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Chow groups of surfaces of lines in cubic fourfolds

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To Claire Voisin, with admiration

Abstract. The surface of lines in a cubic fourfold intersecting a fixed line splits motivically into two parts, one of which resembles a K3 surface. We define the analogue of the Beauville–Voisin class and study the push-forward map to the Fano variety of all lines with respect to the natural splitting of the Bloch–Beilinson filtration introduced by Mingmin Shen and Charles Vial.

Keywords. Cubic fourfolds, Fano variety of lines, Chow groups, Bloch conjecture, Beauville–Voisin class

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Contents

1. Introduction

Fano varieties $F=F(X)$ of lines $L\subset X$ in smooth complex cubic fourfolds $X\subset\mathbb{P}^5$ are hyperkähler manifolds of dimension four and as such have many properties and features in common with K3 surfaces. In particular, the Chow ring $\mathrm{CH}^*(F)$ of the Fano variety is of great interest and has been investigated from various angles by Beauville [Bea07], Voisin [Voi08, Voi16], M. Shen and Vial [SV16], J. Shen, Yin, and Zhao [SYZ20, SY20], and many others.

1.1. The first goal of this note is to study zero-dimensional cycles on F from the perspective of distinguished surfaces contained in F. More specifically, we continue our investigation [Huy21] and consider the surface $F_{L_0}\subset F$ of all lines $L\subset X$ intersecting a given fixed line $L_0\subset X$. The surfaces F_{L_0} and their Chow groups are of great interest in their own right, but here, in order to gain a better understanding of the Chow group of zero-dimensional cycles $\mathrm{CH}^4(F)=\mathrm{CH}_0(F)$ on F, we first have a closer look at the push-forward map $\mathrm{CH}_0(F_{L_0})\to\mathrm{CH}_0(F)$.

The target and source of this map both come with a natural decomposition, and our first result explains how they are related.

- There exists a natural involution $\iota\colon F_{L_0}\xrightarrow{\sim}F_{L_0}$, and its quotient $F_{L_0}\to D_{L_0}\coloneqq F_{L_0}/_{\pm}$ describes a quintic surface $D_{L_0}\subset\mathbb{P}^3$ with 16 nodes. Homologically trivial zero-cycles on D_{L_0} can be identified with ι-invariant homologically trivial zero-cycles on F_{L_0}; *i.e.* $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}}$ is the first summand of the ι-eigenspace decomposition

$$\mathrm{CH}_0\left(F_{L_0}\right)_{\mathrm{hom}}=\mathrm{CH}_0\left(F_{L_0}\right)^+_{\mathrm{hom}}\oplus\mathrm{CH}_0\left(F_{L_0}\right)^-. \tag{1.1}$$

  The second summand gives back the interesting part of the Chow group of the cubic itself, namely $\mathrm{CH}_0(F_{L_0})^-\simeq\mathrm{CH}_1(X)_{\mathrm{hom}}$ via the Fano correspondence.
- On the Fano side, the group $A\coloneqq\mathrm{CH}_0(F(X))$ splits as

$$A=A_4\oplus A_2\oplus A_0. \tag{1.2}$$

  Here, A_0 is generated by a distinguished class c_F, the analogue of the Beauville–Voisin class on a K3 surface, and A_4 is the deepest part of the Bloch–Beilinson filtration. This decomposition was studied in depth by Shen and Vial [SV16]; see Section 2.2 for more details and references.

Our first result clarifies the link between the two decompositions (1.1) and (1.2). The result confirms certain aspects of the Bloch–Beilinson philosophy in the present situation.

Theorem 1.1. *Assume $L_0 \subset X$ is a general line in a smooth cubic fourfold $X \subset \mathbb{P}^5$.*

(i) *The push-forward map of invariant homologically trivial cycles*

$$\mathrm{CH}_0\left(D_{L_0}\right)_{\mathrm{hom}} \simeq \mathrm{CH}_0\left(F_{L_0}\right)^+_{\mathrm{hom}} \to A \tag{1.3}$$

takes image in $A_4 \subset A$. For very general X and very general $L_0 \subset X$, the image is non-trivial. Furthermore, the image of the composition

$$\mathrm{CH}_0\left(D_{L_0}\right) \longrightarrow A \longrightarrow\!\!\!\!\!\rightarrow A/A_4 = A_2 \oplus A_0$$

is spanned by the class $c_F(L_0) := 2c_F - [L_0]_2$, where $[L_0]_2$ denotes the A_2-component of $[L_0]$.

(ii) *The composition of the push-forward map and the projection $A \twoheadrightarrow A_2 \oplus A_0$ describes an isomorphism*

$$\mathrm{CH}_0\left(F_{L_0}\right)^- \xrightarrow{\sim} A_2$$

with its inverse induced by the map $[L] \mapsto (-1/2)[F_L]|_{F_{L_0}}$. Furthermore, if L_0 is very general, the image of $\mathrm{CH}_0(F_{L_0})^- \to A$ is not contained in A_2.

1.2. The second goal of the paper is to investigate the K3 nature of the 'negative half' $F^-_{L_0}$ of the surface F_{L_0}. The surface F_{L_0} 'decomposes' into two parts, and one component behaves very much like a K3 surface. On the level of Chow groups, the two parts correspond to the ι-invariant and the ι-anti-invariant cycles, and for rational Chow motives we have the decomposition $\mathfrak{h}(F_{L_0}) \simeq \mathfrak{h}(F_{L_0})^+ \oplus \mathfrak{h}(F_{L_0})^-$; see the discussion in [Huy21, Section 4.6]. The K3 nature of the anti-invariant part is best seen by the observation that $H^2(F_{L_0}, \mathbb{Z})^-_{\mathrm{pr}}$ is a Hodge structure of K3 type of rank 22, which, up to a factor and a Tate twist, is in fact Hodge isometric to the primitive middle cohomology $H^4(X,\mathbb{Z})_{\mathrm{pr}}$ of the cubic fourfold; *cf.* [Huy21, Theorem 0.2].

One distinguished feature, *cf.* [BV04], of the Chow ring of a complex K3 surface S is that despite $\mathrm{CH}^2(S)$ being big, all intersection products $\alpha_1 \cdot \alpha_2$ of classes $\alpha_1, \alpha_2 \in \mathrm{CH}^1(S)$ are proportional. They are multiples of the Beauville–Voisin class $c_S \in \mathrm{CH}^2(S)$, which is realised by any point contained in a rational curve. Transplanted in our context, this triggers the question whether products $\alpha_1 \cdot \alpha_2$ of two anti-invariant classes $\alpha_1, \alpha_2 \in \mathrm{CH}^1(F_{L_0})^-$ are all multiples of a distinguished class $c_{L_0} \in \mathrm{CH}^2(F_{L_0})$, which would necessarily be invariant. Our second result shows that this is the case for primitive classes and after push-forward to F.

Theorem 1.2. *For a general line $L_0 \subset X$ in a smooth cubic fourfold, the composition*

$$\mathrm{CH}^1\left(F_{L_0}\right)^-_{\mathrm{pr}} \times \mathrm{CH}^1\left(F_{L_0}\right)^-_{\mathrm{pr}} \longrightarrow \mathrm{CH}^2\left(F_{L_0}\right)^+ \longrightarrow A$$

takes image in $\mathbb{Z} \cdot c_F(L_0)$ with $c_F(L_0) = 2c_F - [L_0]_2 \in A_2 \oplus A_0 \subset A$ as above.

This suggests that we should view

$$\mathrm{CH}^*\left(F^-_{L_0}\right) := \mathbb{Z} \oplus \mathrm{CH}^1\left(F_{L_0}\right)^-_{\mathrm{pr}} \oplus \left(\mathrm{CH}^2\left(F_{L_0}\right)^- \oplus \mathbb{Z} \cdot c_F(L_0)\right)$$

as the Chow ring of $F^-_{L_0}$, *i.e.* of the 'K3 half' of the surface F_{L_0}. Coming back to Theorem 1.1, the natural isomorphism $\mathrm{CH}_0(F_{L_0})^- \simeq A_2$ highlights the K3 character of the middle part A_2 of the decomposition (1.2).

Remark 1.3. One could ask whether $\mathrm{CH}^2(F^-_{L_0}) = \mathrm{CH}^2(F_{L_0})^- \oplus \mathbb{Z} \cdot c_F(L_0)$ can be realised simply as the image of $\mathrm{CH}_0(F_{L_0}) \to \mathrm{CH}_0(F)$, so that the hyperkähler fourfold F singles out the 'K3 half' of F_{L_0}. However, this fails due to the slightly counter-intuitive non-triviality of (1.3) in Theorem 1.1(i).

In the opposite direction, one could wonder whether (1.3) is maybe injective. This would have the effect that $c_F(L_0)$ lifts uniquely to a class $c_{L_0} \in \mathrm{CH}_0(F_{L_0})^+$ such that

$$\mathbb{Z} \oplus \mathrm{CH}^1\left(F_{L_0}\right)^-_{\mathrm{pr}} \oplus \left(\mathrm{CH}^2\left(F_{L_0}\right)^- \oplus \mathbb{Z} \cdot c_{L_0}\right) \subset \mathrm{CH}^*\left(F_{L_0}\right)$$

is a subring, *i.e.* such that $\alpha_1 \cdot \alpha_2 \in \mathbb{Z} \cdot c_{L_0}$ for all $\alpha_1, \alpha_2 \in \mathrm{CH}^1(F_{L_0})^-_{\mathrm{pr}}$. The class c_{L_0} on F_{L_0} would then be the geometric realisation of the Beauville–Voisin class for the K3 half $F^-_{L_0}$; see Remark 4.9 for further comments.

The main tool to prove these results is the map $\psi\colon \mathrm{CH}_0(F) \to \mathrm{CH}_2(F)$, $L \mapsto [F_L]$, which has been studied and used before; see [SV16, SYZ20, SY20, Voi16]. Further restricting to surfaces contained in F allows one to transform zero-cycles on F into zero-cycles on surfaces. However, since ψ annihilates the deepest part of the Bloch–Beilinson filtration $A_4 \subset \mathrm{CH}_0(F)$, this technique does not provide control over all zero-cycles, on the Fano variety or on surfaces contained in it.

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2. Zero-cycles on hyperkähler fourfolds

It is expected that the Chow groups of an arbitrary smooth projective variety Z are endowed with natural (Bloch–Beilinson) filtrations

$$\cdots \subset F^2 \subset F^1 \subset \mathrm{CH}^k(Z)$$

that enjoy certain functoriality properties. Also, the filtrations should be finite, more precisely $F^i = 0$ for $i > k$, and the behaviour of algebraic correspondences on the graded parts F^i/F^{i+1} should be determined by their action on the corresponding pieces of the Hodge structure of Z.

For example, any correspondence $\Gamma_*\colon \mathrm{CH}_0(Z_1) \to \mathrm{CH}_0(Z_2)$ should respect the filtration, *i.e.* $\Gamma_*(F^i(Z_1)) \subset F^i(Z_2)$, and the induced map $\mathrm{Gr}^i(\Gamma_*)\colon (F^i/F^{i+1})(Z_1) \to (F^i/F^{i+1})(Z_2)$ should be zero if and only if the cohomological action $\Gamma^*\colon H^0(Z_2,\Omega^i_{Z_2}) \to H^0(Z_1,\Omega^i_{Z_1})$ is trivial. This is often seen as a generalisation of Bloch's conjecture; see [Voi02, Conjecture 23.22].

There are various suggestions for such filtrations proposed by H. Saito [Sai92], S. Saito [Sai96], Nori [Nor93], and Voisin [Voi04], but proving the desired properties in general seems out of reach for now.

Example 2.1. For zero-cycles on a smooth projective surface S, the various requirements determine the Bloch–Beilinson filtration completely: concretely, $F^1\,\mathrm{CH}_0(S)$ is the subgroup of all cycles of degree zero or, equivalently, of those that are algebraically or, still equivalently, homologically trivial, while $F^2 \subset F^1$ is the kernel of the Albanese morphism $F^1 \to \mathrm{Alb}(S)$.

2.1. For a hyperkähler manifold Z of dimension four, the conjectural filtration on zero-cycles would be of the form

$$0 \subset F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = A := \mathrm{CH}_0(Z).$$

Recall that by Roitman's result [Roj80] the group A is torsion-free, and, by using the divisibility of Jacobians of curves contained in Z, the kernel of the degree map $\deg\colon A \to \mathbb{Z}$ is known to be divisible.

Bloch's conjecture applied to the identity correspondence $[\Delta]_* = \mathrm{id}$ leads to the following picture. The vanishings $H^0(Z,\Omega^3_Z) = 0 = H^0(Z,\Omega^1_Z)$ should imply $F^4 = F^3$ and $F^2 = F^1$, and the non-vanishings $H^0(Z,\Omega^4_Z) \neq 0 \neq H^0(Z,\Omega^2_Z)$ should give $F^4 \neq 0$ and $F^3 \neq F^2$. So the conjectural Bloch–Beilinson filtration in this case would actually look like

$$0 \subsetneq F^4 = F^3 \subsetneq F^2 = F^1 \subsetneq F^0 = A = \mathrm{CH}_0(Z). \tag{2.1}$$

In full generality, only $F^2 = F^1$ is rigorously defined. Also note that a generalisation of an argument of Mumford for surfaces shows that F^1 is not trivial and in fact not representable by any finite-type scheme; see [Voi02, Theorem 22.15].

Taking inspiration from the case of K3 surfaces [BV04], Beauville [Bea07] asked whether the hyperkähler structure of Z gives rise to a certain splitting of the conjectural filtration (2.1). This would lead to a decomposition

$$A = A_4 \oplus A_2 \oplus A_0,$$

with $F^4 = F^3 = A_4$, $F^2 = F^1 = A_4 \oplus A_2$, and $\deg\colon A_0 \xrightarrow{\sim} \mathbb{Z}$. The generator of degree one of A_4 is called the Beauville–Voisin class $c_Z \in A$.

Shen and Vial [SV16] suggested to use a certain lift $\tilde{q} \in \mathrm{CH}^2(Z \times Z)$ of the class $q_Z \in S^2H^2(Z,\mathbb{Q}) \subset H^4(Z \times Z,\mathbb{Q})$ induced by the Beauville–Bogomolov–Fujiki pairing as the analogue of the Poincaré bundle for abelian varieties to define the decomposition as

$$A_k := \left\{\alpha \in A \mid \exp(\tilde{q})(\alpha) \in \mathrm{CH}^k(Z)\right\}.$$

An alternative and more geometric definition was proposed by Voisin [Voi16]: One first introduces the orbit $\mathrm{O}_x \subset Z$ of any point $x \in Z$ as the set of all points that are rationally equivalent to x. The orbit turns out to be a countable union of closed subsets of dimensions at most two. Then, $A_0 \subset A$ and $A_2 \oplus A_0$ should be generated by those points with $\dim \mathrm{O}_x \geqslant 2$ and $\dim \mathrm{O}_x \geqslant 1$, respectively.

2.2. Typically, conjectures concerning hyperkähler manifolds (of dimension four) are first checked for the Hilbert scheme $S^{[2]}$ of a K3 surface S and for the Fano variety $F = F(X)$ of lines contained in a smooth cubic fourfold $X \subset \mathbb{P}^5$. This has been undertaken for Beauville's question by Beauville [Bea07] himself, by Voisin [Voi08, Voi16], by M. Shen and Vial [SV16], and more recently by J. Shen, Yin, and Zhao [SY20, SYZ20]. We briefly review some of the results for the Fano variety $F = F(X)$ and its group of zero-cycles $A = \mathrm{CH}_0(F)$.

Firstly, there is a natural candidate for the Beauville–Voisin class, *i.e.* the generator of A_0, namely the unique class $c_F \in \mathrm{CH}_0(F)$ satisfying

$$6 \cdot c_F = g \cdot [C_x].$$

Here, $g \in \mathrm{CH}^1(F)$ is the Plücker polarisation and $C_x := \{L \mid x \in L\} \subset F$, which is a curve for the general point $x \in X$. Observe that, since the cubic X is unirational, the class of the curve $[C_x] \in \mathrm{CH}^3(F)$ does not depend on the choice of the general point x. The class c_F indeed has the desired multiplicative property: according to a result of Voisin [Voi08, Theorem 0.4], the degree four component of any polynomial expression involving only the Chern classes $\mathrm{c}_2(F)$ and $\mathrm{c}_4(F)$ and classes $\alpha \in \mathrm{CH}^1(F)$ is a multiple of c_F.

Secondly, one exploits the Voisin map $f\colon F \dashrightarrow F$ and its induced action on the Chow group $f_*\colon A \to A$. Recall that the Voisin map is the endomorphism of degree 16 that maps a line of the first type L to the residual line L' of the intersection $L \subset P_L \cap X$ with the unique plane $P_L \simeq \mathbb{P}^2$ tangent to $L \subset X$; *i.e.* $P_L \cap X = 2L \cup L'$. Shen and Vial [SV16, Theorem 21.9] proved that the induced action f_* on CH_0 has eigenvalues 4, -2, and 1 and declare the corresponding eigenspaces to be A_4, A_2, and $A_0 = \mathbb{Z} \cdot c_F$.

Thirdly, mapping a line L to the surface F_L induces a map

$$\psi\colon A \longrightarrow \mathrm{CH}_2(F), \quad [L] \longmapsto [F_L], \tag{2.2}$$

and one defines A_4 as the kernel of ψ. Alternatively, A_4 can also be described as the homologically trivial part of the subgroup generated by all triangles, *i.e.* by sums $[L_0] + [L_1] + [L_2]$ of triples of lines $L_0, L_1, L_2 \subset X$ spanning a plane. Yet another possibility is to describe A_4 as the kernel of the Fano correspondence $\varphi\colon A = \mathrm{CH}_0(F) \to \mathrm{CH}_1(X)$ mapping the class of a point in F corresponding to a line in X to the class of that line. In fact, φ factors through an isomorphism:

$$\varphi\colon A \longrightarrow\!\!\!\!\!\rightarrow A/A_4 \simeq A_2 \oplus A_0 \xrightarrow{\sim} \mathrm{CH}_1(X). \tag{2.3}$$

It is *a priori* not clear that the various definitions of the deepest part of the Bloch–Beilinson filtration

$$A_4 = \{\alpha \mid f_*\alpha = 4 \cdot \alpha\} = \ker(\psi) = \ker(\varphi) = \langle [L_0] + [L_1] + [L_2] \rangle_{\mathrm{hom}} \tag{2.4}$$

or of its complement

$$A_2 \oplus A_0 = \{\alpha \mid f_*\alpha = -2 \cdot \alpha\} \oplus \{\alpha \mid f_*\alpha = \alpha\} = \langle [L] \mid \dim \mathrm{O}_L \geqslant 1 \rangle$$

describe the same subgroups. The often geometrically quite intricate arguments have been provided by Shen and Vial [SV16] and Voisin [Voi16]. Note that A_4 and A_2 are both non-trivial, but this is not automatic. In fact, both parts should be thought of as 'big', *i.e.* non-representable, although A_2 is surface-like while A_4 'lives' in dimension four; see below for more on this.

Of the many results in the comprehensive monograph of Shen and Vial [SV16], the following will be important for our discussion:

(i) The first concerns the multiplication $\mathrm{CH}^2(F)\times\mathrm{CH}^2(F)\to\mathrm{CH}^4(F)=A$. For a triangle $L_0\cup L_1\cup L_2\subset X$, the classes $[F_{L_i}]\in\mathrm{CH}^2(F)$ satisfy, *cf.* [SV16, Proposition 20.7],

$$[F_{L_1}]\cdot[F_{L_2}]=6\cdot c_F+[L_0]-[L_1]-[L_2]. \tag{2.5}$$

(ii) The second is about the image of the distinguished class c_F under (2.2). According to [SV16, Lemma A.5], one has

$$3\cdot\psi(c_F)=g^2-\mathrm{c}_2(\mathcal{S}_F)=(1/8)(5g^2+\mathrm{c}_2(F)), \tag{2.6}$$

where $\mathcal{S}_F$ is the universal subbundle on $F\subset\mathbb{G}(1,\mathbb{P}^5)$.

(iii) For every homologically trivial class $\gamma\in\mathrm{Im}(\psi)$, which according to [SV16, Section 21] is equivalent to being an algebraically trivial class in $\mathrm{CH}^2(F)$, and every class $\alpha\in\mathrm{CH}^1(F)$ we have

$$\alpha^2\cdot\gamma\in A_2. \tag{2.7}$$

This follows from combining $A_2=\mathrm{CH}^1(F)_0^{\cdot 2}\cdot\mathrm{CH}^2(F)_2$, see [SV16, Proposition 22.2], $\mathrm{CH}^1(F)_0=\mathrm{CH}^1(F)$, see [SV16, Theorem 2], $\mathrm{CH}^2(F)_2=V_{-2}^2$, see [SV16, Theorem 21.9(iii)], and $V_{-2}^2=\mathcal{A}_{\mathrm{hom}}=\mathrm{Im}(\psi)_{\mathrm{hom}}$, see [SV16, Definition 20.1 and Proposition 21.10].

3. Special surfaces

Apart from the surfaces $F_L\subset F=F(X)$, there are other interesting types of surfaces in F. Most notably, there are the surface $F'\subset F$ of lines of the second type and the surfaces $F(Y)\subset F$ of lines contained in general hyperplane sections $Y=X\cap\mathbb{P}^4$. Before turning to cycles on these surfaces, let us recall a few geometric facts.

All three surfaces F_L, F', and $F(Y)$ are smooth surfaces of general type.(1) Their basic numerical invariants are as follows:

(1) $p_g(F(Y))=10$ and $q(F(Y))=5$; see [Huy23, Chapter 5] for references.
(2) $p_g(F')=449$ and $q(F')=0$; see [GK23] and [Huy23, Section 6.4].
(3) $p_g(F_L)=5$ and $q(F_L)=0$. In fact, $\pi_1(F_L)=\{1\}$, see [Huy21, Lemma 1.2 and Appendix].

3.1. Voisin proved [Voi92, Example 3.7] that the surfaces $F(Y)\subset F$ are Lagrangian; *cf.* [Huy23, Lemma 6.4.5]. In other words, the pull-back map $H^0(F,\Omega_F^2)\to H^0(F(Y),\Omega_{F(Y)}^2)$ is zero.

The Bloch–Beilinson filtration for zero-cycles on the surface $F(Y)$ is of the form

$$0\subset F^2\subset F^1\subset\mathrm{CH}_0(F(Y)),$$

where the Albanese map $F^1/F^2\xrightarrow{\sim}\mathrm{Alb}(F(Y))$ is an isomorphism and F^2 is big, *i.e.* not representable; see Example 2.1.

As always for the push-forward map from a surface, $\mathrm{CH}_0(F(Y))\to A$ respects the filtration. However, since $F(Y)\subset F$ is Lagrangian, the Bloch–Beilinson conjecture predicts that the image of $F^2\mathrm{CH}_0(F(Y))\to A_4\oplus A_2=F^2\subset A$ is actually contained in $F^4=F^3=A_4$. This prediction can be confirmed as follows:(2)

(1)For F' to be smooth, one needs to assume X general.
(2)Thanks to C. Voisin for the argument.

As the Fano correspondences for X and for the hyperplane section $Y \subset X$ are compatible, the natural diagram

$$\begin{array}{ccc} \mathrm{CH}_0(F(Y)) & \longrightarrow & A \\ \downarrow \varphi_Y & & \downarrow \varphi_X \\ \mathrm{CH}_1(Y) & \longrightarrow & \mathrm{CH}_1(X) \end{array}$$

commutes. By the definition of the Bloch–Beilinson filtration for the surface $F(Y)$, we know that $F^2\mathrm{CH}_0(F(Y))$ is the kernel of the Albanese map. Since $\mathrm{Alb}(F(Y)) \simeq \mathrm{CH}_1(Y)_{\mathrm{hom}}$, by a result of Beauville and Murre, *cf.* [Huy23, Corollary 5.3.16], this shows that $F^2\mathrm{CH}_0(F(Y))$ is contained in the kernel of φ_Y. Therefore, its image in A is contained in the kernel of φ_X, which is $F^4 = A_4 \subset A$ by (2.4).

Note that for the general hyperplane section $Y \subset X$, the surface $F(Y) \subset F$ is not a constant cycle surface; *i.e.* the image of $\mathrm{CH}_0(F(Y)) \to A$ is not of rank one. Indeed, otherwise the generator c_Y would stay constant, since Y varies in the projective space $|\mathcal{O}(1)|$, but as every line in X is contained in some hyperplane section, this would imply the triviality of $A_4 \oplus A_2$.

A stronger statement is in fact true. Namely, for the very general hyperplane section Y, the map $F^2\mathrm{CH}_0(F(Y)) \to A_4$ is not trivial. Here is a sketch of the argument:(2) The surfaces $F(Y)$ cover F since every line in X is contained in some hyperplane section. In fact, the family of all $F(Y)$ can be seen as a $\mathbb{P}^3$-bundle $\mathcal{F} \to F$ with a projection $\mathcal{F} \to |\mathcal{O}(1)|$. Restricting to a general two-dimensional family $\pi\colon \mathcal{F}_{\mathbb{P}^2} \to \mathbb{P}^2 \subset |\mathcal{O}(1)|$ gives a fourfold with a dominant morphism $\mathcal{F}_{\mathbb{P}^2} \to F$ inducing an injection $H^{4,0}(F) \hookrightarrow H^{4,0}(\mathcal{F}_{\mathbb{P}^2})$. However, over a dense open subset $U \subset \mathbb{P}^2$, we have $\Omega^4_{\mathcal{F}_U} \simeq \pi^*\Omega^2_U \otimes \Omega^2_\pi$, but if $F^2\mathrm{CH}_0(F(Y)) \to A$ were zero, then the component in Ω^2_π would be trivial.

Remark 3.1. Although for the very general hyperplane section, the surface $F(Y) \subset F$ is not a constant cycle surface, special ones might be. Imitating the study of constant cycle curves in ample linear systems on K3 surfaces, see [Huy14], it should be interesting to study those in more detail.(3) In fact, Voisin [Voi08, Lemma 2.2] shows that the surface of lines contained in a hyperplane section with five nodes is rational (and singular) and hence a constant cycle surface.

3.2. Now let X be general such that the surface $F' \subset F$ of lines of the second type is smooth. Then $F^2 = F^1 \subset \mathrm{CH}_0(F')$, which the push-forward map sends to $F^2 = A_4 \oplus A_2 \subset A$. Since $F' \subset F$ is not Lagrangian, see [Huy23, Section 6.4.4], the projection to the graded part gives a non-trivial map $F^2 \to A_4 \oplus A_2 \twoheadrightarrow A_2$. In fact, according to a result of Shen and Vial [SV16, Proposition 19.5], the surface F' avoids A_4 and covers A_2. More precisely,

$$\mathrm{Im}\left(\mathrm{CH}_0(F') \longrightarrow A\right) = A_2 \oplus A_0 \subset A. \tag{3.1}$$

Again by Mumford's argument, using $p_g(F') > 1$ and that $F' \subset F$ is not Lagrangian, the kernel and image of the push-forward map are not representable. However, a geometric understanding of those cycles on F' that become rationally trivial on F is not available. Also, unlike the case $\mathrm{CH}_0(F_{L_0})^- \subset \mathrm{CH}_0(F_{L_0})$ studied in this paper, there does not seem to be any distinguished subgroup of $\mathrm{CH}_0(F')$ that maps isomorphically onto A_2.

As was observed by Shen and Vial [SV16, Theorem 3], (3.1) also implies $A_4 \neq 0$. Indeed, otherwise $\mathrm{CH}_0(F)$ would be concentrated on the surface F', which by Bloch–Srinivas [BS83] would contradict $H^{4,0}(F) \neq 0$.

Remark 3.2. Observe that (3.1) fits Voisin's description of $A_2 \oplus A_0$ (and was in fact used for its proof). Indeed, the rational endomorphism $f\colon F \dashrightarrow F$ is resolved by a blow-up of F in $F' \subset F$. Thus, for a line of the second type $L \in F'$, the class $f_*[L]$ is realised by all points L' in the image of the exceptional line of the blow-up $\tau\colon \mathrm{Bl}_{F'}(F) \to F$ over L, which clearly satisfy $\dim \mathrm{O}_{L'} \geqslant 1$. Hence, by Voisin's description, we have $f_*[L] \in A_2 \oplus A_0$ and, therefore, $[L] \in A_2 \oplus A_0$. For the other inclusion observe that the image of the

(3) Thanks to E. Sertöz for a related question.

exceptional divisor of τ is an ample divisor in F which therefore intersects any curve. Hence, every line $L'\in F$ with $\dim \mathrm{O}_{L'}\geqslant 1$, and so in particular $[L']\in A_2\oplus A_0$ according to Voisin, gives rise to a class of the form $f_*[L]$ for some $L\in F'$.

4. Lines intersecting a fixed line

Let us now fix a general line $L_0\subset X$ and consider the surface $F_{L_0}\subset F=F(X)$ of lines intersecting L_0. Then the quotient by the standard involution defines a morphism $\pi\colon F_{L_0}\twoheadrightarrow D_{L_0}$ onto a quintic surface $D_{L_0}\subset\mathbb{P}^3$. The 16 fixed points, *i.e.* the 16 lines $L\subset X$ with $\overline{LL_0}\cap X=2L\cup L_0$, give rise to 16 ordinary double points of D_{L_0}; see the original [Voi86] or the discussion in [Huy23, Section 6.4.5] or in [Huy21].

The involution ι acts on $\mathrm{CH}_0(F_{L_0})$, and we define

$$\mathrm{CH}_0\left(F_{L_0}\right)^{\pm}\coloneqq\{\alpha\mid\iota^*\alpha=\pm\alpha\}\subset\mathrm{CH}_0\left(F_{L_0}\right).$$

Then the pull-back map induces isomorphisms

$$\mathrm{CH}_0\left(D_L\right)_{\mathrm{hom}}\xrightarrow{\sim}\mathrm{CH}_0\left(F_L\right)^+_{\mathrm{hom}}\quad\text{and}\quad\mathrm{CH}_0\left(F_{L_0}\right)^-_{\mathrm{hom}}\simeq\mathrm{CH}_0\left(F_{L_0}\right)_{\mathrm{hom}}/\mathrm{CH}_0(D_L)_{\mathrm{hom}}.$$

Moreover, $\mathrm{CH}_0(F_{L_0})^-=\mathrm{CH}_0(F_{L_0})^-_{\mathrm{hom}}=\ker(\pi_*)=\mathrm{Im}(1-\iota^*)$; see [Huy21, Section 4.1].

Of course, $\mathrm{CH}_0(D_{L_0})/\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}}\simeq\mathbb{Z}$, but this isomorphism comes without a canonical split; *i.e.* no analogue of the Beauville–Voisin class exists for the surface D_{L_0}. Theorem 1.1(i) can be seen as a replacement: the image of $\mathrm{CH}_0(D_{L_0})$ in $A/A_4\simeq A_2\oplus A_0$ is generated by a distinguished class depending on L_0.

4.1. Before entering the proof of Theorem 1.1(i), we observe that the correspondence

$$D_{L_0}\twoheadleftarrow F_{L_0}\hookrightarrow F$$

induces the trivial map $H^0(F,\Omega_F^2)\to H^0(D_{L_0},\Omega^2_{D_{L_0}})$ since the restriction of the holomorphic two-form on F to F_{L_0} is anti-invariant; see [Huy21, Section 2.3]. In particular, according to the Bloch–Beilinson philosophy, one expects the map

$$\mathrm{CH}_0\left(D_{L_0}\right)_{\mathrm{hom}}=\left(F^2/F^3\right)\left(D_{L_0}\right)\longrightarrow\left(F^2/F^3\right)(F)=A_2$$

to be trivial. Equivalently, the composition of pull-back and push-forward should map the group of zero-cycles $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}}=\mathrm{CH}_0(F_{L_0})^+_{\mathrm{hom}}$ on D_{L_0} to $F^3=A_4\subset A$. In this sense, the first assertion Theorem 1.1(i) can be seen as a confirmation of the Bloch–Beilinson philosophy.

Proof of Theorem 1.1(i). A point $t\in D_{L_0}$ corresponds to a pair of lines $L_1,L_2\subset X$ such that $L_0\cup L_1\cup L_2$ forms a triangle, *i.e.* all three lines are contained in a single plane.

Under push-forward, the class $[t]\in\mathrm{CH}_0(D_{L_0})$ of the point $t\in D_{L_0}$ is then mapped to $[L_1]+[L_2]\in A$ and, after composing further with $\psi\colon A\to\mathrm{CH}_2(F)$, $[L]\mapsto[F_L]$, to

$$\left[F_{L_1}\right]+\left[F_{L_2}\right]=\varphi\left(h^3\right)-\left[F_{L_0}\right].\tag{4.1}$$

Here, h is the hyperplane class on X, h^3 can be represented by the plane $\mathbb{P}^2=\overline{L_0L_1L_2}$, and $\varphi\colon\mathrm{CH}_1(X)\to\mathrm{CH}_2(F)$ is the Fano correspondence. As $\varphi(h^3)-[F_{L_0}]$ is independent of the point $t\in D_{L_0}$ and $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}}$ is spanned by classes of the form $[t]-[t']$, $t,t'\in D_{L_0}$, we find that the image of $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}}\to A$ is contained in the kernel of ψ, which is A_4.

Next let us show that the image of the composition $\mathrm{CH}_0(D_{L_0})\to A\to A/A_4\simeq A_2\oplus A_0$ is spanned by the class $c_F(L_0)=2c_F-[L_0]_2$, where $[L_0]_2$ is the degree two part of

$$[L_0]=[L_0]_4+[L_0]_2+[L_0]_0\in A_4\oplus A_2\oplus A_0.$$

In order to prove this, observe that the image of $3c_F - ([L_0]_2 + [L_0]_0) = c_F(L_0)$ under the injection $\bar{\psi}\colon A/A_4 \hookrightarrow \mathrm{CH}_2(F)$ can be computed by means of (2.6) as

$$\psi\colon A \longrightarrow \mathrm{CH}_2(F), \quad c_F(L_0) \longmapsto g^2 - c_2(\mathcal{S}_F) - [F_{L_0}].$$

The latter equals $\varphi(h^3) - [F_{L_0}]$ by [Huy23, Proposition 6.4.1], and to conclude we apply (4.1), which proves $\varphi(h^3) - [F_{L_0}] = [F_L] + [F_{\iota(L)}]$ for any $L \in F_{L_0}$. Hence, the classes $c_F(L_0)$ and $[L] + [\iota(L)]$ modulo A_4 have the same image under the injection $\bar{\psi}$. Therefore, the image of

$$\mathbb{Z} \simeq \frac{\mathrm{CH}_0\left(D_{L_0}\right)}{\mathrm{CH}_0\left(D_{L_0}\right)_{\mathrm{hom}}} \longrightarrow A/A_4 \simeq A_2 \oplus A_0$$

is indeed $\mathbb{Z} \cdot c_F(L_0)$.

Before proving the rest of Theorem 1.1(i), we note the following.

Claim. *Assume* $\rho(F) > 1$. *Then the class* $c_F(L_0)$ *is contained in the image of* $\mathrm{CH}_0(D_{L_0}) \to A$.

Proof. It suffices to prove the existence of an invariant class $\eta \in \mathrm{CH}_0(D_{L_0})^+$ with a non-trivial push-forward contained in $A_2 \oplus A_0$. By Lemma 4.7, the proof of which is independent of the rest of the discussion here, this holds for the restriction of the square $\eta = (\alpha|_{F_{L_0}})^2$ of any primitive class $0 \neq \alpha \in \mathrm{CH}^1(F)_{\mathrm{pr}}$. □

It remains to prove that $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}} \to A_4$ is not zero for the very general X and the very general line $L_0 \subset X$. Clearly, it suffices to show this for one cubic, but using the above claim, we will in fact show it for all cubics with $\rho(F) > 1$.

So we assume $\rho(F) > 1$ and suppose that for the very general choice of L_0 and hence for all L_0, the map $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}} \to A_4$ is zero. Then, using that $\mathrm{CH}_0(D_{L_0}) \to A/A_4$ has rank one and the fact that by the claim $c_F(L_0) \in A_2 \oplus A_0$ is contained in the image of the push-forward map, all of $\mathrm{CH}_0(D_{L_0})$ would map to $A_2 \oplus A_0$ (with its image generated by $c_F(L_0)$). Now consider a fixed point $L \in F_{L_0}$ of the covering involution $F_{L_0} \to D_{L_0}$, *i.e.* a line $L \subset X$ such that

$$2L \cup L_0 = \mathbb{P}^2 \cap X \tag{4.2}$$

for some plane $\mathbb{P}^2 \subset \mathbb{P}^5$. Then $2[L] \in \mathrm{CH}_0(D_{L_0}) \subset \mathrm{CH}_0(F_{L_0})^+$, and for L as a point in F, one then has $[L] \in A_2 \oplus A_0$. However, for every line $L \in F$, there exists a line L_0 satisfying (4.2), and for L general L_0 also is general (and so the surface F_{L_0} is smooth). Hence, the triviality of $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}} \to A_4$ for the general L_0 would prove that all points in F define classes in $A_2 \oplus A_0$, *i.e.* $A_4 = 0$, which is absurd. □

Remark 4.1. The last part of the argument also shows that in general the class of a fixed point $L \in F_{L_0}$ of the covering involution ι is not mapped to the distinguished class $(1/2)c_F(L_0) \in A_2 \oplus A_0 \subset A_4 \oplus A_2 \oplus A_0$. Hence, at least for generic choices, none of the 16 fixed points of ι is a candidate for the Beauville–Voisin class $c_{L_0} \in \mathrm{CH}_0(F_{L_0})^+$ in Remark 1.3; see also Remark 4.9.

Remark 4.2. The Beauville–Voisin class on a K3 surface is by definition of degree one, while the analogue $c_F(L_0) \in A_2 \oplus A_0$ is of degree two. There are two reasons for this. First, the projection $F_{L_0} \to D_{L_0}$ is of degree two, and thus the push-forward of any class on F_{L_0} that is pulled back from D_{L_0} has even degree. Second, the intersection pairing $(\alpha_1|_{F_{L_0}}.\alpha_2|_{F_{L_0}})$ of any two primitive classes $\alpha_1, \alpha_2 \in \mathrm{CH}^1(F)_{\mathrm{pr}}$ is even; see [Huy21, Corollary 1.7].

4.2. The proof of Theorem 1.1(ii) starts with the following key technical lemma, which is based on an excess intersection computation.

Lemma 4.3. *We consider the composition*

$$\beta\colon \mathrm{CH}_0\left(F_{L_0}\right) \longrightarrow \mathrm{CH}_0(F) \xrightarrow{\psi} \mathrm{CH}_2(F) \xrightarrow{\mathrm{res}} \mathrm{CH}_0\left(F_{L_0}\right)$$

of the push-forward map $\mathrm{CH}_0(F_{L_0})^- \to \mathrm{CH}_0(F)$*, the map* $\psi\colon \mathrm{CH}_0(F) \to \mathrm{CH}_2(F)$*,* $[L] \mapsto [F_L]$*, and the restriction map* $\mathrm{res}\colon \mathrm{CH}_2(F) \to \mathrm{CH}_0(F_{L_0})$*. Then,* $\beta = -2\cdot \mathrm{id}$*.*

Proof. The group $\mathrm{CH}_0(F_{L_0})^-$ is generated by classes of the form $[L_1]-[L_2]$, where $L_1 \in F_{L_0}$ is any point and $L_2 = \iota(L_1)$ is its image under the covering involution of $F_{L_0} \to D_{L_0}$. The map β sends such a class to the restriction $([F_{L_1}]-[F_{L_2}])|_{F_{L_0}}$.

First observe that the push-forward to A turns $([F_{L_1}]-[F_{L_2}])|_{F_{L_0}}$ into $([F_{L_1}]-[F_{L_2}])\cdot[F_{L_0}]$, which by (2.5) is indeed $-2([L_1]-[L_2]) \in A$. We need to show that this equality holds before pushing forward, *i.e.* $([F_{L_1}]-[F_{L_2}])|_{F_{L_0}} = -2([L_1]-[L_2]) \in \mathrm{CH}_0(F_{L_0})^-$, and for this we have to find a geometric interpretation of $[F_{L_i}]|_{F_{L_0}}$.

The intersection $F_{L_1} \cap F_{L_0}$ is the set of all lines simultaneously intersecting L_1 and L_0. Hence,

$$F_{L_1} \cap F_{L_0} = \{L_2\} \sqcup C_1.$$

Here, $C_1 := \{L \mid x_1 \in L\}$, with x_1 the point of intersection of L_1 and L_0. A similar statement holds for $F_{L_2} \cap F_{L_0}$. Hence,

$$\left([F_{L_1}]-[F_{L_2}]\right)|_{F_{L_0}} = [L_2]-[L_1]+E_1-E_2,$$

where E_1 and E_2 are the contributions from the excess intersections C_1 and C_2. We will show that $E_1 - E_2 = [L_2]-[L_1] \in \mathrm{CH}_0(F_{L_0})$.

To compute E_1, E_2 we recall that the excess intersection $[S]|_T \in \mathrm{CH}_0(S_2)$ of two smooth surfaces $S, T \subset F$ intersecting in a smooth curve $C = S \cap T$ is the push-forward of $\mathrm{c}_1(\mathcal{E})$, where $\mathcal{E}$ is the line bundle $\mathcal{N}_{S/F}|_C/\mathcal{N}_{C/T} \simeq \mathcal{N}_{T/F}|_C/\mathcal{N}_{C/S}$; *cf.* [Ful98, Theorem 9.2] or [EH16, Section 13.3]. In other words, $\mathcal{E}$ is part of a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
\mathcal{N}_{C/T} & \hookrightarrow & \mathcal{N}_{S/F}|_C & \twoheadrightarrow & \mathcal{E} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{T}_T|_C & \hookrightarrow & \mathcal{T}_F|_C & \twoheadrightarrow & \mathcal{N}_{T/F}|_C \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{T}_C & \hookrightarrow & \mathcal{T}_S|_C & \twoheadrightarrow & \mathcal{N}_{C/S}.
\end{array}$$

Hence, as a line bundle on C, one has

$$\begin{aligned}
\mathcal{E} &\simeq \det(\mathcal{N}_{S/F})|_C \otimes \mathcal{N}_{C/T}^* \\
&\simeq \omega_F^*|_C \otimes \omega_S|_C \otimes \omega_T|_C \otimes \omega_C^*.
\end{aligned}$$

In our situation, we let $S = F_{L_i}$ and $T = F_{L_0}$. Their canonical bundles are $\pi_i^*\mathcal{O}(1)$ and $\pi_0^*\mathcal{O}(1)$, where the $\pi_i\colon F_{L_i} \to D_{L_i} \subset \mathbb{P}^3$, $i = 0,1,2$, denote the projections. Hence, the excess contribution for the intersection $[F_{L_i}]|_{F_{L_0}}$ comes from the line bundle

$$\mathcal{E}_i \simeq \pi_i^*\mathcal{O}(1)|_{C_i} \otimes \pi_0^*\mathcal{O}(1)|_{C_i} \otimes \omega_{C_i}^*. \tag{4.3}$$

Claim. *If* $\mathcal{O}_F(1)$ *denotes the Plücker polarisation on* F *and the two lines* L_0, L_i *are considered as points in* C_i*, then*

$$\mathcal{E}_i \simeq \mathcal{O}_F(2)|_{C_i} \otimes \mathcal{O}_{C_i}(-L_0-L_i) \otimes \omega_C^*.$$

Proof. To prove the claim it suffices to show that $\pi_j^*\mathcal{O}(1)|_{C_i} \simeq \mathcal{O}_F(1)|_{C_i} \otimes \mathcal{O}_{C_i}(-L_j)$ for $j=0$ and $j=i$. It is known that $\mathcal{O}_F(1)|_{F_{L_j}} \simeq \pi_j^*\mathcal{O}(1) \otimes \mathcal{O}_{F_j}(C_i)$; see the original [Voi86, Section 3, Lemma 2] or [Huy23, Equation (4.5) in Remark 6.4.13]. Thus, we need to prove that $\mathcal{O}_{F_{L_j}}(C_i)|_{C_i} \simeq \mathcal{O}_{C_i}(L_j)$. To see this, observe that the blow-up of F_{L_j} in the point L_j comes with the projection $q\colon \mathrm{Bl}_L(F_{L_j}) \to L_j$, so that C_i is the fibre over $x_i \in L_j$. In particular, the linear equivalence of $\mathcal{O}_{F_{L_j}}(C_i)$ is independent of the point $x_i \in L_j$. As for two distinct points $x \neq x' \in L_j$, the two curves C_x and $C_{x'}$ only intersect in the point $L_j \in F_{L_j}$ and do so transversally, this concludes the proof of the claim. □

The claim immediately shows $E_1 - E_2 = [L_2] - [L_1] \in \mathrm{CH}_0(F_{L_0})$, which concludes the proof of the lemma. □

Remark 4.4. With the notation in the above proof, we have actually shown that

$$\psi([L_1])|_{F_{L_0}} = [L_2] + E_1 = [L_2] + 2g \cdot C_1 - [L_0] - [L_1] - [\omega_{C_1}].$$

Since $\omega_{C_1} \simeq (\omega_{F_{L_0}} \otimes \mathcal{O}(C_1))|_{C_1} \simeq (\pi_0^* \mathcal{O}(1) \otimes \mathcal{O}(C_1))|_{C_1} \simeq \mathcal{O}_F(1)|_{C_1}$, this gives

$$\psi([L_1] + [L_2])|_{F_{L_0}} = 2(g \cdot C - [L_0]) \in \mathrm{CH}_0\left(F_{L_0}\right),$$

where $C \subset F_{L_0}$ is any curve linearly equivalent to $C_1 \subset F_{L_0}$ or, equivalently, to $C_2 \subset F_{L_0}$.

Corollary 4.5. *The push-forward map*

$$\mathrm{CH}_0\left(F_{L_0}\right)^- \hookrightarrow A$$

and its composition with the projection $A \twoheadrightarrow A/A_4 \simeq A_2 \oplus A_0$

$$\mathrm{CH}_0\left(F_{L_0}\right)^- \hookrightarrow A \twoheadrightarrow A/A_4 \qquad (4.4)$$

are both injective. □

Proof of Theorem 1.1(ii). It remains to prove that for a very general line L_0, the image of the push-forward $\mathrm{CH}_0(F_{L_0})^- \to A$ is not contained in A_2 and that the composition (4.4) induces an isomorphism $\mathrm{CH}_0(F_{L_0})^- \xrightarrow{\sim} A_2 \subset A/A_4$.

For the first assertion, we choose a line L_0 such that for the decomposition

$$[L_0] = [L_0]_4 + [L_0]_2 + [L_0]_0 \in A_4 \oplus A_2 \oplus A_0,$$

we have $[L_0]_4 \neq 0$. Then we view L_0 as a point in F_{L_0} and consider its image $\iota(L_0) \in F_{L_0}$ under the involution. Their difference defines a class $[\iota(L_0)] - [L_0] \in \mathrm{CH}_0(F_{L_0})^-$. By the definition of the Voisin endomorphism $f \colon F \dashrightarrow F$, the push-forward of this class is $f_*[L_0] - [L_0] \in A$, which by the definitions of A_4, A_2, and A_0 equals $3([L_0]_4 - [L_0]_2)$. This proves that the push-forward of $[\iota(L_0)] - [L_0] \in \mathrm{CH}_0(F_{L_0})^-$ is not contained in $A_2 \oplus A_0$.

The surjectivity of the composition (4.4) follows from (2.3), showing $A_2 \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}}$ via the Fano correspondence, and the isomorphism $\mathrm{CH}_0(F_{L_0})^- \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}}$, obtained as the composition of the push-forward and the Fano correspondence; *cf.* [Huy21, Theorem 0.4]. □

Remark 4.6.

(i) Since the image of the other projection

$$\mathrm{CH}_0\left(F_{L_0}\right)^- \hookrightarrow A \twoheadrightarrow A_4 \qquad (4.5)$$

contains the component $[L_0]_4 \in A_4$ and since A_4 is not generated by points in a single surface, the image of (4.5) does indeed depend on L_0. It is natural to wonder whether the composition (4.5), similarly to $\mathrm{CH}_0(D_{L_0})_{\mathrm{hom}} \to A_4$ in Remark 1.3, is actually injective for general L_0.

(ii) As pointed out by Charles Vial, for $[L_0] = c_F$ the map (4.5) is actually trivial and in particular not injective. To prove this one first notices that for $[L_0] = c_F$, the endomorphism $\delta \colon \mathrm{CH}_0(F) \to \mathrm{CH}_0(F)$, $\alpha \mapsto [F_{L_0}] \cdot \psi(\alpha) = (1/3)(g^2 - c_2(\mathcal{S}_F)) \cdot \psi(\alpha)$ preserves A_2. Using the notation of [SV16], this follows from $g, c_2(\mathcal{S}_F) \in \mathrm{CH}^2(F)_0$, see [SV16, Theorem 21.9], $\mathrm{CH}^2(F)_2 = \psi(A_4 \oplus A_2) = \psi(A_2)$, see [SV16, Proposition 21.10], and $\mathrm{CH}^2(F)_0 \cdot \mathrm{CH}^2(F)_2 = A_2$, see [SV16, Proposition 22.3]. Then by Lemma 4.3, δ maps the image $\alpha_4 + \alpha_2 \in A_4 \oplus A_2$ of a class in $\mathrm{CH}_0(F_{L_0})^-$ to $-2 \cdot (\alpha_4 + \alpha_2)$. Since $\delta(\alpha_4 + \alpha_2) = \delta(\alpha_2) \in A_2$, this implies $\alpha_4 = 0$.

In fact, by Lemma 4.3 and the above proof of Theorem 1.1(ii), the push-forward then defines an isomorphism $\mathrm{CH}_0(F_{L_0})^- \xrightarrow{\sim} A_2$, and $\delta|_{A_2} = -2 \cdot \mathrm{id}$. Note that this shows in particular that A_2 is spanned by classes $[L_1] - [L_2]$ forming a triangle with a fixed line L_0 with $[L_0] = c_F$.

(iii) Following a suggestion of Charles Vial, see also [SV16, Theorem 20.2(ii)], we observe that A_4 is spanned by the images of all the maps (4.5) for varying L_0. Indeed, since $A_4 \oplus A_2$ is spanned by classes of the form $[L_1] - [L_2]$ and since for any two lines L_1, L_2, there exists a line L intersecting both, by writing $[L_1] - [L_2] = ([L_1] - [L]) - ([L_2] - [L])$ one reduce to the case that L_1, L_2 are planar. In this case, if L_0 denotes the residual line of $L_1 \cup L_2 \subset \overline{L_1 L_2} \cap X$, then $L_1 = \iota(L_2)$ and, therefore, $([L_1] - [L_2])_4$ is contained in the image of (4.5).

4.3. As the first step towards a proof of Theorem 1.2, we recall from [Huy21, Theorem 0.2], see also [Iza99, Theorem 3] and [She12, Theorem 4.7 and Corollary 4.8], that the restriction of line bundles induces an isomorphism

$$\mathrm{CH}^1(F)_{\mathrm{pr}} \xrightarrow{\sim} \mathrm{CH}^1\left(F_{L_0}\right)^-_{\mathrm{pr}}. \tag{4.6}$$

Here, the primitive parts on the two sides are defined with respect to the Plücker polarisation $\mathcal{O}_F(1)$ on F and its restriction to F_{L_0}. Thus, to establish Theorem 1.2, it suffices to show that for any $\alpha \in \mathrm{CH}^1(F)_{\mathrm{pr}}$, the class $\alpha^2 \cdot [F_{L_0}]$ is a multiple of $c_F(L_0) = 2c_F - [L_0]_2$.

The next step can be seen as a warm-up. The result was used already in the proof of Theorem 1.1(i).

Lemma 4.7. *For any class $\alpha \in \mathrm{CH}^1(F)$, the push-forward of $(\alpha|_{F_{L_0}})^2 \in \mathrm{CH}_0(F_{L_0})$ to F is a class in $A_2 \oplus A_0$.*

Proof. The assertion is equivalent to $\alpha^2 \cdot [F_{L_0}] \in A_2 \oplus A_0$. To prove this, recall that the two classes $\psi([L_0]) = [F_{L_0}]$ and $\psi(c_F) = (1/3)(g^2 - \mathrm{c}_2(\mathcal{S}_F))$ in $\mathrm{CH}^2(F)$ are homologically equivalent; *cf.* [Huy23, Proposition 6.4.1].

Hence, $[F_{L_0}] - \psi(c_F)$ is contained in the homologically trivial part of $\mathrm{Im}(\psi)$. Therefore, by (2.7) we have $\alpha^2 \cdot ([F_{L_0}] - \psi(c_F)) \in A_2$. Now, we apply a result of Voisin [Voi08, Theorem 0.4] and use the well-known fact that $\mathrm{c}_2(\mathcal{S}_F)$ is a linear combination of $\mathrm{c}_2(\mathcal{T}_F)$ and g^2, *cf.* [Huy23, Proposition 6.4.1], to conclude that $3\alpha^2 \cdot \psi(c_F) = \alpha^2 \cdot (g^2 - \mathrm{c}_2(\mathcal{S}_F)) \in A_0$. Hence, $\alpha^2 \cdot [F_{L_0}] \in A_2 \oplus A_0$. □

Proof of Theorem 1.2. We apply Lemma 4.7 to an arbitrary anti-invariant primitive class $\delta \in \mathrm{CH}^1(F_{L_0})^-_{\mathrm{pr}}$, which by (4.6) can be written as $\alpha|_{F_{L_0}}$ for a unique primitive class α on F. Then the push-forward of $\delta^2 \in \mathrm{CH}^2(F_{L_0})^+$ is contained in $A_2 \oplus A_0$. As by Theorem 1.1(i) the image of the composition $\mathrm{CH}^2(D_{L_0}) \to A \twoheadrightarrow A/A_4 = A_2 \oplus A_0$ is spanned by $c_F(L_0)$, this shows that the push-forwards of δ^2 and $c_F(L_0)$ are proportional.

To conclude, note that all numerical intersection products $(\alpha_1|F_{L_0}.\alpha_2|F_{L_0})$ for primitive classes $\alpha_1, \alpha_2 \in \mathrm{CH}^1(F)_{\mathrm{pr}}$ are even; see [Huy21, Corollary 1.7]. This implies that $\alpha_1|_{F_{L_0}} \cdot \alpha_1|_{F_{L_0}} \in \mathrm{CH}^2(F_{L_0})^+$ is contained in the index two subgroup $\mathrm{CH}^2(D_{L_0}) \subset \mathrm{CH}^2(F_{L_0})^+$. □

Remark 4.8. The result cannot be extended to the whole $\mathrm{CH}^1(F_{L_0})$. Indeed, the push-forward of $g^2|_{F_{L_0}}$, *i.e.* $g^2 \cdot [F_{L_0}]$, is not a multiple of $c_F(L_0)$, for by [SV16, Lemma 18.2] we have

$$\begin{aligned} g^2 \cdot [F_{L_0}] &= [f(L_0)] - 4[L_0] + 24c_F \\ &= (4[L_0]_4 - 2[L_0]_2 + [L_0]_0) - 4[L_0]_4 - 4[L_0]_2 - 4[L_0]_0 + 24c_F \\ &= -6[L_0]_2 - 3[L_0]_0 + 24c_F, \end{aligned}$$

which can be a multiple of $c_F(L_0)$ only if $[L_0]_2 = 0$. By virtue of [SYZ20, Theorem 3.4], the latter is in fact equivalent to $[L_0] = c_F$, so excluded for general L_0.

Also, extending to just the full anti-invariant part is not possible. Indeed, as used before, the restriction of the Plücker polarisation satisfies $\mathcal{O}_F(1)|_{F_{L_0}} \simeq \pi_0^*\mathcal{O}(1) \otimes \mathcal{O}(C_x)$ for any point $x \in L_0$. Hence, we have

$(g_{F_{L_0}} - \iota^*(g_{F_{L_0}}))^2 = C_x \cdot C_x + \iota(C_x \cdot C_x) - 2(C_x \cdot \iota(C_x))$ on F_{L_0}, which under push-forward to F becomes $[L_0] + f_*[L_0] - 2(C_x \cdot \iota(C_x) = [L_0] + f_*[L_0] - 2(6c_F - 2[L_0])$, which for general L_0 is not contained in $A_2 \oplus A_0$. The last equality uses a formula in the proof of [SV16, Lemma 18.2].

Remark 4.9. It also seems unlikely that the Beauville–Voisin class $c_F(L_0) \in A_2 \oplus A_0 \subset \mathrm{CH}_0(F)$ can be lifted to a Beauville–Voisin class $c_{L_0} \in \mathrm{CH}_0(F_{L_0})$. Indeed, as a consequence of Remark 4.4, we know that

$$\psi(c_F(L_0))|_{F_{L_0}} = 2(g \cdot C - [L_0]) \in \mathrm{CH}_0\left(F_{L_0}\right)$$

with $C \in |\mathcal{O}_F(1)|_{F_{L_0}} \otimes \pi_0^* \mathcal{O}(-1)|$. So, if there exists a Beauville–Voisin class in $\mathrm{CH}_0(F_{L_0})^+$ as in Remark 1.3, then a natural guess would be that it must be proportional to $g \cdot C - [L_0]$. Note, however, that the push-forward of $g \cdot C - [L_0]$ is typically not contained in $A_2 \oplus A_0$, so not proportional to $c_F(L_0)$. Even worse, the class $g \cdot C - [L_0] \in \mathrm{CH}_0(F_{L_0})$ is not even invariant.

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