two points on it, let P be a general plane containing the points p_1, p_2, and let A_1 be an irreducible conic contained in P and containing the points p_1, p_2, so that the degree of the intersection $A_1 \cdot A_3$ is 2. The linear system of quartics containing A_1 and A_3 is 10-dimensional. Let X be such a general quartic; the intersection of X and P contains A_1; the residual curve A_2 is another smooth conic. By [Har77, Exercise IV.6.1], there exists a unique quadric Q_2 (which is moreover smooth) containing the curve A_3. The intersection of X and Q_2 is the union of A_3 and another degree 4 smooth rational curve A_4. Since

$$8 = 2HA_3 = (A_3 + A_4)A_3,$$

we get $A_3 \cdot A_4 = 10$, and therefore from $(A_3 + A_4)^2 = 16$, one obtains $A_4^2 = -2$. From the construction, the curves $A_1, \dots, A_4$ have the above intersection matrix; thus the Néron–Severi lattice of X contains the lattice $S_{6,1,1}$, and by the general assumption $\mathrm{NS}(X) \simeq S_{6,1,1}$. That construction shows that the moduli space $\mathcal{M}_{S_{6,1,1}}$ is unirational. □

Proposition 3.30. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{6,1,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. □

3.18. The lattice $S_{7,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{7,1,1}$. The surface X contains six (-2)-curves $A_1, \dots, A_6$, with intersection matrix

$$\begin{pmatrix} -2 & 5 & 2 & 5 & 16 & 2 \\ 5 & -2 & 2 & 5 & 2 & 16 \\ 2 & 2 & -2 & 16 & 26 & 26 \\ 5 & 5 & 16 & -2 & 2 & 2 \\ 16 & 2 & 26 & 2 & -2 & 26 \\ 2 & 16 & 26 & 2 & 26 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice, and

$$A_4 \equiv 3A_1 + 3A_2 - 2A_3, \quad A_5 \equiv 4A_1 + 6A_2 - 3A_3, \quad A_6 \equiv 6A_1 + 4A_2 - 3A_3.$$

The divisor

$$D_6 = A_1 + A_2$$

is very ample of square 6. For $j \in \{1, \dots, 6\}$, we have $D_6 \cdot A_j$ equal to, respectively, $3, 3, 4, 10, 18$. The divisors

$$2A_1 + 2A_2 - A_3,\ 3A_1 + 4A_2 - 2A_3,\ 4A_1 + 3A_2 - 2A_3$$

are also very ample of square 6.

Proposition 3.31. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{6,1,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. □

3.19. The lattice $S_{8,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{8,1,1}$. The surface X contains four (-2)-curves $A_1,\dots,A_4$, with intersection matrix

$$\begin{pmatrix} -2 & 6 & 2 & 2 \\ 6 & -2 & 2 & 2 \\ 2 & 2 & -2 & 6 \\ 2 & 2 & 6 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron–Severi lattice. The divisor

$$D_8 = A_1 + A_2 \equiv A_3 + A_4$$

is ample, of square 8, base-point free, non-hyperelliptic, with $D_8 \cdot A_j = 4$ for $j \in \{1,\dots,4\}$.

Proposition 3.32. *The K3 surface is a complete intersection in $\mathbb{P}^5$ with two hyperplane sections which are each the union of two degree 4 rational curves. The moduli space $\mathcal{M}_{S_{8,1,1}}$ is unirational.*

Proof. One can construct these surfaces by taking two degree 4 rational normal curves C_1, C_3 in two different hyperplanes H_1, H_3 but such that the curves C_1, C_3 meet transversely in two fixed points. Let X be a general quartic that contains C_1 and C_3; then the intersections of X with H_1, H_2 are $C_1 + C_2$ and $C_3 + C_4$, where C_2, C_4 are two degree 4 rational normal curves. Curves $C_1,\dots,C_4$ generate a lattice isometric to $S_{8,1,1}$. That construction shows that the moduli space $\mathcal{M}_{S_{8,1,1}}$ is unirational. □

Proposition 3.33. *The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{8,1,1}$ is trivial.*

Proof. We proceed as in the proof of Proposition 3.11. □

3.20. The lattice $S_{10,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{10,1,1}$. The surface X contains eight (-2)-curves $A_1,\dots,A_8$, with intersection matrix

$$\begin{pmatrix} -2 & 18 & 8 & 8 & 2 & 22 & 2 & 22 \\ 18 & -2 & 8 & 8 & 22 & 2 & 22 & 2 \\ 8 & 8 & -2 & 18 & 2 & 22 & 22 & 2 \\ 8 & 8 & 18 & -2 & 22 & 2 & 2 & 22 \\ 2 & 22 & 2 & 22 & -2 & 38 & 18 & 18 \\ 22 & 2 & 22 & 2 & 38 & -2 & 18 & 18 \\ 2 & 22 & 22 & 2 & 18 & 18 & -2 & 38 \\ 22 & 2 & 2 & 22 & 18 & 18 & 38 & -2 \end{pmatrix}.$$

The curves A_1, A_3, A_5 generate the Néron–Severi lattice. The divisor

$$D_2 = A_1 + A_3 - A_5$$

is ample, base-point free, of square 2, with $D_2 \cdot A_j = 4$ for $j \in \{1,2,3,4\}$ and $D_2 \cdot A_j = 6$ for $j \in \{5,6,7,8\}$. We have

$$4D_2 \equiv A_1 + A_2 \equiv A_3 + A_4, \quad 6D_2 \equiv A_5 + A_6 \equiv A_7 + A_8.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 3.34. *The K3 surface is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve C_6 such that there are two quartic cuspidal rational curves Q_4, Q_4' and two sextic cuspidal rational curves Q_6, Q_6' such that the cusps are on C_6 and the intersection multiplicities of these curves with C_6 are even at all intersection points.*

The divisor $D_8 = 2A_1 + A_3 - A_5$ is very ample, of square 8, with $D_8 \cdot A_j = 2, 22, 12, 12, 8, 28, 8, 28$ for $j = 1,\dots,8$.

3.21. The lattice $S_{12,1,1}$

Let X be a K3 surface with Néron–Severi lattice of type $S_{12,1,1}$. The surface X contains six (-2)-curves $A_1,\dots,A_6$, with intersection matrix

$$\begin{pmatrix} -2 & 14 & 2 & 10 & 10 & 2 \\ 14 & -2 & 10 & 2 & 2 & 10 \\ 2 & 10 & -2 & 14 & 2 & 10 \\ 10 & 2 & 14 & -2 & 10 & 2 \\ 10 & 2 & 2 & 10 & -2 & 14 \\ 2 & 10 & 10 & 2 & 14 & -2 \end{pmatrix}.$$

The curves A_1, A_3, A_5 generate the Néron–Severi lattice. The divisor

$$D_6 = A_1 - A_3 + A_5$$

is very ample, of square 6, with $D_2 \cdot A_j = 6$ for $j \in \{1,\dots,6\}$. We have

$$2D_6 \equiv A_1 + A_2 \equiv A_3 + A_4 \equiv A_5 + A_6;$$

thus we obtain the first part of the following proposition.

Proposition 3.35. *The surface X is a degree 6 surface in $\mathbb{P}^4$ such that there are three quadric sections, each of which splits as the union of two degree 6 smooth rational curves.*

The automorphism group of a general K3 surface X with $\mathrm{NS}(X) \simeq S_{12,1,1}$ is trivial.

Proof. For the second part, we proceed as in the proof of Proposition 3.11. □

3.22. The lattice $S'_{4,1,2}$

Let X be a K3 surface with Néron–Severi lattice of type $S'_{4,1,2}$. The surface X contains four (-2)-curves $A_1,\dots,A_4$, with intersection matrix

$$\begin{pmatrix} -2 & 6 & 2 & 2 \\ 6 & -2 & 2 & 2 \\ 2 & 2 & -2 & 6 \\ 2 & 2 & 6 & -2 \end{pmatrix}.$$

This is the same intersection matrix as the four (-2)-curves in Section 3.19, but here the curves A_1, A_2, A_3, A_4 generate only an index 2 subgroup of the Néron–Severi lattice. There is a basis e_1, e_2, e_3 of the Néron–Severi lattice such that the intersection matrix of e_1, e_2, e_3 is

$$\begin{pmatrix} -6 & 2 & 0 \\ 2 & -2 & 4 \\ 0 & 4 & -8 \end{pmatrix}.$$

In that basis, the classes of the curves are

$$A_1 = (2,3,1), \quad A_2 = (0,1,1),$$
$$A_3 = (2,3,2), \quad A_4 = (0,1,0),$$

and the divisor

$$D_2 = (1,2,1)$$

is ample, of square 2, base-point free, with $D_2 \cdot A_j = 2$ for $j \in \{1,2,3,4\}$. We have

$$2D_2 \equiv A_1 + A_2 \equiv A_3 + A_4.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 3.36. *The K3 surface is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve which has two 6-tangent conics.*

Let D_4 be the divisor $D_4 = (1,3,1)$. It is nef, base-point free, with $D_4 \cdot A_j = 4,4,8,0$ for $j = 1,\dots,4$. The linear system $|D_4|$ defines a singular model Y of X which is a quartic in $\mathbb{P}^3$ with a node. Since

$$2D_4 \equiv A_1 + A_2 \equiv A_3 + A_4,$$

there are two quadric sections Q_1, Q_2 such that Q_1 is the union of two smooth rational degree 4 curves which are the images of A_1, A_2 and Q_2 is a degree 8 rational curve (the image of A_3) which contains the node (the image of A_4).

4. Rank 4 lattices

Vinberg's classification

The reference for the classification of rank 4 lattices that are Néron–Severi lattices of K3 surfaces with finite automorphism group is the article of Vinberg [Vin07]. There are two lattices such that the fundamental domain of the Weyl group is compact, and 12 for which the domain is not compact. Geometrically, the fundamental domain is compact if and only if the K3 surface has no elliptic fibration.

4.1. The rank 4 and compact cases

We studied in [Rou19] the two lattices $L(24)$, $L(27)$ of rank 4 such that the K3 surfaces have no elliptic fibrations. The Gram matrices of the lattices $L(24)$, $L(27)$ are, respectively,

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 12 & 2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

The K3 surfaces with such Néron–Severi lattices are double covers of the plane branched over a smooth sextic curve C_6. For completeness, let us recall the results obtained in [Rou19].

Theorem 4.1. *Let X be a K3 surface with Néron–Severi lattice isometric to $L(24)$. The six (-2)-curves on X are pull-backs of three lines tritangent to C_6.*

Let X be a K3 surface with Néron–Severi lattice isometric to $L(27)$. The eight (-2)-curves on X are pull-backs of one line tritangent to C_6 and three conics 6-tangent to C_6.

The K3 surfaces with Néron–Severi lattice isometric to $L(24)$ have some connection with K3 surfaces of type $S_0 \oplus \mathbf{A}_2$, where $S_0 = \left[\begin{smallmatrix} 0 & -3 \\ -3 & -2 \end{smallmatrix}\right]$; see [ACR20].

4.2. The rank 4 and non-compact cases

This is the subject of another paper [ACR20]. The lattices are

$$[8] \oplus \mathbf{A}_1^{\oplus 3},\ U \oplus \mathbf{A}_1^{\oplus 2},\ U(2) \oplus \mathbf{A}_1^{\oplus 2},\ U(3) \oplus \mathbf{A}_1^{\oplus 2},\ U(4) \oplus \mathbf{A}_1^{\oplus 2},$$
$$U \oplus \mathbf{A}_2,\ U(2) \oplus \mathbf{A}_2,\ U(3) \oplus \mathbf{A}_2,\ U(6) \oplus \mathbf{A}_2,$$
$$S_0 \oplus \mathbf{A}_2,\ [4] \oplus [-4] \oplus \mathbf{A}_2,\ U(4) \oplus A_3.$$

We give the number of (-2)-curves in the different cases in the table in Section 16.

For the Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$, there is a unique negative curve s which is such that $s^2 = -n$. We denote by f a fiber of the natural fibration $\mathbb{F}_n \to \mathbb{P}^1$. For completeness, we summarize the constructions obtained in [ACR20].

Theorem 4.2.

- *A K3 surface with* $\mathrm{NS}(X) \simeq [8] \oplus \mathbf{A}_1^{\oplus 3}$ *is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve which has six* 6*-tangent conics.*
- *A K3 surface with* $\mathrm{NS}(X) \simeq U \oplus \mathbf{A}_1^{\oplus 2}$ *is a double cover of a Hirzebruch surface,* $\eta\colon X \to \mathbf{F}_2$*. It is branched over the section* s *and a curve* $B \in |3s+8f|$*, so that* $sB = 2$*.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U(2) \oplus \mathbf{A}_1^{\oplus 2}$ *is the minimal desingularization of the double cover of* $\mathbb{P}^2$ *branched over a sextic curve with three nodal singularities.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U(3) \oplus \mathbf{A}_1^{\oplus 2}$ *is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve with two tritangent lines and two* 6*-tangent conics.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U(4) \oplus \mathbf{A}_1^{\oplus 2}$ *is a quartic in* $\mathbb{P}^3$ *which has four hyperplane sections that decompose into the union of two conics.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U \oplus \mathbf{A}_2$ *is the minimal desingularization of the double cover of* $\mathbf{F}_3$ *with branch locus the section* s *and a reduced curve in* $B \in |3s+10f|$*, so that* $Bs = 1$*.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U(2) \oplus \mathbf{A}_2$ X *is the minimal resolution of the double cover of* $\mathbb{P}^2$ *branched over a sextic with two nodes such that the line through the two nodes is tangent to the sextic curve in a third point.*
- *A K3 surface with* $\mathrm{NS}(X) \simeq U(3) \oplus \mathbf{A}_2$ *is a quartic in* $\mathbb{P}^3$ *which has a hyperplane section that is the union of four lines.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq U(6) \oplus \mathbf{A}_2$ *is a quartic in* $\mathbb{P}^3$ *which has three hyperplane sections that decompose into the union of two conics.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq \left[\begin{smallmatrix} 0 & -3 \\ -3 & -2 \end{smallmatrix}\right] \oplus \mathbf{A}_2$ *is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve which has three tritangent lines.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq [4] \oplus [-4] \oplus \mathbf{A}_2$ *is a double cover of* $\mathbb{P}^2$ *branched over a smooth sextic curve which has two tritangent lines and one* 6*-tangent conic.*
- *A K3 surface* X *with* $\mathrm{NS}(X) \simeq [4] \oplus \mathbf{A}_3$ *is the minimal resolution of a double cover of* $\mathbb{P}^2$ *branched over a sextic curve with one node; through that node go two lines that are tangent to the sextic at every other intersection point.*

We will only add the following result.

Proposition 4.3. *A general K3 surface* X *with a Néron–Severi lattice isometric to one of the lattices*

$$U(4) \oplus \mathbf{A}_1^{\oplus 2},\ U(3) \oplus \mathbf{A}_2,\ U(6) \oplus \mathbf{A}_2$$

has trivial automorphism group.

Proof. For each of these lattices, the Hilbert basis of its nef cone is described in [ACR20]. Then as in Proposition 3.11, one can check that there are no hyperelliptic involutions and conclude that the automorphism group is trivial. □

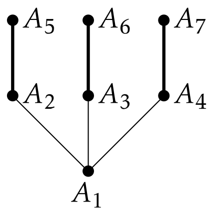
5. Rank 5 lattices

Nikulin's classification for higher ranks

The list of lattices L of rank at least 5 such that the K3 surfaces with $\mathrm{NS}(X) \simeq L$ have finite automorphism group is given in [Nik14]; that list is obtained from the paper [Nik83].

5.1. The lattice $U \oplus \mathbf{A}_1^{\oplus 3}$

There exist seven (-2)-curves $A_1, \ldots, A_7$ on X, with dual graph



The curves $A_1, \ldots, A_5$ generate the Néron–Severi lattice. The divisor

$$D_{18} = 3A_1 + 5A_2 + A_3 + A_4 + 4A_5$$

is ample, with $D_{18} \cdot A_j = 1$ for $j \in \{1,2,3,4\}$ and $D_{18} \cdot A_j = 2$ for $j \in \{5,6,7\}$. The divisor

$$D_2 = 2A_1 + 2A_2 + A_3 + A_4 + A_5$$

is nef, base-point free, of square 2, with $D_2 \cdot A_j = 0$ for $j \in \{1,2,3,4\}$ and $D_2 \cdot A_j = 2$ for $j \in \{5,6,7\}$. We also have

$$D_2 \equiv 2A_1 + A_2 + 2A_3 + A_4 + A_6 \equiv 2A_1 + A_2 + A_3 + 2A_4 + A_7.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 5.1. *The K3 surface is a double cover of* $\mathbb{P}^2$ *branched over a sextic curve with a* $\mathbf{d}_4$ *singularity* q.

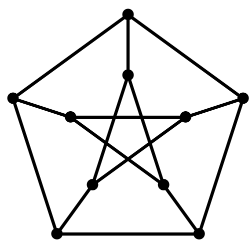
The three curves A_5, A_6, A_7 are mapped by the double cover map to the three lines that are the tangents to the three branches of the singularity q.

5.2. The lattice $U(2) \oplus \mathbf{A}_1^{\oplus 3}$ and the del Pezzo surface of degree 5

The K3 surface X is the minimal resolution of the double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 with four nodes $p_1, \ldots, p_4$ in general position. It is also the double cover branched over the strict transform of C_6 in the degree 5 del Pezzo surface Z which is the blow-up at $p_1, \ldots, p_4$. The surface X contains 10 (-2)-curves denoted by $A_{i,j}$, for $\{i,j\} \subset \{1, \ldots, 5\}$ with $i < j$ (the pull-back of the 10 (-1)-curves on the del Pezzo surface). One has

$$A_{ij} \cdot A_{st} = 2$$

if and only if $\#\{i,j,s,t\} = 4$; else $A_{ij} \cdot A_{st} = 0$ or -2. The dual graph of the configuration is the Petersen graph



with weight 2 on the edges.

Remark 5.2. If C is a general curve of genus 6, then there exists a map $C \to \mathbb{P}^2$ with image a sextic curve with four nodes. Using that property, Artebani and Kondo describe in [AK11] the moduli space of genus 6 curves and their link with K3 surfaces and the quintic del Pezzo surface. They study that moduli space as a quotient of a bounded symmetric domain.

5.3. The lattice $U(4)\oplus \mathrm{A}_1^{\oplus 3}$

The K3 surface X contains 24 (-2)-curves $A_1,\dots,A_{24}$. Up to permutation, one can suppose that A_1, A_3, A_5, A_7, A_9 have the following intersection matrix:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0\\ 0 & -2 & 0 & 0 & 4\\ 0 & 0 & -2 & 2 & 2\\ 0 & 0 & 2 & -2 & 2\\ 0 & 4 & 2 & 2 & -2 \end{pmatrix}.$$

These five curves generate $\mathrm{NS}(X)$. The divisor

$$D_2 = -A_1 + A_3 + A_9$$

is ample, of square 2, with $D_2 \cdot A_j = 2$ for $j \in \{1,\dots,24\}$ and, up to permutation of the indices,

$$2D_2 \equiv A_{2k-1} + A_{2k}$$

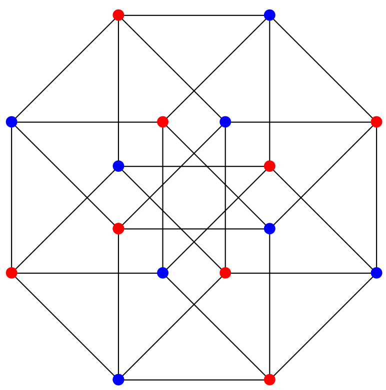
for any $k \in \{1,\dots,12\}$. The classes of $A_{11}, A_{13},\dots,A_{23}$ in the basis A_1, A_3, A_5, A_7, A_9 are

$$\begin{aligned} A_{11} &= (0,-1,1,1,0), & A_{13} &= (-1,0,1,1,0),\\ A_{15} &= (-1,0,0,1,1), & A_{17} &= (-2,1,0,1,1),\\ A_{19} &= (-2,0,1,1,1), & A_{21} &= (-1,0,1,0,1),\\ A_{23} &= (-2,1,1,0,1), \end{aligned}$$

so that we know the 24 classes of (-2)-curves in X. By using the linear system $|D_2|$, we obtain the following.

Proposition 5.3. *The K3 surface X is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve which has 12 6-tangent conics.*

There exists a partition of the 24 (-2)-curves into three sets S_1, S_2, S_3 of 8 curves each such that for curves B, B' in two different sets S, S', one has $BB' = 0$ or 4 and for any $B \in S$, there are exactly 4 curves B' in S' such that $BB' = 4$, and symmetrically for B'. Therefore, S and S' form an 8_4 configuration. This is the so-called Möbius configuration (see [Cox50]). The following graph is the Levi graph of that 8_4 configuration; this is the graph of the 4-dimensional hypercube (see [Cox50]). Vertices in red are curves in S, vertices in blue are curves in S', and an edge links a red curve to a blue curve if and only if their intersection number is 4.



From that graph, we can moreover read the intersection numbers of the curves in S (and S') as follows. For any red curve B, there are four blue curves linked to it by an edge. Consider the complementary set of blue curves; this is another set of four blue curves, all linked through an edge to the same red curve B'. Then we have $BB' = 6$, and for any other red curve $B'' \notin \{B, B'\}$, we have $BB'' = 2$. Symmetrically, the intersection numbers between the blue curves follow the same rule.

In [Nik83, Proof of Theorem 8.1.1], Nikulin studies the lattice $U(4)\oplus \mathbf{A}_1^{\oplus 3}$ in detail, obtaining that it contains only 24 (-2)-curves, and he describes the fundamental polygon of the action of the Weyl group by an embedding in the 4-dimensional Euclidian space. There, the polyhedral formed by the 24 vertices is the dual of the polyhedron formed by the roots of type $\mathbf{D}_4$.

There is a second geometric model which is as follows. The divisor

$$D_4 = -A_1 + 2A_3 + A_9$$

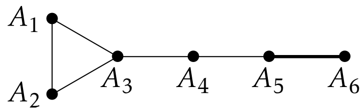
is a nef, non-hyperelliptic divisor of square 4, with $D_4 \cdot A_j = 0$ if and only if $j = 3$. We have, moreover,

$$D_4 \equiv A_1 + A_{10} \equiv A_5 + A_{22} \equiv A_7 + A_{16} \equiv A_{13} + A_{20},$$

and the intersection number of these eight curves with D_4 is 2. Therefore, the linear system $|D_4|$ gives a singular model of X as a quartic in $\mathbb{P}^3$ with a unique node and four hyperplane sections which are unions of two conics. One can check that the eight (-2)-curves which are mapped to the conics and the (-2)-curve which is contracted to the node generate the lattice $U(4)\oplus \mathbf{A}_1^{\oplus 3}$.

5.4. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{A}_2$

The K3 surface X contains six (-2)-curves $A_1,\dots,A_6$; their configuration is



These curves generate the Néron–Severi lattice. The divisor

$$D_{20} = 5A_1 + 5A_2 + 6A_3 + 3A_4 + A_5$$

is ample, of square 20, with $D_{20} \cdot A_j = 1$ for $j \le 5$ and $D_{20} \cdot A_6 = 2$. The divisor

$$D_4 = 2A_1 + 2A_2 + 3A_3 + 2A_4 + A_5$$

is nef, of square 4, base-point free and hyperelliptic since

$$D_4(A_1 + A_2 + A_3) = 2,$$

and $A_1 + A_2 + A_3 \equiv A_5 + A_6$ is a fiber of an elliptic fibration. One has $D_4 \cdot A_1 = D_4 \cdot A_2 = 1$, $D_4 \cdot A_6 = 2$ and $D_4 \cdot A_j = 0$ for $j \in \{3,4,5\}$. Moreover,

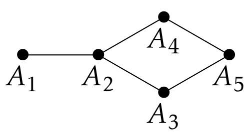
$$D_4 \equiv A_3 + 2A_4 + 3A_5 + 2A_6.$$

Proposition 5.4. *The double cover induced by $|D_4|$ factors through $\eta: X \to \mathbf{F}_2$. The image by η of A_4 is s (where s is the section such that $s^2 = -2$). The branch locus is the union of the section s and $B \in |3s + 8f|$, so that $Bs = 2$. Let f_1, f_2 be the fibers through the points p and q, the intersection points of s and B. The curve f_1 cuts B with multiplicity 2 at another point, and the image by η of the curves A_1, A_2 is f_1, the image of A_6 is f_2, and A_3, A_5 are contracted to p and q, respectively.*

Proof. We apply Theorem 2.3, case a) iii) v). □

5.5. The lattice $U \oplus \mathbf{A}_3$

The K3 surface X contains five (-2)-curves $A_1,\dots,A_5$; their configuration is



These curves generate the Néron–Severi lattice. In that basis, the divisor

$$D_{50} = (5, 11, 9, 9, 8)$$

is ample, of square 50, with $D_{50} \cdot A_j = 1$ for $j \leq 4$ and $D_{50} \cdot A_5 = 2$. We have $D_{50} \cdot A_j = 0$ for $j \leq 4$ and $D_{50} \cdot A_5 = 2$. The divisor

$$D_4 = 2A_1 + 4A_2 + 3A_3 + 3A_4 + 2A_5$$

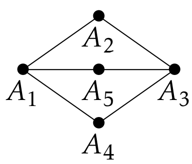
is nef, of square 4, and $|D_4|$ is base-point free hyperelliptic since $D_4(A_2 + A_3 + A_4 + A_5) = 2$; one has $D_4 \cdot A_j = 0$ for $j \leq 4$ and $D_4 \cdot A_5 = 2$.

Proposition 5.5. *The double cover induced by $|D_4|$ factors through $\eta \colon X \to \mathbf{F}_2$. The image by η of A_1 is the section s. The branch locus is the union of the section s and $B \in |3s + 8f|$, so that $Bs = 2$. The intersection of B and s is tangent at one point q, forming an $\mathbf{a}_3$ singularity. The curve A_5 is mapped onto the fiber through q, and the curves A_2, A_3, A_4 are mapped to q.*

Proof. We apply Theorem 2.3, case a) iii) v). □

5.6. The lattice $[4] \oplus \mathbf{D}_4$

The K3 surface X contains five (-2)-curves $A_1, \ldots, A_5$; their configuration is



These curves generate the Néron–Severi lattice. The divisor

$$D_{22} = A_1 + 3\sum_{j=1}^{5} A_j$$

is ample, of square 22, with $D_{22} \cdot A_3 = 3$ and $D_{22} \cdot A_j = 1$ for $j \neq 3$. The divisor

$$D_2 = \sum_{j=1}^{5} A_j$$

is nef, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_3 = 1$, $D_2 \cdot A_j = 0$ for $j \in \{2, 4, 5\}$. By using the linear system $|D_2|$, we obtain the following.

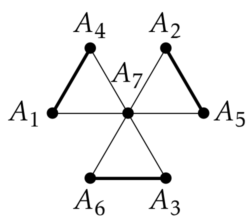
Proposition 5.6. *The K3 surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve with three nodes on a line.*

The image of A_1, A_3 is the line through the three nodes; the curves A_2, A_4, A_5 are contracted to the nodes.

The divisor $D_4 = A_1 + 2A_2 + 3A_3 + 2A_4 + 2A_5$ is nef, non-hyperelliptic, with $D_4 \cdot A_1 = 4$ and $D_4 \cdot A_k = 0$ for $k \geq 2$. It defines a model of the K3 surface as a quartic in $\mathbb{P}^3$ with a $\mathbf{D}_4$ singularity.

5.7. The lattice $[8] \oplus \mathbf{D}_4$

The K3 surface X contains seven (-2)-curves $A_1, \ldots, A_7$; their configuration is



The curves A_1, A_2, A_3, A_4, A_7 generate the Néron–Severi lattice. The divisor

$$D_6 = 2A_1 + 2A_4 + A_7$$

is ample, of square 6, with $D_6 \cdot A_j = 1$ for $j \le 6$ and $D_6 \cdot A_7 = 2$. The divisor

$$D_2 = A_1 + A_4 + A_7$$

is nef, of square 2, with $D_6 \cdot A_j = 1$ for $j \le 6$ and $D_6 \cdot A_7 = 0$. We have

$$D_2 \equiv A_2 + A_5 + A_7 \equiv A_3 + A_6 + A_7.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 5.7. *The surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve with one node, and there are three lines through that node such that each line is tangent to the sextic at its other intersection points.*

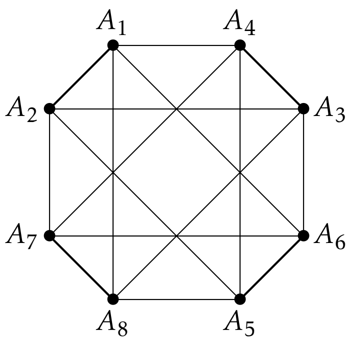
One can obtain another interesting geometric model of the K3 surface X as follows. The divisor $D_8 = 3A_1 + 2A_2 + 2A_3 + A_4 + 4A_7$ is big and nef, base-point free, non-hyperelliptic, with $D_8 \cdot A_k = 0,0,0,8,8,8,0$ for $k = 1,\dots,7$. The image of the K3 surface by the map associated to $|D_8|$ is a degree 8 surface in $\mathbb{P}^5$ with a $\mathbf{D}_4$ singularity.

5.8. The lattice $[16] \oplus \mathbf{D}_4$

The K3 surface X contains eight (-2)-curves $A_1,\dots,A_8$; their intersection matrix is

$$\begin{pmatrix} -2 & 3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 3 & -2 \end{pmatrix}.$$

Their configuration is



where the thick lines have weight 3. The curves A_1, A_2, A_3, A_5, A_7 generate NS(X). The divisor

$$D_2 = A_1 + A_2$$

is ample of square 2, with $D_2 \cdot A_j = 1$ for $j \in \{1,\dots,8\}$. We have also

$$D_2 \equiv A_3 + A_4 \equiv A_5 + A_6 \equiv A_7 + A_8.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 5.8. *The surface is a double cover of $\mathbb{P}^2$ branched over a smooth sextic curve which has four tritangent lines.*

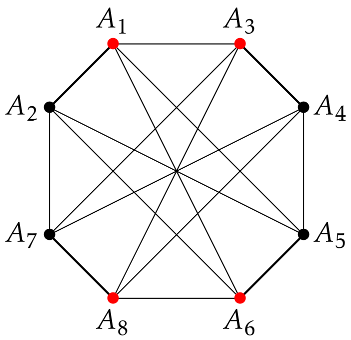
One can obtain another interesting geometric model of the K3 surface X as follows. The divisor $D_{16} = 4A_1 + A_2 + 5A_4 + 3A_5 + 3A_7$ is nef, non-hyperelliptic, of square 16, with $D_{16} \cdot A_k = 0$ for curves A_1, A_4, A_5, A_7 and $D_{16} \cdot A_k = 16$ for the other curves. The image of the K3 surface under the map associated to $|D_{16}|$ is a degree 16 surface in $\mathbb{P}^9$ with a $\mathbf{D}_4$ singularities, which is the image of the curves A_1, A_4, A_5, A_7.

5.9. The lattice $[6] \oplus \mathbf{A}_2^{\oplus 2}$

The K3 surface X contains 10 (-2)-curves $A_1, \dots, A_{10}$; their intersection matrix is

$$\begin{pmatrix} -2 & 3 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\ 3 & -2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & -2 & 3 & 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -2 & 3 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 3 & -2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -2 & 3 & 0 & 2 \\ 1 & 0 & 1 & 0 & 0 & 1 & 3 & -2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & -2 & 6 \\ 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 6 & -2 \end{pmatrix}.$$

The configuration of the first eight curves is



where the thick lines have weight 3, the curves A_k, $k \in \{1,3,6,8\}$ (with a red vertex) are such that $A_k \cdot A_9 = 2$, $A_k \cdot A_{10} = 0$, and the curves A_k, $k \in \{2,4,5,7\}$ (with a black vertex) are such that $A_k \cdot A_{10} = 2$, $A_k \cdot A_9 = 0$. The curves A_1, A_3, A_5, A_7, A_9 generate $\mathrm{NS}(X)$. The divisor

$$D_2 = A_1 + A_3 - A_5 - A_7 + A_9$$

is ample, of square 2, with $D_2 \cdot A_j = 1$ for $j \leq 8$, $D_2 \cdot A_9 = D_2 \cdot A_{10} = 2$. We have

$$D_2 \equiv A_{2k-1} + A_{2k},\ k \in \{1,2,3,4\}$$

and $2D_2 \equiv A_9 + A_{10}$. By using the linear system $|D_2|$, we obtain the following.

Proposition 5.9. *The surface X is the double cover of $\mathbb{P}^2$ branched over a smooth sextic curve which has four tritangent lines and one 6-tangent conic.*

Remark 5.10. The Néron–Severi lattice is also generated by A_1, A_2, A_3, A_5, A_7.

It is interesting to compare this case with the previous lattice case, where four tritangent lines are also involved.

Proposition 5.11. *Let X be a K3 surface with $\mathrm{NS}(X) \simeq [6] \oplus \mathbf{A}_2^{\oplus 2}$. There exist linear forms $\ell_1, \dots, \ell_4 \in H^0(\mathbb{P}^2, \mathcal{O}(1))$, a quadric $q_2 \in H^0(\mathbb{P}^2, \mathcal{O}(2))$ and a cubic $f_3 \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ such that X is the double cover of $\mathbb{P}^2$ branched over the curve*

$$C_6 \colon \ell_1\ell_2\ell_3\ell_4 q_2 - f_3^2 = 0.$$

The moduli space of K3 surfaces X with $\mathrm{NS}(X) \simeq [6] \oplus \mathbf{A}_2^{\oplus 2}$ *is unirational.*

Proof. Let us consider the map

$$\Phi\colon H^0(\mathbb{P}^2, \mathcal{O}(1))^{\oplus 4} \oplus H^0(\mathbb{P}^2, \mathcal{O}(2)) \oplus H^0(\mathbb{P}^2, \mathcal{O}(3)) \to H^0(\mathbb{P}^2, \mathcal{O}(6))$$

defined by

$$w := (\ell_1, \ell_2, \ell_3, \ell_4, q_2, f_3) \mapsto f_{6,w} := \ell_1\ell_2\ell_3\ell_4 q_2 - f_3^2.$$

Suppose that w is general , so that the double cover branched over $C_w = \{f_{6,w} = 0\}$ is a K3 surface. The pull-backs of $\ell_k = 0$, $k = 1, \dots, 4$, and $q_2 = 0$ are pairs of (-2)-curves, for which Example 5.12 below shows that (for a suitable order) these curves intersect according to the above matrix since intersection numbers are preserved for flat families of surfaces.

The map Φ is invariant under the action of the transformations

$$(\ell_1, \ell_2, \ell_3, \ell_4, q_2) \mapsto (\alpha\ell_1, \beta\ell_2, \gamma\ell_3, \delta\ell_4, \epsilon q_2, f_3)$$

for $\alpha\beta\gamma\delta\epsilon = 1$. Since the curves $\ell_j = 0$, $j \in \{1, \dots, 4\}$, and $q_2 = 0$ are the images of the 10 (-2)-curves in the double cover X ramified over $f_{6,w}$, we have

$$\Phi(w) = \Phi(w')$$

if and only if, up to permutation and the action of the above transformations, $w = \lambda w'$ for some $\lambda \neq 0$, where

$$\lambda \cdot (\ell_1, \ell_2, \ell_3, \ell_4, q_2, f_3) = (\lambda\ell_1, \lambda\ell_2, \lambda\ell_3, \lambda\ell_4, \lambda^2 q_2, \lambda^3 f_3).$$

The image of Φ modulo the action of PGL_3 is therefore a unirational space of dimension

$$4 \cdot 3 + 6 + 10 - 4 - 9 = 15,$$

with an open set which is birational to the moduli space of K3 surfaces X with $\mathrm{NS}(X) \simeq [6] \oplus \mathbf{A}_2^{\oplus 2}$. □

Example 5.12. Let us take

$$\begin{aligned}
&\ell_1 = 16y + 29z,\ \ell_2 = 22x + y + 27z,\ \ell_3 = 17y + 29z,\ \ell_4 = 25x + 29y + 11z,\\
&q_2 = 31x^2 + 16xy + 11y^2 + 11xz + 9yz + 17z^2,\\
&f_3 = 6x^3 + 8x^2y + 17xy^2 + 5y^3 + 17x^2z + 18xyz + 19y^2z + 16xz^2 + 8yz^2 + 23z^3.
\end{aligned}$$

Let X be the associated K3 surface, and let X_q be its reduction over the field $\mathbb{F}_q$. Using the Tate conjecture, one computes that the Picard numbers of X_{17^2} and X_{23^2} are both 6, and using the Artin–Tate conjecture and Van Luijk's trick, one obtains that X has Picard number 5. One can check that the 10 (-2)-curves above the lines $\ell_k = 0$ and the conic $q_2 = 0$ have the above intersection matrix and $\mathrm{NS}(X) \simeq [6] \oplus \mathbf{A}_2^{\oplus 2}$.

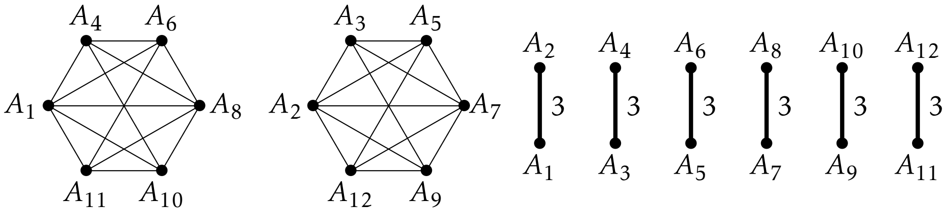
6. Rank 6 lattices

6.1. The lattice $U(3) \oplus \mathbf{A}_2^{\oplus 2}$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U(3) \oplus \mathbf{A}_2^{\oplus 2}.$$

The surface contains 12 (-2)-curves $A_1, \dots, A_{12}$, with configuration



where for clarity we represented the same curve several times.

The Néron–Severi lattice is generated by A_1, A_3, A_5, A_7, A_9, A_{11}. The divisor

$$D_2 = -A_1 + A_3 + A_5 + A_7 + A_9 - A_{11}$$

is ample, base-point free, of square 2, and the 12 (-2)-curves $A_1, \dots, A_{12}$ on X are of degree 1 for D_2. We have

$$D_2 \equiv A_{2k-1} + A_{2k}$$

for $k \in \{1, \dots, 6\}$, and by using the linear system $|D_2|$, we obtain the following.

Proposition 6.1. *The surface X is a double cover $\pi\colon X \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched over a smooth sextic curve C_6. The 12 (-2)-curves on X are pull-backs of 6 lines that are tritangent to the sextic curve. There exist linear forms $\ell_1, \dots, \ell_6 \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ and a cubic $f_3 \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ such that the curve C_6 is given by*

$$\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6 - f_3^2 = 0.$$

The moduli space of K3 surfaces X with $\mathrm{NS}(X) \simeq U(3) \oplus \mathbf{A}_2^{\oplus 2}$ is unirational.

Proof. Let us consider the map

$$\Phi\colon H^0(\mathbb{P}^2, \mathcal{O}(1))^{\oplus 6} \oplus H^0(\mathbb{P}^2, \mathcal{O}(3)) \to H^0(\mathbb{P}^2, \mathcal{O}(6))$$

defined by

$$w := (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, f_3) \mapsto f_{6,w} := \ell_1\ell_2\ell_3\ell_4\ell_5\ell_6 - f_3^2.$$

When the sextic curve $C_w\colon f_{6,w} = 0$ is smooth, the K3 surface Y_w which is the double cover of $\mathbb{P}^2$ branched over C_w contains 12 (-2)-curves over the 6 tritangent lines $\{\ell_k = 0\}$, $k = 1, \dots, 6$. Example 6.2 below shows that these (-2)-curves generate a sublattice isometric to $U(3) \oplus \mathbf{A}_2^{\oplus 2}$. The map Φ is invariant under the action of the transformations

$$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) \mapsto (\alpha\ell_1, \beta\ell_2, \gamma\ell_3, \delta\ell_4, \epsilon\ell_5, \mu\ell_6, f_3)$$

for $\alpha\beta\gamma\delta\epsilon\mu = 1$. Since the curves $\ell_j = 0$, $j \in \{1, \dots, 6\}$, are the images of the 12 (-2)-curves in the double cover ramified over f_6, we have

$$\Phi(w) = \Phi(w')$$

if and only if, up to permutation and the action of the above transformations, $w = \lambda \cdot w'$ for some $\lambda \neq 0$, where

$$\lambda \cdot (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, f_3) = (\lambda\ell_1, \lambda\ell_2, \lambda\ell_3, \lambda\ell_4, \lambda\ell_5, \lambda\ell_6, \lambda^3 f_3).$$

The image of Φ modulo the action of PGL_3 is therefore a unirational space of dimension

$$6 \cdot 3 + 10 - 5 - 9 = 14,$$

with an open set which is birational to the moduli space of K3 surfaces X with $\mathrm{NS}(X) \simeq U(3) \oplus \mathbf{A}_2^{\oplus 2}$. □

Example 6.2. As an example, one can take

$$\begin{aligned}
&\ell_1 = 15x + 5y + 2z,\ \ell_2 = 20x + 12y + 17z,\ \ell_3 = 22x + 22y + 16z,\\
&\ell_4 = 8x + y + 7z,\ \ell_5 = 10x + 15y + 15z,\ \ell_6 = 6x + 22y + 20z,\\
&f_3 = 18x^2y + 13xy^2 + 16y^3 + 16x^2z + 17xyz + 22y^2z + 6xz^2 + 10yz^2 + 20z^3;
\end{aligned}$$

this gives a smooth K3 surface X. Its reduction modulo 23 is a smooth surface X_{23}. Using the Artin–Tate conjecture, one finds that its Picard number is 4 and the discriminant of the Néron–Severi lattice has order 3^4. Using the pull-back of the six lines $\{\ell_k = 0\}$, one can check that the Néron–Severi lattice contains the lattice $U(3) \oplus \mathbf{A}_2^{\oplus 2}$ with discriminant 3^4. Thus, one has $\mathrm{NS}(X_{23}) \simeq U(3) \oplus \mathbf{A}_2^{\oplus 2}$, and that implies that the Néron–Severi lattice of X is also isometric to $U(3) \oplus \mathbf{A}_2^{\oplus 2}$.

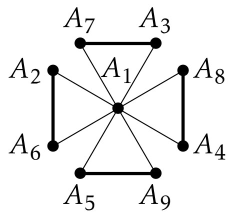
Remark 6.3. In [Nik83, Proof of Theorem 6.4.1], Nikulin constructs some surfaces X with Néron–Severi lattice $U(3)\oplus\mathbf{A}_2^{\oplus 2}$ as the minimal desingularization of a triple cover of a smooth quadric $Q\subset\mathbb{P}^3$ branched over a $(3,3)$-curve C with two singularities $\mathbf{a}_1$. In particular, the automorphism group of such a surface X has order at least 6. According to [Kon89], the general surface with $\mathrm{NS}(X)\simeq U(3)\oplus\mathbf{A}_2^{\oplus 2}$ has automorphism group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

6.2. The lattice $U(4)\oplus\mathbf{D}_4$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X)=U(4)\oplus\mathbf{D}_4.$$

The surface X contains nine (-2)-curves $A_1,\dots,A_9$, with configuration



The class

$$D_6=A_1+2A_2+2A_6$$

is ample with $D_6^2=6$. The divisor $E=A_2+A_6$ is the class of a fiber, and $D_6=2E+A_9$. For $j>1$, one has $EA_j=0$ and $D_6\cdot A_j=1$; moreover, $EA_1=D_6\cdot A_1=2$. The curves $A_2,\dots,A_9$ are such that the divisors

$$A_k+A_{k+4},\ k\in\{2,3,4,5\}$$

are the reducible fibers of the elliptic fibration induced by E.

The divisor

$$D_2=A_1+A_2+A_6=E+A_1$$

is nef, of square 2, base-point free, with $D_2\cdot A_1=0$ and $D_2\cdot A_j=1$ for $j\geq 2$. We have

$$D_2\equiv A_1+A_3+A_7\equiv A_1+A_4+A_8\equiv A_1+A_5+A_9.$$

By using the linear system $|D_2|$, we obtain the following.

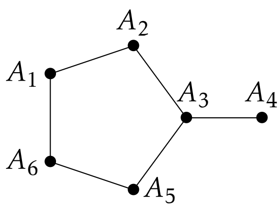
Proposition 6.4. *The K3 surface X is a double cover of the plane branched over a nodal sextic curve such that there exist four lines through the node that are tangent to the sextic at other intersection points.*

6.3. The lattice $U\oplus\mathbf{A}_4$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X)=U\oplus\mathbf{A}_4.$$

The surface X contains six (-2)-curves $A_1,\dots,A_6$ with dual graph



These six curves generate the Néron–Severi lattice. The class

$$D_{50} = (8, 9, 11, 5, 9, 8)$$

in the basis $A_1, \ldots, A_6$ is ample, and $D_{50}^2 = 50$. The six (-2)-curves have degree 1 for D_{50}. The class

$$F = A_1 + A_2 + A_3 + A_5 + A_6$$

is the class of a fiber of type I_5. The curve A_4 is a section of the elliptic fibration. The divisor

$$D_2 = 2F + A_4$$

is nef, of square 2, with $D_2 \cdot A_j = 0$ for $j \neq 3$. It has base points, but $D_8 = 2D_2$ is base-point free.

Proposition 6.5. *The linear system $|D_8|$ defines a morphism $\varphi \colon X \to \mathbf{F}_4$ that is branched over A_4; the image of A_4 is the unique section s of the Hirzebruch surface $\mathbf{F}_4$ with $s^2 = -4$. The branch curve of φ is the disjoint union of s and a curve $B \in |3s + 12f|$. The curve B has a unique singularity q of type $\mathbf{a}_4$ which is the image of A_1, A_2, A_5, A_6, and the image of A_3 by φ is the fiber f through q. The local intersection number of f and B at q is 2; in other words, f is transverse to the branch of B at q.*

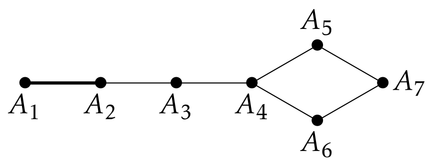
Proof. We apply Theorem 2.3, case a) i). □

6.4. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{A}_3$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus A_1 \oplus A_3.$$

The surface contains seven (-2)-curves $A_1, \ldots, A_7$ with configuration



The class

$$D_{42} = 6A_1 + 8A_2 + 5A_3 + 3A_4 + A_5 + A_6$$

is ample, of square 42. The divisors $F_1 = A_4 + A_5 + A_6 + A_7$ and $F_2 = A_1 + A_2$ are two fibers of an elliptic fibration for which the class A_3 is a section. The class

$$D_2 = A_1 + 2A_2 + 2A_3 + 2A_4 + A_5 + A_6 \equiv A_2 + 2A_3 + 3A_4 + 2A_5 + 2A_6 + A_7$$

is nef, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_7 = 2$ and $D_2 \cdot A_j = 0$ for $j \in \{2, 3, 4, 5, 6\}$. Let $\eta \colon X \to \mathbb{P}^2$ be the associated double cover and C_6 be the branch curve.

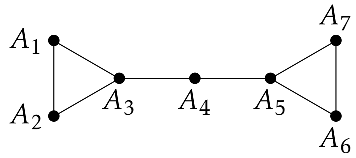
Proposition 6.6. *The curve C_6 has a $\mathbf{d}_5$ singularity q onto which the curves A_j, $j = 2, \ldots, 6$ are contracted. From the above equivalence relation between divisors, we see that the images of A_1 and A_7 are the two tangents to the branches of the singularity q.*

6.5. The lattice $U \oplus \mathbf{A}_2^{\oplus 2}$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{A}_2^{\oplus 2}.$$

The surface contains seven (-2)-curves $A_1, \ldots, A_7$ with configuration



The class

$$D_{20} = 5A_1 + 5A_2 + 6A_3 + 3A_4 + A_5$$

is ample, of square 20, and the curves A_j have degree 1 for D_{20}. The divisor

$$D_4 = 2A_1 + 2A_2 + 3A_3 + 2A_4 + A_5 \equiv A_3 + 2A_4 + 3A_5 + 2A_6 + 2A_7$$

is nef, of square 4, base-point free, with $D_4 \cdot A_j = 1$ for $j \in \{1,2,6,7\}$ and $D_4 \cdot A_j = 0$ for $j \in \{3,4,5\}$. Since $D_4F_1 = 2$, the linear system $|D_4|$ is hyperelliptic. By using the linear system $|D_4|$, we obtain the following.

Proposition 6.7. *The surface is a double cover of $\mathbf{F}_2$ branched over the unique section s with $s^2 = -2$ and a curve $B \in |3s + 8f|$. The fibers f_p, f_q through the two intersection points p, q of s and B are tangent to the curve B at another intersection point with B. The image of A_1, A_2 is f_p, the image of A_6, A_7 is f_q, and the curves A_3, A_5 are mapped to the points p, q.*

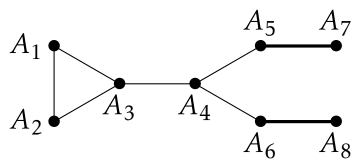
Proof. This comes from Theorem 2.3, case a) iii) v). □

6.6. The lattice $U \oplus \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_2$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_2.$$

The surface contains eight (-2)-curves $A_1, \dots, A_8$, with configuration



The class

$$D_{18} = 4A_1 + 4A_2 + 5A_3 + 3A_4 + A_5 + A_6$$

is ample, of square 18, with $D_{18} \cdot A_j = 1$ for $j \leq 6$ and $D_{18} \cdot A_7 = D_{18} \cdot A_8 = 2$. The three divisors

$$F = A_1 + A_2 + A_3 \equiv A_5 + A_7 \equiv A_6 + A_8$$

are classes of fibers of type I_3 or IV and of type $I_2 + I_2$ or III; the (-2)-curve A_4 is a section of the elliptic fibration. The class

$$D_2 = A_1 + A_2 + 2A_3 + 2A_4 + A_5 + A_6$$

is of square 2 and base-point free. Let $\eta \colon X \to \mathbb{P}^2$ be the associated double cover map, and let C_6 be the branch curve. For $j = 1, \dots, 8$, one has

$$D_2 \cdot A_j = 1, 1, 0, 0, 0, 0, 2, 2,$$

respectively. This leads to the following.

Proposition 6.8. *The morphism η contracts the curves A_3, A_4, A_5, A_6 to a $\mathbf{d}_4$ singularity q, the curve $A_1 + A_2$ is the pull-back of the tangent line of a branch of the singularity, a line which is tangent to the sextic at another intersection point, and the curves A_7, A_8 are pull-backs of the lines tangent to other branches, each of which lines meets C_6 in two other points.*

6.7. The lattice $U(2)\oplus \mathbf{A}_1^{\oplus 4}$ and del Pezzo surfaces of degree 4

The class of square 8

$$D_8 = (2,2,-1,-1,-1,-1) \in U(2)\oplus \mathbf{A}_1^{\oplus 4}$$

has no (-2)-vectors perpendicular to it; thus we have a marking $\mathrm{NS}(X) \simeq U(2)\oplus \mathbf{A}_1^{\oplus 4}$ that maps D_8 to an ample class. There are 16 (-2)-curves $A_1,\dots,A_{16}$ on X; all have degree 2 with respect to D_8. The class D_8 is base-point free and not hyperelliptic; moreover, the linear system $|D_8|$ embeds X as a degree 8 surface in $\mathbb{P}^5$. One can describe the configuration of the (-2)-curves on X as follows:

Let $p_1,\dots,p_5$ be five points in general position in $\mathbb{P}^2$ (by which we mean that no three are on a line), and let C_6 be a sextic curve that has only nodal singularities at each point p_j. Let $\pi\colon X \to \mathbb{P}^2$ be the minimal desingularization of the double cover $Y \to \mathbb{P}^2$ branched over C_6. This is a smooth K3 surface which contains the following 16 (-2)-curves:

- the (-2)-curves $A_1,\dots,A_5$ above the points $p_1,\dots,p_5$,
- the strict transform A_{ij} in X of the line through p_i and p_j for $1 \le i < j \le 5$,
- the strict transform A_0 of the unique conic passing through the five points $p_1,\dots,p_5$.

From that description, one understands easily the configuration of the 16 (-2)-curves A_j, $0 \le j \le 5$, and A_{ij}, $1 \le i < j \le j \le 5$, on X. The pull-backs of a line L and the (-2)-curves $A_0,\dots,A_5$ generate a lattice which is

$$[2]\oplus \mathbf{A}_1^{\oplus 5} \simeq U(2)\oplus \mathbf{A}_1^{\oplus 4}.$$

The involution from the double cover fixes a smooth curve of genus $10-5=5$.

Let $p_1,\dots,p_5$ be five points in general position in the plane, and let C_6 be a general sextic curve with nodes at the points p_i.

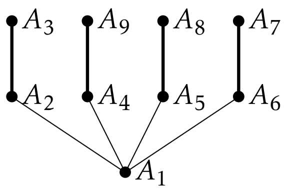
Proposition 6.9. *The double cover of the blow-up of $\mathbb{P}^2$ at the points p_i branched over the strict transform of C_6 is a K3 surface with $\mathrm{NS}(X) \simeq U(2)\oplus \mathbf{A}_1^{\oplus 4}$. The moduli space of these K3 surfaces is therefore unirational.*

6.8. The lattice $U\oplus \mathbf{A}_1^{\oplus 4}$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U\oplus \mathbf{A}_1^{\oplus 4}.$$

The surface contains nine (-2)-curves $A_1,\dots,A_9$, with dual graph



The (-2)-curves $A_1,\dots,A_6$ generate the Néron–Severi lattice, and the classes of the remaining (-2)-curves are

$$A_7 \equiv A_2 + A_3 - A_6, \quad A_8 \equiv A_2 + A_3 - A_5, \quad A_9 \equiv A_2 + A_3 - A_4.$$

The class

$$D_{16} = 3A_1 + 4A_2 + 3A_3 + A_4 + A_5 + A_6$$

is ample and of square 16. The divisors

$$E_1 = A_2 + A_3, A_4 + A_9, A_5 + A_8, A_6 + A_7$$

are the reducible fibers of an elliptic fibration such that A_1 is a section since one has

$$A_j \cdot A_1 = 1 \text{ for } j \in \{2,4,5,6\}.$$

The class

$$E_2 = 2A_1 + A_2 + A_4 + A_5 + A_6$$

is the unique reducible fiber of another fibration. The effective divisor

$$D_2 = 2A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

is nef, of square $D_2^2 = 2$. The system $|D_2|$ is base-point free, and $D_2 \cdot A_j = 0$ if and only if $j \in \{1,3,4,5,6\}$; else $D_2 \cdot A_j = 2$. Let $\pi\colon X \to \mathbb{P}^2$ be the double cover map associated to $|D_2|$. Since $D_2 \cdot A_j = 0$ for $j \in \{1,3,4,5,6\}$, the image of A_2 by the double cover map is a line L_2 such that $D_2 = \pi^* L_2$. The intersection matrix of the curves A_j for $j \in \{1,3,4,5,6\}$ reveals that the sextic branch curve has a $\mathbf{d}_4$ singularity and a node $\mathbf{a}_1$, and L_2 is the line through these two singularities. The node is resolved on the double cover by A_3; the $\mathbf{d}_4$ singularity is resolved by the union of the curves A_1, A_4, A_5, A_6 with $A_1 \cdot A_4 = A_1 \cdot A_5 = A_1 \cdot A_6 = 1$. We have

$$\begin{aligned} D_2 &\equiv 2A_1 + A_4 + A_5 + 2A_6 + A_7, \\ D_2 &\equiv 2A_1 + A_4 + 2A_5 + A_6 + A_8, \\ D_2 &\equiv 2A_1 + 2A_4 + A_5 + A_6 + A_9; \end{aligned}$$

thus the images of A_7, A_8, A_9 are the three tangent lines L_7, L_8, L_9 of the $\mathbf{d}_4$ singularity.

Let C_6 be a general sextic plane curve with a $\mathbf{d}_4$ singularity and a node. We denote by L_2 the line through the node and the $\mathbf{d}_4$ singularity. Let L_7, L_8, L_9 be the three lines tangent to the sextic at the $\mathbf{d}_4$ singularity. Let $Z \to \mathbb{P}^2$ be the embedded desingularization of C_6: this the blow-up of $\mathbb{P}^2$ at the node of C_6 (with the exceptional divisor denoted by L_3), and over the $\mathbf{d}_4$ singularity, there are four blow-ups, producing four exceptional curves L_1, L_4, L_5, L_6 such that

$$L_1^2 = -4, \quad L_1 \cdot L_j = 1,\ L_j^2 = -1,\ L_j \cdot L_k = 0 \quad \forall j,k \in \{4,5,6\} \text{ with } j \neq k.$$

The strict transforms of the L_j ($j \in \{2,7,8,9\}$) are disjoint curves $\bar{L}_j$, and (up to re-ordering)

$$L_4 \cdot \bar{L}_9 = 1, \quad L_5 \cdot \bar{L}_8 = 1, \quad L_6 \cdot \bar{L}_7 = 1.$$

From the above discussion, we obtain the following.

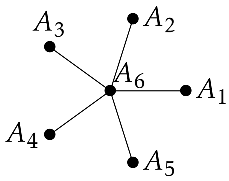
Proposition 6.10. *The curve $\bar{C}_6 + L_1$ is the branch locus of a double cover $Y \to Z$. The surface Y is a smooth K3 surface, and the pull-backs of the curves L_1, $\bar{L}_2$, L_3, L_4 , L_5, L_6, $\bar{L}_7$, $\bar{L}_8$, $\bar{L}_9$ are (-2)-curves $A_1,\dots,A_9$ such that the lattice generated by these curves is $U \oplus \mathbf{A}_1^{\oplus 4}$.*

6.9. The lattice $U(2) \oplus \mathbf{D}_4$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U(2) \oplus \mathbf{D}_4.$$

The surface X contains six (-2)-curves, with dual graph



These six (-2)-curves generate the Néron–Severi lattice. The class

$$D_{22} = 4A_6 + 3\sum_{j=1}^{6} A_j$$

is ample of square 22, and the curves A_j have degree 1 for that polarization.

For $j \in \{1,\dots,5\}$, let $F_j = A_6 + \sum_{k\neq j} A_k$; one has $F_j^2 = 0$, and F_j is a singular fiber of type I_0^* of an elliptic fibration ϕ_j. Moreover, $F_j \cdot A_j = 2$, so that there are no sections.

The class

$$D_2 = A_6 + \sum_{j=1}^{6} A_j$$

has square 2, with $D_2 \cdot A_j = 0$ if and only if $j \in \{1,\dots,5\}$ and $D_2 \cdot A_6 = 1$. By using the linear system $|D_2|$, we obtain the following.

Proposition 6.11. *The K3 surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve which is the union of a line L and a smooth quintic Q such that $L \cup Q$ has normal crossings.*

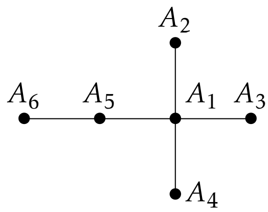
The divisor D_2 is the pull-back of the line L. The six (-2)-curves on X come from the pull-backs of the line and the five exceptional divisors above the nodes. Conversely, the six (-2)-curves on a K3 surface which is the minimal desingularization of the double cover of the plane branched over the union of a line and a quintic necessarily have the same dual graph. The moduli space of K3 surfaces with $\mathrm{NS}(X) = U(2) \oplus \mathbf{D}_4$ is unirational.

6.10. The lattice $U \oplus \mathbf{D}_4$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{D}_4.$$

The dual graph of the six (-2)-curves on X is



The divisor

$$D_{70} = 21A_1 + 10(A_2 + A_3 + A_4) + 13A_5 + 6A_6$$

is ample, of square 70; the curves $A_1,\dots,A_6$ have degree 1 for D_{70}. The divisor

$$F = 2A_1 + A_2 + A_3 + A_4 + A_5$$

is a fiber of type I_0^* of an elliptic fibration of X for which A_6 is the unique section. The divisor

$$D_2 = 2F + A_6$$

is nef, of square 2, and has base points. The divisor $D_8 = 2D_2$ is base-point free and hyperelliptic.

Proposition 6.12. *The linear system $|D_8|$ defines a double cover $\varphi\colon X \to \mathbf{F}_4$, where the branch locus is the disjoint union of the unique section s with $s^2 = -4$ and $B' \in |3s + 12f|$. The curve B' has a $\mathbf{d}_4$ singularity q; the curves A_1, A_2, A_3, A_4 are contracted to q by φ; the image of the curve A_5 is the fiber through q.*

Proof. We apply Theorem 2.3, case a) i). □

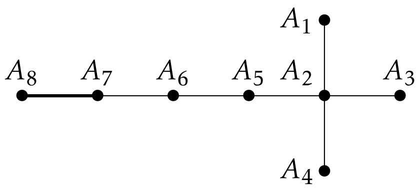
7. Rank 7 lattices

7.1. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1.$$

There exist eight (-2)-curves $A_1, \dots, A_8$ on X with dual graph



The divisor

$$D_{62} = (8, 17, 8, 8, 11, 6, 2)$$

in the basis $A_1, \dots, A_7$ is ample, of square 62. The classes

$$F_1 = A_1 + 2A_2 + A_3 + A_4 + A_5, \quad F_2 = A_7 + A_8$$

are singular fibers of type I_0^* and I_2 or III, respectively, of an elliptic fibration for which A_6 is a section (that determines the class of A_8 in $\mathrm{NS}(X)$). The divisor

$$D_2 = A_1 + 4A_2 + 2A_3 + 2A_4 + 3A_5 + 2A_6 + A_7$$

is base-point free, of square 2, with $D_2 \cdot A_k = 0$ if and only if $k \in \{2, 3, 4, 5, 6, 7\}$, $D_2 \cdot A_1 = D_j \cdot A_8 = 2$ and

$$D_2 \equiv 2A_2 + A_3 + A_4 + 2A_5 + 2A_6 + 2A_7 + A_8.$$

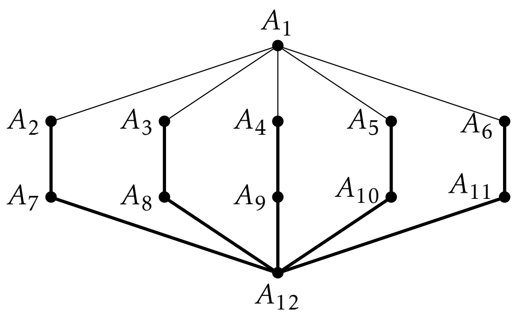
Proposition 7.1. *The linear system $|D_2|$ defines a double cover of the plane $\pi\colon X \to \mathbb{P}^2$ branched over a sextic curve C_6 with a $\mathbf{d}_6$ singularity. The image of the curve A_1 is the line that is tangent to two branches of the $\mathbf{d}_6$ singularity, and the image of A_8 is tangent to the third branch.*

7.2. The lattice $U \oplus \mathbf{A}_1^{\oplus 5}$

Let X be a K3 surface with a Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{A}_1^{\oplus 5}.$$

There exist 12 (-2)-curves $A_1, \dots, A_{12}$ on X, with dual graph



The divisor

$$D_{14} = 2A_1 + 2A_2 + A_7 + \sum_{i=1}^{7} A_i$$

is ample of square 14. The divisors

$$A_{2+j} + A_{7+j},\ j \in \{0,1,2,3,4\},$$

are irreducible fibers of an elliptic fibration ϕ_0 of X. The curve A_1 is a section of ϕ_0, and the curve A_{12} (of class $(2,0,1,1,1,1,-1)$) is a 2-section.

The divisor

$$D_2 = 2A_1 + \sum_{j=2}^{6} A_j$$

has square 2, and $|D_2|$ is base-point free. It defines a 2-to-1 cover $\pi\colon X \to \mathbb{P}^2$ of the plane branched over a sextic curve C_6. For the curves A_j, $j \in \{2,3,4,5,6,12\}$, one has $D_2 \cdot A_j = 0$; thus the images of these disjoint curves are six points in $\mathbb{P}^2$, and the sextic has nodes at these points. We have $D_2 \cdot A_1 = 1$ and

$$D_2 \cdot A_7 = D_2 \cdot A_8 = D_2 \cdot A_9 = D_2 \cdot A_{10} = D_2 \cdot A_{11} = 2.$$

Since $D_2 = 2A_1 + \sum_{j=2}^{6} A_j$, there is a line L_1 in the branch locus such that $D_2 = \pi^* L_1$, and C_6 is the union of L_1 and a nodal quintic. Conversely, we have the following.

Proposition 7.2. *The minimal resolution of a surface which is the double cover of the plane branched over the union of a line and a general quintic with a node is a K3 surface of type* $U \oplus \mathbf{A}_1^{\oplus 5}$.

Such surfaces are studied in [AK11, Section 3.3]. Clearly, the moduli space of K3 surfaces with $\mathrm{NS}(X) = U \oplus \mathbf{A}_1^{\oplus 5}$ is unirational.

7.3. The lattice $U(2) \oplus \mathbf{A}_1^{\oplus 5}$ and cubic surfaces

Let C_6 be a sextic curve in $\mathbb{P}^2$ with six nodes in general position. Let $Z \to \mathbb{P}^2$ be the blow-up of the nodes; it is a degree 3 del Pezzo surface and contains 27 (-1)-curves:

- the 6 exceptional divisors E_i, $i = 1,\dots,6$,
- the strict transforms L_{ij} of the 15 lines through p_i, p_j $(i \neq j)$,
- the strict transforms Q_j of the 6 conics that go through points in $\{p_1,\dots,p_6\} \setminus \{p_j\}$.

The Néron–Severi lattice of Z is generated by the pull-back L' of a line and $E_1,\dots,E_6$; it is the unimodular rank 7 lattice

$$I_1 \oplus I_{-1}^{\oplus 6}.$$

The anti-canonical divisor of the degree 3 del Pezzo surface Z is given by

$$-K_Z = 3L' - (E_1 + \cdots + E_6);$$

this is an ample divisor. The linear system $|-K_Z|$ is base-point free (see [Dem76, Section 3, Theorem 1]), and each of the 30 divisors

$$Q_j + L_{ij} + E_i \quad \text{with } i \neq j,\ i,j \in \{1,\dots,6\}$$

and 15 divisors

$$L_{ij} + L_{kl} + L_{mn} \quad \text{with } \{i,j,k,l,m,n\} = \{1,\dots,6\}$$

belongs to the linear system $|-K_Z|$.

The double cover $f\colon Y \to Z$ branched over the strict transform C_6' of C_6 is a smooth K3 surface. The pull-backs of the 27 (-1)-curves are (-2)-curves. We denote by $A_1,\dots,A_6$ the pull-backs on Y of the curves E_i and by L the pull-back of L'. Naturally, the lattice $f^*\mathrm{NS}(Z)$ is $\left(I_1 \oplus I_{-1}^{\oplus 6}\right)(2)$, which is also the lattice generated by $L, A_1,\dots,A_6$ and is isometric to $U(2) \oplus \mathbf{A}_1^{\oplus 5}$.

Since $f\colon Y \to Z$ is finite, its pull-back $D_6 = f^*(-K_Z)$ is ample, base-point free, non-hyperelliptic, with $D_6^2 = 6$. We thus have the following.

Proposition 7.3. *The image of Y under the map $\pi\colon Y \to \mathbb{P}^4$ obtained from $|D_6|$ is a degree 6 complete intersection surface. There exist 45 hyperplane sections of Y which are each the union of 3 conics.*

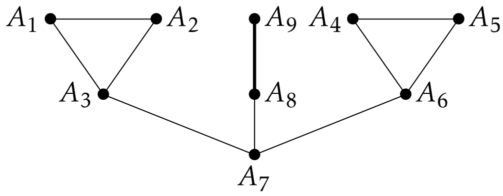
These 45 hyperplane sections correspond to the 45 tritangent planes of a cubic, *i.e.*, to the planes containing 3 lines in the cubic. In fact, by the above discussion, the strict transform in the cubic surface Z of C_6 is a degree 6 curve in $\mathbb{P}^3$ which is the complete intersection of the cubic surface $Y = \{f_3(x,y,z,t) = 0\}$ and a quadric $\{q_2(x,z,y,t) = 0\}$. The K3 surface Y is the complete intersection of $\{f_3(X,Y,Z,T) = 0\}$ and the quadric $\{q_2(X,Y,Z,T) - W^2 = 0\}$ in $\mathbb{P}^4$ (with coordinates X,Y,Z,T,W). From that discussion, we see that the moduli space of K3 surfaces with $\mathrm{NS}(X) \simeq U(2) \oplus \mathbf{A}_1^{\oplus 5}$ is unirational.

7.4. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{A}_2^{\oplus 2}$

Let X be a K3 surface with Néron–Severi lattice

$$\mathrm{NS}(X) = U \oplus \mathbf{A}_1 \oplus \mathbf{A}_2^{\oplus 2}.$$

The dual graph of the nine (-2)-curves on X is



The curves A_1, A_2, A_3, A_4, A_5, A_7, A_8 generate the Néron–Severi lattice. The divisor

$$D_{18} = (5,5,6,-1,-1,3,1)$$

in the above basis is ample, of square 18. One has $D_{18} \cdot A_j = 1$ for $j < 8$ and $D_{18} \cdot A_9 = 2$. The divisors

$$F = A_1 + A_2 + A_3,\ A_4 + A_5 + A_6,\ A_8 + A_9$$

are the reducible fibers of an elliptic fibration of the surface. The divisor

$$D_2 = 2F + A_7$$

is nef, of square 2, with base points and $D_2 \cdot A_3 = D_2 \cdot A_6 = D_2 \cdot A_8 = 1$, $D_2 \cdot A_j = 0$ for $j \neq 0$. By Theorem 2.3, case a) i), the base-point free linear system $|2D_2|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ which is branched over the unique section s of $mathbf{F}_4$ with $s^2 = -4$ and a disjoint curve $B \in |2s + 12f|$. The curve B has a node q (which is the image of A_9) and two $\mathbf{a}_2$ singularities p, p' which are the images of A_1, A_2 and A_4, A_5. The images of A_3, A_6, A_8 are the fibers through p, p' and q, respectively.

One can give another description as follows. The divisor

$$D_2' = A_9 + (A_3 + A_6 + 2A_7 + 2A_8)$$

is nef of square 2, and $|D_2'|$ is base-point free. Moreover, $D_2 \cdot A_j = 0$ if and only if $j \in \{3,6,7,8\}$, and $D_2 \cdot A_9 = 2$ and $D_2 \cdot A_j = 1$ for $j \in \{1,2,4,5\}$. We have

$$D_2' \equiv A_1 + A_2 + (2A_3 + A_6 + 2A_7 + A_8),$$
$$D_2' \equiv A_4 + A_5 + (A_3 + 2A_6 + 2A_7 + A_8).$$

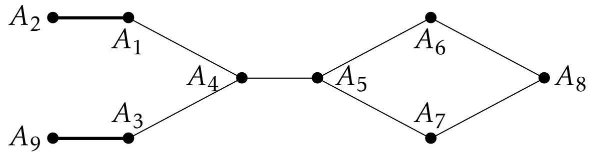
By using the linear system $|D_2'|$, we obtain the following.

Proposition 7.4. *The K3 surface X is the double cover $\eta\colon X \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched over a sextic curve C_6 with a $\mathbf{d}_4$ singularity q onto which the curves A_3, A_6, A_7, A_8 are contracted.*

The image of the curve A_9 is a line that is tangent to a branch of the singularity q and that meets the sextic in two other points. The two lines L_1, L_2 that are tangent to the two other branches of q are tangent to another point of the sextic. The image of A_1, A_2 is L_1, and the image of A_4, A_5 is L_2.

7.5. The lattice $U \oplus \mathbf{A}_1^{\oplus 2} \oplus \mathbf{A}_3$

The dual graph of the nine (-2)-curves $A_1, \dots, A_9$ on X is



The curves $A_1, \dots, A_7$ generate the Néron–Severi lattice. The divisor

$$D_{34} = (6, 4, 2, 5, 3, 1, 1)$$

in the basis $A_1, \dots, A_7$ is ample of square 34. The divisors

$$F_1 = A_1 + A_2, \quad F_2 = A_3 + A_9, \quad F_3 = A_5 + A_6 + A_7 + A_8$$

are the reducible fibers of an elliptic fibration ϕ_1 of X for which A_4 is a section (for that, one can deduce the classes of curves A_8, A_9). The divisor

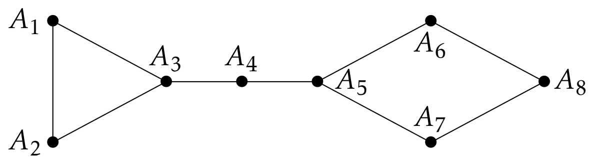
$$D_2 = 2F_1 + A_4$$

is nef, of square 2, with base points. By Theorem 2.3, case i) a), we have the following.

Proposition 7.5. *The linear system $|D_2|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has two nodes p, p' and an $\mathbf{a}_3$ singularity q. The pull-backs of the fibers through p, p', q are the fibers F_1, F_2, F_3.*

7.6. The lattice $U \oplus \mathbf{A}_2 \oplus \mathbf{A}_3$

The dual graph of the eight (-2)-curves $A_1, \dots, A_8$ on X is



The seven (-2)-curves $A_1, \dots, A_7$ generate $\mathrm{NS}(X)$, and the divisor

$$D_{42} = (6, 6, 8, 5, 3, 1, 1)$$

in the basis $A_1, \dots, A_7$ is ample, of square 42. The surface has an elliptic fibration ϕ_1 with reducible fibers the curves

$$F_1 = A_1 + A_2 + A_3, \quad F_2 = A_5 + A_6 + A_7 + A_8,$$

and A_4 is a section. The divisor

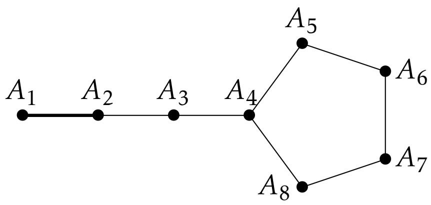
$$D_2 = 2F_1 + A_4$$

is nef, of square 2, with base points. By Theorem 2.3, case i) a), we have the following.

Proposition 7.6. *The linear system $|D_2|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has a cusp p and an $\mathbf{a}_3$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

7.7. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{A}_4$

The dual graph of the eight (-2)-curves $A_1,\dots,A_8$ on X is



In the basis $A_1,\dots,A_7$, the divisor

$$D_{42} = (7,9,5,2,0,-1,-1)$$

is ample, of square 42. The curves

$$F_1 = A_1 + A_2, \quad F_2 = A_4 + A_5 + A_6 + A_7 + A_8$$

are the reducible fibers of an elliptic fibration for which A_3 is a section. The divisor

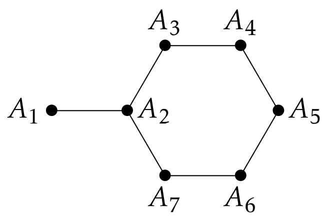
$$D_2 = 2F_1 + A_4$$

is nef, of square 2, with base points. By Theorem 2.3, case i) a), we have the following.

Proposition 7.7. *The linear system $|2D_2|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s+12f|$. The curve B has a node p and an $\mathbf{a}_4$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

7.8. The lattice $U \oplus \mathbf{A}_5$

The dual graph of the seven (-2)-curves $A_1,\dots,A_7$ on X is



The seven (-2)-curves $A_1,\dots,A_7$ generate the Néron–Severi lattice, and in that basis, the divisor

$$D_{84} = (7,15,12,10,9,10,12)$$

is ample, of square 84. We have $D_{84}\cdot A_j = 1$ for $j \in \{1,\dots,7\}\setminus\{5\}$ and $D_{84}\cdot A_5 = 2$. The divisor

$$F = \sum_{j=2}^{7} A_j$$

is the unique reducible fiber of an elliptic fibration of X; the curve A_1 is a section.

The divisor

$$D_2' = (2,4,3,2,1,2,3)$$

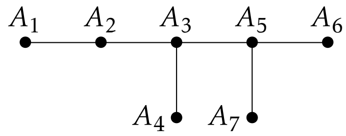
is nef, base-point free, of square 2. One has $D_2\cdot A_j = 0$ for $j \in \{1,\dots,7\}\setminus\{5\}$ and $D_2\cdot A_5 = 2$. The surface X is a double cover of $\mathbb{P}^2$. The branch locus is a sextic curve with an $\mathbf{e}_6$ singularity; there exists a line L through the $\mathbf{e}_6$ singularity which cuts the sextic transversely in two points such that the image of A_5 is L.

One may also describe that surface as a double cover of $\mathbf{F}_4$.

Proposition 7.8. *The linear system $|4F+2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B \in |3s+12f|$. The curve B has an $\mathbf{a}_5$ singularity q. The pull-back of the fiber through q is the fiber F.*

7.9. The lattice $U \oplus \mathbf{D}_5$

The dual graph of the seven (-2)-curves $A_1,\dots,A_7$ on X is



The divisor

$$D_{114} = (8,17,27,13,25,12,12)$$

in the basis $A_1,\dots,A_7$ is ample, of square 114, and every (-2)-curve on X has degree 1 with respect to D_{114}. The divisor

$$F = A_2 + 2A_3 + A_4 + 2A_5 + A_6 + A_7$$

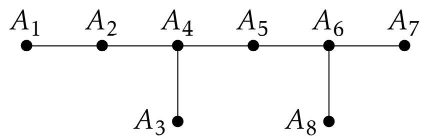
is the unique reducible fiber of an elliptic fibration of X for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 7.9. *The linear system $|4F+2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B \in |3s+12f|$. The curve B has a $\mathbf{d}_5$ singularity q. The pull-back of the fiber through q is the fiber F.*

8. Rank 8 lattices

8.1. The lattice $U \oplus \mathbf{D}_6$

The dual graph of the eight (-2)-curves $A_1,\dots,A_8$ on X is



The divisor

$$D_{220} = (11,24,18,38,35,33,16,16)$$

in the basis $A_1,\dots,A_8$ is ample, of square 220, with $D_{220}\cdot A_j = 1$ if $j \notin \{1,3\}$ and $D_{220}\cdot A_j = 2$ if $j \in \{1,3\}$. The divisor

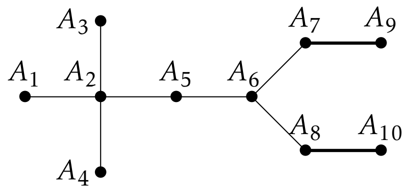
$$F = A_2 + A_3 + 2A_4 + 2A_5 + 2A_6 + A_7 + A_8$$

is a fiber of an elliptic fibration for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.1. *The linear system $|4F+2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B \in |3s+12f|$. The curve B has a $\mathbf{d}_6$ singularity q. The pull-back of the fiber through q is the fiber F.*

8.2. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 2}$

The dual graph of the 10 (-2)-curves $A_1,\dots,A_{10}$ on X is



The first eight curves generate the Néron–Severi lattice. The divisor

$$D_{54} = (6,13,6,6,9,6,2,2)$$

of square 54 is ample with $D_{54} \cdot A_j = 1$ for $j \leq 6$, $D_{54} \cdot A_j = 2$ for $j \in \{7,8\}$ and $D_{54} \cdot A_j = 4$ for $j \in \{9,10\}$. The divisors

$$F_1 = A_2 + \sum_{j=1}^{5} \cdot A_j, \quad F_2 = A_7 + A_9, \quad F_3 = A_8 + A_{10}$$

are fibers of an elliptic fibration φ_2 such that A_6 is a section. For $j \in \{1,3,4\}$, the divisors

$$-A_j + A_2 + A_5 + A_6 + \sum_{k=1}^{8} A_k$$

are fibers of elliptic fibrations φ_j for which A_j and A_9, A_{10} are 2-sections. The divisor

$$D_2 = A_1 + 2A_2 + A_3 + A_4 + 2A_5 + 2A_6 + A_7 + A_8$$

is nef, of square 2 and base-point free. The linear system $|D_2|$ contracts the curves A_j for $j \in \{1,3,\dots,8\}$; moreover, $D_2 \cdot A_2 = 1$ and $D_2 \cdot A_9 = D_2 \cdot A_{10} = 2$.

Proposition 8.2. *The branch curve is the union of a line L and a nodal quintic Q; the line goes through the node transversely (forming a $\mathbf{d}_4$ singularity on the sextic $L \cup Q$, resolved by A_5, A_6, A_7, A_8) and cuts the quintic in three other points (resolved by A_1, A_3, A_4).*

We have $D_2 = \pi^* L$, and the equivalences

$$D_2 \equiv A_5 + 2A_6 + A_8 + A_9$$

and

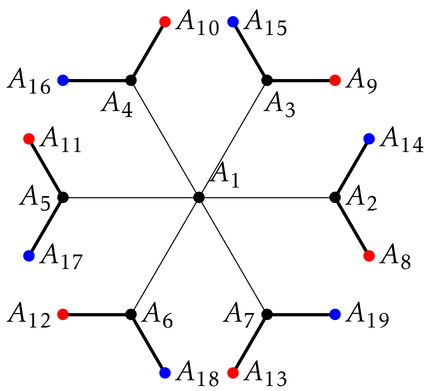
$$D_2 \equiv A_5 + 2A_6 + A_7 + A_{10}$$

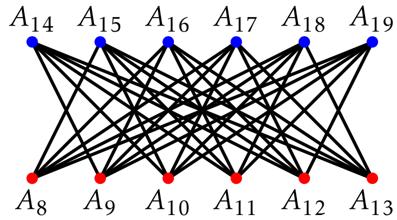
give that the images of A_9 and A_{10} are lines that are tangent to the branches of the node of the quintic Q.

Remark 8.3. One may also describe that surface as the desingularization of the double cover of $\mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$ with a $\mathbf{d}_4$ singularity and two nodes.

8.3. The lattice $U \oplus \mathbf{A}_1^{\oplus 6}$

Let X be a K3 surface with $\mathrm{NS}(X) \simeq U \oplus \mathbf{A}_1^{\oplus 6}$. The surface X contains 19 (-2)-curves; their dual graph is represented by





where we draw the graph of the configuration of the curves $A_8,\dots,A_{19}$ in another part in order for it to be more readable. The first eight curves $A_1,\dots,A_8$ generate the Néron–Severi lattice. The divisor

$$D_{12} = 3A_1 + 2A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8$$

is ample, of square 12, with $D_{12}\cdot A_j = 1$ for $j \le 7$, $D_{12}\cdot A_j = 2$ for $8 \le j \le 13$ and $D_{12}\cdot A_j = 4$ for $14 \le j \le 19$. Let $j \ne k$ be two elements of $\{2,\dots,7\}$; the effective divisor

$$F_{jk} = 2A_1 - A_j - A_k + \sum_{i=2}^{7} A_i$$

is a fiber of type IV of an elliptic fibration φ_{jk}. The divisors

$$A_{j+6} + A_{k+12},\ A_{k+6} + A_{j+12}$$

are the other reducible fibers of that fibration. The divisors

$$A_{2+k} + A_{8+k}, \quad k \in \{0,\dots,5\}$$

are reducible fibers of an elliptic fibration φ_1. The divisors

$$A_{2+k} + A_{14+k}, \quad k \in \{0,\dots,5\}$$

are reducible fibers of an elliptic fibration φ_2. The curve A_1 is a section of φ_1 and φ_2. The divisor

$$D_2 = 2A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

has square 2; it is base-point free, and it contracts $A_2,\dots,A_6,A_{13},A_{19}$. Moreover, $D_2\cdot A_1 = 1$, and for the remaining curves, $D_2\cdot A_j = 2$. From the equivalence relations obtained by the elliptic fibration with fiber $A_2 + A_8$ and the elliptic fibration φ_{27}, we obtain

$$D_2 \equiv A_j + A_{j+6} + A_{19} \equiv A_j + A_{12+6} + A_{13}, \quad \forall j \in \{2,\dots,7\}.$$

Thus we see that there is a line L in the branch locus which is the image of A_1, and the curves $A_2,\dots,A_6$ are mapped to points $p_2,\dots,p_6$ in the intersection of L with residual quintic Q in the branch locus. The curve Q has two nodes p, q above which are A_{13} and A_{19}. The curve A_7 is the strict transform of the line through p and q. For $j \in \{2,3,4,5,6\}$, the curve A_{j+6} (resp. A_{j+12}) is the pull-back of the line through p, p_j (resp. q, p_j). This leads to the following.

Proposition 8.4. *The K3 surface X is the minimal resolution of the double cover of $\mathbb{P}^2$ branched over the union of a line and a quintic with two nodes.*

From that description, the moduli space of K3 surfaces X with $\mathrm{NS}(X) \simeq U \oplus \mathbf{A}_1^{\oplus 6}$ is unirational.

8.4. The lattice $U(2)\oplus \mathbf{A}_1^{\oplus 6}$ and degree 2 del Pezzo surfaces

Let C_6 be a sextic curve in $\mathbb{P}^2$ with nodes through seven points $p_1,\dots,p_7$ in general position. Let $Z\to\mathbb{P}^2$ the blow-up of the nodes; it is a degree 2 del Pezzo surface, and it contains 56 (-1)-curves:

- the 7 exceptional divisors E_i, $i=1,\dots,7$,
- the strict transforms L_{ij} of the 21 lines through p_i, p_j $(i\neq j)$,
- the strict transforms Q_{rs} of the 21 conics that go through points in $\{p_1,\dots,p_7\}\setminus\{p_r,p_s\}$,
- the strict transforms CU_j of the 7 cubics that go through the 7 points p_k, with a double point at one of these points p_j.

The Néron–Severi lattice of Z is generated by the pull-back L' of a line and $E_1,\dots,E_7$; it is the unimodular rank 7 lattice

$$I_1\oplus I_{-1}^{\oplus 7}.$$

The anti-canonical divisor of the del Pezzo surface Z of degree 2 is given by

$$-K_Z=3L'-(E_1+\cdots+E_7);$$

this is an ample divisor. The linear system $|-K_Z|$ is base-point free (see [Dem76, Section 3, Theorem 1]), and we remark that each of the 28 divisors

$$E_i+CU_i,\ L_{ij}+Q_{ij},\quad i,j\in\{1,\dots,7\},\ i\neq j$$

belongs to the system $|-K_Z|$.

Let $Y\to Z$ be the double cover branched over the strict transform C' of C_6 in Z. Since $C'\equiv -2K_Z$, the surface Y is a smooth K3 surface. Since for any (-1)-curve B, $C'B=-2K_ZB=2$, the pull-backs on Y of the curves E_i, L_{ij}, Q_{rs}, CU_j are (-2)-curves. We denote these curves by A_i, B_{ij}, C_{rs}, D_j, respectively. Let L be the pull-back in Y of L'. The lattice generated by L, $A_1,\dots,A_7$ is (isometric to) $U(2)\oplus\mathbf{A}_1^{\oplus 6}=f^*\mathrm{NS}(Z)=\left(I_1\oplus I_{-1}^{\oplus 7}\right)(2)$. Since $f\colon Y\to Z$ is finite and the linear system $|-K_Z|$ is ample and base-point free, its pull-back $D_4=f^*(-K_Z)$ is ample and base-point free, with $D_4^2=4$. The linear system $|D_4|$ is 3-dimensional, and the invariant part is the pencil $f^*|-K_Z|$. We thus have the following.

Proposition 8.5. *The image of Y under the map $\pi\colon Y\to\mathbb{P}^3$ induced from $|D_4|$ is a smooth quartic surface. There exist 28 hyperplane sections of Y which are each the union of 2 conics.*

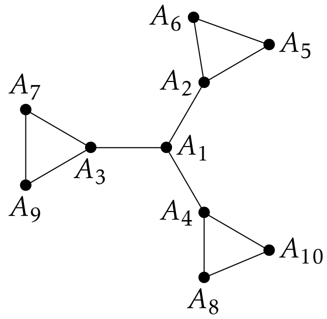
Remark 8.6. The moduli space of K3 surfaces of type $U(2)\oplus\mathbf{A}_1^{\oplus 6}$ is 12-dimensional. In [Kon00], Kondo studies the 6-dimensional moduli space of curves of genus 3 via the periods of the K3 surfaces $X=\{t^4-f_4(x,y,z)=0\}$ which are quadruple covers of the plane branched over smooth quartics curves $C=\{f_4(x,y,z)=0\}$ (thus of genus 3). The double cover of $\mathbb{P}^2$ branched over C is a degree 2 del Pezzo surface. For such K3 surfaces, the pull-backs of the 28 bitangents of the curve C give the 28 hyperplanes sections of Theorem 8.5.

Proposition 8.7. *The moduli space of $U(2)\oplus\mathbf{A}_1^{\oplus 6}$-polarized K3 surfaces is unirational.*

Proof. Seven points $p_1,\dots,p_7$ in general position in $\mathbb{P}^2$ form an open subscheme of $(\mathbb{P}^2)^7$. The space of sextics with the points $p_1,\dots,p_7$ as seven nodes is a 14-dimensional linear subspace of the space of sextics. Therefore, the moduli space of K3 surfaces that are double covers of $\mathbb{P}^2$ branched over a 7-nodal sextic is unirational. □

8.5. The lattice $U\oplus\mathbf{A}_2^{\oplus 3}$

The K3 surface contains 10 (-2)-curves $A_1,\dots,A_{10}$ with dual graph



The first eight curves generate the Néron–Severi lattice. The divisor

$$D_{18} = 3A_1 + 5A_2 + A_3 + A_4 + 4A_5 + 4A_6$$

is ample, of square 18, with $D_{18} \cdot A_j = 1$ for $j \in \{1, \dots, 10\}$. The divisors

$$A_2 + A_5 + A_6, A_3 + A_7 + A_9, A_4 + A_8 + A_{10}$$

are fibers of an elliptic fibration of the K3 surface such that A_1 is a section. For $i \in \{5,6\}$, $j \in \{7,9\}$, $k \in \{8,10\}$, the divisor

$$F_{ijk} = 3A_1 + 2A_2 + 2A_3 + 2A_4 + A_i + A_j + A_k$$

is the unique reducible fiber of type IV^* of an elliptic fibration φ_{ijk}. The divisor

$$D_2 = 2A_1 + 2A_2 + A_3 + A_4 + A_5 + A_6$$

is nef, base-point free, of square 2; it contracts A_1, A_2, A_3, A_4, and the other (-2)-curves have degree 1 for that divisor. We also have

$$D_2 \equiv 2A_1 + A_2 + 2A_3 + A_4 + A_7 + A_9,$$
$$D_2 \equiv 2A_1 + A_2 + A_3 + 2A_4 + A_8 + A_{10}.$$

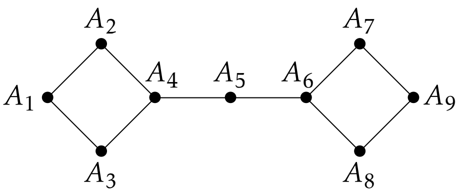
By using the linear system $|D_2|$, we obtain the following.

Proposition 8.8. *The surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve with a $\mathbf{d}_4$ singularity. The three tangents to the three branches of the singularity are tangent to another point of the sextic curve, so that the pull-back of a tangent splits into two (-2)-curves. The $6 = 3 \cdot 2$ (-2)-curves above the tangent lines are $A_5 + A_6$, $A_7 + A_9$ and $A_8 + A_{10}$.*

One can also construct this surface as a double cover of $\mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$ with three cusps singularities.

8.6. The lattice $U \oplus \mathbf{A}_3^{\oplus 2}$

The K3 surface contains nine (-2)-curves; the dual graph is



The eight (-2)-curves $A_1, \dots, A_8$ generate the Néron–Severi lattice. The divisor

$$D_{40} = (5,6,6,8,5,3,1,1)$$

in the basis $A_1, \dots, A_8$ is ample, of square 40, with $D_{40} \cdot A_1 = D_{40} \cdot A_9 = 2$ and $D_{40} \cdot A_j = 1$ for $j \in \{2, \dots, 8\}$. The divisors

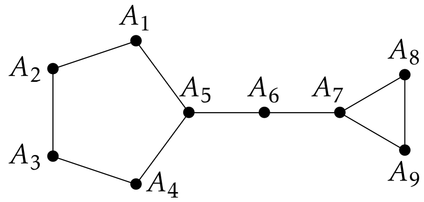
$$F_1 = A_1 + A_2 + A_3 + A_4, \quad F_2 = A_6 + A_7 + A_8 + A_9$$

are reducible fibers of an elliptic fibration, where A_5 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.9. *The linear system* $|4F_1 + 2A_5|$ *defines a morphism* $\varphi\colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$. *The curve* B *has two* $\mathbf{a}_3$ *singularities* p, q. *The pull-backs of the fibers through* p, q *are the fibers* F_1, F_2.

8.7. The lattice $U \oplus \mathbf{A}_2 \oplus \mathbf{A}_4$

The surface X contains nine (-2)-curves, with dual graph



The curves $A_1, \dots, A_8$ generate the Néron–Severi lattice. The divisor

$$D_{42} = (7, 6, 6, 7, 9, 5, 2, 0)$$

in the basis $A_1, \dots, A_8$ is ample, of square 42, with $D_{42} \cdot A_j = 1$ for $j \in \{1, \dots, 7\}$ and $D_{42} \cdot A_8 = D_{42} \cdot A_9 = 2$. The divisors

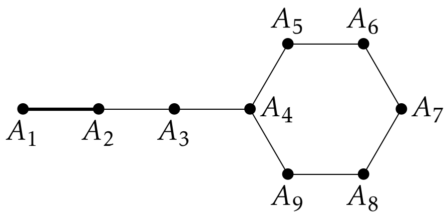
$$F_1 = A_1 + \cdots + A_5, \quad F_2 = A_7 + A_8 + A_9$$

are reducible fibers of an elliptic fibration for which A_6 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.10. *The linear system* $|4F_1 + 2A_6|$ *defines a morphism* $\varphi\colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$. *The curve* B *has one* $\mathbf{a}_4$ *singularity* p *and a cusp* q. *The pull-backs of the fibers through* p, q *are the fibers* F_1, F_2.

8.8. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{A}_5$

The surface X contains nine (-2)-curves, with dual graph



The curves $A_1, \dots, A_8$ generate the Néron–Severi lattice. The divisor

$$D_{76} = (10, 12, 7, 3, 0, -2, -3, -2)$$

in the basis $A_1, \dots, A_8$ is ample, of square 76. The divisors

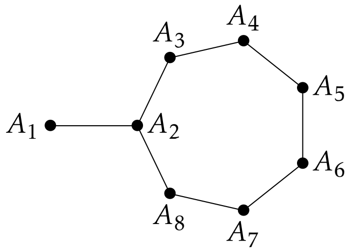
$$F_1 = A_1 + A_2, \quad F_2 = A_4 + A_5 + A_6 + A_7 + A_8 + A_9$$

are the reducible fibers of an elliptic fibration such that A_3 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.11. *The linear system* $|4F_1 + 2A_3|$ *defines a morphism* $\varphi\colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$. *The curve* B *has a node* p *and one* $\mathbf{a}_5$ *singularity* q. *The pull-back of the fibers through* p, q *are the fibers* F_1, F_2.

8.9. The lattice $U \oplus \mathbf{A}_6$

The surface X contains eight (-2)-curves $A_1, \dots, A_8$, with dual graph



These eight curves generate the Néron–Severi lattice; the divisor

$$D_{84} = (7, 15, 12, 10, 9, 9, 10, 12)$$

in the basis $A_1, \dots, A_8$ has square 84, and the curves A_j have degree 1 for D_{84}. The divisor

$$F = A_2 + \cdots + A_8$$

is a fiber of an elliptic fibration for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.12. *The linear system $|4F + 2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has an $\mathbf{a}_6$ singularity q. The pull-back of the fiber through q is the fiber F.*

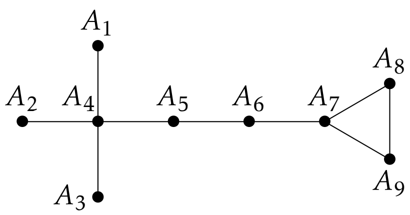
Remark 8.13. The divisor

$$D_2 = (2, 4, 3, 2, 1, 1, 2, 3)$$

in the basis $A_1, \dots, A_8$ is base-point free of square 2. It defines a double cover branched over a sextic with an $\mathbf{e}_6$ singularity.

8.10. The lattice $U \oplus \mathbf{A}_2 \oplus \mathbf{D}_4$

The surface X contains nine (-2)-curves, with dual graph



The curves $A_1, \dots, A_8$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{56} = (7, 7, 7, 15, 10, 6, 3, 1)$$

is ample of square 56, with $D_{56} \cdot A_j = 1$ for $j \in \{1, \dots, 8\}$ and $D_{56} \cdot A_9 = 4$. The divisors

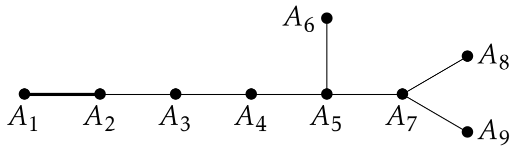
$$F_1 = A_7 + A_8 + A_9, \quad F_2 = A_1 + A_2 + A_3 + A_5 + 2A_4$$

are fibers of an elliptic fibration for which A_6 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.14. *The linear system $|4F_1 + 2A_6|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has a cusp p and one $\mathbf{d}_4$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

8.11. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{D}_5$

The surface X contains nine (-2)-curves, with dual graph



The curves $A_1,\dots,A_8$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{96} = (9,12,8,5,3,1,1,0)$$

is ample of square 96. We have $D_{96}\cdot A_1 = 6$, $D_{96}\cdot A_2 = 2$ and $D_{96}\cdot A_j = 1$ for $j \geq 3$. The divisors

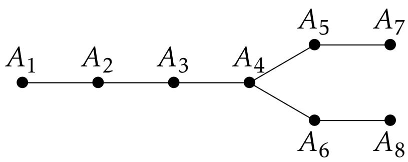
$$F_1 = A_1 + A_2, \quad F_2 = A_4 + A_6 + 2A_5 + 2A_7 + A_8 + A_9$$

are fibers of an elliptic fibration for which A_3 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.15. *The linear system $|4F_1 + 2A_3|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has a node p and one $\mathbf{d}_5$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

8.12. The lattice $U \oplus \mathbf{E}_6$

The surface X contains eight (-2)-curves, with dual graph



These eight curves $A_1,\dots,A_8$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{234} = (12,25,39,54,35,35,17,17)$$

is ample, of square 234, with $D_{234}\cdot A_j = 1$ for $j \in \{1,\dots,8\}$. The divisor

$$F = A_2 + 2A_3 + 3A_4 + 2A_5 + 2A_6 + A_7 + A_8$$

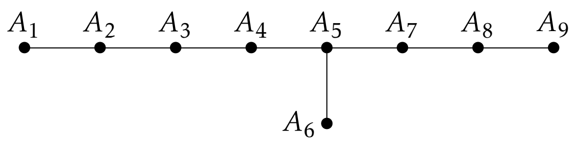
is a fiber of an elliptic fibration for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 8.16. *The linear system $|4F + 2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has an $\mathbf{e}_6$ singularity q. The pull-back of the fiber through q is the fiber F.*

9. Rank 9 lattices

9.1. The lattice $U \oplus \mathbf{E}_7$

The surface X contains nine (-2)-curves, with dual graph



These nine curves $A_1, \ldots, A_9$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{532} = (19, 39, 60, 82, 105, 52, 77, 50, 24)$$

is ample, of square 532, with $D_{532} \cdot A_j = 1$ for $j \leq 8$ and $D_{532} \cdot A_9 = 2$. The divisor

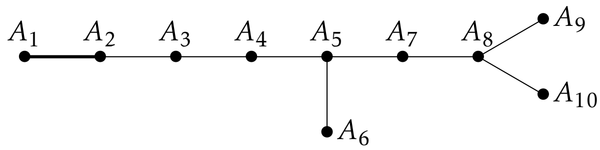
$$F = A_2 + 2A_3 + 3A_4 + 4A_5 + 2A_6 + 3A_7 + 2A_8 + A_9$$

is a fiber of an elliptic fibration, for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 9.1. *The linear system* $|4F + 2A_1|$ *defines a morphism* $\varphi \colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$. *The curve* B *has an* $\mathbf{e}_7$ *singularity* q. *The pull-back of the fiber through* q *is the fiber* F.

9.2. The lattice $U \oplus \mathbf{D}_6 \oplus \mathbf{A}_1$

The surface X contains 10 (-2)-curves, with dual graph



The nine curves $A_1, \ldots, A_9$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{148} = (10, 15, 11, 8, 6, 2, 3, 1, 0)$$

is ample, of square 148, with $D_{148} \cdot A_1 = 10$, $D_{148} \cdot A_6 = 2$ and $D_{148} \cdot A_j = 1$ for $j \neq 1, 6$. The divisors

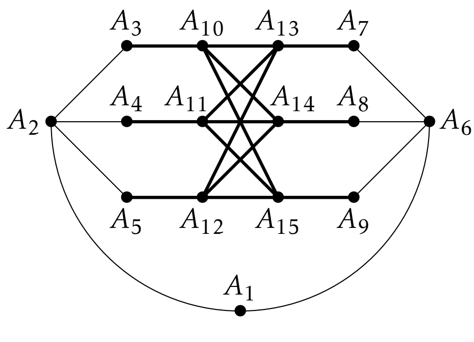
$$F_1 = A_1 + A_2, \quad F_2 = A_4 + A_6 + 2A_5 + 2A_7 + 2A_8 + A_9 + A_{10}$$

are fibers of an elliptic fibration. By Theorem 2.3, case i) a), we have the following.

Proposition 9.2. *The linear system* $|4F_1 + 2A_3|$ *defines a morphism* $\varphi \colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$. *The curve* B *has a node* p *and one* $\mathbf{d}_6$ *singularity* q. *The pull-backs of the fibers through* p, q *are the fibers* F_1, F_2.

9.3. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 3}$

The surface X contains 15 (-2)-curves $A_1, \ldots, A_{15}$, with dual graph



The curves $A_1, \ldots, A_9$ generate the Néron–Severi lattice. In that basis, the divisor

$$D_{44} = (6, 7, 3, 3, 3, 7, 3, 3, 3)$$

is ample, of square 44, with $D_{44} \cdot A_j = 1$ for $j \in \{2, \ldots, 9\}$, $D_{44} \cdot A_1 = 2$ and $D_{44} \cdot A_j = 6$ for $j \in \{10, \ldots, 15\}$. The divisor

$$D_2 = (1, 2, 1, 1, 1, 2, 1, 1, 1)$$

is nef, base-point free, with $D_2 \cdot A_j = 0$ for $j \in \{2, \ldots, 9\}$ and $D_2 \cdot A_j = 2$ for $j \notin \{2, \ldots, 9\}$. Let $\eta \colon X \to \mathbb{P}^2$ be the map associated to the linear system $|D_2|$. One thus has the following.

Proposition 9.3. *The branch curve of η is a sextic with two $\mathbf{d}_4$ singularities p, q. The image of the curve A_1 by the double cover map is the line through p, q. The six lines $C_{10},\dots,C_{15}$ that are tangent to the six branches of the two $\mathbf{d}_4$ singularities are the images of $A_{10},\dots,A_{15}$.*

9.4. The lattice $U\oplus \mathbf{A}_1^{\oplus 7}$

The K3 surface contains 37 (-2)-curves, which we denote by $A_0,A_1,\dots,A_8$ and A_{ij} for $i,j\in\{1,\dots,8\}$ with $i<j$. We have $A_0\cdot A_j=1$ for $j\in\{1,\dots,8\}$, $A_0\cdot A_{ij}=0$ for $j\in\{1,\dots,8\}$ and $i<j$, and we have $A_j\cdot A_{st}=2$ if and only if $j\in\{s,t\}$; else $A_j\cdot A_{st}=0$. Moreover, for $\{u,v\}\neq\{s,t\}$, we have

$$A_{uv}\cdot A_{st}=2(1-|\{u,v\}\cap\{s,t\}|).$$

Proposition 9.4. *The K3 surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve which is the union of a smooth conic C_0 and a quartic Q.*

The curves $A_1,\dots,A_8$ are mapped to the 8 nodes of the sextic curve, the 28 divisors A_{ij} are mapped onto the 28 lines through the 8 nodes, and A_0 is mapped onto the conic C_0. The moduli space of such surfaces is unirational.

9.5. The lattice $U(2)\oplus \mathbf{A}_1^{\oplus 7}$ and degree 1 del Pezzo surfaces

Let C_6 be a sextic curve in $\mathbb{P}^2$ with 8 nodes in general position. Let $Z\to\mathbb{P}^2$ the blow-up of the nodes; it is a degree 1 del Pezzo surface and contains 240 (-1)-curves:

- the 8 exceptional divisors E_i, $i=1,\dots,8$,
- the strict transform L_{ij} of the 28 lines through p_i, p_j $(i\neq j)$,
- the strict transforms QO_{rst} of the 56 conics that go through points in $\{p_1,\dots,p_8\}\setminus\{p_r,p_s,p_t\}$,
- the strict transforms CU_{rj} of the 56 cubics that go through 7 points p_k (with $k\neq r$) with a double point at p_j $(j\neq r)$,
- the strict transforms QA_{rst} of the 56 quartics through the 8 points p_j with double points at p_r, p_s, p_t,
- the strict transforms QI_{ij} of the 28 quintics through the 8 points p_j with double points at p_i, p_j $(i\neq j)$,
- the strict transforms S_j of the 8 sextics through 8 points with double points at all except a single point p_j with multiplicity 3.

The Néron–Severi lattice of Z is generated by the pull-back L' of a line and $E_1,\dots,E_8$; it is the unimodular rank 9 lattice

$$I_1\oplus I_{-1}^{\oplus 8}.$$

The anti-canonical divisor of the del Pezzo surface Z of degree 1 is given by

$$-K_Z=3L'-(E_1+\dots+E_8);$$

this is an ample divisor. Moreover, we remark that each of the 120 divisors

$$E_j+S_j,\ L_{ij}+QI_{ij},\ CU_{ij}+CU_{ji},\ QO_{ijk}+QA_{ijk},\quad \{i,j,k\}\subset\{1,\dots,8\}\text{ with }|\{i,j,k\}|=3,$$

belongs to the base-point free (see [Dem76, Section 3, Theorem 1]) linear system $|-2K_Z|$.

The double cover $f\colon Y\to Z$ branched over the strict transform of C_6 is a smooth K3 surface. The pull-backs of the 240 (-1)-curves are (-2)-curves (the only negative curves on a K3 surface are (-2)-curves). We denote by $A_1,\dots,A_8$ the pull-backs on Y of the curves E_i and by L the pull-back of L. Naturally, the lattice $f^*\mathrm{NS}(Z)$ is $\left(I_1\oplus I_{-1}^{\oplus 8}\right)(2)$, which is also the lattice generated by $L,A_1,\dots,A_8$ and is isometric to $U(2)\oplus\mathbf{A}_1^{\oplus 7}$.

Since $f\colon Y \to Z$ is finite, its pull-back $D_2 = f^*(-K_Z)$ is ample, with $D_2^2 = 2$. Since $|-2K_Z|$ is base-point free, the system $|D_2|$ is non-hyperelliptic; thus it defines a double cover

$$\pi\colon Y \to \mathbb{P}^2$$

branched over a sextic curve $\tilde{C}_6$, which is smooth since D_2 is ample. Let A be the pull-back on Y of a (-1)-curve E; this is a (-2)-curve. We have $D_2A = 2(-K_ZE) = 2$. Moreover, we see from the above description of the 120 divisors in $|-2K_Z|$ that there exists a (-2)-curve B such that $A+B \equiv 2D_2$; in particular, the image of $A+B$ by π is a conic which is 6-tangent to the sextic $\tilde{C}_6$. We thus obtain the following result.

Proposition 9.5. *The branch curve $\tilde{C}_6$ of the morphism π is a smooth sextic curve which possesses 120 6-tangent conics.*

Remark 9.6. Let B be the strict transform in Z of C_6; this is a smooth genus 2 curve such that $B \equiv E_j + S_j$. Thus the ramification locus R of the double cover $f\colon Y \to Z$ is a smooth genus 2 curve with $2R \equiv f^*(E_j + S_j) \equiv 2D_2$, and (since $\mathrm{NS}(X) = \mathrm{Pic}(X)$) the divisor R is in the linear system $|D_2|$, and its image by π is a line in $\mathbb{P}^2$.

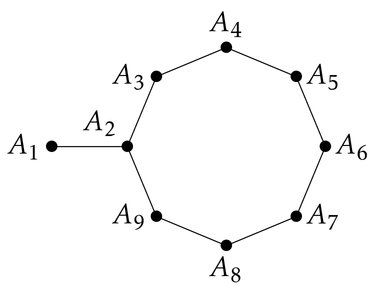
Remark 9.7. The morphism π is branched over a smooth sextic curve $\tilde{C}_6$ (thus of genus 10), whereas f is branched over a smooth genus 2 curve B, so that there exist (at least) two distinct non-symplectic involutions ι_1, ι_2 on the same K3 surface. In fact, according to Kondo [Kon89], the automorphism group of such a K3 surface is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so that we know the generators of $\mathrm{Aut}(X)$.

The curve $\tilde{C}_6$ thus has 120 conics that are 6-tangent. As far as we know, this is the record for a smooth sextic.

Remark 9.8. By a result of Degtyarev [Deg19], a smooth plane sextic curve can have at most 72 tritangent lines. Moreover, there is an example of such a sextic curve by Mukai (see [Muk88b]). In [Elk09], Elkies gives an example of an irreducible sextic curve and 1240 conics that are 6-tangent to it; that curve has a unique node.

9.6. The lattice $U \oplus \mathbf{A}_7$

The surface X contains nine (-2)-curves $A_1, \ldots, A_9$, with dual graph



These nine curves generate the Néron–Severi lattice; the divisor

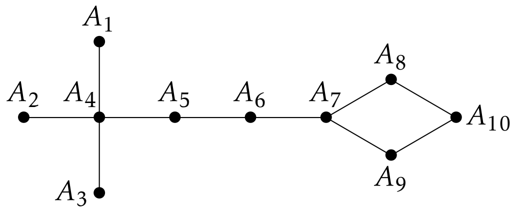
$$D_{120} = (9, 19, 15, 12, 10, 9, 10, 12, 15)$$

in that basis has square 120, is ample, with $D_{120}\cdot A_j = 1$ for $j \neq 6$ and $D_{120}\cdot A_6 = 2$. The divisor $F = \sum_{j=2}^{9} A_j$ is a fiber of an elliptic fibration for which A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 9.9. *The linear system $|4F + 2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has an $\mathbf{a}_7$ singularity q. The pull-back of the fiber through q is the fiber F.*

9.7. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_3$

The surface X contains 10 (-2)-curves, with dual graph



The curves $A_1, \ldots, A_8$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{56} = (7, 7, 7, 15, 10, 6, 3, 1, 0)$$

is ample, with $D_{56} \cdot A_j = 1$ for $j \in \{1, \ldots, 10\} \setminus \{9\}$ and $D_{56} \cdot A_9 = 3$. The divisors

$$F_1 = A_4 + \sum_{j=1}^{5} A_j, \quad F_2 = \sum_{j=7}^{10} A_j$$

are fibers of an elliptic fibration with section A_6. By Theorem 2.3, case i) a), we have the following.

Proposition 9.10. *The linear system $|4F_1 + 2A_6|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has a $\mathbf{d}_4$ singularity p and one $\mathbf{a}_3$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

The divisor

$$D_2 = A_1 + 2A_2 + 2A_3 + 4A_4 + 3A_5 + 2A_6 + A_7$$

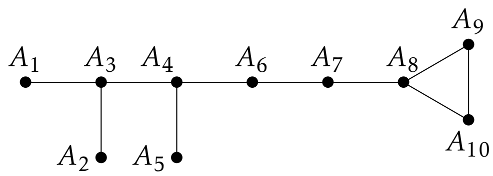
is nef, of square 2 and base-point free. We have $D_2 \cdot A_1 = 2$, $D_2 \cdot A_8 = D_2 \cdot A_9 = 1$ and $D_2 \cdot A_j = 0$ for $j \notin \{1, 8, 9\}$. Moreover,

$$D_2 \equiv A_2 + A_3 + 2A_4 + 2A_5 + 2A_6 + 2A_7 + A_8 + A_9 + A_{10}.$$

Thus the K3 surface is a double cover of the plane branched over a sextic curve with a $\mathbf{d}_6$ singularity and a node $\mathbf{a}_1$.

9.8. The lattice $U \oplus \mathbf{D}_5 \oplus \mathbf{A}_2$

The surface X contains 10 (-2)-curves, with dual graph



The Néron–Severi lattice is generated by $A_1, \ldots, A_9$; moreover,

$$A_{10} = (1, 1, 2, 2, 1, 1, 0, -1, -1)$$

in that base. The divisor

$$D_{88} = (8, 8, 17, 19, 9, 13, 8, 4, 1)$$

is ample, of square 88, with $D_{88} \cdot A_j = 1$ for $j \leq 8$, $D_{88} \cdot A_9 = 2$ and $D_{88} \cdot A_{10} = 5$. The divisors

$$F_1 = A_1 + A_2 + 2A_3 + 2A_4 + A_5 + A_6, \quad F_2 = A_8 + A_9 + A_{10}$$

are fibers of an elliptic fibration with section A_7. By Theorem 2.3, case i) a), we have the following.

Proposition 9.11. *The linear system $|4F_1 + 2A_6|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the negative section s and a curve $B \in |3s + 12f|$. The curve B has a $\mathbf{d}_5$ singularity p and one $\mathbf{a}_2$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

The divisor

$$D_2 = A_1 + A_2 + 3A_3 + 4A_4 + 2A_5 + 3A_6 + 2A_7 + A_8$$

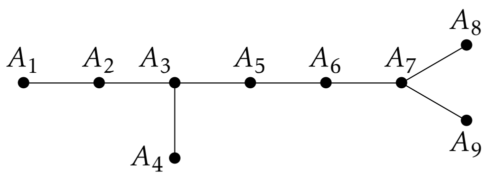
is nef, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_2 = D_2 \cdot A_9 = D_2 \cdot A_{10} = 1$ and $D_2 \cdot A_j = 0$ for $j \notin \{1,2,9,10\}$. Since

$$D_2 \equiv A_3 + 2A_4 + A_5 + 2A_6 + 2A_7 + 2A_8 + A_9 + A_{10},$$

we get that the K3 surface is the minimal resolution of a double cover of $\mathbb{P}^2$ branched over a sextic curve with a $\mathbf{d}_6$ singularity.

9.9. The lattice $U \oplus \mathbf{D}_7$

The surface X contains nine (-2)-curves, with dual graph



The nine curves generate the Néron–Severi lattice, and in that basis, the divisor

$$D_{260} = (13, 27, 42, 20, 38, 35, 33, 16, 16)$$

is ample, of square 260, with $D_{260} \cdot A_2 = 2$ and $D_{260} \cdot A_j = 1$ for $j \neq 4$. The divisor

$$F = A_2 + A_4 + 2(A_3 + A_5 + A_6 + A_7) + A_8 + A_9$$

is a fiber of an elliptic fibration with section A_1. By Theorem 2.3, case i) a), we have the following.

Proposition 9.12. *The linear system $|4F + 2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has a $\mathbf{d}_7$ singularity q. The pull-back of the fiber through q is the fiber F.*

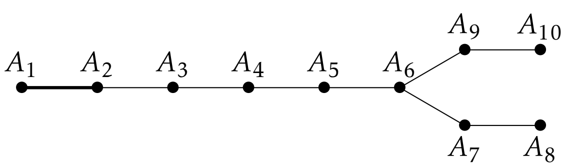
One can also construct that surface as a double plane: the divisor

$$D_2 = 2A_1 + 4A_2 + 6A_3 + 3A_4 + 5A_5 + 4A_6 + 3A_7 + A_8 + A_9$$

is nef, of square 2, with $D_2 \cdot A_j = 0$ for $j \leq 7$ and $D_2 \cdot A_8 = D_2 \cdot A_9 = 1$. The K3 surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve with an $\mathbf{e}_7$ singularity.

9.10. The lattice $U \oplus \mathbf{A}_1 \oplus \mathbf{E}_6$

The surface X contains 10 (-2)-curves, with dual graph



The nine curves $A_1, \ldots, A_9$ generate the Néron–Severi lattice, and in that basis, the divisor

$$D_{184} = (12, 17, 12, 8, 5, 3, 1, 0, 1)$$

is ample, of square 184, with $D_{184} \cdot A_1 = 10$, $D_{184} \cdot A_2 = 2$ and $D_{184} \cdot A_j = 1$ for $j \geq 3$. The divisors

$$F_1 = A_1 + A_2, \quad F_2 = A_4 + A_8 + A_{10} + 2(A_5 + A_7 + A_9) + 3A_6$$

are fibers of an elliptic fibration of X with a section A_3. By Theorem 2.3, case i) a), we have the following.

Proposition 9.13. *The linear system $|4F_1+2A_6|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B \in |3s+12f|$. The curve B has a node p and one $\mathbf{e}_6$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

One can also construct that surface as a double plane: the divisor

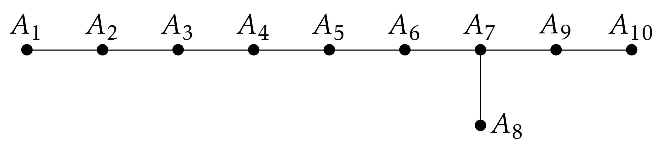
$$D_2 = A_1 + 2(A_2+A_3+A_4+A_5+A_6)+A_7+A_9$$

is nef, base-point free, of square 2. We have $D_2 \cdot A_1 = 2$, $D_2\cdot A_8 = D_2\cdot A_{10} = 1$ and $D_2 \cdot A_j = 0$ for the other curves. The K3 surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve with a $\mathbf{d}_7$ singularity.

10. Rank 10 lattices

10.1. The lattice $U \oplus \mathbf{E}_8$

The K3 surface X contains 10 (-2)-curves with dual graph



These curves generate the Néron–Severi lattice. In that base, the divisor

$$D_{1240} = (30,61,93,126,160,195,231,115,153,76)$$

is ample, of square 1240, with $D_{1240}\cdot A_j = 1$ for $j \in \{1,\dots,10\}$. The divisor

$$F = A_2+2A_3+3A_4+4A_5+5A_6+6A_7+3A_8+4A_9+2A_{10}$$

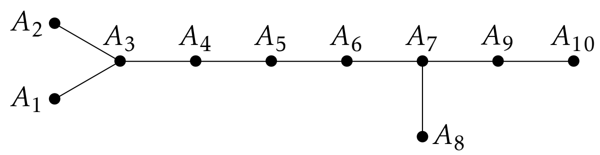
is a fiber of an elliptic fibration and A_1 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 10.1. *The linear system $|4F+2A_1|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B \in |3s+12f|$. The curve B has an $\mathbf{e}_8$ singularity q. The pull-back of the fiber through q is the fiber F.*

Remark 10.2. Using that $U\oplus\mathbf{E}_8$ is unimodular, one can prove that if D is an ample divisor, then $D^2 \geq 1240$.

10.2. The lattice $U \oplus \mathbf{D}_8$

The K3 surface X contains 10 (-2)-curves with dual graph



These curves generate the Néron–Severi lattice. In that base, the divisor

$$D_{280} = (15,15,31,33,36,40,45,22,29,14)$$

is ample, with $D_{280}\cdot A_j = 1$ for $j \in \{1,\dots,10\}$. The divisor

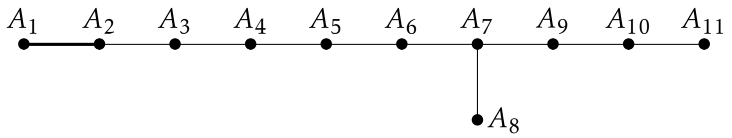
$$F = A_1+A_2+2(A_3+\cdots+A_7)+A_8+A_9$$

is a fiber of an elliptic fibration with section A_{10}. By Theorem 2.3, case i) a), we have the following.

Proposition 10.3. *The linear system $|4F+2A_{10}|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B\in|3s+12f|$. The curve B has a $\mathbf{d}_8$ singularity q. The pull-back of the fiber through q is the fiber F.*

10.3. The lattice $U \oplus \mathbf{E}_7 \oplus \mathbf{A}_1$

The K3 surface X contains 11 (-2)-curves with dual graph



The curves $A_1,\dots,A_{10}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{370}=(15,24,19,15,12,10,9,4,5,2)$$

is ample, of square 370, with $D_{370}\cdot A_j=1$ for $j\in\{2,\dots,10\}$, $D_{370}\cdot A_1=18$ and $D_{370}\cdot A_{11}=2$. The divisors

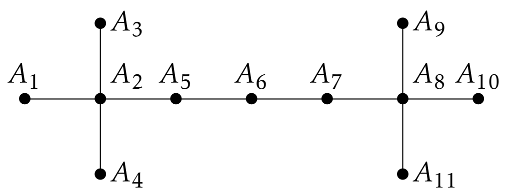
$$F_1=A_1+A_2,\quad F_2=A_4+2A_5+3A_6+4A_7+2A_8+3A_9+2A_{10}+A_{11}$$

are fibers of an elliptic fibration with section A_3. By Theorem 2.3, case i) a), we have the following.

Proposition 10.4. *The linear system $|4F_1+2A_3|$ defines a morphism $\varphi\colon X\to\mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B\in|3s+12f|$. The curve B has a node p and one $\mathbf{e}_7$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

10.4. The lattice $U\oplus\mathbf{D}_4^{\oplus 2}$

The K3 surface X contains 11 (-2)-curves with dual graph



The curves $A_1,\dots,A_{10}$ generate the Néron–Severi lattice. In that base, we have

$$A_{11}=(1,2,1,1,1,0,-1,-2,-1,-1).$$

The divisor

$$D_{56}\equiv(7,15,7,7,10,6,3,1,0,0)$$

is ample of square 56, and the (-2)-curves on X have degree 1 for D_{56}. The divisors

$$F_1=A_2+\sum_{j=1}^{5}A_j,\quad F_2=A_8+\sum_{j=7}^{11}A_j,$$

are fibers of an elliptic fibration with section A_6. By Theorem 2.3, case i) a), we have the following.

Proposition 10.5. *The linear system $|4F_1+2A_6|$ defines a morphism $\varphi\colon X\to\mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B\in|3s+12f|$. The curve B has two $\mathbf{d}_4$ singularities. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

One can give another construction as follows. The divisor

$$D_2=A_1+4A_2+2A_3+2A_4+3A_5+2A_6+A_7$$

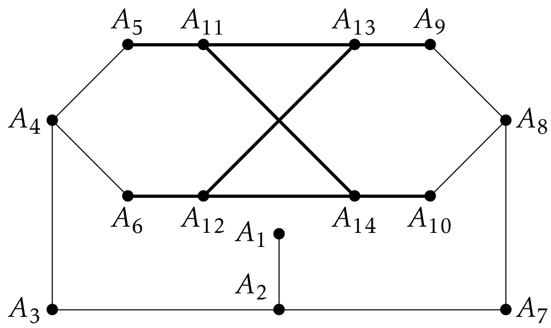
is nef, of square 2, base-point free, with $D_2 \cdot A_1 = 2$, $D_2 \cdot A_8 = 1$ and $D_2 \cdot A_j = 0$ for $j \notin \{2,8\}$. We have

$$D_2 \equiv 2A_2 + A_3 + A_4 + 2A_5 + 2A_6 + 2A_7 + 2A_8 + A_9 + A_{10} + A_{11}.$$

The branch curve of the double cover $\varphi\colon X \to \mathbb{P}^2$ associated to D_2 is a sextic with three nodes and a $\mathbf{d}_6$ singularity q. The curve A_8 is in the ramification locus; its image by φ is a line L. Let Q be the residual quintic curve of the sextic branch locus. The line L cuts Q transversally in an $\mathbf{a}_3$ singularity and in three other points p_1, p_2, p_3. The curves A_9, A_{10}, A_{11} are mapped to these three points. The image by φ of A_1 is the line that is the branch of the $\mathbf{a}_3$ singularity. The curves $A_2, \dots, A_7$ are mapped to q.

10.5. The lattice $U \oplus \mathrm{D}_6 \oplus \mathrm{A}_1^{\oplus 2}$

The K3 surface X contains 14 (-2)-curves with dual graph



The curves $A_1, \dots, A_{10}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{98} = (7, 16, 13, 11, 5, 5, 13, 11, 5, 5)$$

is ample, of square 98, with $D_{98} \cdot A_j = 1$ for $j \in \{2, \dots, 10\}$, $D_{98} \cdot A_1 = 2$ and $D_{98} \cdot A_j = 10$ for $j \in \{11, \dots, 14\}$. The divisor

$$D_2 = (1, 2, 2, 2, 1, 1, 2, 2, 1, 1)$$

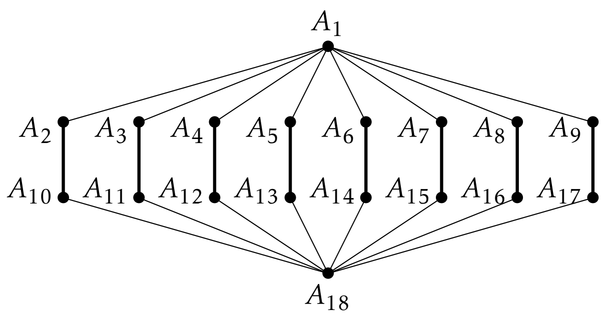
is nef, base-point free, of square 2, with $D_2 \cdot A_2 = 1$, $D_2 \cdot A_j = 2$ for $j \in \{11, \dots, 14\}$ and $D_2 \cdot A_j = 0$ for the remaining curves. By using the linear system $|D_2|$, we obtain the following.

Proposition 10.6. *The branch curve has a node and two $\mathbf{d}_4$ singularities. It is the union of a quintic curve with two nodes q, q' and the line through the points q, q'. The line cuts the quintic in a third point p, forming a node.*

The curve A_1 is sent to p, the curves A_3, A_4, A_5, A_6 are sent to q, and the curves A_7, A_8, A_9, A_{10} are sent to q'. The curves A_{11} and A_{12} are sent to the two branches of the singularity q that are distinct from L; the curves A_{13} and A_{14} are sent to the two branches of the singularity q' that are distinct from L.

10.6. The lattice $U(2) \oplus \mathrm{D}_4^{\oplus 2}$

10.6.1. First involution.— There exist 18 (-2)-curves $A_1, \dots, A_{18}$ on X, with dual graph



The curves $A_1,\dots,A_8,A_{10},A_{18}$ generate the Néron–Severi lattice. The divisor

$$D_8 = A_1 + 3A_2 + 3A_{10} + A_{18}$$

is ample, base-point free, of square 8, with $D_8 \cdot A_j = 1$ for all $j \in \{1,\dots,18\}$. The divisors $A_k + A_{8+k} \equiv F$, $k \in \{2,\dots,9\}$, are singular fibers of an elliptic fibration, for which A_1 and A_{18} are sections. Since $FD_8 = 2$, we see that D_8 is hyperelliptic; thus the image by the associated map φ of X is a rational normal scroll of degree 4 in $\mathbb{P}^5$, *i.e.*, the Hirzebruch surface $\mathbf{F}_2$. The pull-back by φ of the unique negative curve s (the section) is the union of two disjoint (-2)-curves. The branch curve B must satisfy $Bs = 0$; thus $B = v(s + 2f)$, where f is a fiber of the unique fibration of $\mathbf{F}_2$. We have $F = \varphi^* f$; thus the branch curve cuts a general fiber f in four points and $v = 4$.

Proposition 10.7. *The K3 surface X is the double cover of $\mathbf{F}_2$ branched on a smooth curve B of genus 9 in $|4s+8f|$. The curve B is such that there are 8 fibers $f_1,\dots,f_8$ that meet B with even multiplicities at each intersection point. The 18 curves on X are in the pull-back of s (which is $A_1 + A_{18}$) and the pull-backs of the fibers f_i, $i \in \{1,\dots,8\}$.*

10.6.2. Second involution.— The divisor

$$D_2 = 2A_1 + 2A_2 + A_3 + A_4 + A_{10}$$

is nef, of square 2, with $D_2 \cdot A_j = 2$ for $j \in \{5,\dots,12\}$, $D_2 \cdot A_{18} = 1$ and else $D_2 \cdot A_j = 0$. We have

$$D_2 \equiv 2A_1 + A_2 + A_3 + A_4 + A_k + A_{8+k}, \quad k \in \{2,\dots,9\},$$

and

$$D_2 \equiv A_{13} + A_{14} + A_{15} + A_{16} + A_{17} + 2A_{18}.$$

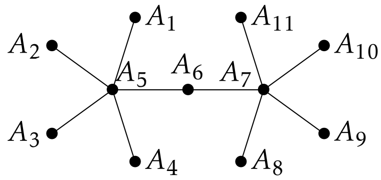
Proposition 10.8. *The K3 surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 with a $\mathbf{d}_4$ singularity q (onto which A_1, A_2, A_3, A_4 are mapped) and five nodes (onto which the five curves $A_{13},\dots,A_{17}$ are mapped).*

The curve C_6 is the union of a line L and a quintic with a $\mathbf{d}_4$ singularity. The double cover is ramified over A_{18}, and the image of A_{18} is L. The images of A_{10}, A_{11}, A_{12} are conics tangent to the three branches of the $\mathbf{d}_4$ singularity; the images of $A_{13},\dots,A_{17}$ are the five lines through the five nodes (intersection of L and the quintic) and q.

Remark 10.9. According to [Kon89], the automorphism group of X is $(\mathbb{Z}/2\mathbb{Z})^2$. The fixed loci of the two involutions we described are distinct; thus we obtained generators of the automorphism group.

10.7. The lattice $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 4}$

The K3 surface X contains 27 (-2)-curves $A_1,\dots,A_{27}$; the curves $A_1,\dots,A_{11}$ have dual graph



and $A_1,\dots,A_{10}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{26} = (3,3,3,3,7,3,1,0,0,0)$$

is ample, of square 26, with $D_{26} \cdot A_j = 1$ for $j \in \{1,\dots,11\}$, $j \neq 5$, $D_{26} \cdot A_5 = 2$ and $D_{26} \cdot A_k = 6$ for the remaining (-2)-curves. The divisors

$$A_1 + A_2 + A_3 + A_4 + 2A_5,\ 2A_7 + A_8 + A_9 + A_{10} + A_{11}$$

are fibers of the same fibration. That gives the class of A_{11} in the basis $A_1,\dots,A_{10}$. The divisor

$$F_5 = A_6 + 2A_7 + A_8 + A_9 + A_{10}$$

is a fiber of an elliptic fibration. For $j \in \{1,\dots,4\}$, one has $A_{11+j} = F_5 - A_j$ (as a class). The divisor

$$F_1 = A_2 + A_3 + A_4 + 2A_5 + A_6$$

is a fiber of an elliptic fibration. For $j \in \{1,\dots,3\}$, one has $A_{15+j} = F_1 - A_{7+j}$ (as a class). The divisor

$$F_2 = A_1 + A_3 + A_4 + 2A_5 + A_6$$

is a fiber of an elliptic fibration. For $j \in \{1,\dots,3\}$, one has $A_{18+j} = F_2 - A_{7+j}$ (as a class). The divisor

$$F_3 = A_1 + A_2 + A_4 + 2A_5 + A_6$$

is a fiber of an elliptic fibration. For $j \in \{1,\dots,3\}$, one has $A_{21+j} = F_3 - A_{7+j}$ (as a class). The divisor

$$F_4 = A_1 + A_2 + A_3 + 2A_5 + A_6$$

is a fiber of an elliptic fibration. For $j \in \{1,\dots,3\}$, one has $A_{24+j} = F_4 - A_{7+j}$ (as a class).

The divisor

$$D_2 = A_1 + A_2 + A_3 + A_4 + 2A_5 + A_6$$

is nef, of square 2, with $D_2 \cdot A_5 = D_2 \cdot A_7 = 1$, $D_2 \cdot A_k = 0$ for $k \in \{1,\dots,4,6,8,\dots,11\}$ and $D_2 \cdot A_j = 2$ for the remaining curves. We also have

$$D_2 \equiv A_6 + 2A_7 + A_8 + A_9 + A_{10} + A_{11}.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 10.10. *The surface X is the double cover of the plane branched over a sextic curve which is the union of two lines L, L' and a quartic Q.*

The images of A_5, A_7 are the two lines. We observe that

$$\begin{aligned}
D_2 &\equiv A_{16} + A_1 + A_8 \equiv A_{17} + A_1 + A_9 \equiv A_{18} + A_1 + A_{10},\\
D_2 &\equiv A_{19} + A_2 + A_8 \equiv A_{20} + A_2 + A_9 \equiv A_{21} + A_2 + A_{10},\\
D_2 &\equiv A_{22} + A_3 + A_8 \equiv A_{23} + A_3 + A_9 \equiv A_{24} + A_3 + A_{10},\\
D_2 &\equiv A_{25} + A_4 + A_8 \equiv A_{26} + A_4 + A_9 \equiv A_{27} + A_4 + A_{10}.
\end{aligned}$$

Thus A_1, A_2, A_3, A_4 are contracted to the nodes in the intersection of Q and L, and so are A_8, A_9, A_{10}. Since

$$D_2 \equiv A_1 + A_{11} + A_{12} \equiv A_2 + A_{11} + A_{13} \equiv A_3 + A_{11} + A_{14} \equiv A_4 + A_{11} + A_{15},$$

the curve A_{11} is contracted to the fourth intersection point of L' and Q. Moreover, we see from these equivalences that the 16 (-2)-curves $A_{12},\dots,A_{27}$ are sent to the 16 lines between points in $L \cap Q$ and points in $L' \cap Q$. The curve A_6 is contracted onto the intersection point of L and L'.

Proposition 10.10 implies that the moduli space of K3 surfaces with Néron–Severi group isometric to $U \oplus \mathbf{D}_4 \oplus \mathbf{A}_1^{\oplus 4}$ is unirational.

10.8. The lattice $U \oplus \mathbf{A}_1^{\oplus 8}$

10.8.1. First involution.— Let us denote by $f_1, f_2, e_1, \dots, e_8$ the canonical basis of $U \oplus \mathbf{A}_1^{\oplus 8}$. In that basis, let

$$D_6 = (7, 5, -2, -2, -2, -2, -2, -2, -2, -2).$$

It has square 6, and no (-2)-classes are perpendicular to it. We thus have a marking such that $U \oplus \mathbf{A}_1^{\oplus 6} \simeq \mathrm{NS}(X)$ which maps D_6 to an ample class. The K3 surface X contains 145 (-2)-curves; with respect to D_6, the curves $A_1,\dots,A_{16}$ have degree 1, the curve A_0 has degree 2, and the remaining curves have degree 4. Let us describe these curves.

For $j \in \{1,\dots,8\}$, one has $A_j = f_1 - e_j$; the divisors

$$A_1 + A_9, \dots, A_8 + A_{16}$$

are fibers of an elliptic fibration $\varphi\colon X \to \mathbb{P}^1$, where the class of a fiber F is

$$F = (4,2,-1,-1,-1,-1,-1,-1,-1,-1)$$

in the canonical basis. The curve A_0 is $A_0 = -f_1 + f_2$, and $A_0 \cdot A_k = 1$ for $1 \le k \le 16$.

For the choice of any three elements $\{i,j,k\}$ in $\{1,\dots,8\}$ (56 possibilities), the classes

$$A_{i,j,k} = 4f_1 + 4f_2 - e_i - e_j - e_k - \textstyle\sum_{l=1}^{8} e_l,$$
$$B_{i,j,k} = 2f_1 + 2f_2 + e_i + e_j + e_k - \textstyle\sum_{l=1}^{8} e_l$$

are classes of (-2)-curves. The 16 classes

$$C_j = 6f_1 + 6f_2 - e_j - 2\textstyle\sum_{l=1}^{8} e_l \quad \text{and} \quad E_j = e_j,\ j \in \{1,\dots,8\}$$

are classes of (-2)-curves. Thus in total, we get 145 (-2)-curves.

The divisor

$$D_2 = (3,3,-1,-1,-1,-1,-1,-1,-1,-1) = F + A_0$$

has square 2, and $D_2 \cdot A_j = 1$ for $j \in \{1,\dots,16\}$, $D_2 \cdot A_0 = 0$ and $D_2 A = 2$ for the remaining (-2)-curves A. Let C_6 be the sextic branch curve of the associated double cover $X \to \mathbb{P}^2$. The curve C_6 has a node q onto which the curve A_0 is contracted. For $j \in \{1,\dots,8\}$, we have

$$A_j + A_{8+j} + A_0 \equiv D_2;$$

thus the divisor $A_j + A_{8+j} + A_0$ is the pull-back of a line through the nodal point q, and that line is tangent to C_6 at every other intersection points. For a set of three elements $\{i,j,k\}$ in $\{1,\dots,8\}$, we have

$$A_{i,j,k} + B_{i,j,k} \equiv 2D_2;$$

thus the 56 divisors $A_{i,j,k} + B_{i,j,k}$ are pull-backs of 56 conics that are 6-tangent to C_6 (and not containing the nodal point of C_6). We have

$$C_j + E_j \equiv 2D_2;$$

thus the eight divisors $C_j + E_j$, $j \in \{1,\dots,8\}$, are also pull-backs of eight conics that are 6-tangent to C_6. Summing up, we have the following.

Proposition 10.11. *The K3 surface X is the double cover of $\mathbb{P}^2$ branched over a nodal sextic curve. Through the node of the sextic, there are eight lines that are tangent to the sextic at other intersection points. Moreover, there are* 64 *conics that are* 6*-tangent to the sextic.*

The Néron–Severi lattice is generated by the classes of $A_0, A_1, \dots, A_8$ and E_8.

Remark 10.12. Let S be the set of the 128 (-2)-curves that are above the 64 conics. One can prove that there exists a partition of S into eight sets $S_1, \dots, S_8$ of 16 curves such that for any such set $S_i = \{B_1, \dots, B_{16}\}$, one has (up to permutation of the indices) $B_{2k-1} \cdot B_{2k} = 6$ for $k \in \{1,\dots,8\}$, and for s, t ($s \neq t$) such that $\{s,t\} \neq \{2k-1, 2k\}$, one has $B_s \cdot B_t = 4$ if $s + t = 1 \bmod 2$; else $B_s \cdot B_t = 0$.

For $j \neq t$ in $\{1,\dots,8\}$, if B is a curve in S_j, there are 10 (-2)-curves B' in S_t such that $BB' = 2$, 3 (-2)-curves such that $BB' = 0$ and 3 (-2)-curves such that $BB' = 4$.

Remark 10.13. The ample divisor D_6 satisfies $D_6 \equiv 2F + A_0$ and $FD_6 = 2$; thus it is hyperelliptic. According to Theorem [Sai74], the morphism associated to $|D_6|$ is a degree 2 map onto a degree 3 rational normal scroll in $\mathbb{P}^4$. That surface is the Hirzebruch surface $\mathbb{F}_1$, which is the blow-up of $\mathbb{P}^2$ in one point, embedded by $|2L - E_0|$, where L is the pull-back of a line and E_0 is the exceptional divisor. The composite of $X \to \mathbb{F}_1$ with the natural map $\mathbb{F}_1 \to \mathbb{P}^2$ is given by the linear system $|D_2|$.

10.8.2. Second involution: A more geometric interpretation of the (−2)-curves.— Let $f_1, f_2, e_1, \dots, e_8$ be the canonical basis of $U \oplus \mathbf{A}_1^{\oplus 8}$. The divisor $D_2' = f_1 + f_2$ is nef, of square 2. We recall that $A_0 = -f_1 + f_2$ is a (-2)-curve; moreover, the divisor $F' = f_1$ is a fiber of an elliptic fibration; thus

$$D_2' = 2F' + A_0.$$

One has $FA_0 = 1$, $D_2'F' = 1$, and therefore the linear system $|D_2'|$ has base points. We have $D_2' \cdot A_0 = 0$ and

$$\begin{array}{rl} D_2' \cdot A_j = 1,\ D_2' \cdot A_{j+8} = 5, & \forall j \in \{1,\dots,8\}, \\ D_2' \cdot A_{i,j,k} = 8, & \forall \{i,j,k\} \subset \{1,\dots,8\} \text{ of order } 3, \\ D_2' \cdot B_{i,j,k} = 4, & \forall \{i,j,k\} \subset \{1,\dots,8\} \text{ of order } 3, \\ D_2' \cdot C_j = 12,\ D_2' \cdot E_j = 0, & \forall k \in \{1,\dots,8\}. \end{array}$$

Let us define $D_8 = 2D_2'$. The linear system $|D_8|$ is base-point free, and it is hyperelliptic since $D_8F' = 2$. By Theorem 2.3, case iii) v), the associated map $\varphi_{D_8}: X \to \mathbb{P}^5$ has image a cone over a rational normal curve in $\mathbb{P}^4$; it factorizes through a surjective map $\varphi': X \to \mathbf{F}_4$, where $\mathbf{F}_4$ is the Hirzebruch surface with a section s such that $s^4 = -4$. The section s is mapped to the vertex of the cone by the map $\mathbf{F}_4 \to \mathbb{P}^5$. Let f denote a fiber of the unique fibration $\mathbf{F}_4 \to \mathbb{P}^1$. The branch locus of φ' is the union of s and a curve C such that $Cs = 0$ and $C \in |3(s+4f)|$ (so that $s + C \in |-2K_{\mathbf{F}_4}|$). The curves $E_1, \dots, E_8$ being contracted by φ', the curve C has eight nodes $p_1, \dots, p_8$, which are the images of $E_1, \dots, E_8$. We have

$$|D_8| = \varphi^*|4f + s|.$$

We moreover have the relations

$$A_j + E_j \equiv F' = \varphi^* f, \quad \forall j \in \{1,\dots,8\};$$

therefore, the curves $A_1, \dots, A_8$ are mapped by φ' to the eight fibers going through the nodes $p_1, \dots, p_8$. Since $D_8 \equiv 4(A_1 + E_1) + 2A_0$, the curve A_0 is in the ramification locus, with image s. Since

$$A_{8+k} + \sum_{j=1, j\neq k}^{j=8} E_j \equiv D_8 + F', \quad \forall k \in \{1,\dots,8\},$$

the image of A_{8+k} belongs in $|5f + s|$; it is a curve which goes through the seven points $\{p_1, \dots, p_8\} \setminus \{p_k\}$. We moreover have the relations

$$\begin{array}{rl} D_8 & \equiv B_{ijk} - (E_i + E_j + E_k) + \sum_{t=1}^8 E_t, \\ 2D_8 & \equiv A_{ijk} + E_i + E_j + E_k + \sum_{t=1}^8 E_s, \\ 3D_8 & \equiv C_k + E_k + 2\sum_{t=1}^8 E_s, \end{array} \tag{10.1}$$

and therefore

- the image of B_{ijk} is a curve in the linear system $|4f + s|$ which goes through the five points distinct from p_i, p_j, p_k;
- the image of A_{ijk} is a curve in the linear system $|2(4f + s)|$ which goes through the eight points $p_1, \dots, p_8$ with double points at p_i, p_j, p_k;
- the image of C_k is a curve in the linear system $|3(4f + s)|$ which goes through eight points with double points at all except at the single point p_k with multiplicity 3.

Let J, J' be subsets of order 3 of $\{1, \dots, 8\}$. Using the relations in (10.1) and $2D_2 \equiv A_J + B_J \equiv C_k + E_k$, one finds that

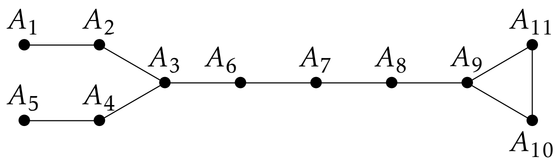
$$\begin{aligned} A_J \cdot A_{J'} &= B_J \cdot B_{J'} = 4 - 2\#(J \cap J'), \\ A_J \cdot B_{J'} &= 2\#(J \cap J'), \\ C_k \cdot C_{k'} &= E_k E_{k'} = -2\delta_{kk'}, \\ C_k \cdot E_{k'} &= 4 + 2\delta_{kk'}, \\ C_k \cdot B_J &= 4 \text{ if } k \in J, \quad C_k \cdot B_J = 2 \text{ if } k \notin J, \\ E_k \cdot B_J &= 0 \text{ if } k \in J, \quad E_k \cdot B_J = 2 \text{ if } k \notin J. \end{aligned}$$

The situation is very much similar to what happens for K3 surfaces Y with Néron–Severi lattice $U(2) \oplus \mathbf{A}_1^{\oplus 7}$, which are double covers of del Pezzo surfaces of degree 1, where the 240 (-2)-curves of Y come from lines, conics, cubics, quartics, quintics and sextics going through 8 points in the plane with various multiplicities. In particular, the 112 (-2)-curves A_J, B_J with $J \subset \{1, \dots, 8\}$ of order 3 has the same configurations as the 112 (-2)-curves on a K3 surface Y which are pull-backs of conics and quartics.

Remark 10.14. According to Kondo [Kon89], the automorphism group of the surface X is $(\mathbb{Z}/2\mathbb{Z})^2$. The branch loci of the two involutions associated to D_2 and D_8 have genus 9 and 2, respectively. Thus these two involutions generate the automorphism group of X.

10.9. The lattice $U \oplus \mathbf{A}_2 \oplus \mathbf{E}_6$

The K3 surface X contains 11 (-2)-curves $A_1, \dots, A_{11}$; their dual graph is



The curves $A_1, \dots, A_{10}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{160} = (10, 21, 33, 21, 10, 25, 18, 12, 7, 3)$$

is ample, of square 160, with $D_{160} \cdot A_j = 1$ for $j \leq 10$ and $D_{160} \cdot A_{11} = 10$. The divisors

$$F_1 = A_9 + A_{10} + A_{11}, \quad F_2 = A_1 + A_7 + A_5 + 2(A_2 + A_4 + A_6) + 3A_3$$

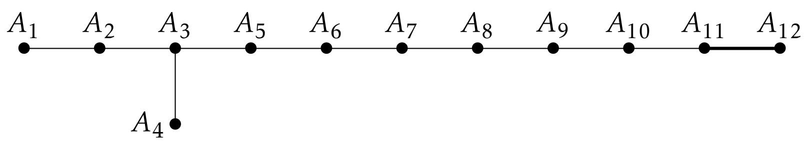
are fibers of an elliptic fibration and A_8 is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 10.15. *The linear system $|4F_1 + 2A_8|$ defines a morphism $\varphi \colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has one $\mathbf{a}_2$ singularity p and one $\mathbf{e}_6$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

11. Rank 11 lattices

11.1. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1$

The K3 surface X contains 12 (-2)-curves $A_1, \dots, A_{12}$; their dual graph is



The curves $A_1,\dots,A_{11}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{848} = (48,97,147,73,125,104,84,65,47,30,14)$$

is ample, of square 848, with $D_{848}\cdot A_j = 1$ for $j\in\{1,\dots,10\}$, $D_{848}\cdot A_{11} = 2$ and $D_{848}\cdot A_{12} = 28$. The divisors

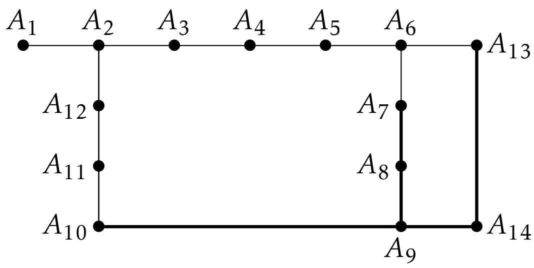
$$F_1 = 2A_1+4A_2+6A_3+3A_4+5A_5+4A_6+3A_7+2A_8+A_9, \quad F_2 = A_{11}+A_{12}$$

are fibers of an elliptic fibration with section A_{10}. By Theorem 2.3, case i) a), we have the following.

Proposition 11.1. *The linear system* $|4F_1+2A_{10}|$ *defines a morphism* $\varphi\colon X\to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2=-4$ *and a curve* $B\in|3s+12f|$. *The curve* B *has one* $\mathbf{e}_8$ *singularity* p *and one node* q. *The pull-backs of the fibers through* p, q *are the fibers* F_1, F_2.

11.2. The lattice $U\oplus \mathbf{D}_8\oplus \mathbf{A}_1$

The K3 surface X contains 14 (-2)-curves with dual graph



The curves $A_1,\dots,A_9,A_{11},A_{12}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{208} = (-23,-45,-35,-24,-12,1,15,15,9,-16,-31)$$

is ample, of square 208; the degrees $D_{208}\cdot A_j$ of the (-2)-curves A_j, $j=1,\dots,14$, are

$$1,1,1,1,1,1,1,18,12,2,1,1,1,18.$$

The divisor $F_1 = A_9+A_{10}$ is the fiber of an elliptic fibration, and there is a second fiber F_2 supported on $A_1,\dots,A_7,A_{12},A_{13}$ of type $\tilde{\mathbf{D}}_8$. The curve A_{11} is a section. By Theorem 2.3, case i) a), we have the following.

Proposition 11.2. *The linear system* $|4F_1+2A_{11}|$ *defines a morphism* $\varphi\colon X\to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2=-4$ *and a curve* $B\in|3s+12f|$. *The curve* B *has one* $\mathbf{d}_8$ *singularity* p *and one node* q. *The pull-backs of the fibers through* p, q *are the fibers* F_2, F_1.

One can also find a construction using the double cover associated to the divisor

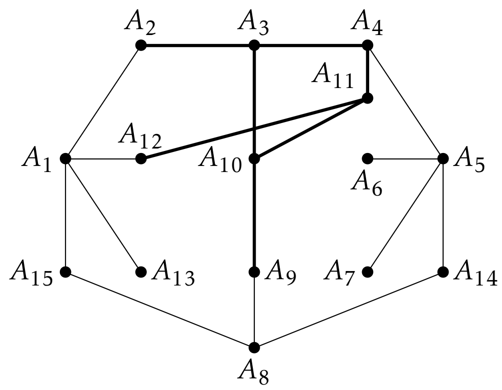
$$D_2 = A_7+A_8+A_9$$

which is nef, base-point free, of square 2, with intersections $D_2\cdot A_j$, $j=1,\dots,14$, equal to, respectively,

$$0,0,0,0,0,1,0,2,0,2,0,0,0,2.$$

11.3. The lattice $U\oplus \mathbf{D}_4\oplus \mathbf{D}_4\oplus \mathbf{A}_1$

The K3 surface contains 22 (-2)-curves $A_1,\dots,A_{22}$. The configuration of the curves $A_1,\dots,A_{15}$ is



The curves $A_1,\dots,A_{11}$ generate the rank 11 Néron–Severi lattice. In that base, the divisor

$$D_{48} = (1,3,3,3,1,0,0,0,1,2,0)$$

is ample, of square 48. We have

$$\begin{aligned} A_{13} &\equiv (-2,-3,-2,-2,-2,-1,-1,2,4,3,1), \\ A_{14} &\equiv (0,0,0,-1,-2,-1,-1,0,1,1,0), \\ A_{15} &\equiv (0,1,1,2,2,1,1,-2,-3,-2,0). \end{aligned}$$

The divisor $D_2 = A_2 + A_3 + A_4$ is nef, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_5 = 1$, $D_2 \cdot A_j = 0$ for $j \in \{2,4,6,7,8,9,12,13,14,15\}$ and else $D_2 \cdot A_j = 2$. We have

$$\begin{aligned} D_2 &\equiv 2A_1 + (A_2 + 2A_8 + A_9 + A_{12} + A_{13} + A_{14} + 2A_{15}), \\ D_2 &\equiv 2A_5 + (A_4 + A_6 + A_7 + 2A_8 + A_9 + 2A_{14} + A_{15}), \\ D_2 &\equiv A_{10} + (2A_8 + 2A_9 + A_{14} + A_{15}) \equiv A_{11} + (A_4 + A_{12}), \\ D_2 &\equiv A_{16} + A_4 + A_{13} \equiv A_{17} + A_2 + A_6 \equiv A_{18} + A_6 + A_{12}, \\ D_2 &\equiv A_{19} + A_{13} + A_6 \equiv A_{20} + A_2 + A_7 \equiv A_{21} + A_7 + A_{12}, \\ D_2 &\equiv A_{22} + A_7 + A_{13}. \end{aligned}$$

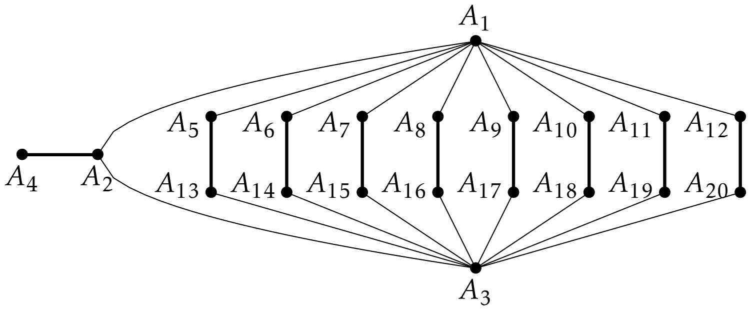
Therefore, the K3 surface is a double cover of $\mathbb{P}^2$ branched over a sextic curve C_6, and the following holds.

Proposition 11.3. *The curves A_1, A_5 are in the ramification locus, and their images are lines L, L' that are components of C_6. Let Q be the residual quartic curve. The lines L, L' meet in a point q which is on Q, so that the sextic has a $\mathbf{d}_4$ singularity at q.*

The curves A_8, A_9, A_{14}, A_{15} are contracted to q. The line L meets the quartic in three other points p_2, p_{12}, p_{13}, which are the images of A_2, A_{12}, A_{13}. The line L' meet the quartic in three other points p_4, p_6, p_7, which are the images of A_4, A_6, A_7. The images of the nine curves A_3, A_{11}, $A_{16},\dots,A_{22}$ are lines through the points (p_2,p_4), (p_4,p_{12}), (p_4,p_{13}), (p_2,p_6), (p_6,p_{12}), (p_6,p_{13}), (p_2,p_7), (p_7,p_{12}), (p_7,p_{13}), respectively. Finally, the image of A_{10} is a line tangent to the third branch of the $\mathbf{d}_4$ singularity.

11.4. The lattice $U \oplus \mathrm{D}_4 \oplus \mathrm{A}_1^{\oplus 5}$

11.4.1. First involution.— The K3 surface contains 90 (-2)-curves. The first 20 curves $A_1,\dots,A_{20}$ have the following configuration:



The curves $A_1, \dots, A_{11}$ generate the Néron–Severi lattice. The divisor

$$D_{18} = A_1 + 5A_2 + 2A_3 + 4A_4$$

is ample, of square 18, with $D_{18} \cdot A_1 = 3$, $D_{18} \cdot A_k = 1$ for $k \in \{2, 3, 5, \dots, 12\}$, $D_{18} \cdot A_k = 2$ for $k \in \{4, 13, \dots, 20\}$ and $D_{18} \cdot A_k = 8$ for $k > 20$.

The divisor

$$D_2 = A_1 + 2A_2 + A_3 + A_4$$

is nef, base-point free, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_2 = D_2 \cdot A_3 = 0$, $D_2 \cdot A_j = 1$ for $j \in \{5, \dots, 20\}$ and $D_2 \cdot A_j = 2$ for $j = 4$ or $j > 20$. In the basis $A_1, \dots, A_{11}$, we have

$$A_{12} = (-2, 4, 2, 4, -1, -1, -1, -1, -1, -1, -1).$$

Moreover, for $k \in \{5, \dots, 12\}$, we have $A_k + A_{k+8} \equiv A_2 + A_4$ (these are fibers of an elliptic fibration); thus

$$D_2 \equiv A_1 + A_2 + A_3 + A_k + A_{k+8}, \quad \forall k \in \{5, \dots, 12\},$$

and we obtain in that way the classes of $A_1, \dots, A_{20}$. Moreover, we see that the surface X is a double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 which has an $\mathbf{a}_3$ singularity q. The curves A_1, A_2, A_3 are contracted to q, the curve A_4 is mapped onto a line that is tangent to the branch of C_6 at q, and the curves A_k, A_{k+8} with $k \in \{5, \dots, 12\}$ are mapped to eight lines going through q which are tangent to the sextic at any other intersection point. For any subset $J = \{i, j, k, l\}$ of $\{5, \dots, 11\}$ of order 4 (there are 35 such choices), let us define

$$A_J = 2A_1 - A_4 + \sum_{t \in J} A_t,$$
$$B_J = 4A_2 + 2A_3 + 3A_4 - \sum_{t \in J} A_t.$$

The classes A_J and B_J are the classes of the remaining 70 (-2)-curves $A_{21}, \dots, A_{90}$. Moreover, we see that

$$2D_2 \equiv A_J + B_J, \quad \forall J = \{i, j, k, l\} \subset \{5, \dots, 11\}, \ \#\{i, j, k, l\} = 4,$$

and therefore there exist 35 conics that are 6-tangent to C_6.

Let J, J' be two subsets of order 4 of $\{5, \dots, 11\}$. The configuration of the curves A_J, $A_{J'}$, B_J, $B_{J'}$ is as follows:

$$A_J \cdot A_{J'} = B_J \cdot B_{J'} = 6 - 2\#(J \cap J'),$$
$$A_J \cdot B_{J'} = -2 + 2\#(J \cap J').$$

11.4.2. Second involution.— The divisor $D_2' = 2A_2 + A_3 + 2A_4$ is nef, of square 2, with $D_2 \cdot A_1 = 2$, $D_2 \cdot A_j = 1$ for $j \in \{2, 13, \dots, 20\}$, $D_2 \cdot A_j = 0$ for $j \in \{3, \dots, 12\}$ and $D_2 \cdot A_j = 4$ for $j \geq 21$. We have

$$D_2' = 2F + A_3,$$

where $F = A_2 + A_4$ is a fiber of an elliptic fibration such that $FA_3 = 1$; thus the linear system $|D_2'|$ has base points. Let $D_8 = 2D_2'$. The linear system $|D_8|$ is base-point free, and it is hyperelliptic since $D_8 F = 2$. One can check easily that

(11.1)
$$\begin{aligned} D_8 &\equiv 2A_1 + \textstyle\sum_{j=5}^{12} A_j, \\ D_8 &\equiv 4(A_k + A_{k+8}) + 2A_3, \quad k \in \{5,\dots,12\}, \\ D_8 &\equiv B_J + A_4 + \textstyle\sum_{t\in J} A_t, \\ D_8 &\equiv A_J + A_4 + A_{12} + \textstyle\sum_{t\in J^c} A_t, \end{aligned}$$

where $J = \{i,j,k,l\}$ is a subset of order 4 of $\{5,\dots,11\}$ and J^c is its complement.

By [Sai74, Equation (5.9.1)], the associated map $\varphi_{D_8}\colon X \to \mathbb{P}^5$ has image a cone over a rational normal curve in $\mathbb{P}^4$; it factorizes through a surjective map $\varphi'\colon X \to \mathbf{F}_4$, where $\mathbf{F}_4$ is the Hirzebruch surface with a section s such that $s^4 = -4$. The section s is mapped to the vertex of the cone by the map $\mathbf{F}_4 \to \mathbb{P}^5$. Let f denote a fiber of the unique fibration $\mathbf{F}_4 \to \mathbb{P}^1$. By [Sai74, Equation (5.9.1)], the branch locus of φ' is the union of s and a curve C such that $Cs = 0$ and $C \in |3(s+4f)|$ (so that $b + C \in |-2K_{\mathbf{F}_4}|$). We have

$$|D_8| = \varphi'^{*}|4f + s|,$$

and therefore from the equivalence relations in (11.1), we get the following:

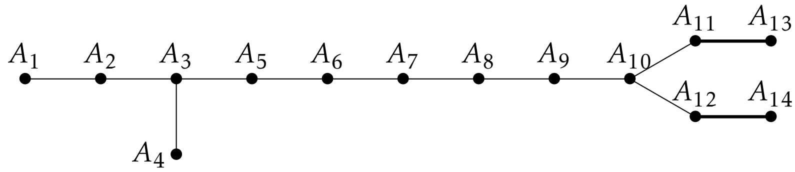
- The curve A_3 is in the ramification locus; the image by φ' of the curve A_3 is the section s.
- The curve A_1 is in the ramification locus. Since $A_1 \cdot A_3 = 0$ and A_1 is a section, the image C_1 of A_1 is in the linear system $|s + 4f|$. Let B' be the curve $B' \in |2s + 8f|$ such that the branch locus of the double cover $\varphi\colon X \to \mathbf{F}_4$ is
$$B = s + C_1 + B' \in |4s + 12f|.$$
The singular points of the branch locus are nodes $p_4,\dots,p_{12}$. The eight points $p_5,\dots,p_{12}$ are the intersection points of C_1 and B'; the image by φ' of $A_5,\dots,A_{12}$ are the eight points $p_5,\dots,p_{12}$. The curve B' has a node at the point p_4 onto which the curve A_4 is contracted by φ'. The curves $A_2, A_{13},\dots,A_{20}$ are sent, respectively, to the fibers passing through $p_4, p_5,\dots,p_{12}$.
- The image of the curve A_J ($J = \{i,j,k,l\}$), is a curve in $|4f + s|$ passing through the points p_t, for $t \in \{4, m, n, o, 12\}$, where $\{i,j,k,l,m,n,o\} = \{5,\dots,11\}$.
- The image of the curve B_J ($J = \{i,j,k,l\}$), is a curve in $|4f + s|$ passing through the points p_t, for $t \in \{4, i, j, k, l\}$.

Remark 11.4. By [Kon89], the automorphism group of X is $(\mathbb{Z}/2\mathbb{Z})^2$. It is generated by the involutions associated to the two double covers we described.

12. Rank 12 lattices

12.1. The lattice $U \oplus \mathrm{E}_8 \oplus \mathrm{A}_1^{\oplus 2}$

The K3 surface X contains 14 (-2)-curves $A_1,\dots,A_{14}$; their dual graph is



The curves $A_1,\dots,A_{12}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{456} = (20, 41, 63, 31, 55, 48, 42, 37, 33, 30, 14, 14)$$

is ample, of square 456, with $D_{456} \cdot A_j = 1$ for $j \le 12$ and $D_{456} \cdot A_j = 28$ for $j \in \{13, 14\}$.

Remark 12.1. The divisors

$$F_1 = 2A_1 + 4A_2 + 6A_3 + 3A_4 + 5A_5 + 4A_6 + 3A_7 + 2A_8 + A_9,$$
$$F_2 = A_{11} + A_{13},$$
$$F_3 = A_{12} + A_{14}$$

are fibers of an elliptic fibration. The number of curves in F_1 counted with multiplicities is 30; thus it is impossible to find an ample divisor D such that $D(A_{11} + A_{13}) < 30$.

By Theorem 2.3, case i) a), we have the following.

Proposition 12.2. *The linear system* $|4F_1 + 2A_{10}|$ *defines a morphism* $\varphi\colon X \to \mathbf{F}_4$ *branched over the unique section* s *with* $s^2 = -4$ *and a curve* $B \in |3s + 12f|$*. The curve* B *has one* $\mathbf{e}_8$ *singularity* p *and two nodes* q, q'*. The pull-backs of the fibers through* p, q, q' *are the fibers* F_1, F_2, F_3.

We can also construct this surface as follows. In the basis $A_1,\dots,A_{12}$, the divisor

$$D_2 = (2,4,6,3,5,4,3,2,2,2,1,1)$$

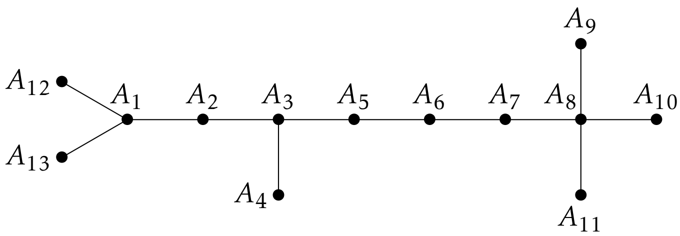
is nef, base-point free, of square 2, with $D_2 \cdot A_8 = 1$, $D_2 \cdot A_{13} = D_2 \cdot A_{14} = 2$ and $D_2 \cdot A_j = 0$ for $j \notin \{8,13,14\}$. Therefore, the K3 surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve which has an $\mathbf{e}_7$ singularity and a $\mathbf{d}_4$ singularity. The map $X \to \mathbb{P}^2$ is ramified over A_8. Since

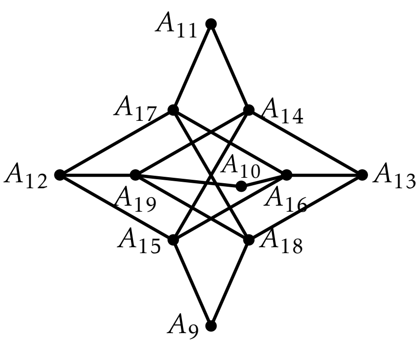
$$D_2 \equiv A_9 + 2A_{10} + 2A_{11} + A_{12} + A_{13} \equiv A_9 + 2A_{10} + A_{11} + 2A_{12} + A_{14},$$

we see that the images of A_{13} and A_{14} are the two tangent lines through the two remaining branches of the $\mathbf{d}_4$ singularities. The sextic curve is the union of a quintic and the line L. The quintic has a nodal and a cusp singularity, so that with the line L, they become an $\mathbf{e}_7$ singularity and a $\mathbf{d}_4$ singularity.

12.2. The lattice $U \oplus \mathbf{D}_8 \oplus \mathbf{A}_1^{\oplus 2}$

The K3 surface X contains 19 (-2)-curves $A_1,\dots,A_{19}$; their dual graph is





The curves $A_1,\dots,A_{12}$ generate the Néron–Severi lattice, and in that basis the divisor

$$D_{136} = (2,5,9,4,10,12,15,19,9,9,6,0)$$

is ample, of square 136, with $D_{136} \cdot A_j = 1$ for $j \le 10$, $D_{136} \cdot A_j = 7,2,2,12,18,18,12,18,18$ for $j = 11,\dots,19$.

The divisor

$$D_2 = (0,1,2,1,2,2,2,2,1,1,1,0)$$

is base-point free, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_8 = 1$, $D_2 \cdot A_j = 0$ for $j \in \{2,\dots,7,9,\dots,13\}$ and $D_2 \cdot A_j = 2$ for $j \in \{14,\dots,19\}$. Moreover,

$$\begin{aligned} D_2 &\equiv A_{11} + A_{12} + A_{17} \equiv A_{11} + A_{13} + A_{14}, \\ D_2 &\equiv A_{10} + A_{12} + A_{19} \equiv A_{10} + A_{13} + A_{16}, \\ D_2 &\equiv A_9 + A_{12} + A_{15} \equiv A_9 + A_{13} + A_{18}, \\ D_2 &\equiv 2A_1 + 3A_2 + 4A_3 + 2A_4 + 3A_5 + 2A_6 + A_7 + A_{12} + A_{13}. \end{aligned}$$

By using the linear system $|D_2|$, we obtain the following.

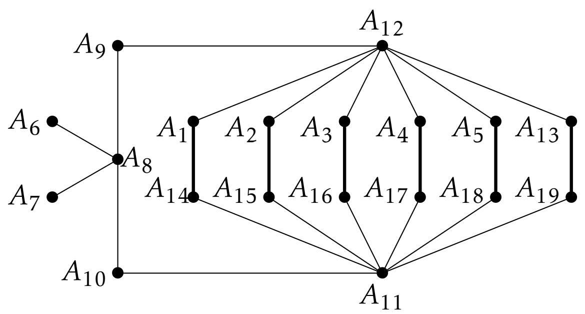
Proposition 12.3. *The K3 surface X is a double cover of $\mathbb{P}^2$ ramified over a sextic curve C_6; the curves A_1 and A_8 are in the ramification locus, and their images are two lines. We denote by Q_4 the residual quartic. The sextic curve has a $\mathbf{d}_6$ singularity (the curves $A_2,\dots,A_7$ are contracted to that singularity) and five nodal singularities p_j to which the curves A_j are contracted, for $j \in \{9,\dots,13\}$.*

The image of A_{15} (resp. A_{17}, A_{19}) is a line through the node p_{12} and the node p_9 (resp. p_{11}, p_{10}) which is tangent to Q_4. The image of A_{14} (resp. A_{16}, A_{18}) is a line through the node p_{13} and the node p_{11} (resp. p_{10}, p_9) which is tangent to Q_4.

12.3. The lattice $U \oplus \mathbf{D}_4^{\oplus 2} \oplus \mathbf{A}_1^{\oplus 2}$

The lattice $U \oplus \mathbf{D}_4^{\oplus 2} \oplus \mathbf{A}_1^{\oplus 2}$ is isometric to $U \oplus \mathbf{D}_6 \oplus \mathbf{A}_1^{\oplus 4}$; see [Kon89].

12.3.1. First involution.— The K3 surface X contains 59 (-2)-curves. The configuration of the first 19 (-2)-curves $A_1,\dots,A_{19}$ is as follows:



The curves $A_1,\dots,A_{12}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{40} = (3,3,3,0,0,2,2,6,6,3,1,7)$$

is ample, of square 40, with $D_{40} \cdot A_6 = D_{40} \cdot A_7 = 2$, $D_{40} \cdot A_j = 1$ for $j \in \{1,2,3,8,\dots,12,17,18,19\}$ and $D_{40} \cdot A_j = 7$ for $j \in \{4,5,13,\dots,16\}$.

The divisor

$$D_2 = (0,0,0,0,0,1,1,3,2,2,1,1)$$

is nef, base-point free, of square 2, with $D_2 \cdot A_j = 0$ for $j \in \{8,\dots,12\}$, $D_2 \cdot A_j = 1$ for $j \in \{1,\dots,7,13,\dots,19\}$ and $D_2 \cdot A_j = 2$ for $j \geq 20$. The K3 surface X is a double cover of the plane branched over a sextic curve C_6 which has an $\mathbf{a}_5$ singularity q; the curves $A_8,\dots,A_{12}$ are contracted to q. We have

$$D_2 \equiv A_8 + A_9 + A_{10} + A_{11} + A_{12} + F,$$

where

$$\begin{aligned} F &\equiv A_6 + A_7 + 2A_8 + A_9 + A_{10} \\ &\equiv A_1 + A_{14} \equiv A_2 + A_{15} \equiv A_3 + A_{16} \\ &\equiv A_4 + A_{17} \equiv A_5 + A_{18} \equiv A_{13} + A_{19}; \end{aligned}$$

thus there exist seven lines L, $L_1,\dots,L_6$ through the $\mathbf{a}_5$ singularity such that L intersects C_6 in that point only (so L is the tangent to the branch curve of the singularity; the strict transform of L is A_6+A_7) and that the lines L_i have even intersection multiplicities at their other intersection points with the sextic.

The (-2)-curves on X are $A_1,\dots,A_{19}$ and the curves

$$\begin{aligned}
B_{ij}^{(1)} &= -A_r - A_s - A_t + 3A_6 + 2A_7 + 6A_8 + 3A_9 + 4A_{10} + 2A_{11},\\
B_{ij}^{(2)} &= -A_r - A_s - A_t + 2A_6 + 3A_7 + 6A_8 + 3A_9 + 4A_{10} + 2A_{11},\\
C_{ij}^{(1)} &= -A_i - A_j - A_{13} + 2A_6 + 3A_7 + 6A_8 + 3A_9 + 4A_{10} + 2A_{11},\\
C_{ij}^{(2)} &= -A_i - A_j - A_{13} + 3A_6 + 2A_7 + 6A_8 + 3A_9 + 4A_{10} + 2A_{11},
\end{aligned}$$

where $\{i,j,r,s,t\} = \{1,2,3,4,5\}$. Using the relation

$$A_{13} = (-1,-1,-1,-1,-1,3,3,6,2,4,2,-2),$$

one can check that $B_{ij}^{(a)} + C_{ij}^{(a)} = 2D_2$; thus the curves $B_{ij}^{(a)}$, $C_{ij}^{(a)}$ are pull-backs of conics which are 6-tangent to the sextic curve C_6.

12.3.2. Second involution.— The divisor $F' = A_6 + A_7 + 2A_8 + A_9 + A_{10}$ is a fiber of an elliptic fibration. The divisor $D_2' = 2F' + A_{11}$ is nef, of square 2, with

$$\begin{aligned}
D_2 \cdot A_j &= 0 \quad \text{for } j \in \{1,2,3,4,5,6,7,8,9,11,13\},\\
D_2 \cdot A_j &= 1 \quad \text{for } j \in \{10,14,15,16,17,18,19\},\\
D_2 \cdot A_j &= 4 \quad \text{for } j \geq 20
\end{aligned}$$

and $D_2' \cdot A_{12} = 2$. The linear system $|D_2'|$ has base points. The linear system $|D_8|$ (where $D_8 = 2D_2'$) is base-point free, and it is hyperelliptic since $D_8F' = 2$. One can check easily that for i, j, r, s, t such that $\{i,j,r,s,t\} = \{1,\dots,5\}$, one has

$$\begin{aligned}
D_8 &\equiv B_{ij}^{(1)} + A_r + A_s + A_t + A_6 + 2A_7 + 2A_8 + A_9,\\
D_8 &\equiv B_{ij}^{(2)} + A_r + A_s + A_t + 2A_6 + A_7 + 2A_8 + A_9,\\
D_8 &\equiv C_{ij}^{(1)} + A_i + A_j + A_{13} + 2A_6 + A_7 + 2A_8 + A_9,\\
D_8 &\equiv C_{ij}^{(1)} + A_i + A_j + A_{13} + A_6 + 2A_7 + 2A_8 + A_9.
\end{aligned}$$

Moreover,

$$\begin{aligned}
D_8 &\equiv 3A_{10} + A_{19} + A_{13} + 3A_6 + 3A_7 + 6A_8 + 3A_9 + 2A_{11},\\
2D_8 &\equiv 2A_{10} + \textstyle\sum_{k=14}^{19} A_k + \sum_{k=1}^{5} A_k + 2(A_6 + A_7 + 2A_8 + A_9 + 2A_{11}) + A_{13},\\
D_8 &\equiv 2A_{12} + \textstyle\sum_{k=1}^{7} A_k + 2A_8 + 2A_9 + A_{13}.
\end{aligned}$$

By [Sai74, Equation (5.9.1)], the map $\varphi_{D_8}\colon X \to \mathbb{P}^5$ associated to $|D_8|$ has image a cone over a rational normal curve in $\mathbb{P}^4$; it factorizes through a surjective map $\varphi'\colon X \to \mathbf{F}_4$, where $\mathbf{F}_4$ is the Hirzebruch surface with a section s such that $s^4 = -4$. The section s is mapped to the vertex of the cone by the map $\mathbf{F}_4 \to \mathbb{P}^5$. Let f denote a fiber of the unique fibration $\mathbf{F}_4 \to \mathbb{P}^1$. By [Sai74, Equation (5.9.1)], the branch locus of φ' is the union of s and a curve C such that $Cs = 0$ and $C \in |3(s+4f)|$ (so that $s + C \in |-2K_{\mathbf{F}_4}|$). We have

$$|D_8| = \varphi'^*|4f + s|,$$

and therefore from the above equivalence relations, we get the following:

- The image of curve A_{11} by φ' is the section s.
- The branch curve is the union of three components: s, B' and C_{12}, where $B' \in |2(s+4f)|$, $C_{12} \in |s+4f|$. The curve B' has a node q, and the curves C_{12} and B' meet at q and at six other points $p_1,\dots,p_5,p_{13}$. The singularity at q of $B'+C_{12}$ has type $\mathbf{d}_4$; the other singular points are nodes.
- The curves $A_6,\dots,A_9$ are mapped by φ' to q.
- The curves $A_1,\dots,A_5,A_{13}$ are sent to $p_1,\dots,p_5,p_{13}$.
- The curve A_{12} is part of the ramification locus; its image is C_{12}.
- The curve A_{11} is part of the ramification locus; its image is s.
- The curves A_{10}, $A_{14},\dots,A_{19}$ are mapped to the fibers through $q,p_1,\dots,p_5,p_{13}$.
- The images of the curves $B^{(1)}_{ij}$ and $B^{(2)}_{ij}$ are curves in $|s+4f|$ passing through p_r, p_s, p_t and through q with certain tangency properties at the branches of the singularity q.
- The images of the curves $C^{(1)}_{ij}$ and $C^{(2)}_{ij}$ are curves in $|s+4f|$ passing through p_i, p_j, p_{13} and through q with certain tangency properties at the branches.

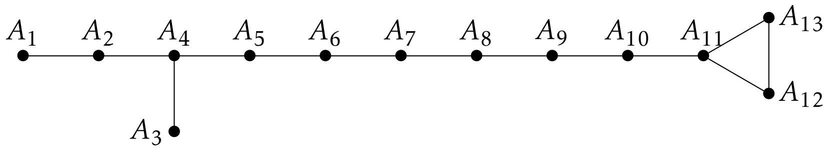
The 10 curves $B^{(1)}_{ij}$ (resp. $B^{(2)}_{ij}$, $C^{(1)}_{ij}$, $C^{(2)}_{ij}$) for $\{i,j\}\subset\{1,2,3,4,5\}$ have the configuration of the Petersen graph, with weight 2 on the edges. Moreover, for $1\le i<j\le 5$ and $1\le s<t\le 5$, the intersections between the four types of curves are as follows:

$$B^{(1)}_{ij}\cdot B^{(2)}_{st} = C^{(1)}_{ij}\cdot C^{(2)}_{st} = 2 + B^{(1)}_{ij}\cdot B^{(1)}_{st},$$
$$B^{(1)}_{ij}\cdot C^{(2)}_{st} = B^{(2)}_{ij}\cdot C^{(1)}_{st} = 2 - B^{(1)}_{ij}\cdot B^{(1)}_{st},$$
$$B^{(1)}_{ij}\cdot C^{(1)}_{st} = B^{(2)}_{ij}\cdot C^{(2)}_{st} = 4 - B^{(1)}_{ij}\cdot B^{(1)}_{st}.$$

Remark 12.4. In [Kon02, Remark 1], Kondo constructed specific surfaces with Néron–Severi lattice isometric to $U\oplus\mathbf{D}_4^{\oplus 2}\oplus\mathbf{A}_1^{\oplus 2}$ as follows. Let C be a smooth curve of genus 2 and q be a point on C. The linear system $|K_C+2q|$ gives a plane quartic curve with a cusp. The minimal resolution Y of the cyclic degree 4 cover of $\mathbb{P}^2$ branched over that curve has an elliptic fibration (obtained by blowing up the cusp) with a $\tilde{\mathbf{D}}_4$ and six $\tilde{\mathbf{A}}_1$ fibers. The automorphism group of such a surface is larger than the general one.

12.4. The lattice $U\oplus\mathbf{A}_2\oplus\mathbf{E}_8$

The K3 surface X contains 13 (-2)-curves $A_1,\dots,A_{13}$; their configuration is as follows:



The curves $A_1,\dots,A_{12}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{698} = (38,77,58,117,100,84,69,55,42,30,19,9)$$

is ample, of square 698, with $D_{698}\cdot A_j = 1$ for $j\le 12$ and $D_{698}\cdot A_{13} = 28$. The divisor

$$F_1 = A_{11} + A_{12} + A_{13}$$

is a fiber of an elliptic fibration with section A_{10}. That fibration has another singular fiber F_2 of type $\tilde{\mathbf{E}}_8$. By Theorem 2.3, case i) a), we have the following.

Proposition 12.5. *The linear system $|4F_1+2A_{10}|$ defines a morphism $\varphi\colon X\to\mathbf{F}_4$ branched over the unique section s with $s^2=-4$ and a curve $B\in|3s+12f|$. The curve B has one cusp p and one $\mathbf{e}_8$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

We can also give another construction as a double cover of $\mathbb{P}^2$: The divisor

$$D_2 = (2,5,4,8,7,6,5,4,3,2,1,0)$$

is nef, of square 2, base-point free, with $D_2 \cdot A_1 = D_2 \cdot A_{12} = D_2 \cdot A_{13} = 1$ and $D_2 \cdot A_j = 0$ for $j \in \{2,\dots,11\}$. The K3 surface is the double cover of $\mathbb{P}^2$ branched over a sextic curve with a $\mathbf{d}_{10}$ singularity. The curve A_1 is in the ramification locus; we denote by L its image and by Q the residual quintic curve. The quintic Q has a node, and L is tangent with multiplicity 4 at a branch of that node. We have

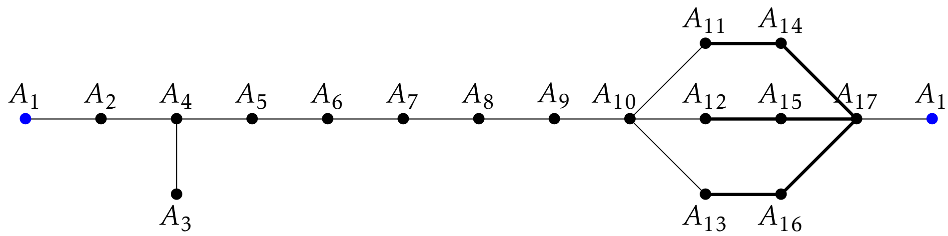
$$D_2 \equiv A_2 + A_3 + 2A_4 + 2A_5 + 2A_6 + 2A_7 + 2A_8 + 2A_9 + 2A_{10} + 2A_{11} + A_{12} + A_{13};$$

thus the image of A_{12} and A_{13} is the line which is the tangent to the other branch of the node, which is moreover tangent to Q at another point.

13. Rank 13 lattices

13.1. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1^{\oplus 3}$

The K3 surface X contains 17 (-2)-curves $A_1,\dots,A_{17}$; their configuration is as follows:



The curves $A_1,\dots,A_{13}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{294} = (2,5,4,9,10,12,15,19,24,30,14,14,9)$$

is ample, of square 294, with $D_{294} \cdot A_j = 1$ for $j \le 10$, $D_{294} \cdot A_j = 2$ for $j \in \{11,12,17\}$, $D_{294} \cdot A_j = 28$ for $j \in \{14,15\}$, $D_{294} \cdot A_{13} = 12$ and $D_{294} \cdot A_{16} = 18$. By considering the elliptic fibration s with section A_{10} and the fibers

$$F = A_{11} + A_{14},\, A_{12} + A_{15},\, A_{13} + A_{16}$$

plus the fiber of type $\tilde{\mathbf{E}}_8$ supported on $A_1,\dots,A_9$, we get the following.

Proposition 13.1. *The surface is a double cover of the Hirzebruch surface* $\mathbf{F}_4$ *branched over the negative section* s *and a curve* $B \in |3(s+4f)|$ *with three nodes and an* $\mathbf{e}_8$ *singularity.*

The double cover η is given by the linear system $|D_8|$, where $D_8 = 4F + 2A_{10}$. One has

$$D_8 = A_{17} + 4A_1 + 7A_2 + 5A_3 + 10A_4 + 8A_5 + 6A_6 + 4A_7 + 2A_8 + A_{14} + A_{15} + A_{16};$$

thus the image by η of the curve A_{17} is a curve in $|s+4f|$ going through the three nodes plus the $\mathbf{e}_8$ singularity and infinitely near points of it.

We can also construct the surface X as follows. The divisor

$$D_2 = (0,1,1,2,2,2,2,2,2,2,1,1,1)$$

is nef, of square 2, base-point free, with $D_2 \cdot A_j = 0$ for $j \in \{2,\dots,9,11,12,13,17\}$. We have the relations

$$D_2 \equiv A_{13} + A_{16} + A_{17} \equiv A_{12} + A_{15} + A_{17} \equiv A_{11} + A_{14} + A_{17}$$

and

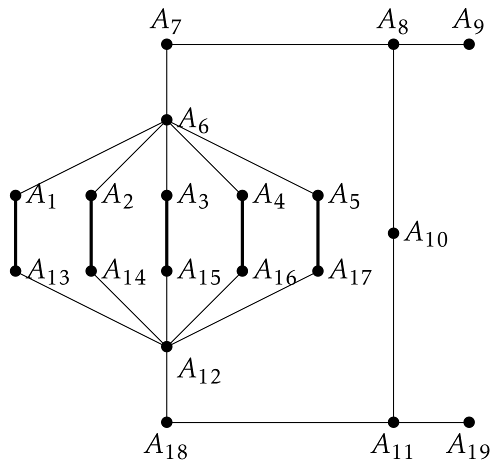
$$D_2 \equiv 2A_1 + 4A_2 + 3A_3 + 6A_4 + 5A_5 + 4A_6 + 3A_7 + 2A_8 + A_9 + A_{17}.$$

The curves A_1 and A_{10} are part of the ramification locus; their images are lines L, L'. Let us denote by Q the residual quartic. The images of the curves A_{14}, A_{15}, A_{16} are lines L_1, L_2, L_3 going through the images of A_{17} and of A_{11}, A_{12}, A_{13}, respectively. The quartic curve is smooth and tangent to the line L (the image of A_1) at the intersection point of L and L', so that the singularity at q is $\mathbf{d}_8$. The branch locus has four nodes, three on the line L' (the image of A_{10}) and one on L (the image of A_1).

13.2. The lattice $U \oplus \mathbf{D}_8 \oplus \mathbf{A}_1^{\oplus 3}$

The lattice $U \oplus \mathbf{D}_8 \oplus \mathbf{A}_1^{\oplus 3}$ is also isometric to $U \oplus \mathbf{E}_7 \oplus \mathbf{A}_1^{\oplus 4}$ and $U \oplus \mathbf{D}_6 \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$; see [Kon89].

13.2.1. The first involution.— The K3 surface X contains 39 (-2)-curves $A_1, \dots, A_{39}$. The configuration of the curves $A_1, \dots, A_{19}$ is as follows:



The curves $A_1, \dots, A_{13}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{86} = (2, 1, 2, 5, 5, 11, 8, 6, 2, 3, 1, 0, 1)$$

is ample, of square 86. The divisor

$$D_2 = (-1, 1, 1, 1, 1, 3, 3, 3, 1, 2, 1, -1, -2)$$

is nef, of square 2, base-point free, with $D_2 \cdot A_j = 0$ for $j \in \{6, 7, 8, 10, 11, 12, 18\}$, $D_2 \cdot A_j = 2$ for $j \geq 20$ and else $D_2 \cdot A_j = 1$. For $k \in \{10, \dots, 19\}$, one has

$$2D_2 \equiv A_{2k} + A_{2k+1},$$

and using the equivalences from the elliptic fibration with fiber $A_1 + A_{13}$, we get

$$\begin{aligned}
D_2 &\equiv A_{13} + A_1 + A_6 + A_7 + A_8 + A_{10} + A_{11} + A_{12} + A_{18}, \\
D_2 &\equiv A_{14} + A_2 + A_6 + A_7 + A_8 + A_{10} + A_{11} + A_{12} + A_{18}, \\
D_2 &\equiv A_{15} + A_3 + A_6 + A_7 + A_8 + A_{10} + A_{11} + A_{12} + A_{18}, \\
D_2 &\equiv A_{16} + A_4 + A_6 + A_7 + A_8 + A_{10} + A_{11} + A_{12} + A_{18}, \\
D_2 &\equiv A_{17} + A_5 + A_6 + A_7 + A_8 + A_{10} + A_{11} + A_{12} + A_{18}, \\
D_2 &\equiv A_9 + A_{19} + A_6 + 2A_7 + 3A_8 + 3A_{10} + 3A_{11} + A_{12} + 2A_{18}.
\end{aligned}$$

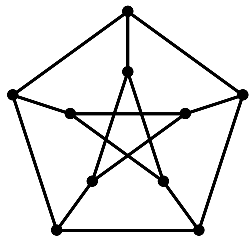
The K3 surface is the double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 with a singularity q of type $\mathbf{a}_7$. The line tangent to the branch of C_6 has no other intersection points with C_6, and the curves A_9, A_{19} map onto that line. Moreover, there exist five lines through q that are bitangent to the sextic at their other intersection points. The images of A_j, A_{j+12} for $j \in \{1, \dots, 5\}$ are these lines.

For $J=\{i,j\}\subset\{1,\dots,5\}$, let

$$A_J=-A_i-A_j+\sum_{t=1}^{5}A_t+2A_6+A_7-A_9.$$

Then A_J is the class of a (-2)-curve, and $B_J=2D_2-A_J$ is also a (-2)-curve. These are the 20 curves $A_{20},\dots,A_{39}$; thus the images by the double cover map of curves A_J,B_J are conics that are 6-tangent to the sextic branch curve.

We have $A_J\cdot A_{J'}=2$ if and only if $\#J\cap J'=0$, and else $A_J\cdot A_{J'}\in\{-2,0\}$. The dual graph of the 10 curves A_J (for $J\subset\{1,\dots,5\}$, $\#J=2$) is the Petersen graph with weight 2 on the edges:



The configuration of the curves B_J ($J\subset\{1,\dots,5\}$, $\#J=2$) is also the Petersen graph with weight 2 on the edges. Using that $D_2\cdot A_J=D_2\cdot B_J=2$, we get that $A_J\cdot A_{J'}=B_J\cdot B_{J'}$ and $A_J\cdot B_{J'}=4-A_J\cdot A_{J'}$ for any subsets J, J' of order 2 of $\{1,\dots,5\}$.

13.2.2. The second involution.— The divisor

$$D_2'=2A_1+A_{12}+2A_{13}$$

is nef, of square 2, with base points. One has $D_2\cdot A_j=0$ for $j\in\{1,\dots 5,7,\dots,12,19\}$, $D_2'\cdot A_6=2$, $D_2'\cdot A_j=1$ for $j\in\{13,\dots,18\}$ and $D_2'\cdot A_j=4$ for $j\geq 20$. Since $D_2'=2F+A_{12}$, where $F=A_1+A_{13}$ is a fiber of an elliptic fibration with $FA_{12}=1$, the linear system $|D_2'|$ has base points, and $|D_8|=|2D_2'|$ is base-point free and hyperelliptic. We have

$$\begin{aligned}
D_8&\equiv A_{13}+A_{14}+A_{15}+A_{16}+A_1+A_2+A_3+A_4+2A_{12},\\
D_8&\equiv A_{15}+A_{16}+A_{17}+A_{18}+A_3+A_4+A_5+A_7+2A_8\\
&\quad+A_9+2A_{10}+2A_{11}+2A_{12}+A_{19},\\
D_8&\equiv B_J+\textstyle\sum_{t=1,t\notin J}^{5}A_j+A_7+2A_8+A_9+2A_{10}+2A_{11}+2A_{19},\\
D_8&\equiv A_J+\textstyle\sum_{t\in J}A_t+2A_7+4A_8+3A_9+3A_{10}+2A_{11},\\
D_8&=2A_6+\textstyle\sum_{t=1}^{5}A_t+3A_7+4A_8+2A_9+3A_{10}+2A_{11}+A_{19}.
\end{aligned}\tag{13.1}$$

The map $\varphi_{D_8}\colon X\to\mathbb{P}^5$ associated to $|D_8|$ has image a cone over a rational normal curve in $\mathbb{P}^4$; it factorizes through a surjective map $\varphi'\colon X\to\mathbf{F}_4$, where $\mathbf{F}_4$ is the Hirzebruch surface with a section s such that $s^4=-4$. The section s is mapped to the vertex of the cone by the map $\mathbf{F}_4\to\mathbb{P}^5$. Let f denote a fiber of the unique fibration $\mathbf{F}_4\to\mathbb{P}^1$. By Theorem 2.3, the branch locus of φ' is the union of s and a curve C such that $Cs=0$ and $C\in|3(s+4f)|$ (so that $b+C\in|-2K_{\mathbf{F}_4}|$). We have

$$|D_8|=\varphi'^*|4f+s|,$$

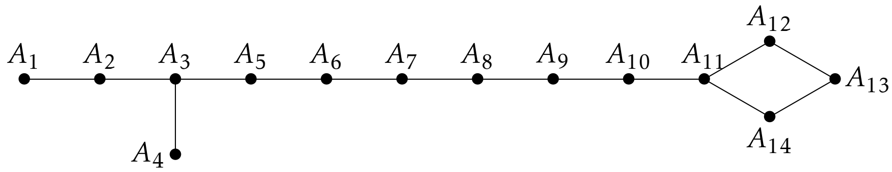
and therefore from the equivalence relations in (13.1), we get the following claims.

- The curve A_{12} is in the ramification locus; its image is s.
- The curves A_7, A_8, A_9, A_{10}, A_{11}, A_{19} are contracted to a $\mathbf{d}_6$ singularity q of C.
- The curve C is the union of a curve s' in $|s+4f|$ and a curve B' in $|2(s+4f)|$ (thus $s'B'=8$). The curve B' has a node at q, and s' is tangent to one of the branches of B' at q, so that the intersection multiplicity of s' and B' at q is 3 and the singularity of $C=B'+s'$ has type $\mathbf{d}_6$.

- The other intersection points of s' and B' are nodes $p_1, \dots, p_5$ to which the curves $A_1, \dots, A_5$ are mapped.
- The curve A_6 is in the ramification locus; its image is s'.
- The curves $A_{13}, \dots, A_{17}, A_{18}$ are mapped to the fibers through $p_1, \dots, p_5$ and q, respectively.
- The curves A_J and B_J are mapped to curves in the linear system $|s + 4f|$ passing through the points $p_1, \dots, p_5$ and points infinitely near q with certain multiplicities.

13.3. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{A}_3$

The K3 surface X contains 14 (-2)-curves $A_1, \dots, A_{14}$; their configuration is as follows:



The curves $A_1, \dots, A_{13}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{506} = (46, 93, 141, 70, 120, 100, 81, 63, 46, 30, 15, 1, -12)$$

is ample, of square 506, with $D_{506} \cdot A_j = 1$ for $j \leq 12$, $D_{506} \cdot A_{13} = 25$ and $D_{506} \cdot A_{14} = 3$. The surface has an elliptic fibration such that A_{10} is a section with singular fiber

$$F_1 = A_{11} + A_{12} + A_{13} + A_{14}$$

and a fiber F_2 of type $\tilde{\mathbf{E}}_8$. By Theorem 2.3, case i) a), we have the following.

Proposition 13.2. *The linear system $|4F_1 + 2A_8|$ defines a morphism $\varphi\colon X \to \mathbf{F}_4$ branched over the unique section s with $s^2 = -4$ with $s^2 = -4$ and a curve $B \in |3s + 12f|$. The curve B has one $\mathbf{a}_3$ singularity p and one $\mathbf{e}_8$ singularity q. The pull-backs of the fibers through p, q are the fibers F_1, F_2.*

In order to construct X, one can also use the divisor

$$D_2 = (2, 5, 8, 4, 7, 6, 5, 4, 3, 2, 1, 0, 0),$$

which is nef, of square 2, base-point free, with $D_2 \cdot A_j = 0$ for $j \in \{2, \dots, 11, 13\}$ and $D_2 \cdot A_j = 1$ for $j \in \{1, 12, 14\}$. Since

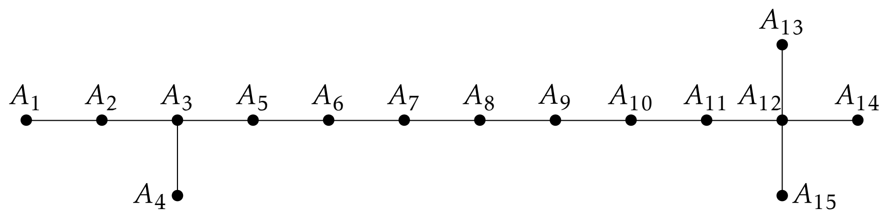
$$D_2 \equiv A_2 + 2A_3 + A_4 + 2A_5 + 2A_6 + 2A_7 + 2A_8 + A_9 + 2A_{10} + 2A_{11} + A_{12} + A_{13} + A_{14},$$

we obtain that the K3 surface is the double cover of $\mathbb{P}^2$ branched over a sextic curve which is the union of a line L and a quintic Q such that A_1 is in the ramification locus and its image is L. The sextic curve has a singularity q of type $\mathbf{d}_{10}$ and a node. The image of A_{12} and A_{14} is the line L. The situation is a specialization of Section 12.4.

14. Rank 14 lattices

14.1. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{D}_4$

The K3 surface X contains 15 (-2)-curves $A_1, \dots, A_{15}$; their dual graph is



The curves $A_1,\dots,A_{14}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{506} = (46, 93, 141, 70, 120, 100, 81, 63, 46, 30, 15, 1, -12, 0)$$

is ample, of square 506, with $D_{506} \cdot A_{13} = 25$ and $D_{506} \cdot A_j = 1$ for $j \neq 13$. The divisor

$$D_2 = (2, 5, 8, 4, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0)$$

is nef, base-point free, of square 2, with $D_2 \cdot A_1 = D_2 \cdot A_{12} = 1$ and $D_2 \cdot A_j = 0$ for $j \notin \{1, 12\}$. We have

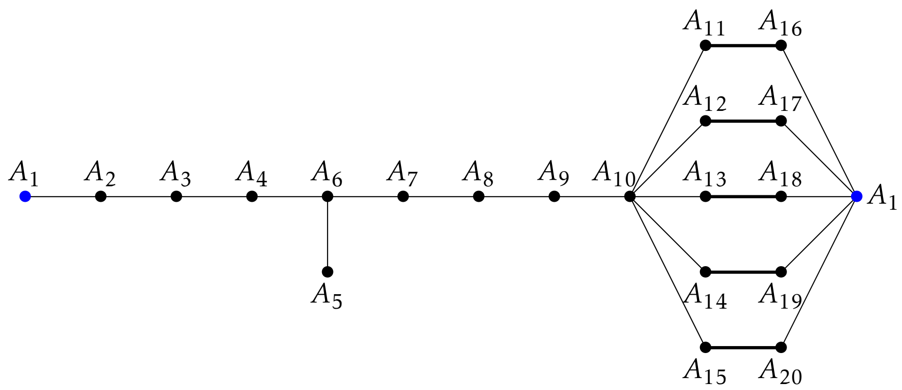
$$D_2 \equiv A_2 + 2A_3 + A_4 + 2(A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} + A_{11} + A_{12}) + A_{13} + A_{14} + A_{15};$$

thus, using the linear system $|D_2|$, we obtain the following.

Proposition 14.1. *The K3 surface is a double cover of $\mathbb{P}^2$ which is ramified over A_1 and A_{12}; the images of these curves are lines in the sextic branch curve. The sextic curve has a $\mathbf{d}_{10}$ singularity at a point q and three nodal singularities. The residual quartic curve is smooth at q. The tangent at q is the line L, and the order of tangency at q is 4. The other nodes are the intersections of the line L' with the quartic.*

14.2. The lattice $U \oplus \mathbf{D}_8 \oplus \mathbf{D}_4$

14.2.1. First involution.— The K3 surface X contains 20 (-2)-curves $A_1,\dots,A_{20}$; their configuration is as follows:



The curves $A_1,\dots,A_{14}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{154} = (1, 3, 6, 10, 7, 15, 14, 14, 15, 17, 8, 8, 0, 4)$$

is ample, of square 154, with $D_{154} \cdot A_j = 1$ for $j \in \{1, 1, \dots, 12, 18, 20\}$, $D_{154} \cdot A_j = 9$ for $j \in \{14, 19\}$ and $D_{154} \cdot A_j = 17$ for $j \in \{15, 16, 17\}$. The divisor

$$D_2 = (1, 2, 3, 4, 2, 5, 4, 3, 2, 1, 0, 0, 0, 0)$$

is nef, base-point free, of square 2, with $D_2 \cdot A_j = 0$ for $j \in \{1, \dots, 4, 6, \dots, 10\}$ and else $D_2 \cdot A_j = 1$. Using the equivalences from the elliptic fibration with fiber $A_{11} + A_{16}$, we obtain that

$$D_2 \equiv A_{10+k} + A_{15+k} + \sum_{j=1, j\neq 5}^{10} A_j, \ \forall k \in \{1, 2, 3, 4, 5\}.$$

The linear system $|D_2|$ defines the K3 surface X as the double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 with an $\mathbf{a}_9$ singularity q. The cover is ramified above A_5; the image of A_5 is a line, a component of C_6 which has no other intersection points with the residual quintic curve Q. The images of $A_{10+k} + A_{15+k}$, $k \in \{1, \dots, 5\}$, are lines through q which are bitangent to Q. The quintic is smooth, thus of genus 6.

14.2.2. Second involution.— The divisor

$$D_2' = (0,1,2,4,3,6,5,4,3,2,1,0,0,0)$$

is nef, base-point free, of square 2. It satisfies $D_2 \cdot A_j = 0$ for $j \in \{2,4,\ldots,11,17,\ldots,20\}$. Using equivalence relations obtained from the elliptic fibration with fiber $A_{11} + A_{16}$, we have

$$\begin{aligned} D_2' &\equiv A_{12} + (A_{17} + A_4 + A_5 + 2\textstyle\sum_{t=6}^{10} A_t + A_{11}),\\ D_2' &\equiv A_{13} + (A_{18} + A_4 + A_5 + 2\textstyle\sum_{t=6}^{10} A_t + A_{11}),\\ D_2' &\equiv A_{14} + (A_{19} + A_4 + A_5 + 2\textstyle\sum_{t=6}^{10} A_t + A_{11}),\\ D_2' &\equiv A_{15} + (A_{20} + A_4 + A_5 + 2\textstyle\sum_{t=6}^{10} A_t + A_{11}),\\ D_2' &\equiv A_{16} + (A_{11} + A_4 + A_5 + 2\textstyle\sum_{t=6}^{10} A_t + A_{11}); \end{aligned}$$

moreover,

$$D_2' \equiv 2A_1 + A_2 + \sum_{j=17}^{20} A_j.$$

We have the following assertions.

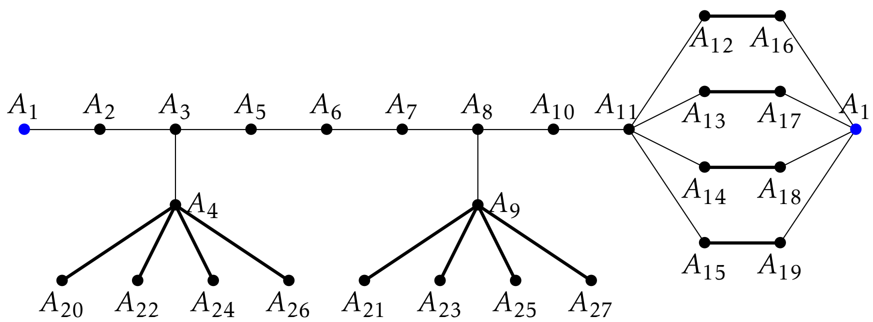
- The branch curve of the corresponding double cover is a sextic with a $\mathbf{d}_8$ singularity at a point q onto which the curves $A_4,\ldots,A_{11}$ are contracted and five nodes $p_2, p_{17},\ldots,p_{20}$ onto which $A_2, A_{17},\ldots,A_{20}$ are contracted. The curves A_1, A_3 are in the ramification locus; their images are lines L_1, L_3. Let us denote by Q' the residual quartic curve.
- The line L_1 cuts Q' (resp. L_3) at the points $p_{17},\ldots,p_{20}$ (resp. the point p_2).
- The quartic Q' has a node at q, and the line L_3 is tangent with multiplicity 3 at one of the branches of the node.
- For $k \in \{12,\ldots,15\}$, the image of A_k is a line through q and p_{k+5}.
- The image of A_{16} is a line passing through q and tangent to Q' at another point.

Remark 14.2. We constructed the surface X as two double covers. The associated involutions fix curves of geometric genus of 6 and 2, respectively; thus these involution generate the automorphism group of X, which is $(\mathbb{Z}/2\mathbb{Z})^2$ according to [Kon89].

14.3. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1^{\oplus 4}$

The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1^{\oplus 4}$ is also isometric to $U \oplus \mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$; see [Kon89].

14.3.1. First involution.— The K3 surface X contains 27 (-2)-curves $A_1,\ldots,A_{27}$. The dual graph of $A_1,\ldots,A_{19}$ and the intersections of the curves $A_1,\ldots,A_{11}$ with the curves A_j for $j \geq 20$ is given by



The curves $A_1,\ldots,A_{14}$ generate the Néron–Severi lattice. In that base, the divisor

$$D_{132} = (2,5,9,4,10,12,15,19,9,15,12,5,5,0)$$

is ample, of square 132, with $D_{132}\cdot A_j = 1$ for $j \leq 11$, $D_{132}\cdot A_j = 2$ for $j \in \{12,13,18,19\}$, $D_{132}\cdot A_j = 12$ for $j \in \{15,\dots,18\}$, $D_{132}\cdot A_j = 18$ for $j \in \{21,23,24,26\}$ and $D_{132}\cdot A_j = 28$ for $j \in \{20,22,25,27\}$. The divisor

$$D_2 = (1,2,3,1,3,3,3,3,1,2,1,0,0,0)$$

is nef, of square 2, base-point free. We have $D_2\cdot A_j = 0$ for $j \in \{1,2,3,5,6,7,8,10,11\}$, $D_2\cdot A_j = 1$ for $j \in \{4,9,12,\dots,19\}$ and $D_2\cdot A_j = 2$ for $j \in \{20,\dots,27\}$. Using the elliptic fibration with fiber $A_{12}+A_{16}$, we get

$$D_2 \equiv A_{11+k} + A_{15+k} + \left(-A_4 - A_9 + \sum_{j=1}^{11} A_j\right), k \in \{1,2,3,4\}.$$

Moreover, one has

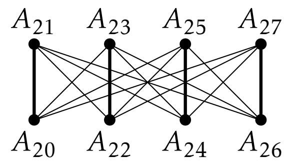
$$2D_2 \equiv A_{2k} + A_{2k+1} \quad \text{for } k \in \{10,11,12,13\};$$

thus the K3 surface is the double cover branched over a sextic curve with an $\mathbf{a}_9$ singularity q. The image of A_4 and A_9 is the line tangent to the branch of that singularity. The curves $A_{11+k}+A_{15+k}$, $k \in \{1,2,3,4\}$, are mapped onto lines going through q that are tangent to the sextic curve in two other points. The curves $A_{2k}+A_{2k+1}$ for $k \in \{10,11,12,13\}$ are mapped onto conics that are 6-tangent to the sextic curve.

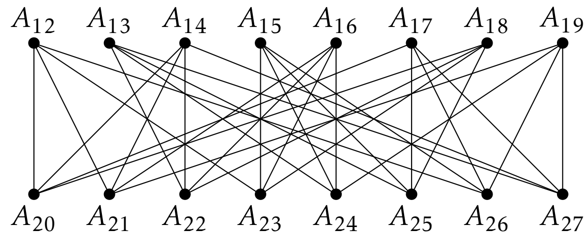
The classes of the curves A_{20}, A_{22}, A_{24}, A_{26} are

$$A_{20} = (0,0,0,-1,1,2,3,4,2,3,2,0,0,1), \quad A_{22} = (2,4,6,2,6,6,6,6,3,3,0,-1,-1,-1)$$
$$A_{24} = (0,0,0,-1,1,2,3,4,2,3,2,1,0,0), \quad A_{26} = (0,0,0,-1,1,2,3,4,2,3,2,0,1,0),$$

and the dual graph of the curves $A_{20},\dots,A_{27}$ is



where a thick line has weight 6 and a thin line has weight 4. For completeness, we also give the intersections between the curves $A_{12},\dots,A_{19}$ and the curves $A_{20},\dots,A_{27}$:



where a thin line has weight 2. In fact, one can check that the above graph is another occurrence of the Levi graph of the Möbius configuration.

14.3.2. Second involution.— The divisor

$$D_2' = (1,2,4,2,4,4,4,4,2,2,0,0,0,0)$$

is nef, of square 2, with base points and with $D_2\cdot A_j = 0$ for $j \in \{1,3,\dots,10,12,\dots,15\}$ and $D_2\cdot A_j = 1$ for $j \in \{2,16,17,18,19\}$, $D_2\cdot A_j = 4$ for $j \geq 20$. Let $D_8 = 2D_2'$; it is base-point free and hyperelliptic. It defines a morphism $\varphi'\colon X \to \mathbb{P}^5$ onto a degree 4 surface. That morphism factors through the Hirzebruch surface $\mathbf{F}_4$ by a map denoted by φ; the map $\mathbf{F}_4 \to \mathbb{P}^5$ contracts the section s of the Hirzebruch surface. The divisor

$$F_1 = A_2 + A_4 + 2(A_3 + A_5 + A_6 + A_7 + A_8) + A_9 + A_{10}$$

is the fiber of an elliptic fibration for which the curves $A_{11+k}+A_{15+k}$, $k\in\{1,2,3,4\}$, are also fibers. One has $D_2'=2F_1+A_1$; thus

$$D_8\equiv\sum_{j=12}^{19}A_j+2A_1.$$

Moreover, we have

$$\begin{aligned}
D_8 &\equiv 2A_{11}+2A_3+A_4+3A_5+4A_6+5A_7+6A_8+3A_9+4A_{10},\\
D_8 &\equiv A_{20}+2\textstyle\sum_{j=3}^{8}A_j+A_9+A_{10}+A_{12}+A_{13}+A_{15},\\
D_8 &\equiv A_{22}+2\textstyle\sum_{j=3}^{8}A_j+A_9+A_{10}+A_{12}+A_{13}+A_{14},\\
D_8 &\equiv A_{24}+2\textstyle\sum_{j=3}^{8}A_j+A_9+A_{10}+A_{13}+A_{14}+A_{15},\\
D_8 &\equiv A_{26}+2\textstyle\sum_{j=3}^{8}A_j+A_9+A_{10}+A_{12}+A_{14}+A_{15},\\
D_8 &\equiv A_{21}+2A_3+A_4+3A_5+4A_6+5A_7+6A_8+4A_9+3A_{10}+A_{14},\\
D_8 &\equiv A_{23}+2A_3+A_4+3A_5+4A_6+5A_7+6A_8+4A_9+3A_{10}+A_{15},\\
D_8 &\equiv A_{25}+2A_3+A_4+3A_5+4A_6+5A_7+6A_8+4A_9+3A_{10}+A_{12},\\
D_8 &\equiv A_{27}+2A_3+A_4+3A_5+4A_6+5A_7+6A_8+4A_9+3A_{10}+A_{13}.
\end{aligned}$$

We therefore have the following claims.

- The curves $A_3,\dots,A_{10}$ are contracted to a $\mathbf{d}_8$ singularity q of the branch curve B of φ; the curves $A_{12},\dots,A_{15}$ are mapped to nodes $p_{12},\dots,p_{15}$ of B.
- The curve B is the union of s and a curve C such that $Cs=0$ and $C\in|3(s+4f)|$ (so that $s+C\in|-2K_{\mathbf{F}_4}|$). We have

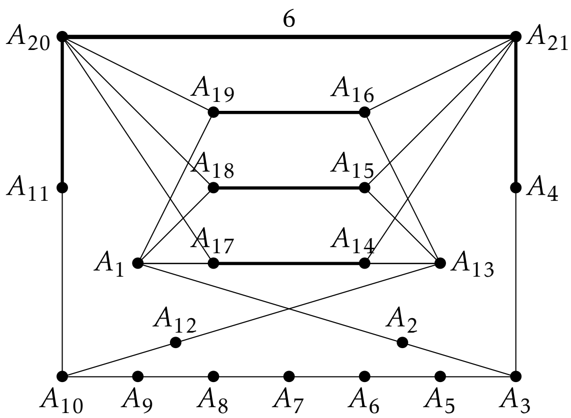
$$|D_8|=\varphi^*|4f+s|$$

  and therefore from the above equivalence relations, we get the following:
- The image of curve A_1 by φ is the section s.
- The curve B is the union of three components: s, B' and C_{11}, where $B'\in|2(s+4f)|$ and $C_{11}\in|s+4f|$. The curve B' has a node q, and the curves C_{11} and B' meet at q in such a way that C_{11} intersect one of the branches with multiplicity 3 (so that the singularity at q of $B'+C_{11}$ has type $\mathbf{d}_8$). The four points $p_{12},\dots,p_{15}$ are nodes; they are the remaining intersection points of C_{11} and B' (so that $B'C_{11}=8$).
- The images of the curves $A_{16},\dots,A_{19}$ are the fibers through $p_{12},\dots,p_{15}$,
- The images of the curves A_{20}, A_{22}, A_{24}, A_{26} by φ are curves in the linear system $|s+4f|$ passing through three of the four points $p_{12},\dots,p_{15}$, and through q and points infinitely near q with certain multiplicities.
- The images of the curves A_{21}, A_{23}, A_{25}, A_{27} by φ are curves in the linear system $|s+4f|$ passing through one of the four points $p_{12},\dots,p_{15}$ and through q and points infinitely near q with certain multiplicities.

15. Rank at least 15 lattices

15.1. The lattice $U\oplus \mathrm{E}_8\oplus \mathrm{D}_4\oplus \mathrm{A}_1$

The K3 surface X contains 21 (-2)-curves $A_1,\dots,A_{21}$. The dual graph of these curves is given in [Kon89, Section 4]; we reproduce it here:



The curves $A_1,\dots,A_{15}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{242} = (2,5,9,4,10,12,15,19,24,30,14,23,17,8,4)$$

is ample, of square 242, with $D_{242}A_j = 1$ for $j \in \{1,\dots,14\} \setminus \{11\}$.

The divisor

$$D_2 = (1,2,3,1,3,3,3,3,3,3,1,2,1,0,0)$$

is nef, base-point free, with $D_2^2 = 2$, $D_2 \cdot A_{20} = D_2 \cdot A_{21} = 2$, $D_2 \cdot A_j = 1$ for $j \in \{4,11,14,\dots,19\}$ and else $D_2 \cdot A_j = 0$. We have

$$2D_2 \equiv A_{20} + A_{21}$$

and

$$D_2 \equiv \sum_{k\in\{1,\dots,13\}\setminus\{4,11\}} A_k + A_{13+j} + A_{16+j},\ j \in \{1,2,3\}.$$

By using the linear system $|D_2|$, we obtain the following.

Proposition 15.1. *The K3 surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve C_6 with an $\mathbf{a}_{11}$ singularity q; the curves A_j with $j \in \{1,\dots,13\} \setminus \{4,11\}$ are mapped to q. The image of the curves A_4 and A_{11} is the line L tangent to the branch of C_6 at q. The images of A_{13+j} and A_{16+j}, $j \in \{1,2,3\}$, are lines L_1, L_2, L_3 through q that are bitangent to the sextic at other intersection points. The image of A_{20}, A_{21} is a conic which is 6-tangent to the sextic.*

Remark 15.2. The branch curve is irreducible, with geometric genus 4.

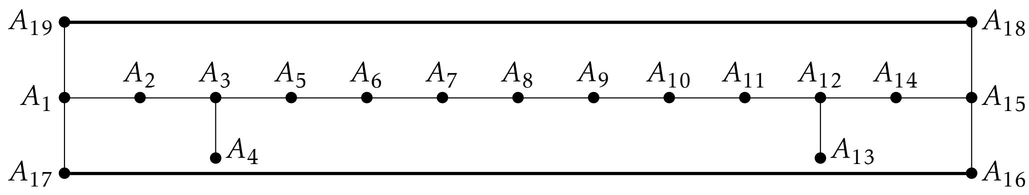
The divisor $D_2' = (0,1,2,1,2,2,2,2,3,4,2,3,2,1,0)$ is nef, base-point free of square 2, with

$$D_2 \cdot A_1 = D_2 \cdot A_8 = 1, D_2 \cdot A_{15} = D_2 \cdot A_{16} = D_2 \cdot A_{17} = 2, D_2 \cdot A_{20} = D_2 \cdot A_{21} = 4.$$

Therefore, the linear system $|D_2'|$ induces a double cover $\varphi'\colon X \to \mathbb{P}^2$ which is ramified above the line $L' = \varphi'(A_8)$; the curves A_2, A_3, A_5, A_6, A_7 are contracted to a singularity of the branch curve, and the curves $A_9,\dots,A_{14}$ are contracted to another singularity. In that way, we obtain a second involution acting on the K3 surface. By [Kon89], we know that the automorphism group of the general K3 is $(\mathbb{Z}/2\mathbb{Z})^2$.

15.2. The lattice $U \oplus \mathrm{E}_8 \oplus \mathrm{D}_6$

The K3 surface X contains 19 (-2)-curves $A_1,\dots,A_{19}$. The dual graph of these curves is (see [Kon89, Section 4])



The curves $A_1,\dots,A_{16}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{304} = (20,41,63,31,55,48,42,37,33,30,28,27,13,14,2,-9)$$

is ample, of square 304, with $D_{304}\cdot A_j = 1$ for $j \le 15$, $D_{304}\cdot A_{16} = D_{304}\cdot A_{19} = 20$ and $D_{304}\cdot A_{17} = D_{304}\cdot A_{18} = 2$. The divisor

$$D_2 = (1,2,3,1,3,3,3,3,3,3,3,3,1,2,1,0)$$

is nef, base-point free, of square 2, with $D_2\cdot A_j = 1$ for $j \in \{4,13,16,17,18,19\}$ and else $D_2\cdot A_j = 0$. One has

$$D_2 \equiv \left(-A_4 - A_{13} + \textstyle\sum_{j=1}^{15} A_j\right) + A_{16} + A_{17},$$
$$D_2 \equiv \left(-A_4 - A_{13} + \textstyle\sum_{j=1}^{15} A_j\right) + A_{18} + A_{19}.$$

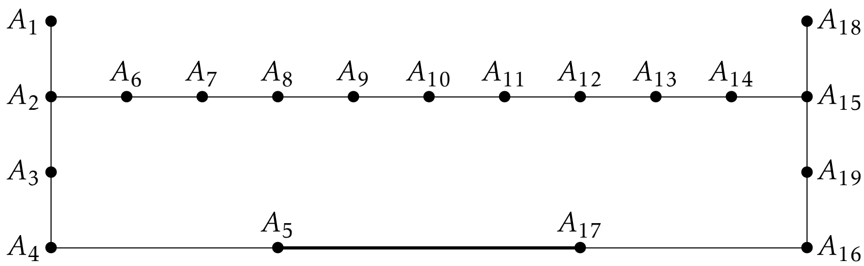
By using the linear system $|D_2|$, we obtain the following.

Proposition 15.3. *The K3 surface is the double cover of $\mathbb{P}^2$ branched over a sextic curve with an $\mathbf{a}_{13}$ singularity. The image of A_4 and A_{13} is a line, so are the images of A_{16} and A_{17} and of A_{18} and A_{19}.*

The automorphism group of the general K3 surface is $(\mathbb{Z}/2\mathbb{Z})^2$ (see [Kon89, Section 4]).

15.3. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{E}_7$

The K3 surface X contains 19 (-2)-curves $A_1,\dots,A_{19}$. The dual graph of these curves is (see [Kon89, Section 4])



The curves $A_1,\dots,A_{17}$ generate the Néron–Severi lattice; in that basis, the divisor

$$D_{538} = (32,66,46,27,9,55,45,36,28,21,15,10,6,3,1,0,1)$$

is ample, of square 538, with $D_{538}\cdot A_j = 1$ for $j \ne 1,4,17$ and $D_{538}\cdot A_j = 2,11,16$ for $j = 1,4,17$.

The automorphism group of the general K3 surface is $(\mathbb{Z}/2\mathbb{Z})^2$ (see [Kon89, Section 4]).

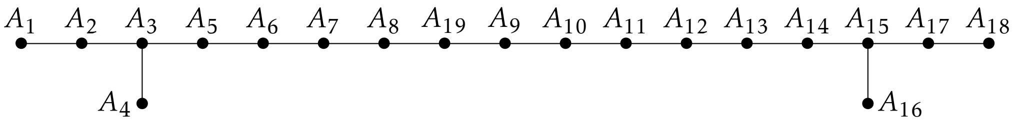
The coarse moduli space $\mathcal{M}^N$ of $N = U \oplus \mathbf{E}_8 \oplus \mathbf{E}_7$-polarized K3 surfaces is studied in [CD12], where it is proved that this moduli space can be seen naturally as the open set

$$\mathcal{M}^N = \{[\alpha,\beta,\gamma,\delta] \in \mathbb{WP}^3(2,3,5,6) \,|\, \gamma \ne 0 \text{ or } \delta \ne 0\}$$

of a weighted projective space; thus in particular that moduli is rational. In [CD12] is also given a (singular) model of the K3 surfaces in $\mathcal{M}^N$ as a quartic surface in $\mathbb{P}^3$. The K3 surfaces with lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{E}_7$ also belong among the “famous 95” families discussed in Section 2.6.

15.4. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8$

The set of (-2)-curves $A_1,\dots,A_{19}$ on the K3 surface X and their configuration had been determined in the classical work of Vinberg [Vin75] (see also [Kon89, Section 4]):



The curves $A_1,\dots,A_{18}$ generate the Néron–Severi lattice; in that base, the divisor

$$D_{620} = (-46,-91,-135,-68,-110,-84,-57,-29,30,61,93,126,160,195,231,115,153,76)$$

is ample, of square 620, with $D_{620}\cdot A_j = 1$ for $1 \le j \le 19$. The divisor

$$D_2 = (-5,-10,-15,-8,-12,-9,-6,-3,3,6,9,12,15,18,21,10,14,7)$$

is nef, of square 2, with $D_2\cdot A_4 = D_2\cdot A_{16} = 1$ and $D_2\cdot A_j = 0$ for $j \neq 4,16$. Using $|D_2|$, we see that the K3 surface X is the double cover of $\mathbb{P}^2$ branched over a sextic curve with an $\mathbf{a}_{17}$ singularity. We have

$$D_2 \equiv A_1 + 2A_2 + 3A_3 + A_4 + 3\left(\sum_{j=5}^{15} A_j\right) + A_{16} + 2A_{17} + A_{18} + 3A_{19};$$

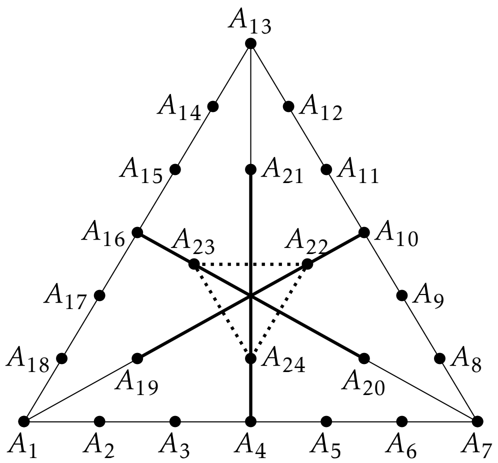
thus the image of A_4, A_{16} by the double cover map is a line through the singularity.

The automorphism group of the general K3 surface is $(\mathbb{Z}/2\mathbb{Z})^2$ (see [Kon89, Section 4]).

The rich geometry of $U\oplus\mathbf{E}_8\oplus\mathbf{E}_8$-polarized K3 surfaces is studied in [CD07]. These K3 surfaces also belong among the "famous 95" families discussed in Section 2.6.

15.5. The lattice $U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8 \oplus \mathbf{A}_1$

The K3 surface X contains 24 (-2)-curves $A_1,\dots,A_{24}$. The dual graph of these curves is (see [Kon89, Section 4])



Here the dotted thick segments indicate an intersection number equal to 6 between the curves. The automorphism group of the general K3 surface is $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$ (see [Kon89, Section 4]).

16. Table: Number of (−2)-curves

Lattice	#(-2)		Aut		Lattice	#(-2)		Aut	
Rank 3					**Rank** 3				
S_1	6	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$S_{1,3,1}$	3	**u**	$\{1\}$	
S_2	6	**u**	$\{1\}$		$S_{1,4,1}$	4	**u**	$\mathbb{Z}/2\mathbb{Z}$	n
S_3	4	**u**	$\{1\}$		$S_{1,5,1}$	6	**u**	$\mathbb{Z}/2\mathbb{Z}$	n
S_4	4		$\mathbb{Z}/2\mathbb{Z}$	n	$S_{1,6,1}$	4	**u**	$\{1\}$	
S_5	4	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$S_{1,9,1}$	9		$\{1\}$	
S_6	6	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$S_{4,1,1}$	3	**u**	$\{1\}$	
$S_{1,1,1}$	3	ℵ **u**	$\mathbb{Z}/2\mathbb{Z}$	t	$S_{5,1,1}$	4	**u**	$\mathbb{Z}/2\mathbb{Z}$	n
$S_{1,1,2}$	3	ℵ **u**	$\mathbb{Z}/2\mathbb{Z}$	t	$S_{6,1,1}$	4	**u**	$\{1\}$	
$S_{1,1,3}$	4	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$S_{7,1,1}$	6		$\{1\}$	
$S_{1,1,4}$	4	**u**	$\{1\}$		$S_{8,1,1}$	4	**u**	$\{1\}$	
$S_{1,1,6}$	6		$\mathbb{Z}/2\mathbb{Z}$	n	$S_{10,1,1}$	8		$\mathbb{Z}/2\mathbb{Z}$	n
$S_{1,1,8}$	8		$\{1\}$		$S_{12,1,1}$	6		$\{1\}$	
$S_{1,2,1}$	3	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$S'_{4,1,2}$	4		$\mathbb{Z}/2\mathbb{Z}$	n
Rank 4					**Rank** 5				
$L(24)$	6		$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_1^{\oplus 3}$	7		$\mathbb{Z}/2\mathbb{Z}$	t
$L(27)$	8		$\mathbb{Z}/2\mathbb{Z}$	n	$U(2)\oplus\mathbf{A}_1^{\oplus 3}$	10		$\mathbb{Z}/2\mathbb{Z}$	t
$[8]\oplus\mathbf{A}_1^{\oplus 3}$	12	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$U(4)\oplus\mathbf{A}_1^{\oplus 3}$	24		$\mathbb{Z}/2\mathbb{Z}$	n
$U\oplus\mathbf{A}_1^{\oplus 2}$	5	∠ **u**	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{A}_1\oplus\mathbf{A}_2$	6	∠★	$\mathbb{Z}/2\mathbb{Z}$	n
$U(2)\oplus\mathbf{A}_1^{\oplus 2}$	6	**u**	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{A}_3$	5	∠★	$\mathbb{Z}/2\mathbb{Z}$	n
$U(3)\oplus\mathbf{A}_1^{\oplus 2}$	8	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$[4]\oplus\mathbf{D}_4$	5		$\mathbb{Z}/2\mathbb{Z}$	n
$U(4)\oplus\mathbf{A}_1^{\oplus 2}$	8	**u**	$\{1\}$		$[8]\oplus\mathbf{D}_4$	7		$\mathbb{Z}/2\mathbb{Z}$	n
$U\oplus\mathbf{A}_2$	4	ℵ★ **u**	$\mathbb{Z}/2\mathbb{Z}$	n	$[16]\oplus\mathbf{D}_4$	8		$\mathbb{Z}/2\mathbb{Z}$	n
$U(2)\oplus\mathbf{A}_2$	4	**u**	$\mathbb{Z}/2\mathbb{Z}$	n	$[6]\oplus\mathbf{A}_2^{\oplus 2}$	10	**u**	$\mathbb{Z}/2\mathbb{Z}$	n
$U(3)\oplus\mathbf{A}_2$	4	∠ **u**	$\{1\}$						
$U(6)\oplus\mathbf{A}_2$	6	**u**	$\{1\}$						
$L_{12}=S_0\oplus\mathbf{A}_2$	6	**u**	$\mathbb{Z}/2\mathbb{Z}$	n					
$[4]\oplus[-4]\oplus\mathbf{A}_2$	6		$\mathbb{Z}/2\mathbb{Z}$	n					
$[4]\oplus\mathbf{A}_3$	5	**u**	$\mathbb{Z}/2\mathbb{Z}$	n					

Lattice	#(-2)		Aut		Lattice	#(-2)		Aut	
Rank 6					**Rank** 9				
$U(3)\oplus\mathbf{A}_2^{\oplus 2}$	12	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{E}_7$	9	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U(4)\oplus\mathbf{D}_4$	9		$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_6\oplus\mathbf{A}_1$	10		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_4$	6	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1^{\oplus 3}$	15		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_1\oplus\mathbf{A}_3$	7	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_1^{\oplus 7}$	37	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_2^{\oplus 2}$	7	$\angle\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U(2)\oplus\mathbf{A}_1^{\oplus 7}$	240	$\mathbf{u}$	$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U\oplus\mathbf{A}_1^{\oplus 2}\oplus\mathbf{A}_2$	8	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_7$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n
$U(2)\oplus\mathbf{A}_1^{\oplus 4}$	16	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_4\oplus\mathbf{A}_3$	10	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n
$U\oplus\mathbf{A}_1^{\oplus 4}$	9		$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_5\oplus\mathbf{A}_2$	10	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n
$U(2)\oplus\mathbf{D}_4$	6	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_7$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{D}_4$	6	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{A}_1\oplus\mathbf{E}_6$	10	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n
Rank 7					**Rank** 10				
$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1$	8		$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{E}_8$	10	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_1^{\oplus 5}$	12	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_8$	10		$\mathbb{Z}/2\mathbb{Z}$	t
$U(2)\oplus\mathbf{A}_1^{\oplus 5}$	27	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{E}_7\oplus\mathbf{A}_1$	11	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_1\oplus\mathbf{A}_2^{\oplus 2}$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_4^{\oplus 2}$	11	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_1^{\oplus 2}\oplus\mathbf{A}_3$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_6\oplus\mathbf{A}_1^{\oplus 2}$	14		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_2\oplus\mathbf{A}_3$	8	$\angle\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U(2)\oplus\mathbf{D}_4^{\oplus 2}$	18	$\aleph\,\mathbf{u}$	$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U\oplus\mathbf{A}_1\oplus\mathbf{A}_4$	8	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1^{\oplus 4}$	27	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_5$	7	$\angle\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_1^{\oplus 8}$	145		$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U\oplus\mathbf{D}_5$	7	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_2\oplus\mathbf{E}_6$	11	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n
Rank 8					**Rank** 11				
$U\oplus\mathbf{D}_6$	8		$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{E}_8\oplus\mathbf{A}_1$	12	$\aleph\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1^{\oplus 2}$	10		$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_8\oplus\mathbf{A}_1$	14		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_1^{\oplus 6}$	19	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_4^{\oplus 2}\oplus\mathbf{A}_1$	22		$\mathbb{Z}/2\mathbb{Z}$	t
$U(2)\oplus\mathbf{A}_1^{\oplus 6}$	56	$\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{D}_4\oplus\mathbf{A}_1^{\oplus 5}$	90		$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U\oplus\mathbf{A}_2^{\oplus 3}$	10	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n					
$U\oplus\mathbf{A}_3^{\oplus 2}$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n					
$U\oplus\mathbf{A}_2\oplus\mathbf{A}_4$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	**Rank** 12				
$U\oplus\mathbf{A}_1\oplus\mathbf{A}_5$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	t	$U\oplus\mathbf{E}_8\oplus\mathbf{A}_1^{\oplus 2}$	14		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_6$	8	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_8\oplus\mathbf{A}_1^{\oplus 2}$	19		$\mathbb{Z}/2\mathbb{Z}$	t
$U\oplus\mathbf{A}_2\oplus\mathbf{D}_4$	9	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{D}_4^{\oplus 2}\oplus\mathbf{A}_1^{\oplus 2}$	59		$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U\oplus\mathbf{A}_1\oplus\mathbf{D}_5$	9	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U\oplus\mathbf{A}_2\oplus\mathbf{E}_8$	13	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n
$U\oplus\mathbf{E}_6$	8	$\aleph\star\,\mathbf{u}$	$\mathbb{Z}/2\mathbb{Z}$	n					

Lattice	#(-2)		Aut		Lattice	#(-2)		Aut	
Rank 13					**Rank** ≥ 15				
$U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1^{\oplus 3}$	17	$\aleph$ **u**	$\mathbb{Z}/2\mathbb{Z}$	t	$U \oplus \mathbf{E}_8 \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$	21		$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U \oplus \mathbf{D}_8 \oplus \mathbf{A}_1^{\oplus 3}$	39	$\aleph$ **u**	$(\mathbb{Z}/2\mathbb{Z})^2$	tn	$U \oplus \mathbf{E}_8 \oplus \mathbf{D}_6$	19	$\aleph$ **u**	$(\mathbb{Z}/2\mathbb{Z})^2$	tn
$U \oplus \mathbf{E}_8 \oplus \mathbf{A}_3$	14	$\star$	$\mathbb{Z}/2\mathbb{Z}$	n	$U \oplus \mathbf{E}_8 \oplus \mathbf{E}_7$	19	$\aleph$ **u**	$(\mathbb{Z}/2\mathbb{Z})^2$	tn
					$U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8$	19	$\aleph$ **u**	$(\mathbb{Z}/2\mathbb{Z})^2$	tn
Rank 14					$U \oplus \mathbf{E}_8 \oplus \mathbf{E}_8 \oplus \mathbf{A}_1$	24	$\aleph$ **u**	$\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$	tn
$U \oplus \mathbf{E}_8 \oplus \mathbf{D}_4$	15	$\aleph$ **u**	$\mathbb{Z}/2\mathbb{Z}$	t					
$U \oplus \mathbf{D}_8 \oplus \mathbf{D}_4$	20	$\aleph$ **u**	$(\mathbb{Z}/2\mathbb{Z})^2$	tn					
$U \oplus \mathbf{E}_8 \oplus \mathbf{A}_1^{\oplus 4}$	27		$(\mathbb{Z}/2\mathbb{Z})^2$	t					

This table give the number of (-2)-curves. A $\aleph$ means that the lattice is among the 95 famous families; a $\angle$ means that the lattice is a mirror of one of the 95 famous ones, but not one of the 95 (see Section 2.6). A $\star$ means that their (-2)-curve configuration is predicted by Theorem 2.6. A **u** means that the unirationality of the moduli space is proved (the absence of **u** does not exclude the possibility of unirationality). The column Aut gives the automorphism group of the general K3 surface. A t means that the action on the Néron–Severi lattice of a hyperelliptic involution is trivial, an n means that the action of a hyperelliptic involution is not trivial, and a tn means that both cases exists. From that data, one can recover the kernel of the map $\mathrm{Aut}(X) \to O(\mathrm{NS}(X))$.

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