

Étale and crystalline companions, I

Kiran S. Kedlaya

Abstract. Let X be a smooth scheme over a finite field of characteristic p . Consider the coefficient objects of locally constant rank on X in ℓ -adic Weil cohomology: these are lisse Weil sheaves in étale cohomology when $\ell \neq p$, and overconvergent F -isocrystals in rigid cohomology when $\ell = p$. Using the Langlands correspondence for global function fields in both the étale and crystalline settings (work of Lafforgue and Abe, respectively), one sees that on a curve, any coefficient object in one category has “companions” in the other categories with matching characteristic polynomials of Frobenius at closed points. A similar statement is expected for general X ; building on work of Deligne, Drinfeld showed that any étale coefficient object has étale companions. We adapt Drinfeld’s method to show that any crystalline coefficient object has étale companions; this has been shown independently by Abe–Esnault. We also prove some auxiliary results relevant for the construction of crystalline companions of étale coefficient objects; this subject will be pursued in a subsequent paper.

Keywords. crystalline cohomology, overconvergent F -isocrystals, étale cohomology, lisse sheaves, companions, compatible systems

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1. Introduction

1.1. Coefficients, companions, and conjectures

Throughout this introduction (but not beyond; see §2.1), let k be a finite field of characteristic p and let X be a smooth scheme over k . When studying the cohomology of motives over X , one typically fixes a prime $\ell \neq p$ and considers étale cohomology with ℓ -adic coefficients; in this setting, the natural coefficient objects of locally constant rank are the *lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaves*. However, étale cohomology with p -adic coefficients behaves poorly in characteristic p ; for $\ell = p$, the correct choice for a Weil cohomology with ℓ -adic coefficients is Berthelot’s *rigid cohomology*, wherein the natural coefficient objects of locally constant rank are the *overconvergent F -isocrystals*. For the purposes of this introduction, the exact nature of these objects is not material; we suggest our recent survey [Ked22a] as a starting point for this topic.

For \mathcal{E} a coefficient object on X and x a closed point of X , let \mathcal{E}_x denote the fiber of \mathcal{E} at x . This is a finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector space carrying an invertible linear action by (geometric) Frobenius, whose reverse characteristic polynomial we denote by $P(\mathcal{E}_x, T)$.

For the remainder of this introduction, let ℓ, ℓ' be two primes (which may or may not equal each other and/or p), and fix an isomorphism of the algebraic closures of \mathbb{Q} within $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_{\ell'}$ (which may be nontrivial in case $\ell = \ell'$). We say that an ℓ -adic coefficient object \mathcal{E} and an ℓ' -adic coefficient object \mathcal{E}' are *companions* if for all x , the characteristic polynomials of Frobenius on \mathcal{E}_x and \mathcal{E}'_x coincide; given this relation, each object determines the other up to semisimplification (see Theorem 4.17).

In his work on the Weil conjectures, Deligne introduced a far-reaching set of conjectures [Del81, Conjecture 1.2.10] motivated by the idea that coefficient objects should only exist for “geometric reasons” (more on which below); we state here a lightly modified version of this conjecture. (Compare [Cad20, Conjecture 1.1].)

Conjecture 1.1 (after Deligne). *Let \mathcal{E} be an ℓ -adic coefficient object which is irreducible with determinant of finite order. (Remember that ℓ may or may not equal p .)*

- (i) \mathcal{E} is pure of weight 0: for every algebraic embedding of $\overline{\mathbb{Q}}_\ell$ into \mathbb{C} , for all x , the roots of $P(\mathcal{E}_x, T)$ in \mathbb{C} all have complex absolute value 1. (One can avoid having to embed $\overline{\mathbb{Q}}_\ell$ into \mathbb{C} by reading (ii) first.)
- (ii) For some number field E , \mathcal{E} is E -algebraic: for all x , we have $P(\mathcal{E}_x, T) \in E[T]$.
- (iii) \mathcal{E} is p -plain: for all x , the roots of $P(\mathcal{E}_x, T)$ have trivial λ -adic valuation at all finite places of λ of E not lying above p . (The terminology here follows D’Addezio [D’Ad20, Definition 2.0.2]; the terminology used by Chin [Chi04] is plain of characteristic p .)
- (iv) For every place λ of E above p , for all x , the roots of $P(\mathcal{E}_x, T)$ have λ -adic valuation at most $\frac{1}{2} \text{rank}(\mathcal{E})$ times the valuation of $\#\kappa(x)$ (the order of the residue field).
- (v) For every prime $\ell' \neq p$, there exists an ℓ' -adic coefficient object \mathcal{E}' which is irreducible with determinant of finite order and is a companion of \mathcal{E} .

(vi) As in (v), but with $\ell' = p$. (We separate this case to maintain parallelism with [Del81, Conjecture 1.2.10].)

By way of comparison with [Del81, Conjecture 1.2.10], we restrict to X smooth (not just normal) but do not require X to be connected; we allow $\ell = p$ in the initial data (which is relevant for our main result); and in (vi) we incorporate Crew’s proposal to interpret Deligne’s undefined phrase “petits camarades cristallins” to mean “overconvergent F -isocrystals” (a concept which did not exist at the time of [Del81]; see [Cre92, Conjecture 4.13] and [Abel8b, Conjecture D]).

The purpose of this paper is to consolidate and extend our knowledge towards this conjecture; this includes extensive work by other authors (attributed in this introduction) as well as original results as noted. To summarize, when $\dim(X) = 1$ everything is known; when $\dim(X) > 1$ everything except part (vi) is known; and part (vi) will be addressed in a subsequent paper [Ked22b].

1.2. Curves and the Langlands correspondence

A key step in the proof of Conjecture 1.1 is the construction of the ℓ -adic Langlands correspondence for the group GL_n over a global function field of positive characteristic. This was already anticipated by Deligne in [Del81]; at the time, the Langlands correspondence was available for $n = 2$, $\ell \neq p$ by the work of Drinfeld [Dri78]. Deligne observed that Drinfeld’s construction not only established the requisite bijection between ℓ -adic coefficient objects on a curve and automorphic representations on the associated function field, but also realized the latter as the realizations of certain motives appearing in moduli spaces of shtukas; this then provides a geometric origin for these coefficient objects.

Following Drinfeld’s approach, L. Lafforgue [Laf02] established the Langlands correspondence for GL_n for arbitrary n for $\ell \neq p$. The case $\ell = p$ had to be excluded at the time due to the limited foundational development of p -adic cohomology at the time; this issue has subsequently been remedied (see [Ked22a] for a summary), and this enabled Abe [Abel18a] to extend Lafforgue’s work to the case $\ell = p$. This immediately implies all of Conjecture 1.1 in the case where $\dim(X) = 1$ [Laf02, Théorème VII.6], [Abel18a, §4.4].

To sum up, we have in hand the following statement, which we will use as a black box in what follows; see §3.3 for the proof.

Theorem 1.2. *Conjecture 1.1 holds in all cases where $\dim(X) = 1$.*

1.3. Avoiding residual representations

In order to make further progress without losing the case $\ell = p$, one must replace several key arguments in étale cohomology that make direct use of the representation-theoretic interpretation of étale coefficient objects. These cannot be used for $\ell = p$ because only a rather small subcategory of coefficient objects can be described in terms of representations of the étale fundamental group (namely the *unit-root* objects; see §2.2). Such arguments can sometimes be circumvented using cohomological considerations, as in the following key examples.

- For $\ell \neq p$, the uniqueness of ℓ -adic companions up to semisimplification is usually obtained by applying the Chebotarëv density theorem to $\mathrm{mod}\text{-}\ell^n$ representations (as in the proof of [Laf02, Proposition VI.11]). For $\ell = p$, we use instead an argument of Tsuzuki based on Deligne’s theory of weights (see Theorem 4.17). This result is already used crucially in the work of Abe [Abel18a, Proposition A.4.1], and is akin to the argument of Deligne described below to control coefficient fields (see especially [Dell2, Proposition 2.5]).
- The extension of Conjecture 1.1 from curves to higher-dimensional varieties makes frequent use of a fact which we call the *Lefschetz slicing principle*: for any closed point $x \in X$ and any irreducible coefficient object \mathcal{E} on X , after possibly making a finite base extension of k (depending on x and \mathcal{E}), one can find a curve C in X containing x such that the restriction of \mathcal{E} to C is again irreducible. For $\ell \neq p$, this is shown by an analysis of the images of representations, as in [Dell2, §1.7] (correcting the

proof of [Laf02, Proposition VII.7]; see also [EK12, Appendix B]). Using cohomological arguments, we are able to make a fairly similar argument (Lemma 4.14); a related but distinct approach has been taken by Abe–Esnault (see below).

We mention briefly one further example for which cohomological considerations do not suffice. For $\ell \neq p$, one can eliminate all wild monodromy at the boundary using a single finite étale cover (by trivializing a residual representation); this plays an important role in the methods of Deligne and Drinfeld to attack aspects of Conjecture 1.1 (see below). The analogous statement for $\ell = p$ is the *semistable reduction theorem for overconvergent F -isocrystals* (see Proposition 2.37): for every p -adic coefficient object, one can pull back by a suitable alteration (in the sense of de Jong [deJ]) and then make a logarithmic extension to some good compactification. The proof of this cannot be considered a “cohomological argument” because in fact the foundations of rigid cohomology are deeply rooted in this fact; see [Ked22a, §11] or the introduction to [Abel8a].

1.4. Deligne’s conjecture for $\dim(X) > 1$

We now return to Conjecture 1.1. By analogy with a conjecture of Simpson that quasi-unipotent rigid local systems on smooth complex varieties should have a geometric/motivic origin (see [Sim91, Conjecture 4] for the case of a proper variety), it is generally expected that coefficient objects on a general X should all arise from motives over X ; for any given coefficient object, such a description would imply Conjecture 1.1 for this object. (This conjecture is attributed to Deligne by Esnault–Kerz [EK11, Conjecture 2.3]; see also [Dri12, Question 1.4], [KP21, Remark 1.5].)

Unfortunately, there seems to be no analogue of the Langlands correspondence which would provide geometric origins for coefficient objects on higher-dimensional varieties, except in the case of weight 1 where some results are known (as in [Kri22, KP21]), and some isolated cases which can be described completely in terms of weight-1 data and hence might be tractable by similar methods (*e.g.* weight-2 motives of K3 surfaces, via the Kuga–Satake construction). Instead, all known results towards Conjecture 1.1 in cases where $\dim(X) > 1$ proceed by using Theorem 1.2 as a black box, restricting to curves in X to get a system of interrelated coefficient objects (such a system is called⁽¹⁾ a *2-skeleton sheaf* by Esnault–Kerz).

- Statements (i), (iii), (iv) are pointwise conditions on \mathcal{E} , so they immediately promote to general X using the Lefschetz slicing principle.
- Statement (ii) was shown by Deligne [Del12] for $\ell \neq p$ using a numerical argument to give a uniform bound for the number field generated by the Frobenius traces over all curves in X . The argument transposes to $\ell = p$, using semistable reduction for p -adic coefficient objects (see above) to provide the requisite control of wild ramification.
- Statement (v) was shown by Drinfeld [Dri12] for $\ell \neq p$ using Deligne’s work on (ii) plus an argument to piece together companions on curves; the latter draws on the use of Hilbert irreducibility in the work of Wiesend [Wie06]. The argument transposes to $\ell = p$, again using semistable reduction for p -adic coefficient objects.

To sum up, we have in hand the following statement; see Corollary 4.29 for the proof.

Theorem 1.3. *Parts (i)–(v) of Conjecture 1.1 hold as written (whether or not $\ell = p$).*

At this point, we gratefully acknowledge the recent joint paper of Abe–Esnault [AE19], in which a result which is essentially equivalent to Theorem 1.3 is proved. That paper and this one constitute in some sense a virtual joint project, in that many of the ideas were discussed among the two sets of authors at an embryonic stage; we have opted to write the proofs up separately to illustrate different ways to assemble the main ingredients. See §4.7 for a direct comparison.

⁽¹⁾As per [Cad20, §4], this terminology is in fact due to Kindler.

We also wish to emphasize here that we are assuming that X is smooth, rather than normal. While parts (i)–(iv) are indifferent to this distinction (as discussed in detail in [Cad20]), parts (v) and (vi) are not; the examples of [Dri12, §6] show that one cannot directly apply the method of skeleton sheaves when X is not smooth, and so some additional ideas are needed.

1.5. Towards crystalline companions

Going forward, we are left to address part (vi) of Conjecture 1.1, or equivalently the following conjecture.

Conjecture 1.4. *Any ℓ -adic coefficient object on X which is E -algebraic for some number field E admits an ℓ' -adic companion.*

In light of Theorems 1.2 and 1.3, only the case $\ell' = p$, $\dim(X) > 1$ remains open; it is moreover harmless to also assume $\ell \neq p$. We resolve this conjecture in a sequel paper [Ked22b].

In the opposite direction, we point out that this paper itself has a “companion” in the form of the expository article [Ked22a]; that paper presents the basic properties of p -adic cohomology that will be used herein. Our general approach to citations in this paper is to refer to [Ked22a] rather than original sources; moreover, some notation and terminology from [Ked22a] will be used with limited explanation. For these reasons (among others), it is strongly recommended to read [Ked22a] before proceeding with this paper.

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2. Coefficient objects

We begin by resetting some basic notation and terminology from the introduction, then gathering some basic facts about coefficient objects in Weil cohomologies.

Notation 2.1. Throughout this paper, let k be a finite field of characteristic $p > 0$; let ℓ denote an arbitrary prime *different from* p ; let K denote the fraction field of the ring $W(k)$ of p -typical Witt vectors with coefficients in k ; and let X denote a smooth scheme of finite type over k . Let X° denote the set of closed points of X . For $x \in X^\circ$, let $\kappa(x)$ denote the residue field of x and let $q(x)$ denote the cardinality of $\kappa(x)$.

By a *curve* over k , we mean a scheme which is smooth of relative dimension 1 and geometrically irreducible, but not necessarily proper, over k . A *curve in X* is a locally closed subscheme of X which is a curve over k .

For n a positive integer, let k_n be the unique extension of k of degree n within a fixed algebraic closure of k and put $X_n := X \times_k k_n$.

2.1.  tale and crystalline companions

We first introduce the relevant Weil cohomology theories and their coefficient objects. For crystalline coefficients, see [Ked22a] for a more thorough background discussion.

Notation 2.2. For L an algebraic extension of \mathbb{Q}_ℓ , let $\mathbf{Weil}(X)$ (resp. $\mathbf{Weil}(X) \otimes L$) denote the category of lisse Weil \mathbb{Q}_ℓ -sheaves (resp. lisse Weil L -sheaves) on X . The concrete definition of this category is not immediately relevant, so we postpone recalling it until Definition 2.21.

Let $\mathbf{F-Isoc}^+(X)$ denote the category of overconvergent F -isocrystals over X . For L an algebraic extension of \mathbb{Q}_p , let $\mathbf{F-Isoc}^+(X) \otimes L$ denote the category of overconvergent F -isocrystals over X with coefficients in L , in the sense of [Ked22a, Definition 9.2].

By a *coefficient object* on X , we will mean an object of one of the categories $\mathbf{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$ or $\mathbf{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. We will use this definition frequently to make uniform statements; when it becomes necessary to separate these statements into their two constituents, we will distinguish these as, respectively, the *étale case* and the *crystalline case*. We also refer to ℓ (resp. p) as the *coefficient prime*, to \mathbb{Q}_ℓ (resp. \mathbb{Q}_p) as the *base coefficient field*, and to $\overline{\mathbb{Q}}_\ell$ (resp. $\overline{\mathbb{Q}}_p$) as the *full coefficient field*. Note that the definition of a coefficient object is functorial in X alone, without its structure morphism to $\mathrm{Spec}(k)$; this means we can generally assume that X is geometrically irreducible in what follows.

We say that a coefficient object on X is *constant* if it is the pullback of a coefficient object on $\mathrm{Spec}(k)$. This definition also does not depend on k . The operation of tensoring with a constant object of rank 1 will be referred to as a *constant twist*.

Lemma 2.3. *Let U be an open dense subscheme of X . Then the restriction functors from coefficient objects on X to coefficient objects on U are fully faithful and preserve irreducibility; moreover, they are equivalences whenever $X \setminus U$ has codimension at least 2 in each component of X .*

Proof. In the étale case, these assertions are straightforward consequences of the interpretation of lisse sheaves (when X is connected) in terms of the geometric étale fundamental group, plus Zariski–Nagata purity [Stacks, Tag 0BMB] in the case of the final assertion. In the crystalline case, more difficult arguments are required; see [Ked22a, Theorems 5.1, 5.3, 5.11] for attributions. \square

Lemma 2.4. *Assume that X is connected. For every coefficient object \mathcal{E} on X , there exists a constant twist of \mathcal{E} whose determinant is of finite order.*

Proof. This reduces easily to the case where \mathcal{E} is itself of rank 1. For this, see [Dell2, 1.2] in the étale case and [Ked22a, Lemma 9.14] in the crystalline case. \square

Definition 2.5. For \mathcal{E} a coefficient object on X , let $H^i(X, \mathcal{E})$ denote the cohomology groups of \mathcal{E} (without supports); these are finite-dimensional vector spaces over the full coefficient field (see [Ked22a, Theorem 8.4] in the crystalline case).

Definition 2.6. For \mathcal{E} a coefficient object on X and $x \in X^\circ$, let \mathcal{E}_x be the pullback of \mathcal{E} to x ; it admits a linear action of the $q(x)$ -power geometric Frobenius, whose reverse characteristic polynomial we denote by $P(\mathcal{E}_x, T)$. (See [Ked22a, Definition 9.5] for discussion of the crystalline case.)

We say that \mathcal{E} is *algebraic* if for all $x \in X^\circ$, $P(\mathcal{E}_x, T)$ has coefficients in the field of algebraic numbers. We say that \mathcal{E} is *uniformly algebraic* if these coefficients all belong to a single number field, or *E -algebraic* if this field can be taken to be E . For example, if \mathcal{E} is of rank 1 and of finite order, then the roots of the polynomials $P(\mathcal{E}_x, T)$ are all roots of unity of bounded order, so \mathcal{E} is uniformly algebraic.

Suppose that \mathcal{E} is algebraic and fix a finite place λ of the algebraic closure of \mathbb{Q} in the full coefficient field. For $x \in X^\circ$, we define the *normalized Newton polygon* of \mathcal{E} at x with respect to λ , denoted $N_x(\mathcal{E}, \lambda)$, to be the boundary of the lower convex hull of the set of points

$$\left\{ \left(i, \frac{1}{[\kappa(x):k]} v_\lambda(a_i) \right) \mid i = 0, \dots, d \right\} \subset \mathbb{R}^2.$$

Definition 2.7. Let \mathcal{E} and \mathcal{F} be two coefficient objects, possibly in different categories (and possibly for different values of ℓ), and fix an identification ι of the algebraic closures of \mathbb{Q} within the respective full coefficient fields. We say that \mathcal{E} and \mathcal{F} are *companions* (with respect to ι) if for each $x \in X^\circ$, the polynomials $P(\mathcal{E}_x, T)$ and $P(\mathcal{F}_x, T)$ are equal to the same element of $\overline{\mathbb{Q}}[T]$; in particular, this implies that \mathcal{E} and \mathcal{F} are both algebraic on each connected component of X . We will see later that within their respective categories, \mathcal{E} and \mathcal{F} uniquely determine each other up to semisimplification (Theorem 4.17). In case \mathcal{F} has coefficient prime ℓ (resp. p), we also call it an *étale companion* (resp. a *crystalline companion*) of \mathcal{E} (again with respect to ι).

Definition 2.8. For \mathcal{E} a coefficient object with coefficient prime $*$, let

$$L(\mathcal{E}, T) := \prod_{x \in X^\circ} P(\mathcal{E}_x, T^{[\kappa(x):k]})^{-1} \in \overline{\mathbb{Q}}_*[[T]]$$

be its associated L -function. The Lefschetz trace formula for Frobenius (see [Ked22a, Theorem 9.6] in the crystalline case) asserts that

$$(2.1) \quad L(\mathcal{E}, T) = \prod_{i=0}^{2d} \det(1 - (\#k)^{-d} F^{-1} T, H^i(X, \mathcal{E}^\vee))^{(-1)^{i+1}}.$$

(Note the dualization here, caused by working with cohomology without supports.) Consequently, the Euler characteristic

$$\chi(\mathcal{E}) := \sum_{i=0}^{2 \dim(X)} (-1)^i \dim_{\overline{\mathbb{Q}}} H^i(X, \mathcal{E})$$

of \mathcal{E} equals the order of vanishing of $L(\mathcal{E}^\vee, T)$ at $T = \infty$. In particular, any two companions have the same L -function and hence the same Euler characteristic.

2.2. Slopes of isocrystals

We next review some crucial properties of Newton polygons in the crystalline case. For this, we must distinguish between convergent and overconvergent F -isocrystals; again, see [Ked22a] for a more detailed discussion of this fundamental dichotomy.

Definition 2.9. For L an algebraic extension of \mathbb{Q}_p , let $\mathbf{F}\text{-Isoc}(X)$ (resp. $\mathbf{F}\text{-Isoc}(X) \otimes L$) denote the category of convergent F -isocrystals on X (resp. convergent F -isocrystals on X with coefficients in L). There is a restriction functor $\mathbf{F}\text{-Isoc}^\dagger(X) \otimes L \rightarrow \mathbf{F}\text{-Isoc}(X) \otimes L$ which is an equivalence of categories when X is proper (so in particular when $X = \text{Spec}(k)$), and fully faithful in general (see below). We may thus extend definitions pertaining to convergent F -isocrystals, such as Newton polygons, to overconvergent F -isocrystals via restriction.

Lemma 2.10. *The restriction functor $\mathbf{F}\text{-Isoc}^\dagger(X) \otimes L \rightarrow \mathbf{F}\text{-Isoc}(X) \otimes L$ is fully faithful. Moreover, for U an open dense subscheme of X , the restriction functor $\mathbf{F}\text{-Isoc}(X) \otimes L \rightarrow \mathbf{F}\text{-Isoc}(U) \otimes L$ is fully faithful.*

Proof. See [Ked22a, Theorem 5.3]. □

Definition 2.11. For $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X)$, we define the *Newton polygon* $N_x(\mathcal{E})$ for $x \in X$ (not necessarily a closed point) using the Dieudonné–Manin classification theorem, as in [Ked22a, §3]. We extend the definition to $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$ by renormalization. By convention, $N_x(\mathcal{E})$ is convex with left endpoint at $(0, 0)$ and right endpoint at $(\text{rank}_{\overline{\mathbb{Q}}_p}(\mathcal{E}), *)$; its vertices belong to $\mathbb{Z} \times [L : \mathbb{Q}_p]^{-1} \mathbb{Z}$ where L is a finite extension of \mathbb{Q}_p for which $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X) \otimes L$. We say \mathcal{E} is *unit-root* (or *étale*) if for all $x \in X$, $N_x(\mathcal{E})$ has all slopes equal to 0.

Lemma 2.12. *For $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$ and $x \in X^\circ$, write $P(\mathcal{E}_x, T) = \sum_{i=0}^d a_i T^i \in \overline{\mathbb{Q}}_p[[T]]$ with $a_0 = 1$. Then $N_x(\mathcal{E})$ equals the boundary of the lower convex hull of the set of points*

$$\left\{ \left(i, \frac{1}{[\kappa(x):k]} v_p(a_i) \right) \mid i = 0, \dots, d \right\} \subset \mathbb{R}^2.$$

In particular, if $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X)^\dagger \otimes \overline{\mathbb{Q}}_p$, then $N_x(\mathcal{E}) = N_x(\mathcal{E}, \lambda)$ where λ is the place defined by the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Proof. We may assume at once that $X = \{x\}$. Using the Dieudonné–Manin decomposition, we may further reduce to the case where \mathcal{E} is unit-root; we may then also assume that $L = \mathbb{Q}_p$. In this case, an object of $\mathbf{F}\text{-Isoc}(X)$ is a finite-dimensional K -vector space V equipped with an isomorphism $F : \sigma^* V \rightarrow V$, and

the unit-root condition corresponds to the existence of a $W(k)$ -lattice T in V such that F induces an isomorphism $\sigma^*T \cong T$. Using such a lattice to compute $P(\mathcal{E}_x, T)$, we see that the latter has Newton polygon with all slopes equal to 0; this proves the claim. \square

Proposition 2.13. *Assume that X is connected and fix a geometric point \bar{x} of X . For each finite extension L of \mathbb{Q}_p , the following statements hold.*

- (a) *The category of unit-root objects in $\mathbf{F}\text{-Isoc}(X) \otimes L$ is equivalent to the category of continuous representations of $\pi_1(X, \bar{x})$ on finite-dimensional L -vector spaces.*
- (b) *Under the equivalence in (a), the unit-root objects in $\mathbf{F}\text{-Isoc}^\dagger(X) \otimes L$ correspond to representations which are potentially unramified. This condition means that after passing from X to some connected finite étale cover X' through which $\bar{x} \rightarrow X$ factors, the restriction from $\pi_1(X', \bar{x})$ to the inertia group of any divisorial valuation is the trivial representation.*

Proof. See [Ked22a, Theorem 9.4]. \square

Corollary 2.14. *Every coefficient object on X which is algebraic of rank 1 admits companions in all categories.*

Proof. We are free to check the claim after a constant twist, so by Lemma 2.4 we may ensure that \mathcal{E} is of finite order. Fix a geometric point \bar{x} of X . By definition (in the étale case) or Proposition 2.13 (in the crystalline case), \mathcal{E} corresponds to a continuous homomorphism $\rho : \pi_1(X, \bar{x}) \rightarrow \mu_\infty$ for the discrete topology on the target; we may then reverse the logic in any other category. \square

Remark 2.15. Proposition 2.13 also has closely related integral and truncated versions. For instance, the category of finite projective \mathcal{O}_X -modules equipped with isomorphisms with their φ_k -pullbacks is isomorphic to the category of continuous representations of $\pi_1(X)$ on finite-dimensional k -vector spaces [Kat73, Proposition 1.1].

Proposition 2.16. *For $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$, the function $x \mapsto N_x(\mathcal{E})$ is upper semicontinuous and the right endpoint of $N_x(\mathcal{E})$ is locally constant on X . In particular, if X is a curve over k with generic point η , then there are only finitely many $x \in X^\circ$ for which $N_x(\mathcal{E}) \neq N_\eta(\mathcal{E})$.*

Proof. This reduces immediately to the case $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X)$, for which see [Ked22a, Theorem 3.12]. \square

Proposition 2.17. *Choose $\mathcal{E} \in \mathbf{F}\text{-Isoc}(X) \otimes L$ for some finite extension L of \mathbb{Q}_p .*

- (a) *Suppose the point $(m, n) \in \mathbb{Z} \times \mathbb{Q}$ is a vertex of $N_x(\mathcal{E})$ for each $x \in X$. Then there exists a short exact sequence*

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

in $\mathbf{F}\text{-Isoc}(X) \otimes L$ with $\text{rank } \mathcal{E}_1 = m$ such that for each $x \in X$, $N_x(\mathcal{E}_1)$ is the portion of $N_x(\mathcal{E})$ from $(0, 0)$ to (m, n) .

- (b) *Suppose that the function $x \mapsto N_x(\mathcal{E})$ is constant. Then there exists a unique filtration*

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

in $\mathbf{F}\text{-Isoc}(X) \otimes L$ with the property that for some sequence $s_1 < \cdots < s_l$, for each i the quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ has constant Newton polygon with all slopes equal to s_i . This filtration is called the slope filtration of \mathcal{E} .

Proof. See [Ked22a, Theorem 4.1, Corollary 4.2]. \square

Proposition 2.18 (Drinfeld–Kedlaya, Kramer–Miller). *For X irreducible with generic point η and $\mathcal{E} \in \mathbf{F}\text{-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$ indecomposable, no two consecutive slopes of $N_\eta(\mathcal{E})$ differ by more than 1.*

Proof. By Lemma 2.10 this reduces to the corresponding assertion in $\mathbf{F}\text{-Isoc}(X) \otimes \overline{\mathbb{Q}}_p$, for which see [DK17, Theorem 1.1.5]. An alternate proof has been communicated to the author by Kramer–Miller; it will appear elsewhere. \square

2.3. Monodromy groups

We next introduce the formalism of monodromy groups, following [Ked22a, §9] and [D'Ad20, §3.2] in the crystalline case. (The original reference in the crystalline case is [Cre92, §5], but it has somewhat restrictive hypotheses which are not suitable for our present discussion; see [Ked22a, Remark 9.9].)

Hypothesis 2.19. Throughout §2.3, assume that X is connected, and fix a closed point $x \in X^\circ$ and a geometric point $\bar{x} = \text{Spec}(\bar{k})$ of X lying over x . In the crystalline case, also choose an embedding $W(\bar{k}) \hookrightarrow \overline{\mathbb{Q}}_p$.

Definition 2.20. On any category of coefficient objects, let ω_x be the neutral (over the full coefficient field) fiber functor defined by pullback from X to x . Note that in this construction, we implicitly discard the Frobenius structure: coefficient objects on x correspond to vector spaces equipped with a fixed automorphism (see Definition 2.6), and ω_x does not retain the data of the automorphism. (In the crystalline case, this definition depends on the choice of the embedding $W(\bar{k}) \hookrightarrow \overline{\mathbb{Q}}_p$; see [Ked22a, Definition 9.5].)

For \mathcal{E} a coefficient object on X , let $G(\mathcal{E})$ denote the automorphism group of ω_x on the Tannakian category generated by \mathcal{E} . The group $G(\mathcal{E})$ is the *arithmetic monodromy group* associated to \mathcal{E} . This group turns out not to be especially useful for our purposes; compare with Definition 2.22.

In the étale case, the construction of Definition 2.20 has a rather concrete interpretation which we recall here.

Definition 2.21. Suppose that X is geometrically irreducible (by enlarging k if necessary), and consider the exact sequence

$$1 \rightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow \pi_1^{\text{ét}}(X, \bar{x}) \rightarrow G_k \rightarrow 1.$$

The group G_k is isomorphic to $\widehat{\mathbb{Z}}$; let us normalize this isomorphism so that geometric Frobenius corresponds to 1. We may then pull back from $\widehat{\mathbb{Z}}$ to \mathbb{Z} to obtain an exact sequence

$$1 \rightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \rightarrow W_X \rightarrow \mathbb{Z} \rightarrow 1;$$

the group W_X is the *Weil group* associated to X , and for any algebraic extension L of \mathbb{Q}_ℓ , the category $\text{Weil}(X) \otimes L$ may be identified with (or even defined as) the category of continuous representations of W_X on finite-dimensional L -vector spaces. (To use this as the definition when L is not finite over \mathbb{Q}_ℓ , one must check that the definition commutes with colimits; for this, see [KS99, Remark 9.0.7].) For any object \mathcal{E} in this category, $G(\mathcal{E})$ may then be naturally identified with the Zariski closure of the image of W_X in the associated representation.

Definition 2.22. Retain notation as in Definition 2.20. In the étale case, we define the *geometric monodromy group* $\overline{G}(\mathcal{E})$ to be the Zariski closure of the image of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ in the associated representation. Equivalently, define the category of *geometric coefficient objects* by forming the category of continuous representations of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ on finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces, then extracting the Tannakian subcategory generated by coefficient objects. We may then equivalently define $\overline{G}(\mathcal{E})$ as the automorphism group of ω_x on the Tannakian subcategory of geometric coefficient objects generated by \mathcal{E} .

In the crystalline case, we define the category of *geometric coefficient objects* by forming the category of overconvergent isocrystals (without Frobenius structure) with coefficients in $\overline{\mathbb{Q}}_p$ (by a suitable 2-colimit over finite extensions of \mathbb{Q}_p , as described in [Ked22a, Definition 9.2] in the presence of Frobenius structures) and then extracting the Tannakian subcategory generated by coefficient objects. We then define $\overline{G}(\mathcal{E})$ again as the automorphism group of ω_x on the Tannakian subcategory of geometric coefficient objects generated by \mathcal{E} . We may then turn around and define the *Weil group* of \mathcal{E} as the semidirect product of $\overline{G}(\mathcal{E})$ by \mathbb{Z} via the action of Frobenius.

Remark 2.23. The definition of monodromy groups given here is essentially [D'Ad20, Definition 3.2.4]; in the crystalline case, see also [Ked22a, Definition 9.8] and subsequent remarks. In particular, as pointed out in [D'Ad20, Remark 3.2.5], the definition of $\overline{G}(\mathcal{E})$ agrees with Crew's definition of the differential Galois group [Cre92, §2] when X is a curve and $x \in X(k)$.

Remark 2.24. For any open dense subset U of X containing x , the inclusion $\overline{G}(\mathcal{E}|_U) \rightarrow \overline{G}(\mathcal{E})$ is an isomorphism. This is straightforward in the étale case; in the crystalline case, see [Ked22a, Remark 9.10]. (See also Lemma 4.38 below.)

Remark 2.25. By definition, any coefficient object \mathcal{E} on X belongs to $\mathbf{Weil}(X) \otimes L$ or $\mathbf{F-Isoc}^+(X) \otimes L$ for some finite extension L of the base coefficient field. Choose such an L , then suppose that $\overline{G}(\mathcal{E})$ is finite. In the étale case, it is obvious that \mathcal{E} gives rise to a discrete L -representation of $\pi_1(X_{\overline{k}}, \overline{x})$. It is a key observation of Crew that this also holds in the crystalline case; namely, the underlying isocrystal of \mathcal{E} admits a canonical unit-root Frobenius structure (see [Ked22a, Corollary 9.17]) and then Proposition 2.13 applies (see [Ked22a, Lemma 9.18]).

Remark 2.26. As a special case of Remark 2.25, note that a coefficient object \mathcal{E} on X is constant if and only if $\overline{G}(\mathcal{E})$ is trivial. In particular, this condition passes to subobjects.

Remark 2.27. As the terminology suggests, the definition of the geometric monodromy group is invariant under base extension on k . However, it is not the case that the geometric monodromy group coincides with the arithmetic monodromy group after a sufficiently large base extension. For example, if \mathcal{E} is constant of rank 1 but not of finite order, then $G(\mathcal{E}_{k'}) = \mathbb{G}_m$ for all finite extensions k' of k , whereas $\overline{G}(\mathcal{E})$ is trivial.

One consequence of this is that it is not immediately obvious whether a coefficient object which is *absolutely irreducible* (by which we mean it remains irreducible after any finite extension of the base field) is *geometrically irreducible* (by which we mean it is irreducible as a geometric coefficient object, or equivalently that $\overline{G}(\mathcal{E})$ acts irreducibly on its tautological representation). This does turn out to be true; moreover, for any irreducible coefficient object \mathcal{E} on X , there exists a positive integer n such that on X_n , \mathcal{E} splits as a direct sum of geometrically irreducible subobjects $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ over X_n which are cyclically permuted by the action of the k -Frobenius. See [Ked22a, Remark 9.13] and references therein.

In light of Remark 2.27, we may refine Lemma 2.4 using the following statement.

Lemma 2.28. *Suppose that \mathcal{E} is irreducible and $\det(\mathcal{E})$ is of finite order. Then for any positive integer n , every irreducible constituent of the pullback of \mathcal{E} to X_n has determinant of finite order.*

Proof. By Remark 2.27, \mathcal{E} splits as a direct sum of irreducible subobjects $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$ over X_n which are cyclically permuted by the action of the k -Frobenius. By Lemma 2.4, we can choose a coefficient object \mathcal{L}' on k_n of rank 1 such that $\mathcal{E}_1 \otimes \mathcal{L}'$ has determinant of finite order. By extracting a suitable n -th root, we may realize \mathcal{L} (not uniquely) as the pullback of a coefficient object \mathcal{L} on k of rank 1. The products $\mathcal{E}_1 \otimes \mathcal{L}, \dots, \mathcal{E}_n \otimes \mathcal{L}$ are then cyclically permuted by the action of the k -Frobenius; consequently, since the first one has finite order, so do the others. In particular, $\det(\mathcal{E} \otimes \mathcal{L})$ is of finite order, as then is \mathcal{L} , as then are $\det(\mathcal{E}_1), \dots, \det(\mathcal{E}_n)$. \square

Proposition 2.29. *For any coefficient object \mathcal{E} on X , there is a natural surjective continuous homomorphism $\psi_{\mathcal{E}} : \pi_1^{\text{ét}}(X, \overline{x}) \rightarrow \pi_0(G(\mathcal{E}))$, which induces a surjective continuous homomorphism $\overline{\psi}_{\mathcal{E}} : \pi_1^{\text{ét}}(X_{\overline{k}}, \overline{x}) \rightarrow \pi_0(\overline{G}(\mathcal{E}))$.*

Proof. In the étale case, this is apparent from Definition 2.21. For the crystalline case, see [D'Ad20, Proposition 3.3.4]. \square

The following is a highly nontrivial result of Grothendieck in the étale case and Crew in the crystalline case. See [D'Ad20] for numerous consequences of this statement.

Proposition 2.30. *For any coefficient object \mathcal{E} on X , the radical of $\overline{G}(\mathcal{E})^\circ$ is unipotent. Consequently, if \mathcal{E} is semisimple, then $\overline{G}(\mathcal{E})^\circ$ is semisimple.*

Proof. See [D’Ad20, Theorem 3.4.5] for the first statement (or [Ked22a, Theorem 9.19] for the crystalline case). The passage from the first statement to the second is purely group-theoretic; see [Cre92, Corollary 4.10]. \square

Remark 2.31. One can also consider geometric monodromy groups of convergent F -isocrystals; in fact, the relationship between the geometric monodromy group of an overconvergent F -isocrystal and that of its underlying convergent F -isocrystal provides much useful information. However, we will not make explicit use of this here. See [D’Ad20] for a modern treatment.

2.4. Tame and docile coefficients

In much of the study of companions, a crucial role is played by coefficient objects for which the local monodromy at the boundary is tamely ramified and unipotent.

Definition 2.32. Let $X \hookrightarrow \overline{X}$ be an open immersion with dense image. Let D be an irreducible divisor of \overline{X} with generic point η . Let \mathcal{E} be a coefficient object on X . In the étale case, we say that \mathcal{E} is *tame* (resp. *docile*) along D if the action of the inertia group of η on \mathcal{E} (arising from Definition 2.21) is tamely ramified (resp. tamely ramified and unipotent).

In the crystalline case, we say that \mathcal{E} is *tame* (resp. *docile*) along D if \mathcal{E} has \mathbb{Q} -unipotent monodromy in the sense of [Shi10, Definition 1.3] (resp. unipotent monodromy in the sense of [Ked07, Definition 4.4.2]) along D .

We say that \mathcal{E} is *tame* (resp. *docile*) if it is so with respect to every choice of \overline{X} and D . By definition, this property is stable under arbitrary pullbacks.

Remark 2.33. The category of tame (resp. docile) coefficient objects on X is stable under subquotients and extensions. This is clear in the étale case; in the crystalline case, see [Ked07, Proposition 3.2.20, Proposition 6.5.1].

To make this definition more useful in practice, we need the concept of a *good compactification*.

Definition 2.34. A *smooth pair* over k is a pair (Y, Z) in which Y is a smooth k -scheme and Z is a strict normal crossings divisor on Y ; we refer to Z as the *boundary* of the pair. (Note that $Z = \emptyset$ is allowed.) A *good compactification* of X is a smooth pair (\overline{X}, Z) over k with \overline{X} projective (not just proper) over k , together with an isomorphism $X \cong \overline{X} \setminus Z$; we will generally treat the latter as an identification.

If \overline{X} is a good compactification of X with boundary Z , then a coefficient object on X is tame (resp. docile) if and only if it is so with respect to each component of Z . Namely, this follows from Zariski–Nagata purity in the étale case, and from [Ked07, Theorem 6.4.5] in the crystalline case.

For general X , the existence of a good compactification is (to the best of our understanding) not known; it would follow from resolution of singularities in characteristic p . As a workaround, we use de Jong’s theorem on alterations.

Definition 2.35. An *alteration* of X is a morphism $f : X' \rightarrow X$ which is proper, surjective, and generically finite étale. This corresponds to a *separable alteration* in the sense of de Jong [deJ].

Proposition 2.36 (de Jong). *There exists an alteration $f : X' \rightarrow X$ such that X' admits a good compactification. (Beware that X' is not guaranteed to be geometrically irreducible over k .)*

Proof. Keeping in mind that k is perfect, see [deJ, Theorem 4.1]. \square

We will frequently make use of the fact that the tame and docile conditions can always be achieved after pullback along a suitable alteration.

Proposition 2.37. *For any coefficient object \mathcal{E} on X , there exists an alteration $f : X' \rightarrow X$ such that X' admits a good compactification and $f^*\mathcal{E}$ is docile. (As in Proposition 2.36, we cannot guarantee that X' is geometrically irreducible over k .)*

Proof. We first treat the étale case. We may assume that X is connected and choose a geometric basepoint \bar{x} . By Definition 2.21, \mathcal{E} corresponds to a representation of W_X on some finite-dimensional L -vector space V . The action of $\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$ on V has compact image in $\text{GL}(V)$, so we may choose a lattice T stable under this action. By taking f to trivialize the mod- ℓ action on T (using Proposition 2.36), we enforce that the actions of inertia are all tame. They are also quasi-unipotent by the usual argument of Grothendieck: the eigenvalues of Frobenius form a multiset of length at most $r := \text{rank}(\mathcal{E})$ which is stable under taking p -th powers, so this multiset must consist entirely of roots of unity. To upgrade from quasi-unipotency to unipotency, it suffices to further trivialize the action on T modulo some power of ℓ , which may be bounded either in terms of p and r (by the previous consideration) or k (because k only contains finitely many ℓ -power roots of unity).

In the crystalline case, we instead apply the semistable reduction theorem for overconvergent F -isocrystals [Ked22a, Theorem 7.6]. \square

Remark 2.38. In the étale case, the proof of Proposition 2.37 also shows that there exists a finite étale cover $f_0 : X'_0 \rightarrow X$ such that $f_0^*\mathcal{E}$ is docile (but we cannot then guarantee the existence of a good compactification without a further alteration). After establishing the existence of étale companions, this will also hold in the crystalline case; see Remark 4.30.

Remark 2.39. In the crystalline case, our notion of a tame coefficient object matches that of Abe–Esnault [AE19, §1.2]. Note that in either the étale or crystalline case, a coefficient object on X is tame if and only if it becomes docile after pullback along some finite étale cover of X which is tamely ramified at the boundary.

Lemma 2.40. *A coefficient object on X is tame (resp. docile) if and only if its restriction to every curve in X is tame (resp. docile).*

Proof. In the étale case, this is a straightforward consequence of Zariski–Nagata purity. In the crystalline case, a result of Shiho [Ked22a, Theorem 7.4] implies the docile version of the desired result; the tame version then follows using Remark 2.39. \square

3. Curves

We now introduce the use of the Langlands correspondence to prove Theorem 1.2, thus resolving Conjecture 1.1 when $\dim(X) = 1$; this amounts to a recitation of the works of L. Lafforgue and Abe. We then make some additional calculations on curves in preparation for the study of higher-dimensional varieties.

Hypothesis 3.1. Throughout §3, assume that X is a curve over k . Let \bar{X} be the unique smooth compactification of X and put $Z := \bar{X} \setminus X$.

3.1. The Langlands correspondence for curves

We begin this discussion by recalling the statement of the Langlands correspondence for curves. For compatibility with the literature in the étale case, we must introduce an additional definition.

Definition 3.2. Suppose that X is connected (but not necessarily a curve, for this definition only) and let \bar{x} be a geometric point of X . For L a finite extension of \mathbb{Q}_ℓ , a *lisse étale L -sheaf* is a continuous representation of $\pi_1^{\text{ét}}(X, \bar{x})$ on a finite-dimensional L -vector space. By taking the 2-colimit over L , we obtain the category of *lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaves*. (We may then extend this definition to disconnected X in an obvious fashion.)

Note that a continuous representation of W_X extends to $\pi_1^{\text{ét}}(X, \bar{x})$ in at most one way (because Z is dense in G_k), and does so at all if and only if its image has compact closure; from this observation, it follows

that the lisse étale L -sheaves constitute a subcategory of $\text{Weil}(X) \otimes L$ which is closed under extensions and subquotients. By a similar group-theoretic argument [Del81, Proposition 1.3.14], an irreducible lisse Weil L -sheaf is a lisse étale sheaf if and only if its determinant is a lisse étale sheaf; in particular, this is the case when the determinant is of finite order. See also Lemma 4.27.

Remark 3.3. In the statement of Theorem 3.4, we refer to *local Euler factors* and *local ϵ -factors* associated to coefficient objects; these are associated to a coefficient object \mathcal{E} on X and a closed point $x \in \overline{X}$. The local Euler factor is defined by pushing coefficient objects into a larger (derived) category of constructible objects and then pulling back to x ; see [Abel18a, §A.3] for a detailed description. The construction of local ϵ -factors is due to Laumon [Lau87] in the étale case (extending work of Langlands and Deligne) and Abe–Marmora [AM15] in the crystalline case.

We will also need a more explicit description of the local Euler factor at x in the case where \mathcal{E} is a crystalline coefficient object which is docile at x : it is the reverse characteristic polynomial of F_x on the kernel of the residue map at x . See for example [Car06, Théorème 3.4.1].

Theorem 3.4 (L. Lafforgue, Abe). *Fix an embedding of $\overline{\mathbb{Q}}$ into the full coefficient field. Then for each positive integer r , there is a bijection between irreducible coefficient objects of rank r on X with determinant of finite order, and cuspidal automorphic representations of $\text{GL}_r(\mathbb{A}_{k(X)})$ with central character of finite order which are unramified in X . Moreover, this bijection may be chosen so that Frobenius eigenvalues at points of X correspond to Hecke eigenvalues at unramified places, while local Euler factors and ϵ -factors at points of Z correspond to the analogous quantities at ramified places.*

Proof. See [Laf02] for the étale case and [Abel18a, Theorem 4.2.2] for the crystalline case. \square

Remark 3.5. As noted earlier, Theorem 3.4 involves a p -adic replication of Lafforgue’s geometric realization of the Langlands correspondence for GL_r in the cohomology of moduli spaces of shtukas. Consequently, it requires not just the rigid cohomology of smooth varieties with coefficients in overconvergent F -isocrystals, but a much more detailed cohomological formalism: constructible sheaves, complexes, nonsmooth varieties, and even algebraic stacks. However, since we take Theorem 3.4 as a black box, we will not encounter any of these subtleties; the reader interested in them may start with the discussion at the end of [Ked22a].

Corollary 3.6. *Suppose that X is proper. For any positive integer r and any category of coefficient objects on X , there are only finitely many isomorphism classes of irreducible objects of rank r up to twisting by constant objects of rank 1.*

Proof. By Lemma 2.4, the problem does not change if we consider only objects with determinant of finite order. For such objects, we may apply Theorem 3.4 to equate the problem to a corresponding finiteness statement for automorphic representations, for which an argument of Harder applies [Har74, Theorem 1.2.1] (compare the discussion in [Yu22, II.2.1]). \square

Remark 3.7. In light of Corollary 3.6 (and Corollary 3.18 below), there has been much interest in the question of counting lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaves on curves with various prescribed properties, starting with a celebrated note of Drinfeld [Dri81]. Subsequent work in this direction can be found in [Dell15, DF13, Fli15, Fli17, EK12, Kon09, Yu22]. Most previous progress has been made using automorphic trace formulas; the existence of crystalline companions may provide an alternate approach through the geometry of moduli spaces of vector bundles. For a concrete problem in this direction, we suggest to recover the result of [Dri81] by counting F -isocrystals.

3.2. Companions of algebraic coefficient objects

We next address algebraicity and the existence of companions, corresponding to parts (ii), (v), and (vi) of Conjecture 1.1. It is convenient to construct companions not only for coefficient objects which are irreducible of finite determinant, but also for algebraic coefficient objects.

Theorem 3.8. *Every coefficient object on X which is irreducible with determinant of finite order is uniformly algebraic and admits companions in all categories of coefficient objects, which are again irreducible with determinant of finite order. Moreover, the companion in a given category of coefficient objects, for a given embedding of $\overline{\mathbb{Q}}$ into the full coefficient field, is unique up to isomorphism.*

Proof. See [Abel18a, Theorems 4.4.1, 4.4.5]. Note that [Abel18a, Theorem 4.4.5] is stated in a conditional form, but is in fact unconditional when $\dim(X) = 1$. \square

It is useful to extract the following corollary.

Corollary 3.9. *Let \mathcal{E} be a coefficient object on X which is E -algebraic for some Galois number field E . Then for each $\tau \in \text{Gal}(E/\mathbb{Q})$, there exists a coefficient object \mathcal{E}_τ on X such that for each $x \in X^\circ$, $P((\mathcal{E}_\tau)_x, T) = \tau(P(\mathcal{E}_x, T))$ (where τ acts coefficientwise on $E[T]$).*

Proof. Recall that if E' is a Galois number field containing E , then $\text{Gal}(E'/\mathbb{Q})$ surjects onto $\text{Gal}(E/\mathbb{Q})$; this means that there is no harm in enlarging E during the course of the proof. By Lemma 2.4, there exists a constant coefficient object \mathcal{L} of rank 1 such that $\det(\mathcal{E} \otimes \mathcal{L})$ is of finite order. Since \mathcal{E} is E -algebraic for some E , \mathcal{L} is E -algebraic for some (possibly larger) E ; we may thus twist by \mathcal{L} to reduce to the case where \mathcal{E} is irreducible and $\det(\mathcal{E})$ is of finite order. At this point, the claim is immediate from Theorem 3.4, or more precisely from the fact that there is such a correspondence for each embedding of $\overline{\mathbb{Q}}$ into the full coefficient field (see [Del12, (2.2)] for the corresponding discussion in the étale case). \square

Corollary 3.10. *For \mathcal{E} a coefficient object on X , the following conditions are equivalent.*

- (i) \mathcal{E} is algebraic.
- (ii) \mathcal{E} is uniformly algebraic.
- (iii) Each irreducible constituent of \mathcal{E} is the twist of an object with determinant of finite order by an algebraic object of rank 1 pulled back from k .
- (iv) There exists a positive integer n such that the pullback of \mathcal{E} to X_n satisfies (iii) (with k replaced by k_n).

Proof. It is obvious that (ii) implies (i) and that (iii) implies (iv). By Theorem 3.8, (iv) implies (ii). By Lemma 2.4, (i) implies (iii). \square

Corollary 3.11. *Conjecture 1.4 holds when $\dim(X) = 1$ (for any ℓ, ℓ').*

Proof. We may reduce immediately to the case of an irreducible object; by Lemma 2.4, it must have the form $\mathcal{E} \otimes \mathcal{L}$ where \mathcal{E} is a coefficient object on X with determinant of finite order and \mathcal{L} is a constant algebraic coefficient object of rank 1. Note that \mathcal{E} must itself be irreducible; hence by Theorem 3.8, \mathcal{E} admits a companion. Meanwhile, \mathcal{L} trivially admits a companion, as thus does $\mathcal{E} \otimes \mathcal{L}$. \square

The following statement resembles [Dri12, Lemma 2.7] in the étale case; we recast the argument so that it works uniformly, but without taking care to achieve the same bound on the field extension as in *op. cit.* (Note that we do not claim that $L_1 = L_0$.)

Corollary 3.12. *Fix a category of coefficient objects and an embedding of $\overline{\mathbb{Q}}$ into the full coefficient field. Let E be a number field and let L_0 be the completion of E in the full coefficient field. Let \mathcal{E} be an E -algebraic coefficient object on X (in the specified category) of rank r . Then there exists a finite extension L_1 of L_0 , depending only on L_0 and r (but not on X or \mathcal{E} or E), for which \mathcal{E} can be realized as an object of $\mathbf{Weil}(X) \otimes L_1$ or $\mathbf{F-Isoc}^\dagger(X) \otimes L_1$.*

Proof. There is no harm in enlarging L_0 to ensure that $\mu_r \subset L_0$. We may further assume that \mathcal{E} is irreducible (of rank r) with determinant of finite order, as then for general \mathcal{E} we may decompose into irreducibles and apply Lemma 2.4 to obtain the original statement (but with a larger bound on L_1). Let n be the order of $\det(\mathcal{E})$; since L_0 contains only finitely many roots of unity, n is bounded in terms of L_0 alone.

For L a finite extension of L_0 , let \mathcal{C}_L denote the category $\mathbf{Weil}(X) \otimes L$ or $\mathbf{F-Isoc}^\dagger(X) \otimes L$ within the full category of coefficient objects under consideration. By definition, we can realize \mathcal{E} in \mathcal{C}_L for some finite

extension L of L_0 , which we may assume is Galois over L_0 . Since \mathcal{E} remains irreducible upon enlarging L , by Schur's lemma the endomorphism algebra of \mathcal{E} in \mathcal{C}_L is equal to L .

Put $G := \text{Gal}(L/L_0)$; then G acts on the category \mathcal{C}_L , so we may form the pullback $\tau^*\mathcal{E}$ for any $\tau \in G$. By the uniqueness aspect of Theorem 3.8, we may choose an isomorphism $\iota_\tau : \tau^*\mathcal{E} \cong \mathcal{E}$ for each τ . For $\tau_1, \tau_2 \in G$, the isomorphism $\iota_{\tau_1\tau_2} : (\tau_1\tau_2)^*\mathcal{E} \rightarrow \mathcal{E}$ differs from the composition $\iota_{\tau_1} \circ \tau_1^*(\iota_{\tau_2})$ by multiplication by a nonzero scalar $c(\tau_1, \tau_2) \in L$. The c_{τ_1, τ_2} form a 2-cocycle and thus represent a class in $H^2(G, L^\times)$; since $\det(\mathcal{E})$ has order n , this class in fact belongs to $H^2(G, \mu_{rn})$. By local class field theory, this class (which represents an rn -torsion element of the Brauer group of L_0) can be trivialized by passing from L_0 to any finite extension L_1 of L_0 of degree rn (e.g. the unramified one); this proves the claim. \square

3.3. Proof of Theorem 1.2

We now complete the proof of Theorem 1.2. For this, in light of Theorem 3.8 we could simply assume $\ell \neq p$ and appeal to [Laf02, Théorème VII.6]; however, we prefer to spell out a bit more explicitly how the various aspects of Conjecture 1.1 are addressed. Let r be the rank of \mathcal{E} .

(i) By Theorem 3.8, we may assume that $\ell \neq p$ and that \mathcal{E} is uniformly algebraic. Using Corollary 3.9, we may take the direct sum of the Galois conjugates of \mathcal{E} to obtain a lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} which is \mathbb{Q} -algebraic. In particular, for any algebraic embedding ι of $\overline{\mathbb{Q}}_\ell$ into \mathbb{C} , \mathcal{F} is ι -real, and so we may apply [Del81, Théorème 1.5.1] to deduce that its constituents are all ι -pure. In particular \mathcal{E} is ι -pure of some weight w ; its determinant is then pure of weight rw , but this forces $w = 0$ because the determinant is of finite order.

(ii) This is included in Theorem 3.8.

(iii) By Theorem 3.8, \mathcal{E} admits an ℓ' -adic companion where ℓ' is the rational prime below λ . Since this companion is irreducible with finite determinant, it must be a lisse étale $\overline{\mathbb{Q}}_{\ell'}$ -sheaf (see Definition 3.2), and this proves the claim.

(iv) By Theorem 3.8, we may assume that $\ell \neq p$. Applying [Laf02, Théorème VII.2] yields an upper bound of the desired form, except that the desired factor of $\frac{r}{2}$ is replaced by the larger factor $\frac{(r-1)^2}{r}$.

On the other hand, combining Theorem 3.4 with [Laf11, Corollaire 2.2] (which makes the calculation on the automorphic side) yields a bound with the factor $\frac{r-1}{2}$; moreover, it is shown that for $i = 0, \dots, r$, the sum of the i largest valuations is at most $\frac{i(r-i)}{2}$ times the valuation of $\#\kappa(x)$. Another way to get the same bound is to apply Theorem 3.8 to reduce to the case $\ell = p$ and then invoke Proposition 2.18; see [DK17, Remark 1.3.6].

(v, vi) These are both included in Theorem 3.8.

3.4. Compatibility of ramification

We now spell out in more detail what the compatibility assertion of Theorem 3.4 says about the local behavior of companions. For this, we need to introduce local monodromy representations, particularly in the crystalline case.

Definition 3.13. For $x \in \overline{X}$, let I_x denote the inertia group of \overline{X} at x . For \mathcal{E} a coefficient object, denote by $\rho_x(\mathcal{E})$ the *semisimplified local monodromy representation* of I_x associated to \mathcal{E} . In the étale case, this is simply the semisimplification of the restriction of the associated Weil group representation; in the crystalline case, see [Ked22a, Remark 4.12]. We need the following properties of the construction.

- \mathcal{E} is tame at x if and only if $\rho_x(\mathcal{E})$ is at most tamely ramified. In particular, if we write $\text{Swan}_x(\mathcal{E})$ for the Swan conductor of $\rho_x(\mathcal{E})$, then \mathcal{E} is tame at x if and only if $\text{Swan}_x(\mathcal{E}) = 0$.
- \mathcal{E} is docile at x if and only if $\rho_x(\mathcal{E})$ is unramified. In particular, this is true if $x \in X$.
- The construction of ρ_x is functorial in the following sense: if $f : \overline{X}' \rightarrow \overline{X}$ is a finite flat morphism and $f(y) = x$, then $\rho_y(f^*\mathcal{E})$ is the restriction of $\rho_x(\mathcal{E})$ along $I_y \rightarrow I_x$.

For curves, the Euler characteristic is related to local monodromy via the Grothendieck–Ogg–Shafarevich formula.

Proposition 3.14. *For any coefficient object \mathcal{E} on X , we have*

$$\chi(\mathcal{E}) = \chi(X) \operatorname{rank}(\mathcal{E}) - \sum_{x \in Z} [\kappa(x) : k] \operatorname{Swan}_x(\mathcal{E}).$$

Proof. See [Gro77] in the étale case and [Ked06b, Theorem 4.3.1] in the crystalline case. \square

Corollary 3.15. *Let \mathcal{E} and \mathcal{F} be coefficient objects on X (possibly in different categories) which are companions. Then \mathcal{E} is tame (resp. docile) if and only if \mathcal{F} is. (By Lemma 2.40, this remains true even without the same conclusion holds without Hypothesis 3.1, i.e., allowing X to be smooth but not necessarily one-dimensional.)*

Proof. From Definition 2.8 and Proposition 3.14, we can read off the (weighted) sum of Swan conductors of \mathcal{E} from $L(\mathcal{E}^\vee, T)$; in particular, the L -function determines whether or not this sum is zero, and hence whether or not \mathcal{E} is tame. Since the duals of companions have matching L -functions, this proves the claim in the tame case.

In the docile case, we may assume at once that both \mathcal{E} and \mathcal{F} are tame; we may also assume that they are both irreducible with determinant of finite order. For each $x \in \bar{X}$, by Theorem 3.4 the local Euler factors of \mathcal{E} and \mathcal{F} at x coincide. The degree of the local Euler factor computes the dimension of the kernel of either the full local monodromy representation in the étale case, or the monodromy operator in the crystalline case (see Remark 3.3). Let d_1 denote this dimension, let d_2 denote the corresponding dimension for \mathcal{E}^\vee , and let r denote the common rank of \mathcal{E} and \mathcal{F} . Then the degree of the local Euler factor of $\mathcal{E}^\vee \otimes \mathcal{E}$ at x is at least $d_1 r + d_2 r - d_1 d_2$, with equality if and only if \mathcal{E} is docile. (This quantity computes the contribution of the generalized eigenspace for the eigenvalue 1; any other eigenvalue that occurs makes a positive contribution.) By applying the same logic with \mathcal{E} replaced by \mathcal{F} , we deduce the claim. \square

We next formally promote Corollary 3.15 to a statement about ramification at individual points.

Lemma 3.16. *Let $f : \bar{X}' \rightarrow \bar{X}$ be a finite flat morphism which is étale over X , and choose $x \in Z$. Then there exists a finite flat morphism $g : \bar{X}'' \rightarrow \bar{X}$ with the following properties.*

(i) *g is étale over x .*

(ii) *For each $x' \in Z \setminus \{x\}$, write $Z_{x'}$ for the formal completion of \bar{X} along x' ; then the base extension $\bar{X}'' \times_{\bar{X}} Z_{x'} \rightarrow Z_{x'}$ of g factors through the base extension $\bar{X}' \times_{\bar{X}} Z_{x'} \rightarrow Z_{x'}$ of f .*

(Note that we do not require g to be étale over X .)

Proof. It suffices to achieve (i) and (ii) for a single x' , as we may then take a fiber product of the resulting covers to achieve the desired result. For this, first choose a finite flat morphism from \bar{X} to \mathbb{P}_k^1 carrying x and x' to 1 and 0, respectively. We may then achieve the result using the Katz–Gabber canonical extension theorem [Kat86]: any extension of $k((t))$ can be uniquely realized as the base extension of a finite flat cover of \mathbb{P}_k^1 which is tamely ramified at $t = \infty$ and étale away from $t = 0, \infty$. \square

Proposition 3.17. *Let \mathcal{E} and \mathcal{F} be coefficient objects on X (possibly in different categories) which are companions. Then for $x \in Z$, the following statements hold.*

(a) *We have $\operatorname{Swan}_x(\mathcal{E}) = \operatorname{Swan}_x(\mathcal{F})$. In particular, \mathcal{E} is tame at x if and only if \mathcal{F} is.*

(b) *\mathcal{E} is docile at x if and only if \mathcal{F} is.*

Proof. Apply Remark 2.38 to construct a finite flat morphism $f : \bar{X}' \rightarrow \bar{X}$ which is étale over X such that both $f^*\mathcal{E}$ and $f^*\mathcal{F}$ are docile. (More precisely, by Corollary 3.11 and Corollary 3.15 we may do this assuming that both \mathcal{E} and \mathcal{F} are étale coefficient objects, and then Remark 2.38 applies.) Define another morphism $g : \bar{X}'' \rightarrow \bar{X}$ as in Lemma 3.16. Then $g^*\mathcal{E}$ and $g^*\mathcal{F}$ are both tame at every point of \bar{X}'' not lying over x ; meanwhile, for $y \in g^{-1}(x)$, we have $\operatorname{Swan}_y(g^*\mathcal{E}) = \operatorname{Swan}_x(\mathcal{E})$ and $\operatorname{Swan}_y(g^*\mathcal{F}) = \operatorname{Swan}_x(\mathcal{F})$. We may thus

argue using Proposition 3.14 (as in the proof of Corollary 3.15) to deduce (a). By a similar argument, we may reduce (b) to Corollary 3.15. \square

Corollary 3.18. *For any positive integer r and any category of coefficient objects on X , there are only finitely many isomorphism classes of irreducible tame objects of rank r up to twisting by constant objects of rank 1.*

Proof. By Lemma 2.4, the problem does not change if we consider only objects with determinant of finite order. By Theorem 3.8 and Proposition 3.17, we need only consider the étale case. In this case, an irreducible coefficient object of rank r corresponds via Theorem 3.8 to an automorphic representation. If the original coefficient object is tame, then by local-global compatibility, the resulting automorphic representation has Iwahori level, which in particular depends only on r ; hence [Har74, Theorem 1.2.1] again applies. \square

4. Higher-dimensional varieties

In this section, we prove Theorem 1.3, thus resolving Conjecture 1.1(i)–(v) for general X . As noted in the introduction, this includes some duplication of results of Abe–Esnault [AE19]; we comment at the end of the section on differences between the two approaches (see §4.7).

Convention 4.1. We will frequently do the following: pick some closed points $x_1, \dots, x_m \in X^\circ$, choose a positive integer n , and choose a “curve in X_n containing x_1, \dots, x_m .” By this, we mean a curve (over k_n) in X_n containing some points lying over x_1, \dots, x_m ; we remind the reader that we do not insist that curves be proper, so the existence of such a curve is elementary (that is, it does not depend on anything as difficult as Poonen’s Bertini theorem [Poo04]).

4.1. Weights

We first address part (i) of Conjecture 1.1 by consolidating Deligne’s theory of weights from [Del81] using the Langlands correspondence in the form of Theorem 1.2. In the process, we will make a first use of the geometric setup that will recur in the discussion of Lefschetz slicing (§4.2).

Hypothesis 4.2. Throughout §4.1, fix a coefficient object \mathcal{E} on X , and let ι be an embedding of the full coefficient field of \mathcal{E} into \mathbb{C} . (This choice is for expediency only; see Remark 4.13.) Also, let q denote the cardinality of k .

Definition 4.3. For $x \in X^\circ$, the *multiset of ι -weights of \mathcal{E} at x* is the multiset consisting of $-2 \log_{\#\kappa(x)} |\iota(\lambda)|$ as λ varies over the roots of $P(\mathcal{E}_x, T)$ (counted with multiplicity). Recall that these roots are reciprocals of eigenvalues of Frobenius.

We say that \mathcal{E} is *ι -pure of weight w* if for all $x \in X^\circ$, the multiset of ι -weights of \mathcal{E} at x consists of the single element w . If the determinant of \mathcal{E} is of finite order, we must then have $w = 0$.

We will make crucial use of the following form of “Weil II”.

Lemma 4.4. *Suppose that \mathcal{E} is ι -pure of weight w . Then the eigenvalues of the Frobenius φ on $H^1(X, \mathcal{E})$ all have ι -absolute value at least $q^{(w+1)/2}$.*

Proof. See [Del81, Théorème 3.3.1] in the étale case and [Ked22a, Theorem 10.3] in the crystalline case. \square

As in [Del81, Théorème 3.4.1(3)], we have the following corollary. (See also [D’Ad20, Corollary 3.5.2].)

Corollary 4.5. *Suppose that \mathcal{E} is ι -pure.*

- (a) *In the category of geometric coefficient objects, \mathcal{E} is semisimple.*
- (b) *In the category of coefficient objects, \mathcal{E} admits an isotypical decomposition: a direct sum decomposition in which each summand is a successive extension of a single irreducible object.*

Proof. We prove both assertions at once by induction on $\text{rank}(\mathcal{E})$. Suppose first that \mathcal{E} is irreducible. Let \mathcal{F} be the sum of all irreducible subobjects of \mathcal{E} in the category of geometric coefficient objects; then \mathcal{F} is stable under Frobenius, and hence a subobject of \mathcal{E} in the category of coefficient objects. It follows that $\mathcal{E} = \mathcal{F}$, proving (a); meanwhile, (b) is vacuous.

Suppose next that \mathcal{E} is reducible. Then there exists an exact sequence

$$(4.1) \quad 0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

of coefficient objects in which \mathcal{E}_1 is irreducible and \mathcal{E}_2 is nonzero. If this sequence splits, then we may apply the induction hypothesis to \mathcal{E}_2 to deduce both (a) and (b). Otherwise, from the Hochschild–Serre spectral sequence (compare [Del81, Lemme 3.4.2] in the étale case), we obtain an exact sequence

$$(4.2) \quad H^0(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)_\varphi \rightarrow \text{Ext}(\mathcal{E}_2, \mathcal{E}_1) \rightarrow H^1(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)^\varphi$$

where the Ext group is defined in the category of coefficient objects. Since $\mathcal{E}_2^\vee \otimes \mathcal{E}_1$ is ι -pure of weight 0, the term on the right vanishes by Lemma 4.4. It follows that the extension (4.1) splits in the category of geometric coefficient objects, so we may deduce (a) from the induction hypothesis. Meanwhile, the nonvanishing of the middle term of (4.2) implies that the finite-dimensional vector space $H^0(X, \mathcal{E}_2^\vee \otimes \mathcal{E}_1)$ has nonzero φ -coinvariants and hence nonzero φ -invariants; that is, there is a nonzero homomorphism $\mathcal{E}_2 \rightarrow \mathcal{E}_1$ in the category of coefficient objects (not just geometric coefficient objects). By the induction hypothesis, \mathcal{E}_2 admits an isotypical decomposition, which must admit a nonzero summand corresponding to \mathcal{E}_1 . Let \mathcal{E}'_1 be the preimage of this summand in \mathcal{E} ; then the same argument applied to the exact sequence

$$0 \rightarrow \mathcal{E}'_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow 0$$

shows that this sequence must split. Since \mathcal{E}'_1 is isotypical, we may apply the induction hypothesis to \mathcal{E}'_2 to deduce (b). \square

The following lemma will occur in passing in the construction of the weight filtration, but will be used much more heavily in the study of uniqueness of companions (§4.3).

Lemma 4.6. *Suppose that X is irreducible of pure dimension d and that \mathcal{E} is ι -pure of weight 0. Then the pole order of $L(\mathcal{E}, T)$ at $T = q^{-d}$ equals the multiplicity of 1 as an eigenvalue of Frobenius on $H^0(X, \mathcal{E}^\vee)$ (viewed as a vector space over the full coefficient field).*

Proof. In the Lefschetz trace formula (2.1), the factor $i = 0$ contributes the predicted value to the pole order. Meanwhile, by Lemma 4.4, the eigenvalues of F on $H^i(X, \mathcal{E}^\vee)$ for $i > 0$ all have absolute value at least $q^{1/2}$, so the corresponding factor of (2.1) only has zeroes or poles in the region $|T| \geq q^{-d+1/2}$. This proves the claim. \square

By probing X with curves, we gain some initial data about weights.

Lemma 4.7. *Suppose that X is irreducible.*

- (a) *The multiset of ι -weights of \mathcal{E} at $x \in X^\circ$ is independent of x . We may thus refer to it as the multiset of ι -weights of \mathcal{E} .*
- (b) *There exist a constant twist \mathcal{E}' of \mathcal{E} and a positive integer N such that the multiset of ι -weights of \mathcal{E}' has least element 0 and for each $x \in X^\circ$, the product of the roots of $P(\mathcal{E}'_x, T)$ of ι -weight 0 equals a root of unity of order dividing N .*

Proof. To check (a), it suffices to compare two points $x, y \in X^\circ$; this may be done by restricting to a curve in X through x, y . That is, we may assume that X is a curve; in this case, we may further assume that \mathcal{E} is irreducible and (by making a constant twist dictated by Lemma 2.4) its determinant is of finite order. We may then apply Theorem 1.2 to deduce that \mathcal{E} is ι -pure of weight 0.

To check (b), we may first twist to ensure that for some $x \in X^\circ$, the product in question equals 1. Given any other point $y \in X^\circ$, choose a curve C in X containing x and y ; by Theorem 1.2 again, the eigenvalues

in question arise from some irreducible constituents of $\mathcal{E}|_C$. By Lemma 2.4, we see that the product of the eigenvalues at y is a root of unity; since these eigenvalues belong to an extension of \mathbb{Q}_ℓ or \mathbb{Q}_p of degree bounded independently of x , the order of the root of unity is also bounded. \square

Corollary 4.8. *Suppose that for some $w \in \mathbb{R}$, there exists an open dense subset U of X such that $\mathcal{E}|_U$ is ι -pure of weight w . Then \mathcal{E} is also ι -pure of weight w .*

Proof. This is immediate from Lemma 4.7(a). \square

We now introduce the geometric setup for Lefschetz slicing. We recall here a commonly used construction found in [Art73, Proposition 3.3]. (One can make a similar construction without enlarging k using Poonen’s Bertini theorem; see §4.7.)

Lemma 4.9. *Pick $x \in X(k)$ and let \tilde{X} be the blowup of X at (the reduced closed subscheme supported at) x . Then for some finite extension k' of k , there exists an open neighborhood U of x in $X_{k'}$ whose inverse image \tilde{U} in $\tilde{X}_{k'}$ can be expressed as a fibration $f : \tilde{U} \rightarrow S$ in curves over some smooth proper k' -scheme S .*

Proof. Since we are free to replace X with an open subscheme containing x , we may ensure that X is affine of dimension n . Let \bar{X} be the Zariski closure of X in some projective embedding $X \hookrightarrow \mathbb{P}_k^N$. For some choice of k' , we can choose sections s_0, \dots, s_{n-1} of $\mathcal{O}(1)$ on $\mathbb{P}_{k'}^N$ whose zero loci H_0, \dots, H_{n-1} on $\mathbb{P}_{k'}^N$ have the following properties (because each property holds generically).

- The intersection $L = H_0 \cap \dots \cap H_{n-1}$ has dimension $N - n$ and contains x .
- The intersection $L \cap \tilde{X}_{k'}$ is transversal at x .
- The intersection $L \cap (\bar{X}_{k'} \setminus X_{k'})$ is empty.

Put $Z := (L \cap \tilde{X}_{k'}) \setminus \{x\}$; then the sections s_0, \dots, s_{n-1} define a map $f : \tilde{X}_{k'} \setminus Z \rightarrow S := \mathbb{P}_{k'}^{n-1}$. The exceptional divisor of $\tilde{X}_{k'} \rightarrow X_{k'}$ is the image of a section $g : S \rightarrow \tilde{X}_{k'}$ of f . Each fiber of f is isomorphic to an intersection of $X_{k'} \setminus Z$ with $n - 1$ hyperplanes which is transversal at x ; consequently, there is a unique irreducible component of this fiber containing x , and this component is smooth of dimension 1 at x . We now take \tilde{U} to be the set obtained from $\tilde{X}_{k'} \setminus Z$ by removing:

- the union of all irreducible components of fibers of f which do not meet $g(S)$; and
- the set of all singular points of irreducible components of fibers of f which do meet $g(S)$.

This yields an open subscheme \tilde{U} of $\tilde{X}_{k'} \setminus Z$ not meeting $g(S)$; we take U to be the image of \tilde{U} in $X_{k'}$. \square

Remark 4.10. We iterate a key point from the proof of Lemma 4.9: the exceptional divisor of the blowup $\tilde{U} \rightarrow U$ is the image of a section $g : S \rightarrow \tilde{U}$ of f on which the pullback of \mathcal{E} is constant. We will combine this geometric construction with the following observation: for any coefficient object \mathcal{F} in the same category as \mathcal{E} , there exists an open dense subscheme V of S (depending on \mathcal{F}) such that $f_*\mathcal{F}$ exists as a coefficient object on V and its formation commutes with arbitrary base change on V . (See [Ked06a, Theorem 7.3.3] for the crystalline case.)

We finally obtain the main theorem on weights.

Theorem 4.11. *The following statements hold.*

- (a) *If \mathcal{E} is irreducible, then it is ι -pure of some weight.*
- (b) *There exists a unique filtration*

$$0 = \mathcal{E}_0 \subset \dots \subset \mathcal{E}_l = \mathcal{E}$$

for which there exists an increasing sequence $w_1 < \dots < w_l$ of real numbers for which $\mathcal{E}_i/\mathcal{E}_{i-1}$ is ι -pure of weight w_i . We call this filtration the weight filtration of \mathcal{E} .

Proof. For any given X , part (a) of Theorem 4.11 implies part (b) using Lemma 4.4; see [Ked22a, Corollary 10.5] or [D’Ad20, Corollary 3.5.2]. For $\dim(X) = 1$, we may apply Theorem 1.2 to obtain (a).

It thus remains to prove (a) for general X using the truth of (b) for curves. For this, we may assume that X is irreducible; moreover, by Corollary 4.8, we may check the claim after replacing X with an open dense subscheme. We may also make a constant twist on \mathcal{E} ; hence by Lemma 4.7, we may assume that the multiset of ι -weights of \mathcal{E} has least element 0 with some multiplicity m , and that for each $x \in X^\circ$ the product of the corresponding roots of $P(\mathcal{E}_x, T)$ is a root of unity of order dividing some fixed positive integer N .

After shrinking X and enlarging k , we may apply Lemma 4.9 and Remark 4.10 to $\mathcal{F} := \wedge^{mN}(\mathcal{E}^{\oplus N})$. On each fiber of f , we may apply the induction hypothesis to produce a weight filtration of \mathcal{F} ; the first step of this filtration is an object of rank 1 with all Frobenius eigenvalues equal to 1, so by Lemma 4.6 it is trivial. On some open subscheme V of S , we may form the pushforward $f_*\mathcal{F}$ in a manner compatible with arbitrary base change (Remark 4.10); this pushforward must therefore be trivial of rank 1, and we may pull back to obtain a trivial rank-1 subobject of \mathcal{F} (initially on $f^{-1}(V)$, but by Lemma 2.3 this subobject extends over X). By multilinear algebra, we recover a subobject of \mathcal{E} of rank m ; this is a contradiction unless $m = \text{rank}(\mathcal{E})$, meaning that \mathcal{E} is ι -pure of weight 0. \square

Remark 4.12. An alternate approach to proving Theorem 4.11 would be to use the fact that part (a) can be checked after an alteration; in light of Proposition 2.37, this means that it suffices to check (a) under the additional assumption that \mathcal{E} is tame. In this context, one can obtain very strong versions of Lefschetz slicing; see §4.7.

Remark 4.13. We again remind the reader of [Del81, Remarque 1.2.11]: the use of an algebraic embedding ι in Theorem 4.11 is merely for expository convenience. *A posteriori*, we will only need to embed various number fields into \mathbb{C} .

On a similar note, the weight of an irreducible coefficient object may depend on the choice of ι , but in light of Lemma 2.4 this reduces to a rank-1 issue. In any case, note that changing ι may change the order of the successive quotients in the weight filtration, in which case some splitting must occur.

4.2. Lefschetz slicing and valuations of eigenvalues

As discussed in the introduction, a key tool in reducing the study of general coefficient objects to the case of curves is the preservation of irreducibility of a coefficient object under restriction to a suitable curve, but the usual proof of this in the étale case does not apply in the crystalline case. We give a replacement argument here using weights; we make no effort to strengthen the final result because this can easily be done *a posteriori* (see §4.7). We then deduce parts (iii) and (iv) of Conjecture 1.1, as well as a weak version of (ii).

Lemma 4.14. *Let \mathcal{E} be a geometrically irreducible coefficient object on X and choose $x \in X^\circ$. Then for some positive integer n , there exists a curve C in X_n containing x such that $\mathcal{E}|_C$ is irreducible.*

Proof. Instead of keeping track of the extension degree n , we adopt the convention that we may enlarge k freely during the argument, replacing x with some closed point lying over it. To begin with, we may assume that $x \in X(k)$.

Apply Lemma 4.9 and Remark 4.10 with \mathcal{F} equal to the trace-zero component of $\mathcal{E}^\vee \otimes \mathcal{E}$. By Remark 2.27, $\overline{G}(\mathcal{E})$ must act irreducibly on its standard representation, so $H^0(X, \mathcal{E}^\vee \otimes \mathcal{E})$ is the span of the identity map and $H^0(X, \mathcal{F}) = 0$.

Since $g^*\mathcal{E}$ is a trivial coefficient object, so is $g^*\mathcal{F}$. By adjunction, $f^*f_*\mathcal{F}$ corresponds to a subobject of \mathcal{F} , so $g^*f^*f_*\mathcal{F}$ corresponds to a nonzero subobject of $g^*\mathcal{F}$; hence $g^*f^*f_*\mathcal{F}$ is trivial. Since g is a section of f , there is a natural identification $g^*f^*f_*\mathcal{F} \cong f_*\mathcal{F}$; hence $f_*\mathcal{F}$ is trivial, as then is $f^*f_*\mathcal{F}$. In particular, the dimension of $H^0(X, f^*f_*\mathcal{F})$ equals the rank of $f_*\mathcal{F}$; since $H^0(X, f^*f_*\mathcal{F})$ is a subspace of $H^0(X, \mathcal{F}) = 0$, we conclude that $f_*\mathcal{F} = 0$.

Choose a point $y \in S(k)$ and take $C := f^{-1}(y)$. The vanishing of $f_*\mathcal{F}$ implies that $H^0(C, \mathcal{F}|_C) = 0$. Pick an embedding ι of the coefficient field of \mathcal{E} into \mathbb{C} ; by Theorem 4.11, \mathcal{E} is ι -pure of some weight, and likewise

after restricting to C . By Corollary 4.5, $\mathcal{E}|_C$ admits an isotypical decomposition; combined with the vanishing of $H^0(C, \mathcal{F}|_C)$, this implies first that $\mathcal{E}|_C$ is isotypical, and second that $\mathcal{E}|_C$ is irreducible. \square

Remark 4.15. Another way to establish Lemma 4.14, which does not require a prior study of weights, is to show that the inclusion $\overline{G}(\mathcal{E}|_C) \rightarrow \overline{G}(\mathcal{E})$ is an isomorphism by applying Lemma 4.9 and Remark 4.10 with \mathcal{F} being a sufficiently large object in the Tannakian category generated by \mathcal{E} . This is in the spirit of Corollary 4.39 below, but requires slightly more care because the open set U varies with \mathcal{F} ; one must thus make some argument to give an *a priori* choice of \mathcal{F} that suffices to rule out all unwanted possibilities for $\overline{G}(\mathcal{E}|_C)$. (As in Lemma 4.7, the key point is that an extension of \mathbb{Q}_ℓ or \mathbb{Q}_p of bounded degree contains a bounded number of roots of unity.)

Lemma 4.16. *Let \mathcal{E} be a coefficient object on X which is irreducible with determinant of finite order. Let λ be a reciprocal root of $P(\mathcal{E}_x, T)$ for some $x \in X^\circ$.*

- (a) λ is algebraic over \mathbb{Q} .
- (b) At any (finite or infinite) place of $\mathbb{Q}(\lambda)$ not lying above p , λ has absolute value 1. (For infinite places, this is included in Theorem 4.11.)
- (c) At any place of $\mathbb{Q}(\lambda)$ lying above p , λ has absolute value at least that of $\kappa(x)$ to the power $\frac{1}{2} \text{rank}(\mathcal{E})$.

In particular, parts (i), (iii), and (iv) of Conjecture 1.1 hold.

Proof. We start with some initial reductions. For n a positive integer, let $f_n : X_n \rightarrow X$ be the canonical morphism. For $x_n \in X_n^\circ$ lying over $x \in X^\circ$, the roots of $P((f_n^* \mathcal{E})_{x_n}, T)$ are the $[\kappa(x_n) : \kappa(x)]$ -th powers of the roots of $P(\mathcal{E}_x, T)$; consequently, for any particular x , checking the claims at x is equivalent to checking them at x_n .

This means that by Remark 2.27 and Lemma 2.28, we may reduce to the case where \mathcal{E} is geometrically irreducible. Then for each $x \in X^\circ$, we may apply Lemma 4.14 to find a curve C in X_n through x , for some n depending on x , for which $\mathcal{E}|_C$ is again irreducible. We may then invoke Theorem 1.2 to deduce that the desired assertions hold at x . \square

4.3. Uniqueness of companions via weights

For étale coefficient objects, the uniqueness of a companion in a given category is an easy consequence of the Chebotarëv density theorem applied to residual representations. In order to obtain similar assertions that also cover crystalline coefficient objects, one must replace the use of Chebotarëv density with a weight-theoretic argument. The argument we have in mind is due to Tsuzuki (as presented in [Abel18a, Proposition A.4.1], [Ked22a, Theorem 10.8]), but it must be noted that similar arguments occur in étale cohomology in the context of independence-of- ℓ statements, as in Larsen–Pink [LP95, Proposition 2.1] (see also [Pál15, Proposition 8.20]).

Theorem 4.17 (Tsuzuki). *Let $\mathcal{E}_1, \mathcal{E}_2$ be two algebraic coefficient objects on X which are companions.*

- (a) If \mathcal{E}_1 is irreducible, then so is \mathcal{E}_2 .
- (b) If $\mathcal{E}_1, \mathcal{E}_2$ are in the same category, then \mathcal{E}_1 and \mathcal{E}_2 have the same semisimplification.

Proof. In both cases, we may assume that X is irreducible of pure dimension d , and (by Theorem 4.11) that $\mathcal{E}_1, \mathcal{E}_2$ are both pure of weight 0 and semisimple.

In case (a), note that by Schur's lemma, \mathcal{E}_i is irreducible if and only if $H^0(X, \mathcal{E}_i^\vee \otimes \mathcal{E}_i)^\varphi$ is one-dimensional. We may thus apply Lemma 4.6 to $\mathcal{E}_1^\vee \otimes \mathcal{E}_1, \mathcal{E}_2^\vee \otimes \mathcal{E}_2$ to conclude.

In case (b), it suffices to check that any irreducible subobject \mathcal{F} of \mathcal{E}_1 (which must also be pure of weight 0) also occurs as a summand of \mathcal{E}_2 . To this end, note that \mathcal{F} occurs as a summand of \mathcal{E}_i if and only if $H^0(X, \mathcal{F}^\vee \otimes \mathcal{E}_i)^\varphi \neq 0$; we may thus apply Lemma 4.6 to $\mathcal{E}_1^\vee \otimes \mathcal{F}, \mathcal{E}_2^\vee \otimes \mathcal{F}$ to conclude. \square

Corollary 4.18. *Let \mathcal{E} and \mathcal{F} be algebraic coefficient objects on X (not necessarily in the same category). If there exists an open dense subscheme U of X such that $\mathcal{E}|_U$ and $\mathcal{F}|_U$ are companions, then \mathcal{E} and \mathcal{F} are companions.*

Proof. It suffices to check the equality $P(\mathcal{E}_x, T) = P(\mathcal{F}_x, T)$ at an arbitrary $x \in X^\circ$. By restricting to a suitable curve in X containing x , we may further assume that $\dim(X) = 1$. At this point, we may assume that \mathcal{E} and \mathcal{F} are semisimple. By Corollary 3.11, \mathcal{E} admits a semisimple companion \mathcal{F}' in the same category as \mathcal{F} . On U , the restrictions $\mathcal{F}|_U$ and $\mathcal{F}'|_U$ are semisimple (by Lemma 2.3) companions of each other; by Theorem 4.17(b), there exists an isomorphism $\mathcal{F}|_U \cong \mathcal{F}'|_U$. By Lemma 2.3 again, this isomorphism extends to X ; this yields the desired result. \square

Corollary 4.19. *Let U be an open dense subscheme of X . Let \mathcal{E} and \mathcal{F} be algebraic coefficient objects on X and U , respectively (not necessarily in the same category), with \mathcal{F} semisimple. If $\mathcal{E}|_U$ and \mathcal{F} are companions, then \mathcal{F} extends to a coefficient object on X which is a companion of \mathcal{E} .*

Proof. By Corollary 4.18, we only need to check that \mathcal{F} extends to X . Using the “cut-by-curves” criterion (see [Ked22a, Theorem 5.16] in the crystalline case), it suffices to check that for any morphism $f : C \rightarrow X$ over k where C is a curve over k , the pullback $f^*\mathcal{F}$ extends from $C \times_X U$ to C . To this end, apply Corollary 3.11 to construct a semisimple crystalline companion \mathcal{F}' of $f^*\mathcal{E}$ on C . By Lemma 2.3, the restriction of \mathcal{F}' to $C \times_X U$ is again semisimple. Now $f^*\mathcal{F}$ and $\mathcal{F}'|_{C \times_X U}$ are semisimple companions of each other in the same category, so by Theorem 4.17(b) they are isomorphic; this proves the claim. \square

Corollary 4.20. *Let \mathcal{E} and \mathcal{F} be algebraic coefficient objects on X (not necessarily in the same category). If $\det(\mathcal{E})$ is of finite order, then so is $\det(\mathcal{F})$.*

Proof. Since $\det(\mathcal{E})$ and $\det(\mathcal{F})$ are again companions, we may assume at once that \mathcal{E} and \mathcal{F} are of rank 1. Choose a positive integer n for which $\mathcal{E}^{\otimes n}$ is trivial; then $\mathcal{F}^{\otimes n}$ is a companion of the trivial object, and so by Theorem 4.17(b) is itself trivial. \square

Corollary 4.21. *The conclusion of Corollary 3.12, which was originally stated for $\dim(X) = 1$, holds for general X .*

Proof. We may replace the application of Theorem 3.8 in the proof of Corollary 4.21 with Theorem 4.17, and then the rest of the argument remains unchanged. \square

Remark 4.22. A related assertion to Theorem 4.17(b) is that \mathcal{E}_1 and \mathcal{E}_2 are forced to have the same semisimplification as soon as the equality $P(\mathcal{E}_{1,x}, T) = P(\mathcal{E}_{2,x}, T)$ holds for x running over a certain finite subset of X° determined by X , the common rank of \mathcal{E}_1 and \mathcal{E}_2 , and the wild monodromy of \mathcal{E}_1 and \mathcal{E}_2 . For the case of curves, this is made explicit by Deligne in [Dell2, Proposition 2.5]; this plays a crucial role in the proof of Theorem 4.25.

Remark 4.23. While Theorem 4.17 is formulated in terms of an algebraically closed full coefficient field, it also implies a corresponding assertion for a restricted coefficient field. Namely, suppose that \mathcal{E}, \mathcal{F} are two objects in $\mathbf{Weil}(X) \otimes L$ or $\mathbf{F-Isoc}^+(X) \otimes L$, where L is a finite extension of the base coefficient field, which are companions of each other. Suppose further that \mathcal{E} is irreducible as a coefficient object (*i.e.*, it remains irreducible after replacing L with any finite extension). Then Theorem 4.17 implies that \mathcal{E} and \mathcal{F} are isomorphic as objects of $\mathbf{Weil}(X) \otimes \bar{L}$ or $\mathbf{F-Isoc}^+(X) \otimes \bar{L}$, but the hypothesis on \mathcal{E} ensures that the space of homomorphisms from \mathcal{E} to \mathcal{F} as coefficient objects is one-dimensional over \bar{L} . Moreover, this space is the base extension from L to \bar{L} of the space of homomorphisms from \mathcal{E} to \mathcal{F} in the original category; consequently, there must exist a nonzero homomorphism from \mathcal{E} to \mathcal{F} in the original category, which must then be an isomorphism.

4.4. Uniform algebraicity

We next apply the method of Deligne [Dell2] to address part (ii) of Conjecture 1.1. This discussion constitutes our first use of *uniformity* across curves within a fixed variety, as a technique for studying coefficient objects on higher-dimensional varieties; this technique will prove to be extremely fruitful as we continue.

Lemma 4.24. *Let U be an open dense subscheme of X , let E be a number field, and let \mathcal{E} be a coefficient object on X . If $\mathcal{E}|_U$ is E -algebraic, then so is \mathcal{E} .*

Proof. Since E -algebraicity is a pointwise condition for any given E , to check it at some $x \in X^\circ$ it suffices to do so after restriction to some curve in X containing x . We may thus assume hereafter that X is a curve over k . We may further assume that \mathcal{E} is irreducible.

We first check that \mathcal{E} is uniformly algebraic (but not necessarily E -algebraic). By Lemma 2.4, there exists a constant coefficient object \mathcal{F} of rank 1 such that $\det(\mathcal{E} \otimes \mathcal{F})$ has finite order. By restricting to U , we see that $\mathcal{F}^{\otimes \text{rank}(\mathcal{E})}$ is (uniformly) algebraic, as then is \mathcal{F} . Consequently, to check that \mathcal{E} is uniformly algebraic, we may reduce to the case where \mathcal{E} is irreducible with finite determinant, in which case Theorem 3.8 applies.

By the previous paragraph, we can find a Galois number field E' containing E such that \mathcal{E} is E' -algebraic. By Corollary 3.9, for each automorphism $\tau \in \text{Gal}(E'/E)$, we can find another coefficient object \mathcal{E}_τ in the same category as \mathcal{E} whose Frobenius characteristic polynomials are related to those of \mathcal{E} by application of τ . By hypothesis, $\mathcal{E}|_U$ and $\mathcal{E}_\tau|_U$ are companions, as then are \mathcal{E} and \mathcal{E}_τ by Corollary 4.18. This proves the claim. \square

Theorem 4.25. *Any algebraic coefficient object on X is uniformly algebraic.*

Proof. We use the axiomatized form of Deligne's argument described in [Dell2, Remarque 3.10] (see also [EK12, Theorem 5.1] for an alternate exposition). By Lemma 4.24, there is no harm in replacing X with an open dense subscheme during the argument.

Let \mathcal{E} be an algebraic coefficient object on X . For each positive integer n , consider the function t_n assigning to each $x \in X(k_n)$ the trace of F^n on \mathcal{E}_x . These functions have the following properties.

- (a) For each morphism $f : Z \rightarrow X$ with Z smooth of dimension 1 (but not necessarily geometrically irreducible over k), the functions $t_n \circ f$ arise from an étale coefficient object \mathcal{F}_Z on Z . This follows from Corollary 3.11.
- (b) There exists a connected finite étale cover X' of X such that in (a), if f factors through X' , then \mathcal{F}_Z is tame. In light of Proposition 2.37 and Corollary 3.15, this becomes true after replacing X with a suitable open dense subscheme (but see Remark 4.30 below).

Given these properties, [Dell2, Remarque 3.10] implies the claim. \square

Corollary 4.26. *For \mathcal{E} a coefficient object on X , the following conditions are equivalent.*

- (i) \mathcal{E} is algebraic.
- (ii) \mathcal{E} is uniformly algebraic.
- (iii) Each irreducible constituent of \mathcal{E} is the twist of an object with determinant of finite order by an algebraic object of rank 1 pulled back from k .
- (iv) There exists a positive integer n such that the pullback of \mathcal{E} to X_n satisfies (iii) (with k replaced by k_n).

In other words, the conclusion of Corollary 3.10 (which applies when X is a curve over k) holds for arbitrary X . In particular, part (ii) of Conjecture 1.1 holds.

Proof. It is again obvious that (ii) implies (i) and that (iii) implies (iv). By Lemma 2.4, (i) implies (iii). By Lemma 4.16(a), (iv) implies (i). By Theorem 4.25, (i) implies (ii). \square

4.5. Étale companions

We address part (v) of Conjecture 1.1 by adapting the method of Drinfeld [Dri12] to include the crystalline case.

Lemma 4.27. *Let \mathcal{E} be an object of $\text{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$. Then \mathcal{E} is a lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf if and only if for each $x \in X^\circ$, the roots of $P(\mathcal{E}_x, T)$ all have ℓ -adic valuation 0; in particular, this holds if \mathcal{E} is p -plain.*

Proof. We may assume that \mathcal{E} is irreducible. If \mathcal{E} is a lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf, then the image of W_X in the corresponding representation is compact; this implies that the roots of $P(\mathcal{E}_x, T)$ all have ℓ -adic valuation 0.

To check the converse implication, as per Definition 3.2 we may further assume that \mathcal{E} is of rank 1. By Lemma 2.4, we may write $\mathcal{E} = \mathcal{F} \otimes \mathcal{L}$ where \mathcal{F} is of finite order and \mathcal{L} is constant. We may then check the claim after replacing \mathcal{E} with \mathcal{L} ; that is, we may assume that $X = \text{Spec}(k)$, for which the desired statement is obvious. (Compare [D'Ad20, Proposition 3.1.16].) \square

Theorem 4.28. *Let \mathcal{E} be a coefficient object on X .*

- (a) *If \mathcal{E} is algebraic, then it has an étale companion.*
- (b) *If \mathcal{E} is irreducible with determinant of finite order, then it has an étale companion which is an irreducible lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf with determinant of finite order.*

In particular, part (v) of Conjecture 1.1 holds.

Proof. By Lemma 2.4, it suffices to prove (b). It was shown by Drinfeld [Dri12, Theorem 2.5] that a lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf on X can be constructed out of any family of Frobenius characteristic polynomials at closed points with the following properties (analogous to the ones used in the proof of Theorem 4.25).

- (a) On any curve in X , there is a (not necessarily irreducible) lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf giving rise to the specified characteristic polynomials.
- (b) The sheaves occurring in (a) obey a uniformity property: their pullbacks along some fixed alteration of X are all tame.

The polynomials $P(\mathcal{E}_x, T)$ satisfy these conditions: condition (a) follows from Corollary 3.11, noting that Lemma 4.16(b) and Lemma 4.27 together guarantee that we get a lisse étale $\overline{\mathbb{Q}}_\ell$ -sheaf, not just a lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaf; condition (b) follows from Proposition 2.37 and Corollary 3.15.

This proves that \mathcal{E} has an étale companion \mathcal{F} , which by Theorem 4.17(a) is again irreducible. By Corollary 4.20, $\det(\mathcal{F})$ is of finite order. \square

Corollary 4.29. *Conjecture 1.4 holds when $\ell' \neq p$, and Theorem 1.3 holds.*

Proof. Theorem 4.28(a) implies that Conjecture 1.4 holds when $\ell' \neq p$. As for Theorem 1.3, see Theorem 4.11(a) for part (i); see Corollary 4.26 for part (ii); see Lemma 4.16 for parts (iii), (iv); and see Theorem 4.28(b) for part (v). \square

Remark 4.30. As noted above, the semistable reduction theorem for overconvergent F -isocrystals [Ked22a, Theorem 7.6] was originally proved in terms of an uncontrolled alteration, which only becomes finite étale over some open dense subscheme of the original variety. However, for k finite, Theorem 4.28 makes it possible (in conjunction with Remark 2.38) to choose a finite étale cover of X with the property that any alteration factoring through this cover suffices to achieve semistable reduction; see [AE19, Remark 4.4] or [Ked22a, Remark 7.9].

4.6. Reductions for crystalline companions

One unexpected side benefit of the construction of étale companions is that it helps with some key reductions in the construction of crystalline companions. These reductions will be used in [Ked22b].

The following is essentially [D'Ad20, Proposition 4.3.6].

Lemma 4.31. *Let \mathcal{E}, \mathcal{F} be two algebraic coefficient objects which are companions. Then there is a bijection between the irreducible constituents of \mathcal{E} and those of \mathcal{F} , in which any two objects in correspondence are again companions.*

Proof. Suppose first that at least one of \mathcal{E} or \mathcal{F} is an étale coefficient object, say \mathcal{F} . Apply Theorem 4.28 to each irreducible constituent of \mathcal{E} to obtain objects in the same category as \mathcal{F} ; by Theorem 4.17(a), these objects are all irreducible. Taking their sum yields another companion \mathcal{F}' of \mathcal{E} in the same category as \mathcal{F} ; by Theorem 4.17(b), \mathcal{F} and \mathcal{F}' have the same semisimplification.

Suppose next that \mathcal{E} and \mathcal{F} are both crystalline objects. By Theorem 4.28, we may insert an étale companion \mathcal{G} between \mathcal{E} and \mathcal{F} , and then apply the previous paragraph to pass from \mathcal{E} to \mathcal{G} to \mathcal{F} . \square

Lemma 4.32. *Let $f : X' \rightarrow X$ be a finite étale morphism.*

- (a) *Let \mathcal{E} be an irreducible algebraic coefficient object on X , and let \mathcal{F} be a semisimple companion of $f^*\mathcal{E}$. Then there exist a companion \mathcal{F}_0 of \mathcal{E} and an isomorphism $f^*\mathcal{F}_0 \cong \mathcal{F}$.*
- (b) *Let \mathcal{E} be a semisimple algebraic coefficient object on X' , and let \mathcal{F} be a semisimple companion of $f_*\mathcal{E}$. Then there exists a companion \mathcal{F}_0 of \mathcal{E} which is a direct summand of $f^*\mathcal{F}$.*

Proof. We first treat (a). Since $f_*\mathcal{F}$ is a companion of $f_*f^*\mathcal{E}$ and \mathcal{E} arises as a direct summand of $f_*f^*\mathcal{E}$, we may apply Lemma 4.31 to deduce that \mathcal{E} has an irreducible companion \mathcal{F}_0 which is a constituent of $f_*\mathcal{F}$. By Proposition 2.30, $f^*\mathcal{F}_0$ is semisimple (note that $\overline{G}(\mathcal{F}_0)$ and $\overline{G}(f^*\mathcal{F}_0)$ have the same connected part thanks to Proposition 2.29). Since $f^*\mathcal{F}_0$ and \mathcal{F} are both semisimple companions of $f^*\mathcal{E}$, by Theorem 4.17 they are isomorphic to each other.

We next treat (b). By Proposition 2.30 again, $f^*\mathcal{F}$ is semisimple. Since $f^*\mathcal{F}$ is a semisimple companion of $f^*f_*\mathcal{E}$ and \mathcal{E} occurs as a direct summand of $f^*f_*\mathcal{E}$, we may again apply Lemma 4.31 to deduce that every constituent of \mathcal{E} has an irreducible companion which is a direct summand of $f^*\mathcal{F}$. This proves the claim. \square

Corollary 4.33. *Let $f : X' \rightarrow X$ be an alteration. Let \mathcal{E} be an algebraic coefficient object on X . If $f^*\mathcal{E}$ admits a crystalline companion, then so does \mathcal{E} .*

Proof. We may assume from the outset that \mathcal{E} is irreducible. Suppose first that f is flat (and hence finite and separable). Let \mathcal{F} be a crystalline companion of $f^*\mathcal{E}$. Let U be the open dense subscheme of X over which f is étale (and hence finite étale because f is proper). By Lemma 4.32(a), there exist a crystalline companion \mathcal{F}_0 for $\mathcal{E}|_U$ and an isomorphism $f^*\mathcal{F}_0 \cong \mathcal{F}$. This isomorphism gives rise to a descent datum for \mathcal{F} on the hypercovering of U generated by $f|_U$; by the full faithfulness of restriction (Lemma 2.3), this descent datum extends over X . By faithfully flat descent for modules, the descent datum arises from an object of $\mathbf{F}\text{-Isoc}^+(X) \otimes \overline{\mathbb{Q}}_p$ whose restriction to U is isomorphic to \mathcal{F}_0 ; by Lemma 2.3 again, \mathcal{F}_0 must itself extend to X . By Corollary 4.18, \mathcal{F}_0 is a crystalline companion of \mathcal{E} , as desired.

We now treat the general case. Let U be the open dense subscheme of X over which f is flat; since X' is smooth, the complement $Z := X \setminus U$ has codimension at least 2 in X . By the previous paragraph, $\mathcal{E}|_U$ admits a crystalline companion \mathcal{F} ; by Lemma 2.3, \mathcal{F} extends uniquely to X . By Corollary 4.18, \mathcal{F} is a crystalline companion of \mathcal{E} , as desired. \square

Corollary 4.34. *To prove Conjecture 1.1, it suffices to prove the following: for every X admitting a smooth compactification \overline{X} and every docile étale coefficient object \mathcal{E} on X which is irreducible with finite determinant, there exists an open dense subscheme U of X such that $\mathcal{E}|_U$ has a crystalline companion.*

Proof. By Lemma 2.3 (for preservation of irreducibility upon restriction to an open dense subscheme), Remark 2.33 (preservation of docileness upon passage to subquotients), and Corollary 4.19, the hypothesis would imply that every docile ℓ -adic coefficient object has a crystalline companion. By Proposition 2.37 and Corollary 4.33, we would then obtain Conjecture 1.4 for $\ell' = p$. By Corollary 4.20 and Lemma 4.31, the crystalline companion of an étale coefficient object which is irreducible of finite determinant is again irreducible of finite determinant. In light of Theorem 1.3, this proves the claim. \square

4.7. Comparison with Abe–Esnault and Lefschetz slicing

As mentioned above, there is a strong overlap between the results described in this section and those of Abe–Esnault [AE19]; the main differences are in the handling of Lefschetz slicing. We first indicate the main differences, then indicate how to reconcile the results.

Remark 4.35. For tame crystalline coefficient objects, Abe–Esnault prove a form of the slicing principle without recourse to weights [AE19, Theorem 2.5]. This then yields the existence of étale companions for tame crystalline coefficient objects [AE19, Proposition 2.7] using the arguments of Deligne and Drinfeld described above.

Abe–Esnault then use the existence of étale companions in the tame case to establish a form of the slicing principle without the tame restriction [AE19, Theorem 3.10]. This in turn yields the existence of étale companions as above.

Remark 4.36. Note that the forms of the slicing principle obtained in [AE19] are *a priori* stronger than Lemma 4.14, in that there is no base extension necessary. On the other hand, with Theorem 1.3 in hand, one can transfer *any* version of the Lefschetz slicing principle from the étale case to the crystalline case; in particular, one can recover these stronger statements *a posteriori* by working on the étale side, as demonstrated by Theorem 4.37 and Theorem 4.41 below. However, even in the étale case some care is required; for example, an incorrect version of the slicing principle is asserted in [Laf02, VII.7]. See [Esn17, Theorem 8.1] for further discussion of this point, and the rest of that article for a broader discussion of Lefschetz slicing.

The following result reproduces [AE19, Theorem 2.5], modulo a theorem of Poonen [Poo04, Theorem 1.1] which can be used to produce curves as described passing through any finite set of closed points. Compare [Dri12, Proposition C.1]. (See [Esn17, Lemma 5.4] for an example showing that the tame hypothesis cannot be omitted.)

Theorem 4.37. *Let U be an open dense affine subscheme of X . Let H_1, \dots, H_{n-1} be very ample divisors on U which intersect transversely. Let \mathcal{E} be a tame irreducible coefficient object on X . Then the restriction of \mathcal{E} to $C := H_1 \cap \dots \cap H_{n-1}$ is irreducible.*

Proof. By Lemma 2.3, $\mathcal{E}|_U$ is irreducible; we may thus assume hereafter that $U = X$. Using Lemma 2.4, we may reduce to the case where \mathcal{E} has determinant of finite order; by Corollary 4.26, \mathcal{E} is algebraic. By Theorem 4.28 and Lemma 4.31, \mathcal{E} admits an irreducible ℓ -adic companion for any $\ell \neq p$; we may thus further reduce to the étale case. In this setting, the claim follows from the fact that for any connected finite étale cover W of X which is tamely ramified at the boundary, $W \times_X C$ is also connected; this follows from the Bertini connectedness theorem [FL81, Theorem 2.1(A)] as in [Dri12, Lemma C.2]. \square

We now turn around and invoke a lemma from [AE19] that allows one to formally upgrade Theorem 4.37.

Lemma 4.38 (Abe–Esnault). *Let $f : Y \rightarrow X$ be a morphism of smooth connected k -schemes. Let \mathcal{E} be a coefficient object on X . Suppose that for each positive integer n , for $\mathcal{F} := (\mathcal{E} \oplus \mathcal{E}^\vee)^{\otimes n}$, the map $H^0(X, \mathcal{F}) \rightarrow H^0(Y, f^* \mathcal{F})$ is an isomorphism. Then the morphism $\overline{G}(\mathcal{E}) \rightarrow \overline{G}(f^* \mathcal{E})$ (defined with respect to a consistent choice of base points) is an isomorphism.*

Proof. This is essentially [AE19, Proposition 1.9] modulo a slight change of hypotheses; we sketch the argument to show that nothing is affected. Let $[\mathcal{E}]$ be the Tannakian category of geometric coefficient objects generated by \mathcal{E} . By hypothesis, f^* restricts to a fully faithful functor on $[\mathcal{E}]$. This is not enough to prove the claim; however, a quick group-theoretic argument [AE19, Lemma 1.7] shows that this does hold provided that every $\mathcal{G} \in [\mathcal{E}]$ of rank 1 is of finite order. By Remark 2.27, \mathcal{G} promotes to a true coefficient object on X_n for some n ; we may thus apply Lemma 2.4 to deduce the claim. \square

Corollary 4.39. *With notation as in Theorem 4.37, the induced morphism $\overline{G}(\mathcal{E}) \rightarrow \overline{G}(\mathcal{E}|_C)$ (defined with respect to a consistent choice of base points) is an isomorphism.*

Proof. By Theorem 4.37, for every constituent of $(\mathcal{E} \oplus \mathcal{E}^\vee)^{\otimes n}$ as a coefficient object, the pullback to C is again an irreducible coefficient object. By Lemma 2.28, the same is true for geometric coefficient objects. We may thus apply Lemma 4.38 to conclude. \square

On a related note, we mention the following result.

Corollary 4.40. *For any positive integer r , any category of coefficient objects only contains finitely many isomorphism classes of irreducible tame coefficient objects of rank r on X up to twisting by constant objects of rank 1.*

Proof. By Theorem 4.28, it suffices to check the case of a category of étale coefficient objects. By Lemma 2.3, we may assume that X is affine. By Corollary 4.39 and the aforementioned theorem of Poonen, we can find a curve C in X such that restriction of coefficient objects from X to C is fully faithful; by Corollary 3.18, we deduce the claim. \square

Without the tameness hypothesis, one has the following result which reproduces [AE19, Theorem 3.10].

Theorem 4.41. *Let \mathcal{E} be an irreducible coefficient object on X . There is a dense open subscheme U of X such that for any finite subset $S \subseteq U^\circ$, there exist a curve C and a morphism $f : C \rightarrow X$ over k such that f admits a section on S and $f^*\mathcal{E}$ is irreducible.*

Proof. We may reduce to the étale case as in the proof of Theorem 4.37; then [Esn17, Theorem 8.1] gives the desired result. (This amounts to applying Hilbert irreducibility in the form of [Dril2, Theorem 2.15].) \square

4.8. On a conjecture of de Jong

The following conjecture is due to de Jong, as reported in [ES18, Shi14] where some special cases are established.

Conjecture 4.42 (de Jong). *Let X be a smooth, proper, geometrically connected variety over a (not necessarily finite) perfect field k of characteristic p with trivial geometric étale fundamental group. Then every convergent isocrystal on X is constant.*

Using étale companions, one obtains the following result in the direction of this conjecture. (Compare [AE19, Remark 4.5].)

Proposition 4.43. *Suppose that (k is finite and) X is proper over k and has trivial geometric étale fundamental group. Then the functor*

$$2\text{-}\varinjlim_{n \rightarrow \infty} \mathbf{F}\text{-Isoc}^\dagger(\mathrm{Spec} k_n) \otimes \overline{\mathbb{Q}}_p \rightarrow 2\text{-}\varinjlim_{n \rightarrow \infty} \mathbf{F}\text{-Isoc}^\dagger(X_n) \otimes \overline{\mathbb{Q}}_p$$

is an equivalence of categories.

Proof. We are free to enlarge k , so we may assume that X contains a k -rational point; we may also assume that X is irreducible. Since X contains a k -rational point, the functor $\mathbf{F}\text{-Isoc}^\dagger(\mathrm{Spec} k) \otimes \overline{\mathbb{Q}}_p \rightarrow \mathbf{F}\text{-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$ is fully faithful; moreover, it preserves extension groups. Consequently, it remains to check that the essential image contains an arbitrary irreducible $\mathcal{E} \in \mathbf{F}\text{-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$, possibly after enlarging k . At this point, there is no harm in twisting by a constant rank-1 object; by Lemma 2.4, we may thus reduce to the case where $\det(\mathcal{E})$ is of finite order. By Theorem 4.28, \mathcal{E} arises as a crystalline companion of some lisse Weil $\overline{\mathbb{Q}}_\ell$ -sheaf. By hypothesis, the latter must arise by pullback from k , as then must \mathcal{E} by Theorem 4.17(b). \square

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