
Maximality of moduli spaces of vector bundles on curves

Erwan Brugallé and Florent Schaffhauser

Abstract. We prove that moduli spaces of semistable vector bundles of coprime rank and degree over a non-singular real projective curve are maximal real algebraic varieties if and only if the base curve itself is maximal. This provides a new family of maximal varieties, with members of arbitrarily large dimension. We prove the result by comparing the Betti numbers of the real locus to the Hodge numbers of the complex locus and showing that moduli spaces of vector bundles over a maximal curve actually satisfy a property which is stronger than maximality and that we call Hodge-expressivity. We also give a brief account on other varieties for which this property was already known.

Keywords. Maximal real algebraic varieties, Hodge numbers, moduli spaces of vector bundles

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Notation

A real algebraic variety is a pair (X, τ) where X is a complex algebraic variety equipped with an antiholomorphic involution $\tau : X \rightarrow X$. We denote by $\mathbb{R}X$ the set of its real points, that is to say the fixed point set of τ . Throughout this note, complex algebraic varieties are endowed with their Euclidean topology. Except otherwise stated, we consider cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Given a smooth compact manifold M and a non-singular projective complex algebraic variety X , we denote by $P_t(M)$ the mod 2 Poincaré polynomial of M and by $H_{(x,y)}(X)$ the Hodge polynomial of X , *i.e.*

$$P_t(M) = \sum_{i \geq 0} b_i(M)t^i \quad \text{and} \quad H_{(x,y)}(X) = \sum_{i,j \geq 0} h^{i,j}(X)x^i y^j.$$

1. Introduction

1.1. Maximal and Hodge-expressive real algebraic varieties

Let X be a real algebraic variety (not necessarily projective nor smooth). It is a consequence of Smith theory (see for example [Man17, Section 3.3]) that the mod 2 Betti numbers of $\mathbb{R}X$ and X satisfy the following Smith–Thom inequality:

$$\sum_{i=0}^{\dim X} b_i(\mathbb{R}X) \leq \sum_{i=0}^{2 \dim X} b_i(X).$$

When equality holds, the real algebraic variety X is said to be *maximal*. Maximal varieties constitute extremal objects in real algebraic geometry that enjoy special properties. We refer to the book [Man17] or to the survey [DK00] for an account on the subject.

To date, not many maximal real algebraic varieties are known. It is standard that Grassmannians equipped with their standard real structure (*e.g.* real projective spaces) are maximal, as well as non-singular (or mildly singular) real toric varieties [BFMvH06, Fra22]. One also easily constructs maximal real algebraic (hyperelliptic) curves of arbitrary genus. Furthermore, the Jacobian of a real algebraic curve with non-empty real part is maximal if and only if the curve is maximal [GH81]. Knowledge starts to be more fragmented in the case of surfaces, we refer the interested reader to [Man17, Chapter 4] or [DK00, Section 3], as well as to Section 3. Things are of course getting worse when increasing the dimension; the knowledge of the authors essentially reduces to the classifications of maximal cubic real hypersurfaces of dimension 3 and 4 [Kra09, FK10], and to the unpublished construction by Itenberg and Viro, dating back to the 1990’s, of maximal real projective hypersurfaces of arbitrary degree and dimension.

The goal of this note is to provide a new family of maximal real algebraic varieties with members of arbitrarily large dimension: moduli spaces of vector bundles of coprime rank and degree over a maximal real algebraic curve.

Poincaré polynomials of the complex and real part of these moduli spaces have been computed in [AB83] and [LS13] respectively (see also [Bai14]). It turns out, however, that equating both sums of Betti numbers directly is quite intricate (see [LS13] for the rank 2 case). So, in Theorem 1.2, we prove a stronger statement instead: the i^{th} Betti number of the real part of the moduli space is equal to the i^{th} ascending diagonal sum of the Hodge diamond of its complex part (see Figure 1). This implies maximality since the moduli spaces in question have torsion-free integral cohomology [AB83].

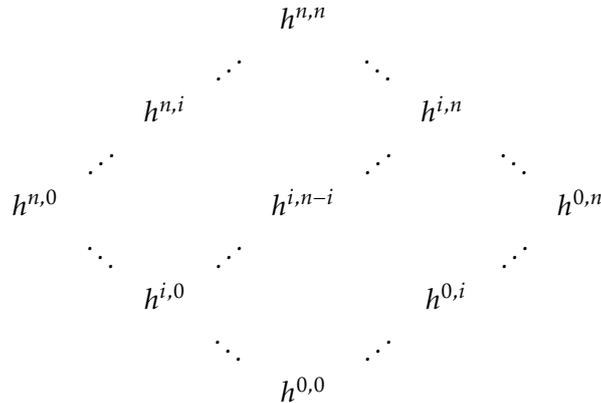


Figure 1. Ascending diagonals of the Hodge diamond.

The idea of deducing maximality from this stronger statement comes from the old empirical observation that, although few maximal real algebraic varieties are currently known, a seemingly large proportion of them, that we suggest to call *Hodge-expressive*, in fact satisfy this stronger property.

Definition 1.1. A non-singular projective real algebraic variety X is called Hodge-expressive if $H^*(X; \mathbb{Z})$ is torsion free and

$$P_t(\mathbb{R}X) = H_{(t,1)}(X).$$

Clearly, a Hodge-expressive variety is maximal. As mentioned above, such varieties already appear in the literature and, in an informal way, one may say that Hodge-expressive real algebraic varieties constitute the basic maximal real algebraic varieties. In Section 3, we provide a brief panorama of known Hodge-expressive varieties. Usually, Hodge-expressivity of a real algebraic variety is a consequence of a prior proof of its maximality. Here, we propose to go the other way round: maximality is established as a consequence of Hodge-expressivity. From what we know, this is also the strategy used in the aforementioned unpublished construction by Itenberg and Viro.

Note that Hodge-expressive varieties satisfy $\chi = \sigma$ (in the terminology of [Bru22]), since

$$\chi(\mathbb{R}X) = P_{-1}(\mathbb{R}X) = H_{(-1,1)}(X) = \sigma(X),$$

where the last equality is the Hodge index Theorem.

1.2. Moduli spaces of vector bundles over real algebraic curves

Let \mathcal{C} be a non-singular connected projective complex algebraic curve of genus $g \geq 1$. Given two integer numbers $r \geq 1$ and $d \in \mathbb{Z}$, we denote by $\mathcal{M}_{\mathcal{C}}(r, d)$ the moduli space of semistable holomorphic vector bundles of rank r and degree d over \mathcal{C} . When r and d are coprime, the space $\mathcal{M}_{\mathcal{C}}(r, d)$ is a non-singular complex projective variety of dimension $r^2(g-1) + 1$. For the rest of this discussion, we assume that $g \geq 2$. Using the number-theoretic approach to the cohomology of moduli spaces of vector bundles developed by Harder and Narasimhan in [HN75], Desale and Ramanan obtained in [DR75] a recursive formula for the rational Poincaré polynomial of $\mathcal{M}_{\mathcal{C}}(r, d)$. Then, using gauge theory, Atiyah and Bott proved in [AB83] that

the integral cohomology of $\mathcal{M}_{\mathcal{C}}(r, d)$ is torsion-free and gave an alternative proof of the recursive formula by Desale and Ramanan. Later on, Zagier obtained a closed formula for $P_t(\mathcal{M}_{\mathcal{C}}(r, d))$ in [Zag96], and Earl and Kirwan used a finite-dimensional analogue of the Atiyah–Bott approach to prove a recursive formula for the Hodge polynomial $H_{(x,y)}(\mathcal{M}_{\mathcal{C}}(r, d))$ in [EK00]. A consequence of these formulas (either recursive or closed) is that Betti and Hodge numbers of $\mathcal{M}_{\mathcal{C}}(r, d)$ depend only on the topological data g and r (and not d).

When \mathcal{C} is a non-singular real projective curve, the anti-holomorphic involution $\tau : \mathcal{C} \rightarrow \mathcal{C}$ induces a real structure $\mathcal{E} \mapsto \overline{\tau^* \mathcal{E}}$ on $\mathcal{M}_{\mathcal{C}}(r, d)$, turning the moduli space $\mathcal{M}_{\mathcal{C}}(r, d)$ into a real algebraic variety as well. In fact, if r and d are coprime and \mathcal{C} has real points, the real locus of $\mathcal{M}_{\mathcal{C}}(r, d)$ consists exactly of isomorphism classes of geometrically stable real vector bundles of rank r and degree d (cf. [Sch12]), where geometric stability of a real bundle (\mathcal{E}, τ) means that the underlying complex bundle \mathcal{E} is stable. In [LS13], Liu and Schaffhauser developed a real analogue of the Atiyah–Bott approach to obtain a recursive formula, as well as a closed formula, computing $P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d))$. As in the complex situation, the Betti numbers of $\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)$ are seen *a posteriori* not to depend on d but only on g, r and $b_0(\mathbb{R}\mathcal{C})$. Note that $b_0(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) = 2^{b_0(\mathbb{R}\mathcal{C})-1}$ by [Sch12].

Given a line bundle Λ of degree d on \mathcal{C} , one may also consider the moduli space $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ of semistable holomorphic vector bundles of rank r with fixed determinant Λ . By definition, this space is a fiber of the determinant map $\mathcal{M}_{\mathcal{C}}(r, d) \rightarrow \text{Pic}^d(\mathcal{C})$. When r and d are coprime, the space $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ is a non-singular complex projective variety of dimension $(r^2 - 1)(g - 1)$, which is real if both \mathcal{C} and Λ are real. Both polynomials $H_{(x,y)}(\mathcal{M}_{\mathcal{C}}(r, \Lambda))$ and $P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda))$ can be deduced from those of $\mathcal{M}_{\mathcal{C}}(r, d)$, see Section 2.3.

The next statement is the main result of this note.

Theorem 1.2. *Let \mathcal{C} be a maximal non-singular real projective curve of genus $g \geq 1$. Let $r \geq 1$ and $d \in \mathbb{Z}$ be coprime integers, and let Λ be a real line bundle of degree d on \mathcal{C} . Then both moduli spaces $\mathcal{M}_{\mathcal{C}}(r, d)$ and $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ are Hodge-expressive.*

In particular, these moduli spaces are maximal when \mathcal{C} is maximal. The next proposition shows that the converse is true for the moduli spaces $\mathcal{M}_{\mathcal{C}}(r, d)$.

Proposition 1.3. *Let \mathcal{C} be a non-singular real projective curve of genus $g \geq 1$ and such that $\mathbb{R}\mathcal{C} \neq \emptyset$. Let $r \geq 1$ and $d \in \mathbb{Z}$ be coprime integers. If \mathcal{C} is not maximal, then neither is $\mathcal{M}_{\mathcal{C}}(r, d)$.*

Note that $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ is maximal when either $g = 1$ or $r = 1$ (provided $\mathbb{R}\mathcal{C} \neq \emptyset$), since in this case it is reduced to a point. Whether $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ is non-maximal when \mathcal{C} is non-maximal and $g, r \geq 2$ remains an open question. As noted in [LS13], the explicit formulas for the mod 2 Betti numbers of $\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ quickly become too complex to be evaluated at $t = 1$ (and similar remarks apply for the Hodge numbers of $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$). It was however checked in [LS13], using a computer, that this is indeed the case up until $r = 6$. As is visible from Section 2.3, our proof of Proposition 1.3 does not allow us to conclude that $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ is not maximal when \mathcal{C} is not maximal and $g, r \geq 2$.

1.3. Outline of the paper

Section 2 is devoted to the proof of Theorem 1.2 and Proposition 1.3. In Section 3, we give a panorama of known maximal and Hodge-expressive real algebraic varieties.

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2. Hodge-expressivity for moduli spaces of vector bundles

In this section we prove Theorem 1.2 and Proposition 1.3. We first treat the particular cases $g = 1$ and $r = 1$ in Sections 2.1 and 2.2 respectively. As a consequence, we prove Proposition 1.3 in Section 2.3 and show that it suffices to prove Theorem 1.2 for $\mathcal{M}_{\mathcal{C}}(r, d)$. This is achieved in Sections 2.4 and 2.5.

2.1. The case $g = 1$

If \mathcal{C} is of genus 1 and $\gcd(r, d) = 1$, the determinant map $\mathcal{M}_{\mathcal{C}}(r, d) \rightarrow \text{Pic}^d(\mathcal{C})$ is an isomorphism of algebraic varieties [Tu93]. In particular, one gets an isomorphism $\mathcal{M}_{\mathcal{C}}(r, d) \simeq \mathcal{C}$ after a point $x_0 \in \mathcal{C}$ has been chosen. In the case when \mathcal{C} is real and $x_0 \in \mathbb{R}\mathcal{C}$, this isomorphism is real by [BS16], proving Theorem 1.2 and Proposition 1.3 in the case $g = 1$.

From now on, we suppose that $g \geq 2$.

2.2. The case $r = 1$

The case of moduli spaces of line bundles over \mathcal{C} is well known, since these moduli spaces are the Picard varieties $\text{Pic}^d(\mathcal{C})$. In particular one has

$$H_{(x,y)}(\text{Pic}^d(\mathcal{C})) = (1+x)^g(1+y)^g.$$

On the other hand by [GH81], one has

$$P_t(\mathbb{R}\text{Pic}^d(\mathcal{C})) = 2^{b_0(\mathbb{R}\mathcal{C})-1} P_t((\mathbb{S}^1)^g) = 2^{b_0(\mathbb{R}\mathcal{C})-1} (1+t)^g$$

when $\mathbb{R}\mathcal{C} \neq \emptyset$. Since \mathcal{C} is maximal if and only if $b_0(\mathbb{R}\mathcal{C}) = g+1$, this proves Theorem 1.2 and Proposition 1.3 in the case $r = 1$.

2.3. Preliminaries

Recall that the groups $H^*(\mathcal{M}_{\mathcal{C}}(r, d); \mathbb{Z})$ and $H^*(\mathcal{M}_{\mathcal{C}}(r, \Lambda); \mathbb{Z})$ are torsion-free by [AB83]. Hence it is enough to prove the two identities

$$P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) = H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, d)) \quad \text{and} \quad P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda)) = H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, \Lambda)).$$

Moreover when $\gcd(r, d) = 1$, it is proved in [EK00] and in [Bai20], respectively, that

$$H_{(x,y)}(\mathcal{M}_{\mathcal{C}}(r, d)) = H_{(x,y)}(\text{Pic}^d(\mathcal{C}))H_{(x,y)}(\mathcal{M}_{\mathcal{C}}(r, \Lambda))$$

and

$$P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) = P_t(\mathbb{R}\text{Pic}^d(\mathcal{C}))P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda)).$$

In view of Section 2.2, this gives

$$(2.1) \quad H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, d)) = 2^g(1+t)^g H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, \Lambda)),$$

and

$$P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) = 2^{b_0(\mathbb{R}\mathcal{C})-1} (1+t)^g P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda)).$$

In particular, applying the Smith–Thom inequality to $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$, we get:

$$\begin{aligned} \sum_{i \geq 0} b_i(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) &= P_1(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) \\ &= 2^{b_0(\mathbb{R}\mathcal{C})-1+g} P_1(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, \Lambda)) \\ &\leq 2^{2g} P_1(\mathcal{M}_{\mathcal{C}}(r, \Lambda)) = \sum_{i \geq 0} b_i(\mathcal{M}_{\mathcal{C}}(r, d)) \end{aligned}$$

and equality can only hold here if $b_0(\mathbb{R}\mathcal{C}) = g + 1$, that is to say if \mathcal{C} is maximal. This proves Proposition 1.3. \square

Next, when \mathcal{C} is maximal, we see from (2.1) that $\mathcal{M}_{\mathcal{C}}(r, d)$ is Hodge-expressive if and only if $\mathcal{M}_{\mathcal{C}}(r, \Lambda)$ is Hodge-expressive. To prove Theorem 1.2, it is therefore sufficient to prove

$$(2.2) \quad P_t(\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)) = H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, d)).$$

In order to do so, we will show in the next two sections, using results from [EK00, LS13], that the polynomials $H_{(t,1)}(\mathcal{M}_{\mathcal{C}}(r, d))$ and $P_t(\mathcal{M}_{\mathcal{C}}(r, d))$ satisfy the same recursion relation.

2.4. Hodge numbers of $\mathcal{M}_{\mathcal{C}}(r, d)$

Here we recast the computation from [AB83, EK00] of Poincaré and Hodge polynomials of the moduli spaces $\mathcal{M}_{\mathcal{C}}(r, d)$. For all $(r, d) \in \mathbb{Z}_{>0} \times \mathbb{Z}$, we denote by $Bun_{\mathcal{C}}(r, d)$ the moduli stack of all vector bundles of rank r and degree d on \mathcal{C} . It contains, as an open substack, the moduli stack of semistable vector bundles of rank r and degree d on \mathcal{C} , which we denote by $Bun_{\mathcal{C}}^{ss}(r, d)$. If we fix a C^∞ complex vector bundle E of rank r and degree d over \mathcal{C} , there is an isomorphism of stacks

$$Bun_{\mathcal{C}}(r, d) \simeq [\mathcal{A}_E/\mathcal{G}_E],$$

where \mathcal{A}_E is the set of Dolbeault operators (*i.e.* holomorphic structures) on E and \mathcal{G}_E is the automorphism group of E (*i.e.* the gauge group). In particular, we can think of the cohomology of the stack $Bun_{\mathcal{C}}(r, d)$ in the sense of [Beh05] simply as the \mathcal{G}_E -equivariant cohomology of the affine space \mathcal{A}_E . By [AB83], the integral cohomology of $Bun_{\mathcal{C}}(r, d)$ is torsion-free. Moreover, its Poincaré series does not depend on d and is given by

$$P_t(Bun_{\mathcal{C}}(r, d)) = \frac{(1+t)^{2g}}{1-t^2} \prod_{i=2}^r \frac{(1+t^{2i-1})^{2g}}{(1-t^{2i-2})(1-t^{2i})}.$$

By [HN75], an algebraic vector bundle \mathcal{E} over \mathcal{C} admits a unique filtration, called the *Harder-Narasimhan filtration*,

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_\ell = \mathcal{E}$$

such that:

- for all $i \in \{1, \dots, \ell\}$, the vector bundle $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable;
- setting $r_i := \text{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$ and $d_i := \text{deg}(\mathcal{E}_i/\mathcal{E}_{i-1})$, we have

$$\frac{d_1}{r_1} > \dots > \frac{d_\ell}{r_\ell}.$$

The topological invariants $(r_i, d_i)_{1 \leq i \leq \ell}$ of the successive quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ constitute the *Harder-Narasimhan type* (or HN type) of the vector bundle \mathcal{E} . If \mathcal{E} is of rank r and degree d , then its HN type is subject to the following constraints:

- (1) $d_1 + \dots + d_\ell = d$,
- (2) $r_1 + \dots + r_\ell = r$,
- (3) $\frac{d_1}{r_1} > \dots > \frac{d_\ell}{r_\ell}$.

By definition, the vector bundle \mathcal{E} is semistable if and only if it is of HN type (r, d) , which we denote by μ_{ss} . We denote by $I_{r,d}$ the set of all possible tuples of integers $\mu = (r_i, d_i)_{1 \leq i \leq \ell_\mu}$ satisfying properties (1), (2), (3) above. Note that this set is infinite as soon as $r \geq 2$. For instance:

$$I_{2,d} = \left\{ (2, d), (1, k, 1, d-k) \mid k > \frac{d}{2} \right\}.$$

For all $\mu \in I_{r,d}$, there exists an algebraic substack $Bun_{\mathcal{C}}(\mu)$ of $Bun_{\mathcal{C}}(r,d)$, parameterizing vector bundles of type μ , with $Bun_{\mathcal{C}}(\mu_{\text{ss}}) = Bun_{\mathcal{C}}^{\text{ss}}(r,d)$. The codimension of $Bun_{\mathcal{C}}(\mu)$ is finite and, for all $\mu \neq \mu_{\text{ss}}$, it is equal to

$$d_{\mu} := \sum_{1 \leq i < j \leq l(\mu)} (r_j d_i - r_i d_j + r_i r_j (g-1)).$$

Moreover, there is an isomorphism of \mathbb{Q} -vector spaces

$$H^*(Bun_{\mathcal{C}}(\mu); \mathbb{Q}) \simeq \bigotimes_{i=1}^{\ell_{\mu}} H^*(Bun_{\mathcal{C}}^{\text{ss}}(r_i, d_i); \mathbb{Q}).$$

In particular, we have, for all $\mu \in I_{r,d}$,

$$P_t(Bun_{\mathcal{C}}(\mu)) = \prod_{i=1}^{\ell_{\mu}} P_t(Bun_{\mathcal{C}}^{\text{ss}}(r_i, d_i)).$$

Finally, the stratification of the moduli stack $Bun_{\mathcal{C}}(r,d)$ by the substacks $Bun_{\mathcal{C}}(\mu)$ is *perfect* in the sense that the associated Gysin long exact sequence breaks up into short exact sequences [AB83]. This implies that

$$P_t(Bun_{\mathcal{C}}(r,d)) = \sum_{\mu \in I_{r,d}} t^{2d_{\mu}} P_t(Bun_{\mathcal{C}}(\mu)),$$

yielding the recursive formula

$$P_t(Bun_{\mathcal{C}}^{\text{ss}}(r,d)) = P_t(Bun_{\mathcal{C}}(r,d)) - \sum_{\mu \neq \mu_{\text{ss}}} t^{2d_{\mu}} \prod_{i=1}^{\ell_{\mu}} P_t(Bun_{\mathcal{C}}^{\text{ss}}(r_i, d_i)).$$

When r and d are coprime, all semistable vector bundles of rank r and degree d are stable. Atiyah and Bott have shown that, in this case, the Poincaré polynomial of $\mathcal{M}_{\mathcal{C}}(r,d)$ is related to the Poincaré series of $Bun_{\mathcal{C}}^{\text{ss}}(r,d)$ via the identity

$$P_t(\mathcal{M}_{\mathcal{C}}(r,d)) = (1-t^2) P_t(Bun_{\mathcal{C}}^{\text{ss}}(r,d)).$$

Altogether, setting $Q_t^{\mathbb{C}}(r,d) = (1-t^2) P_t(Bun_{\mathcal{C}}^{\text{ss}}(r,d))$, one obtains the recursive formula

$$(2.3) \quad Q_t^{\mathbb{C}}(r,d) = (1+t)^{2g} \prod_{i=2}^r \frac{(1+t^{2i-1})^{2g}}{(1-t^{2i-2})(1-t^{2i})} - \sum_{\mu \in I_{r,d} \setminus \{\mu_{\text{ss}}\}} \frac{t^{2d_{\mu}}}{(1-t^2)^{\ell_{\mu}-1}} \prod_{i=1}^{\ell_{\mu}} Q_t^{\mathbb{C}}(r_i, d_i),$$

expressing all polynomials $Q_t^{\mathbb{C}}(r,d)$ in terms of the initial term

$$Q_t^{\mathbb{C}}(1,d) = P_t(\mathcal{M}_{\mathcal{C}}(1,d)) = P_t(\text{Pic}^d(\mathcal{C})) = (1+t)^{2g}.$$

In [EK00], Earl and Kirwan obtained a similar recursive formula for the Hodge polynomial of $\mathcal{M}_{\mathcal{C}}(r,d)$ when r and d are coprime. Namely, setting $Q_{(x,y)}(r,d) = H_{(x,y)}(\mathcal{M}_{\mathcal{C}}(r,d))$ and using the construction of $\mathcal{M}_{\mathcal{C}}(r,d)$ as a GIT quotient, they proved that

$$(2.4) \quad Q_{(x,y)}(r,d) = (1-xy) G_{(x,y)}(r,d) - \sum_{\mu \in I_{r,d} \setminus \{\mu_{\text{ss}}\}} \frac{(xy)^{d_{\mu}}}{(1-xy)^{\ell_{\mu}-1}} \prod_{i=1}^{\ell_{\mu}} Q_{(x,y)}(r_i, d_i),$$

expressing recursively all polynomials $Q_{(x,y)}(r,d)$ in terms of the power series

$$G_{(x,y)}(r,d) := \frac{(1+x)^g (1+y)^g}{1-xy} \prod_{i=2}^r \frac{(1+x^{i-1}y^i)^g (1+x^i y^{i-1})^g}{(1-x^{i-1}y^{i-1})(1-x^i y^i)}$$

and the initial term

$$Q_{(x,y)}(1, d) = (1 - xy)G_{(x,y)}(1, d) = (1 + x)^g(1 + y)^g = H_{(x,y)}(\text{Pic}^d(\mathcal{C})).$$

Plugging $x = y = t$ in equation (2.4), one finds again the recursion (2.3). And plugging $x = t$ and $y = 1$ in equation (2.4), one obtains the following recursion:

$$(2.5) \quad Q_{(t,1)}(r, d) = 2^g(1 + y)^g \prod_{i=2}^r \frac{(1 + t^{i-1})^g(1 + t^i)^g}{(1 - t^{i-1})(1 - t^i)} - \sum_{\mu \in I_{r,d} \setminus \{\mu_{ss}\}} \frac{t^{d_\mu}}{(1 - t)^{\ell_\mu - 1}} \prod_{i=1}^{\ell_\mu} Q_{(t,1)}(r_i, d_i),$$

which will be useful to prove Theorem 1.2.

Remark 2.1. Although this is not how Earl and Kirwan prove their result, the power series $G_{(x,y)}(r, d)$ can be interpreted as the Hodge series of the moduli stack $Bun_{\mathcal{C}}(r, d)$ [Tel98, BD07].

2.5. Betti numbers of $\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)$

When the curve \mathcal{C} is defined over the reals, so is the moduli stack $Bun_{\mathcal{C}}(r, d)$. Here, we recast the computation of the mod 2 Poincaré polynomial of $\mathbb{R}\mathcal{M}_{\mathcal{C}}(r, d)$ from [LS13]. We denote by $n := b_0(\mathbb{R}\mathcal{C})$ the number of connected components of $\mathbb{R}\mathcal{C}$, and we assume that $n > 0$. The stack $\mathbb{R}Bun_{\mathcal{C}}(r, d)$ is a disjoint union

$$\mathbb{R}Bun_{\mathcal{C}}(r, d) = \bigsqcup_s Bun_{\mathcal{C}}^{\mathbb{R}}(r, d, s)$$

of 2^{n-1} connected components, each of which is indexed by the possible real invariants of a real vector bundle (\mathcal{E}, τ) of rank r and degree d , namely the first Stiefel-Whitney class

$$s = (s_1, \dots, s_n) \in (\mathbb{Z}/2\mathbb{Z})^n$$

of the vector bundle $\mathbb{R}\mathcal{E}$ over $\mathbb{R}\mathcal{C} = \sqcup_{i=1}^n \mathbb{S}^1$, subject to the condition $s_1 + \dots + s_n = d \pmod{2}$. The substack $Bun_{\mathcal{C}}^{\mathbb{R}}(r, d, s)$ is the stack of all real vector bundles of rank r , degree d and real type s . Fixing a C^∞ real vector bundle (E, τ) with these invariants (r, d, s) , we get an isomorphism of stacks

$$Bun_{\mathcal{C}}^{\mathbb{R}}(r, d, s) \simeq [\mathcal{A}_E^\tau / \mathcal{G}_E^\tau],$$

where $\mathcal{A}^\tau \subset \mathcal{A}$ is the set of all τ -fixed Dolbeault operators (for the real structure on \mathcal{A}_E induced by the real structure of E), and $\mathcal{G}_E^\tau \subset \mathcal{G}_E$ is the group of gauge transformations of E that commute with τ . It turns out that the mod 2 Poincaré series of $Bun_{\mathcal{C}}^{\mathbb{R}}(r, d, s)$ is independent of d and s [LS13]. As a consequence, the space $\mathbb{R}Bun_{\mathcal{C}}(r, d)$ has mod 2 Poincaré series

$$P_t(\mathbb{R}Bun_{\mathcal{C}}(r, d)) = 2^{n-1} \frac{(1+t)^g}{1-t} \prod_{i=2}^r \frac{(1+t^{2i-1})^{g+1-n} (1+t^{i-1})^{n-1} (1+t^i)^{n-1}}{(1-t^{i-1})(1-t^i)}.$$

Because of the uniqueness of the destabilizing sub-bundle, semistability over \mathbb{R} is equivalent to semistability over \mathbb{C} and all sub-bundles $(\mathcal{E}_i)_{1 \leq i \leq \ell}$ in the Harder-Narasimhan filtration of a real algebraic vector bundle \mathcal{E} over \mathcal{C} are real. So the substack $Bun_{\mathcal{C}}(\mu)$ is defined over \mathbb{R} and $\mathbb{R}Bun_{\mathcal{C}}(\mu)$ is the stack of all real vector bundles of HN type μ . Moreover, for all $\mu \in I_{r,d}$, there is an isomorphism of $(\mathbb{Z}/2\mathbb{Z})$ -vector spaces

$$H^*(\mathbb{R}Bun_{\mathcal{C}}(\mu); \mathbb{Z}/2\mathbb{Z}) \simeq \bigotimes_{i=1}^{\ell_\mu} H^*(\mathbb{R}Bun_{\mathcal{C}}^{ss}(r_i, d_i); \mathbb{Z}/2\mathbb{Z}).$$

In particular, we have, for all $\mu \in I_{r,d}$,

$$P_t(\mathbb{R}Bun_{\mathcal{C}}(\mu)) = \prod_{i=1}^{\ell_\mu} P_t(\mathbb{R}Bun_{\mathcal{C}}^{ss}(r_i, d_i)).$$

Finally, the substacks $(\mathbb{R}Bun_{\mathcal{E}}(\mu))_{\mu \in I_{r,d}}$ form a stratification of $\mathbb{R}Bun_{\mathcal{E}}(r,d)$ which is perfect over $\mathbb{Z}/2\mathbb{Z}$, so

$$P_t(\mathbb{R}Bun_{\mathcal{E}}(r,d)) = \sum_{\mu \in I_{r,d}} t^{d_{\mu}} P_t(\mathbb{R}Bun_{\mathcal{E}}(\mu))$$

yielding the recursive formula

$$P_t(\mathbb{R}Bun_{\mathcal{E}}^{ss}(r,d)) = P_t(\mathbb{R}Bun_{\mathcal{E}}(r,d)) - \sum_{\mu \in I_{r,d} \setminus \{\mu_{ss}\}} t^{d_{\mu}} \prod_{i=1}^{\ell_{\mu}} P_t(\mathbb{R}Bun_{\mathcal{E}}^{ss}(r_i, d_i)).$$

When r and d are coprime, one has the identity

$$P_t(\mathbb{R}\mathcal{M}_{\mathcal{E}}(r,d)) = (1-t) P_t(\mathbb{R}Bun_{\mathcal{E}}^{ss}(r,d)).$$

Altogether, setting $Q_t^{\mathbb{R}}(n,r,d) = (1-t) P_t(\mathbb{R}Bun_{\mathcal{E}}^{ss}(r,d))$, one obtains the recursive formula

$$(2.6) \quad \begin{aligned} Q_t^{\mathbb{R}}(n,r,d) &= 2^{n-1} (1+t)^g \prod_{i=2}^r \frac{(1+t^{2i-1})^{g+1-n} (1+t^{i-1})^{n-1} (1+t^i)^{n-1}}{(1-t^{i-1})(1-t^i)} \\ &\quad - \sum_{\mu \neq \mu_{ss}} \frac{t^{d_{\mu}}}{(1-t)^{\ell_{\mu}-1}} \prod_{i=1}^{\ell_{\mu}} Q_t^{\mathbb{R}}(n, r_i, d_i), \end{aligned}$$

expressing all polynomials $Q_t^{\mathbb{R}}(n,r,d)$ in terms of the initial term

$$Q_t^{\mathbb{R}}(n,1,d) = P_t(\mathbb{R}\text{Pic}^d(\mathcal{E})) = 2^{n-1} (1+t)^g.$$

Thus it follows from relations (2.5) and (2.6) that, for all r and d , one has

$$Q_t^{\mathbb{R}}(g+1,r,d) = Q_t(r,d).$$

Since, when d and r are coprime,

$$Q_t^{\mathbb{R}}(g+1,r,d) = P_t(\mathbb{R}\mathcal{M}_{\mathcal{E}}(r,d)) \quad \text{and} \quad Q_t(r,d) = H_{(t,1)}(\mathcal{M}_{\mathcal{E}}(r,d)),$$

by results from [EK00, LS13], we see that relation (2.2), hence also Theorem 1.2, is proved. \square

3. A brief panorama of Hodge-expressive varieties

We list below a few basic examples of Hodge-expressive varieties and discuss their relevance among maximal real algebraic varieties.

3.1. Handy real algebraic varieties

Real projective spaces, *i.e.* $\mathbb{C}P^n$ equipped with the standard complex conjugation, constitute the simplest projective real algebraic varieties. One checks easily that they are all Hodge-expressive.

More generally, the standard complex conjugation in \mathbb{C} induces a real structure on the Grassmannian variety $\mathbf{Gr}(d,n)$ of d -planes in \mathbb{C}^n . Schubert cells provide a stratification of $\mathbf{Gr}(d,n)$ by real affine spaces [MS74], implying that for all $i \geq 0$

$$b_{2i+1}(\mathbf{Gr}(d,n)) = 0 \quad \text{and} \quad b_i(\mathbb{R}\mathbf{Gr}(d,n)) = b_{2i}(\mathbf{Gr}(d,n)) = h^{i,i}(\mathbf{Gr}(d,n)).$$

Hence all Grassmannian varieties are Hodge-expressive.

Toric varieties constitute another generalization of projective spaces. A toric variety carries a standard real structure inherited once again from the standard complex conjugation in \mathbb{C}^* , and all these non-singular real projective toric varieties are Hodge-expressive by [BFMvH06, Fra22].

Grassmannian and toric varieties are particular cases of *balanced varieties*, *i.e.* non-singular projective complex manifold X such that $h^{i,j}(X) = 0$ whenever $i \neq j$. By definition, such a variety satisfies $P_t(\mathbb{C}X) = H_{(t,t)}(X) = H_{(t^2,1)}(X)$. In particular if X is both balanced and Hodge-expressive, then $P_t(\mathbb{C}X) = P_{t^2}(\mathbb{R}X)$.

Hodge-expressivity is preserved by certain elementary operations on real algebraic varieties, for instance taking products or projectivizing a vector bundle over the variety. This follows from the fact that both Hodge and Betti numbers satisfy the same relations under these elementary operations: on the one hand, the Hodge polynomial extends to a motivic invariant from the Grothendieck ring $K_0(\text{Var}_{\mathbb{C}})$ of complex algebraic varieties to $\mathbb{Z}[x,y]$ [PS08, Remark 5.56], and on the other hand, the Künneth formula (or more generally the Leray–Hirsch Theorem, [Hat02, Theorem 4D.1]) implies that the Poincaré polynomial behaves analogously in basic situations. More precisely, we observe the following:

- The product of Hodge-expressive real algebraic varieties X_1, X_2, \dots, X_k , equipped with the product real structure, is Hodge-expressive. Indeed, one deduces from the multiplicativity of the Hodge polynomial and from the Künneth formula that

$$H_{(x,y)}\left(\prod_{i=1}^k X_i\right) = \prod_{i=1}^k H_{(x,y)}(X_i), \quad \text{and} \quad P_t\left(\prod_{i=1}^k \mathbb{R}X_i\right) = \prod_{i=1}^k P_t(\mathbb{R}X_i).$$

- Similarly, if E is a real algebraic vector bundle of rank r over a Hodge-expressive real algebraic variety X , the real projective bundle $\mathbb{P}(E)$ is also Hodge-expressive. By the scissor relations on $K_0(\text{Var}_{\mathbb{C}})$, we get that

$$H_{(x,y)}(\mathbb{P}(E)) = H_{(x,y)}(X)H_{(x,y)}(\mathbb{C}\mathbb{P}^{r-1}).$$

And since the first Chern class (resp. the first Stiefel–Whitney class) of the tautological line bundle on $\mathbb{P}(E)$ (resp. $\mathbb{R}\mathbb{P}(E)$) restricts to the generator of $H^*(\mathbb{C}\mathbb{P}^{r-1})$ (resp. $H^*(\mathbb{R}\mathbb{P}^{r-1})$), we deduce from the Leray–Hirsch theorem that $H^*(\mathbb{P}(E); \mathbb{Z})$ is torsion-free if X is, and that

$$P_t(\mathbb{R}\mathbb{P}(E)) = P_t(X)P_t(\mathbb{R}\mathbb{P}^{r-1}).$$

- We deduce in particular from the last item that the blow-up \widetilde{X} of a Hodge-expressive variety X along a Hodge-expressive subvariety Y is Hodge-expressive as well. Using again the scissor relations on $K_0(\text{Var}_{\mathbb{C}})$, we have

$$H_{(x,y)}(\widetilde{X}) = H_{(x,y)}(X) + (x^{r-1}y^{r-1} + x^{r-2}y^{r-2} + \dots + xy)H_{(x,y)}(Y),$$

where r is the codimension of Y in X . The computation of Betti numbers of \widetilde{X} from [GH78, Chapter 4, Section 6] carries over word for word to the real part. In particular, we obtain that $H^*(\widetilde{X}; \mathbb{Z})$ is torsion-free if X and Y are, and that

$$P_t(\mathbb{R}\widetilde{X}) = P_t(\mathbb{R}X) + (t^{r-1} + t^{r-2} + \dots + t)P_t(\mathbb{R}Y).$$

One may also consider symmetric powers of real algebraic varieties, or more generally quotients X^k/Γ of a k^{th} product X^k by the canonical action of a subgroup Γ of the symmetric group \mathfrak{S}_k . The product and the symmetric power correspond to the two extremal cases $\Gamma = \{\text{Id}\}$ and $\Gamma = \mathfrak{S}_k$, respectively. Any real structure on X extends canonically to X^k/Γ , and Franz proved in [Fra18] that X^k/Γ is a maximal real algebraic variety as soon as X is (note that the converse does not hold in general, as shown by the second symmetric power of $\mathbb{C}\mathbb{P}^1$ equipped with the antipodal involution). It would be interesting to investigate whether X^k/Γ is Hodge-expressive as soon as it is non-singular and X is Hodge-expressive. When $\Gamma = \mathfrak{S}_k$, the k^{th} symmetric power $X^{[k]}$ of X is projective and non-singular if and only if X is a non-singular projective curve. In this case partial results regarding Hodge-expressivity can be obtained, as we shall see in Section 3.2 below.

3.2. Curves

Let \mathcal{C} be a non-singular real projective curve of genus g . In this case the Smith–Thom inequality reduces to the Harnack–Klein inequality

$$b_0(\mathbb{R}\mathcal{C}) \leq g + 1,$$

with equality if and only if \mathcal{C} is maximal. Since

$$H_{(x,y)}(\mathcal{C}) = 1 + g(x + y) + xy,$$

and

$$P_t(\mathbb{R}\mathcal{C}) = b_0(\mathbb{R}\mathcal{C})(1 + t) \leq (g + 1)(1 + t) = H_{(t,1)}(\mathcal{C}),$$

the curve \mathcal{C} is Hodge-expressive if and only if it is maximal.

By the Abel–Jacobi Theorem, we have $\text{Pic}^d(\mathcal{C}) \simeq \text{Jac}(\mathcal{C})$ for all $d \in \mathbb{Z}$, so

$$H_{(x,y)}(\text{Pic}^d(\mathcal{C})) = (1 + x)^g (1 + y)^g.$$

By [GH81], if $b_0(\mathbb{R}\mathcal{C}) > 0$ one has

$$P_t(\mathbb{R}\text{Pic}^d(\mathcal{C})) = 2^{b_0(\mathbb{R}\mathcal{C})-1} (1 + t)^g \leq 2^g (1 + t)^g = H_{(t,1)}(\text{Pic}^d(\mathcal{C})).$$

Since $\text{Pic}^d(\mathcal{C})$ has torsion-free integral cohomology, we deduce that

$$(3.1) \quad \text{Pic}^d(\mathcal{C}) \text{ Hodge-expressive} \iff \text{Pic}^d(\mathcal{C}) \text{ maximal} \iff \mathcal{C} \text{ maximal}$$

as soon as $b_0(\mathbb{R}\mathcal{C}) > 0$.

Remark 3.1. When $\mathbb{R}\mathcal{C} = \emptyset$, one has $\mathbb{R}\text{Pic}^d(\mathcal{C}) = \emptyset$ if g and d are both odd [GH81]. And if g is odd and d is even, the Poincaré polynomial of $\mathbb{R}\text{Pic}^d(\mathcal{C})$ is $2(1 + t)^g$, so its value at $t = 1$ is strictly smaller than 2^{2g} , unless $g = 1$ (in which case $\text{Pic}^{2d}(\mathcal{C})$ is maximal even though $\mathbb{R}\mathcal{C} = \emptyset$). Finally, if g is even, the Poincaré polynomial of $\mathbb{R}\text{Pic}^d(\mathcal{C})$ is $(1 + t)^g$, so its value at $t = 1$ is strictly smaller than 2^{2g} , unless $g = 0$.

From the utmost right equivalence in (3.1) and the discussion in Section 3.1, one deduces that the k^{th} symmetric power $\mathcal{C}^{[k]}$ of \mathcal{C} is Hodge-expressive if \mathcal{C} is maximal and $k \geq 2g - 1$, which refines [BDM17, Theorem 3.1]. Indeed, the Riemann–Roch and Abel–Jacobi theorems imply that $\mathcal{C}^{[k]}$ can be expressed as the projectivization of a real algebraic vector bundle over $\text{Jac}(\mathcal{C})$, see [ACGH85, Chapter IV]. Furthermore, from the explicit computation of Betti numbers of $\mathbb{R}\mathcal{C}^{[k]}$ for $k = 2, 3$ in [BDM17], one also sees that $\mathcal{C}^{[k]}$ is Hodge-expressive when \mathcal{C} is maximal and $k = 2, 3$ (see for example [LS11] for Hodge numbers of $\mathcal{C}^{[k]}$). It would be interesting to investigate whether $\mathcal{C}^{[k]}$ is Hodge-expressive as soon as \mathcal{C} is.

Hodge-expressivity may also be used to determine “elementary embeddings” of real algebraic curves in real algebraic surfaces. As an example, let us consider Harnack curves in the real projective plane. They were originally constructed by Harnack in [Har76], and constituted the first family of maximal curves of arbitrary degree in $\mathbb{R}\mathbb{P}^2$. Since then, Harnack curves have also been obtained as elementary applications of a large proportion of construction methods of maximal real algebraic curves (e.g. [IV96]). Surprisingly, they also appeared in several contexts other than pure real algebraic geometry (e.g. [KO06]). To summarize informally, a maximal real algebraic curve in $\mathbb{R}\mathbb{P}^2$ is a Harnack curve, except if it has a good reason not to be so.

The topological type of the pair $(\mathbb{R}\mathbb{P}^2, \mathbb{R}\mathcal{C})$ for a Harnack curve \mathcal{C} in $\mathbb{R}\mathbb{P}^2$ is depicted in Figure 2 below. Suppose that a Harnack curve \mathcal{C} of even degree $2k$ in $\mathbb{R}\mathbb{P}^2$ is given by the real equation $P(x, y, z) = 0$, where the real polynomial $P(x, y, z)$ is positive on the non-orientable connected component of $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}\mathcal{C}$. Then, on the one hand, one easily sees from Figure 2 that the real algebraic surface X with equation $w^2 - P(x, y, z)$ in the weighted projective space $\mathbb{C}\mathbb{P}(1, 1, 1, k)$ satisfies

$$b_0(\mathbb{R}X) = b_2(\mathbb{R}X) = \frac{(k-1)(k-2)}{2} + 1 \quad \text{and} \quad b_1(\mathbb{R}X) = 3k(k-1) + 2.$$



Figure 2. Harnack curves of degree d in \mathbb{RP}^2 , up to isotopy and in an affine chart.

On the other hand, the Hodge diamond of X is (see for example [DK00, Section 2.5])

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 0 & & 0 \\
 \frac{(k-1)(k-2)}{2} & & 3k(k-1) + 2 & & \frac{(k-1)(k-2)}{2} \\
 & & 0 & & 0 \\
 & & 1 & &
 \end{array} .$$

Note that $H^*(X; \mathbb{Z})$ is torsion-free by the Lefschetz hyperplane section Theorem. So the basic nature of a Harnack curve \mathcal{C} may be reflected in the fact that X is Hodge-expressive.

3.3. Surfaces

We refer to [DK00, Section 3] and [Man17, Chapter 4] for more details and references regarding the various classifications of real algebraic surfaces discussed in this section.

It follows from the classifications by Comessati that all maximal real Abelian surfaces are Hodge-expressive. For examples, the Hodge diamond of an Abelian surface is

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 2 & & 2 \\
 1 & & 4 & & 1 \\
 & & 2 & & 2 \\
 & & 1 & &
 \end{array} ,$$

and such a real surface X is maximal if and only if $\mathbb{R}X$ is the disjoint union of four tori $\mathbb{S}^1 \times \mathbb{S}^1$.

There exist maximal real algebraic $K3$ -surfaces that are not Hodge-expressive. As shown by Kharlamov [Kha76], the three possible topological types for the real part of a maximal real $K3$ -surface are: the disjoint union of a sphere S^2 and a surface of genus 10, the disjoint union of five spheres with a surface of genus 6, and the disjoint union of nine spheres with a surface of genus 2. Since a $K3$ surface has torsion-free integral cohomology and has the following Hodge diamond

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & 0 & & 0 \\
 1 & & 20 & & 1 \\
 & & 0 & & 0 \\
 & & 1 & &
 \end{array} ,$$

only the first topological type corresponds to a Hodge-expressive variety.

Note that there exist maximal real algebraic Enriques surfaces satisfying

$$b_i(\mathbb{R}X) > \sum_{j \geq 0} h^{i,j}(X) \quad \forall i \in \{0, 1, 2\},$$

see [DK00, Man17]. This shows the necessity of the torsion-freeness assumption in the implication

$$\left(P_t(\mathbb{R}X) = H_{(t,1)}(X) \right) \implies X \text{ maximal.}$$

The first family of maximal real algebraic surfaces of arbitrary degree in $\mathbb{C}\mathbb{P}^3$ was constructed by Viro in [Vir79], generalizing Harnack’s construction. A non-singular hypersurface in $\mathbb{C}\mathbb{P}^n$ has torsion-free integral cohomology by the Lefschetz hyperplane section Theorem, and it turns out that all surfaces in Viro’s family are Hodge-expressive.

3.4. Higher dimension

As mentioned earlier, our knowledge of maximal real algebraic varieties of dimension at least 3 is quite restricted. By [Kra09], there exists a unique topological type of maximal real cubic 3-folds in $\mathbb{C}\mathbb{P}^4$, and one checks that it is Hodge-expressive. By [FK10], there exist three topological types of maximal real cubic 4-folds in $\mathbb{C}\mathbb{P}^5$, only one of them being Hodge-expressive.

All maximal real algebraic projective hypersurfaces from Itenberg and Viro’s unpublished construction are Hodge-expressive. As mentioned earlier, this is even a crucial point in their argumentation, since they prove that the real algebraic hypersurfaces that they construct are maximal by showing that they are Hodge-expressive. Note that the Itenberg–Viro’s construction uses primitive combinatorial patchworking. Renaudineau and Shaw recently proved [RS18] that all maximal real algebraic hypersurfaces of a non-singular compact toric variety that are obtained by primitive combinatorial patchworking are Hodge-expressive, thus confirming a long standing conjecture of Itenberg’s.

We are not aware of any general construction of maximal real hypersurfaces in $\mathbb{C}\mathbb{P}^n$ that are not Hodge-expressive.

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