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K-stability for varieties with a big anticanonical class

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To Claire Voisin on the occasion of the conference in her honor

Abstract. We extend the algebraic K-stability theory to projective klt pairs with a big anticanonical class. While in general such a pair could behave pathologically, it is observed in this note that the K-semistability condition will force them to have a klt anticanonical model, whose stability property is the same as that of the original pair.

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Contents

1. Introduction

There has been tremendous progress in algebraic K-stability theory of log Fano pairs (see [Xu21] for a survey of the topic). In the recent works [DZ22] and [DR22], the Kähler–Einstein problem is considered for a Kähler manifold (X,ω) such that $-K_X$ is big. More precisely, in [DZ22] the authors prove a transcendental Yau–Tian–Donaldson theorem for twisted big Kähler–Einstein metrics. As a consequence of their result, in the algebraic setting, uniform K-stability of X with a big anticanonical class implies the existence of a Kähler–Einstein metric. In this note we want to show that the K-stability theory in this setting, *i.e.* a projective klt pair with a big anticanonical class, essentially follows from the original (log) Fano case.

In general, there could be pathological examples in projective varieties X with a big anticanonical class $-K_X$; *e.g.* the anticanonical ring $R(X,-K_X)=\bigoplus_{m\in\mathbb{N}}H^0(X,-mK_X)$ is not necessarily finitely generated (see Example 3.8). However, we will show that the K-stability condition implies that X is of log Fano type.

Theorem 1.1. *Let (X,Δ) be a klt projective pair with $-K_X-\Delta$ big. Assume $\delta(X,\Delta)\geq 1$. Then there exists an effective $\mathbb{Q}$-divisor Γ such that $(X,\Delta+\Gamma)$ is a log Fano pair, i.e. $(X,\Delta+\Gamma)$ is klt and $-K_X-\Delta-\Gamma$ is ample. In particular,*

$$R(X,-r(K_X+\Delta))=\bigoplus_{m\in r\cdot\mathbb{N}}H^0(X,-m(K_X+\Delta))$$

is finitely generated for any r such that $r(K_X+\Delta)$ is Cartier.

Here $\delta(X,\Delta)$ is defined in the exactly same fashion as in the case when $-K_X-\Delta$ is ample (see [FO18,BJ20]), *i.e.*

$$\delta(X,\Delta)=\inf_E\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)}.$$

For the stronger and more precise statement, see Theorem 3.4. We note that the above finite generation is asked in [DR22].

The above observation makes it possible to use existing birational geometry techniques to study K-stability questions for X with a big anticanonical class. In fact, without too much difficulty, it reduces K-stability questions for (X,Δ) to K-stability questions for its anticanonical model (Z,Δ_Z), as we can see from the following statement.

Theorem 1.2. *Let (X,Δ) be a klt projective pair with $-K_X-\Delta$ big. Assume $R=\bigoplus_{m\in r\cdot\mathbb{N}}H^0(-m(K_X+\Delta))$ is finitely generated, and denote by (Z,Δ_Z) the anticanonical model. Then (X,Δ) is K-semistable (resp. K-stable, uniformly K-stable) if and only (Z,Δ_Z) is K-semistable (resp. K-stable, uniformly K-stable). In particular, uniform K-stability of (X,Δ) is the same as K-stability of (X,Δ).*

Remark 1.3. In [DR22], Ding stability notions for a projective klt pair (X,Δ) with big $-K_X-\Delta$ are developed. If one assumes $R=\bigoplus_{m\in r\cdot\mathbb{N}}H^0(-m(K_X+\Delta))$ is finitely generated and denotes by (Z,Δ_Z) the anticanonical model, then one can show a similar statement to Theorem 1.2; *i.e.* the Ding stability notions for (X,Δ) are equivalent to the notions for (Z,Δ_Z).

Notation and Convention.— Throughout this paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0. We follow the standard terminology from [KM98, Kol13].

For a normal log pair (X,Δ) such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier and a divisor E over X, we denote by $A_{X,\Delta}(E)$ the log discrepancy of E with respect to (X,Δ).

We say a klt projective pair (X,Δ) is *log Fano* if (X,Δ) is klt and $-K_X-\Delta$ is ample, and a klt projective pair (X,Δ) is *of log Fano type* if there exists an effective $\mathbb{Q}$-divisor D such that $(X,\Delta+D)$ is a log Fano pair.

We say an effective $\mathbb{Q}$-divisor Γ on a projective log pair (X,Δ) is an *N-complement* for a positive integer N if $N(K_X+\Delta+\Gamma)\sim 0$ and $(X,\Delta+\Gamma)$ is log canonical. A *$\mathbb{Q}$-complement* is an N-complement for some N.

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2. S-invariants

Let (X,Δ) be an n-dimensional projective normal pair such that $-K_X-\Delta$ is big. For any prime divisor E which appears on a birational model $\mu\colon Y\to X$, the S-invariant is defined as

$$S_{X,\Delta}(E):=\frac{1}{\operatorname{vol}(-K_X-\Delta)}\int_0^\infty \operatorname{vol}(-\mu^*(K_X+\Delta)-tE)\,dt.$$

Definition 2.1. If (X,Δ) is klt, we define

$$\delta(X,\Delta):=\inf_E\frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)},$$

where E runs through all valuations over (X,Δ). We say (X,Δ) is uniformly K-stable (resp. K-semistable), if $\delta(X,\Delta)>1$ (resp. $\delta(X,\Delta)\geq 1$). We say (X,Δ) is K-stable if $A_{X,\Delta}(E)>S_{X,\Delta}(E)$ for any E over X.

Remark 2.2. When (X,Δ) is log Fano, the equivalence between this way of defining K-stability notions using valuations and the original one using test configurations, called the Fujita–Li criterion, is proved in [Fuj19], [Li17] and [BX19]. For (X,Δ) with a big anticanonical class, the current definition is formulated in [DZ22].

Remark 2.3. Theorem 1.2 says K-stability is indeed the same as uniform K-stability. For a log Fano pair, this is proved in [LXZ22] (see [XZ22] for a different proof).

Fix $m\in r\cdot\mathbb{N}$, let $R_m=H^0(X,-m(K_X+\Delta))$, and assume $N_m:=\dim H^0(X,-m(K_X+\Delta))>0$. Following [FO18], we say a $\mathbb{Q}$-divisor D is an *m-basis type divisor* if

$$\frac{1}{m\cdot N_m}\operatorname{ord}_E\Big(\operatorname{div}(s_1)+\cdots+\operatorname{div}(s_{N_m})\Big)$$

for a basis $\{s_1,\dots,s_{N_m}\}$ of R_m. In particular, $D\sim_{\mathbb{Q}}-K_X-\Delta$.

We define $S_{X,\Delta,m}(E)$ (or $S_m(E)$ if (X,Δ) is clear) for any E over X as follows: E yields a decreasing filtration $\mathcal{F}_E^\lambda$ $(\lambda\in\mathbb{R})$ on $R_m:=H^0(X,-m(K_X+\Delta))$ by

$$\mathcal{F}_E^\lambda R_m=\Big\{s\in H^0(X,-m(K_X+\Delta))\mid \operatorname{ord}_E(s)\geq\lambda\Big\},$$

and

$$S_m(E)=\frac{1}{m\cdot N_m}\operatorname{ord}_E\Big(\operatorname{div}(s_1)+\cdots+\operatorname{div}(s_{N_m})\Big)$$

for any basis $\{s_1,\dots,s_{N_m}\}$ of R_m compatible with $\mathcal{F}_E^\lambda$ ($\lambda\in\mathbb{R}$). Here the basis $\{s_1,\dots,s_{N_m}\}$ is compatible with $\mathcal{F}_E^\lambda$ ($\lambda\in\mathbb{R}$) if for any λ, all the elements s_i contained in $\mathcal{F}_E^\lambda R_m$ span $\mathcal{F}_E^\lambda R_m$. Then

$$S_m(E)=\frac{1}{m\cdot N_m}\sum_{\lambda\in\mathbb{N}}\lambda\cdot\dim \mathrm{Gr}_E^\lambda R_m,$$

where $\mathrm{Gr}_E^\lambda R_m:=\mathcal{F}_E^\lambda R_m/\mathcal{F}_E^{\lambda+1}R_m$.

We also define

$$\delta_m(X,\Delta):=\inf_E\frac{A_{X,\Delta}(E)}{S_m(E)}.$$

The following are basic properties proved in [BJ20].

Theorem 2.4. *Keep the notation as above.*

(1) *We have* $\lim_{m\to\infty}S_m(E)=S(E)$.
(2) *For any* $\varepsilon>0$, *there exists an* m_0 *such that for any* E *over* X *and* $m\ge m_0$ *with* $m\in r\cdot\mathbb{N}$,

$$S_m(E)\le(1+\varepsilon)S(E).$$

(3) *We have* $\delta_m(X,\Delta)=\inf_D \mathrm{lct}(X,\Delta;D)$, *where* D *runs through all* m*-basis type divisors.*
(4) *We have* $\lim_{m\to\infty}\delta_m(X,\Delta)=\delta(X,\Delta)$.

Proof. Statement (1) follows from [BJ20, Lemma 2.9] and (2) from [BJ20, Corollary 2.10]. Statement (3) is [BJ20, Proposition 4.3], and (4) is [BJ20, Theorem 4.4]. □

We can consider more general filtrations.

Definition 2.5. By a (linearly bounded) filtration $\mathcal{F}$ on $R(X,-r(K_X+\Delta))=\bigoplus_{m\in r\cdot\mathbb{N}}R_m$, we mean the data of a family $\mathcal{F}^\lambda R_m\subseteq R_m$ of $\mathbb{k}$-vector subspaces of R_m for $m\in r\cdot\mathbb{N}$ and $\lambda\in\mathbb{R}$, satisfying

(1) $\mathcal{F}^{\lambda'}R_m\subseteq\mathcal{F}^\lambda R_m$ when $\lambda\ge\lambda'$;
(2) $\mathcal{F}^\lambda R_m=\bigcap_{\lambda'<\lambda}\mathcal{F}^{\lambda'}R_m$ for any λ;
(3) there exist $e_-,e_+\in\mathbb{R}$ such that $\mathcal{F}^{me_-}R_m=R_m$ and $\mathcal{F}^{me_+}R_m=0$ for any m;
(4) $\mathcal{F}^\lambda R_m\cdot\mathcal{F}^{\lambda'}R_{m'}\subseteq\mathcal{F}^{\lambda+\lambda'}R_{m+m'}$.

For any filtration $\mathcal{F}$ on R, we can define $S_m(\mathcal{F})$ and $S(\mathcal{F})$ as in [BJ20, Sections 2.5 and 2.6, pp. 15–16], and we have

$$\lim_{m\to\infty}S_m(\mathcal{F})\longrightarrow S(\mathcal{F}); \tag{2.1}$$

see [BJ20, Lemma 2.9].

Lemma 2.6. *If* A *is an effective ample* $\mathbb{Q}$*-divisor on* X *such that* $-K_X-\Delta-A$ *is pseudoeffective, then* $S_{X,\Delta}(A)\ge\frac{1}{n+1}$.

Proof. Since $-K_X-\Delta-A$ is pseudoeffective, for any $t\ge0$, we have

$$\mathrm{vol}(-K_X-\Delta-tA)\ge\mathrm{vol}((1-t)(-K_X-\Delta)).$$

Thus

$$\begin{aligned}S(A)&=\frac{1}{\mathrm{vol}(-K_X-\Delta)}\int_0^{+\infty}\mathrm{vol}(-K_X-\Delta-tA)dt\\&\ge\frac{1}{\mathrm{vol}(-K_X-\Delta)}\int_0^{1}\mathrm{vol}((1-t)(-K_X-\Delta))dt\\&=\frac{1}{(n+1)}.\end{aligned}$$

□

3. Finite generation

3.1. $\mathbb{Q}$-complements and finite generation

For a $\mathbb{Q}$-divisor D with $|rD| \neq 0$ and any $m \in \mathbb{N}$, we denote by $\mathrm{Bs}(|mrD|)$ the base ideal. We can define the log canonical threshold of the asymptotic linear series as follows:

$$\mathrm{lct}(X,\Delta;\|-K_X-\Delta\|) := \sup_{\ell} \mathrm{lct}\Big(X,\Delta;\frac{1}{\ell r}\mathrm{Bs}\,|\ell r(-K_X-\Delta)|\Big).$$

We can define a sequence of multiplier ideals

$$\mathcal{I}\Big(X,\Delta;\frac{1}{r}\mathrm{Bs}(|rD|)\Big) \subseteq \mathcal{I}\Big(X,\Delta;\frac{1}{2r}\mathrm{Bs}(|2rD|)\Big) \subseteq \cdots \subseteq \mathcal{I}\Big(X,\Delta;\frac{1}{\ell! r}\mathrm{Bs}(|\ell! rD|)\Big) \subseteq \cdots.$$

By the ascending chain condition of ideals, this sequence will stabilize. We denote the maximal element by $\mathcal{I}(X,\Delta;\|-K_X-\Delta\|)$ and call it the *asymptotic multiplier ideal sheaf of* D. For more background, see [Laz04, Section 11.1]. Recall that for any ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, we have $\mathrm{lct}(X,\Delta;\mathfrak{a}) > 1$ if and only if $\mathcal{I}(X,\Delta;\mathfrak{a}) > 1$.

Lemma 3.1. *Assume (X,Δ) is a projective pair with $-K_X-\Delta$ big. If*

$$\mathrm{lct}(X,\Delta;\|-K_X-\Delta\|) > 1 \quad (\textit{or equivalently } \mathcal{I}(X,\Delta;\|-K_X-\Delta\|) = \mathcal{O}_X),$$

then (X,Δ) is of log Fano type.

Proof. From the assumption, there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -(K_X+\Delta)$ such that $(X,\Delta+D)$ is klt. Since D is big, then $D \sim_{\mathbb{Q}} A+E$ for an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E. Set

$$\Gamma := (1-\varepsilon)D + \varepsilon E$$

for $0 < \varepsilon \ll 1$; then $(X,\Delta+\Gamma)$ is klt, and $-K_X-\Delta-\Gamma \sim \varepsilon A$ is ample. Thus (X,Δ) is of log Fano type. ☐

Definition 3.2. For any projective pair (X,Δ), we define the constant $a(X,\Delta)$ by

$$a(X,\Delta) = \sup_{t\in\mathbb{R}} \left\{ \begin{array}{c} \text{there exists an ample divisor } A \text{ such that } A - t(K_X+\Delta) \\ \text{is ample and } -K_X-\Delta-A \text{ is pseudoeffective} \end{array} \right\}. \tag{3.1}$$

If $-K_X-\Delta$ is big, then $a(X,\Delta) > 0$; if $-K_X-\Delta$ is ample, then $a(X,\Delta) = +\infty$.

Assumption 3.3. Let (X,Δ) be an n-dimensional klt projective pair with $-K_X-\Delta$ big. Assume

$$\delta(X,\Delta) > \frac{n+1}{n+1+a_0}, \quad \text{where } a_0 = a(X,\Delta). \tag{3.2}$$

Now we can show the following.

Theorem 3.4. *Let (X,Δ) satisfy Assumption 3.3; then (X,Δ) is of log Fano type. In particular, any Cartier divisor E on X satisfies that $R(X,E) := \bigoplus_{m\in\mathbb{N}} H^0(X,mE)$ is finitely generated.*

Proof. Let us first prove this when $\delta(X,\Delta) > 1$ as it is quite straightforward. By Theorem 2.4, we know that for a sufficiently large m and any m-basis type divisor D,

$$\mathrm{lct}(X,\Delta;D) \geq \delta_m(X,\Delta) > 1.$$

Thus we can apply Lemma 3.1.

In the general case, we may assume $\delta(X,\Delta) \leq 1$, and we need some perturbation argument. By our definition of $a(X,\Delta)$, for any $t \in (0, a(X,\Delta))$, there exists an ample $\mathbb{Q}$-divisor A such that

$$-K_X-\Delta-A \sim_{\mathbb{Q}} E_1 \quad \text{and} \quad A - t(K_X+\Delta) \sim_{\mathbb{Q}} A_0,$$

where E_1 is an effective $\mathbb{Q}$-divisor and A_0 is an ample $\mathbb{Q}$-divisor. Moreover, by (3.2) we may assume

$$1-\delta(X,\Delta)<\frac{t}{n+1}\delta(X,\Delta). \tag{3.3}$$

Fix $m_0\in\mathbb{N}$ such that $|m_0A|$ is base-point-free. Then for any prime divisor $H\in|m_0A|$, by Lemma 2.6,

$$\begin{aligned} S(H) &= \frac{1}{\mathrm{vol}(-K_X-\Delta)}\int \mathrm{vol}(-K_X-\Delta-tH)dt \\ &= \frac{1}{m_0}S_{X,\Delta}(A) \\ &\geq \frac{1}{m_0(n+1)}. \end{aligned}$$

We can choose an m-basis type $\mathbb{Q}$-divisor D_m compatible with H, so we can write $D_m=F_m+b_mH$, where

$$\lim_{m\to\infty} b_m=\lim_{m\to\infty} S_m(H)=S(H)\geq\frac{1}{m_0(n+1)}. \tag{3.4}$$

By (3.3), (3.4), and the equality $\lim_m \delta_m(X,\Delta)=\delta(X,\Delta)$, we can find a sufficiently large m and a positive δ' such that $\delta'<\min\{\delta_m(X,\Delta),1\}$ and

$$1-\delta'<tm_0b_m\delta'. \tag{3.5}$$

Then $(X,\Delta+\delta'F_m)$ is klt, as $(X,\Delta+\delta'D_m)$ is klt and $D_m=F_m+b_mH$. Moreover,

$$-K_X-\Delta-\delta'F_m\sim_{\mathbb{Q}}-(1-\delta')(K_X+\Delta)+\delta'b_mH,$$

which implies $(X,\Delta+\delta'F_m)$ is a log Fano pair since

$$\begin{aligned} -(1-\delta')(K_X+\Delta)+\delta'b_mH &\sim_{\mathbb{Q}} (1-\delta')\left(-(K_X+\Delta)+\frac{1}{t}A\right)+\left(\delta'b_mm_0-\frac{1-\delta'}{t}\right)A \\ &\sim_{\mathbb{Q}} \frac{1-\delta'}{t}A_0+\left(\delta'b_mm_0-\frac{1-\delta'}{t}\right)A \end{aligned}$$

is ample by (3.5).

The last statement then follows from [BCH+10]. □

Corollary 3.5. *Let (X,Δ) satisfy Assumption 3.3. Let $r(K_X+\Delta)$ be Cartier and $Z:=\mathrm{Proj}\ R(X,-r(K_X+\Delta))$. Denote by Δ_Z the birational transform of Δ on Z; then (Z,Δ_Z) is a log Fano pair.*

Proof. We know $f\colon X\dashrightarrow Z$ is a birational contraction; *i.e.* $\mathrm{Ex}(f^{-1})$ does not contain any divisor, and $f_*(K_X+\Delta)=K_Z+\Delta_Z$ is antiample.

It follows from Theorem 3.4 that there exists a $\mathbb{Q}$-complement Γ for (X,Δ) such that $(X,\Delta+\Gamma)$ is klt. Then $(Z,\Delta_Z+f_*\Gamma)$ is klt as the pullbacks of $K_Z+\Delta_Z+f_*\Gamma$ and $K_X+\Delta+\Gamma$ on a common resolution are equal. So (Z,Δ_Z) is klt. □

3.2. K-stability of the anticanonical model

Let (X,Δ) be a projective log pair with big $-K_X-\Delta$. Let (Z,Δ_Z) be its anticanonical model; *i.e.* $Z=\mathrm{Proj}\ R(X,-r(K_X+\Delta))$, and Δ_Z is the birational transform of Δ on Z. Let Y be a common resolution.

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{\mu} & & \searrow^{\pi} & \\ (X,\Delta) & & \overset{f}{\dashrightarrow} & & (Z,\Delta_Z). \end{array} \tag{3.6}$$

Then

$$\pi^*(K_Z+\Delta_Z)-\mu^*(K_X+\Delta)=B\geq 0$$

Lemma 3.6. *Let* (X,Δ) *satisfy Assumption 3.3. Then for any prime divisor* E *over* X,

$$A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E) + \mathrm{ord}_E(B) \text{ and } S_{X,\Delta}(E) = S_{Z,\Delta_Z}(E) + \mathrm{ord}_E(B).$$

Proof. For the log discrepancy function, this follows directly from the definition. Since

$$|\mu^*(-m(K_X+\Delta))| = |\pi^*(-m(K_Z+\Delta_Z))| + mB,$$

we have $S_{X,\Delta,m}(E) = S_{Z,\Delta_Z,m}(E) + \mathrm{ord}_E(B)$. Therefore, the same is true for the S-function. □

Lemma 3.7. *If* (Z,Δ_Z) *is klt, there exists a* $t>0$ *depending on* Z (*but not* E) *such that for any divisor* E *over* X

$$A_{Z,\Delta_Z}(E) \geq t\cdot \mathrm{ord}_E(B).$$

Proof. Since (Z,Δ_Z) is klt, we know that there exists a $t>0$ such that if we write $\pi^*(K_Z+\Delta_Z) = K_Y+\Delta_1$, then $(K_Y+\Delta_1+tB)$ is sub-lc for some $t>0$; *i.e.* for any E,

$$A_{Z,\Delta_Z}(E) \geq t\cdot \mathrm{ord}_E(B).$$ □

Proof of Theorem 1.2. Since

$$\delta(X,\Delta) = \inf_E \frac{A_{Z,\Delta_Z}(E)+\mathrm{ord}_E(B)}{S_{Z,\Delta_Z}(E)+\mathrm{ord}_E(B)},$$

it is clear that $\delta(X,\Delta)\geq 1$ if and only if $A_{Z,\Delta_Z}(E)\geq S_{Z,\Delta_Z}(E)$, *i.e.* (Z,Δ) is klt and $\delta(Z,\Delta_Z)\geq 1$. Moreover,

$$A_{X,\Delta}(E) = A_{Z,\Delta_Z}(E)+\mathrm{ord}_E(B) > S_{Z,\Delta_Z}(E)+\mathrm{ord}_E(B) = S_{X,\Delta}(E)$$

if and only if $A_{Z,\Delta_Z}(E) > S_{Z,\Delta_Z}(E)$.

Assume $\delta(X,\Delta)>1$; then $\delta(Z,\Delta_Z)\geq\delta(X,\Delta)$. Conversely, if $\delta(Z,\Delta_Z)>1$, an easy calculation shows that

$$\delta(X,\Delta) \geq \frac{\delta(Z,\Delta_Z)(t+1)}{\delta(Z,\Delta_Z)+t} > 1,$$

where t is the constant from Lemma 3.7. □

Example 3.8. This example has appeared in several works to present pathological phenomena, see *e.g.* [Gon12]: Let S be the blowup of $\mathbb{P}^2$ at nine very general points. Then $-K_S$ is known to be nef but not semiample. In fact, there will be a unique cubic curve passing through these nine points, and if we denote by E its birational transform on S, then for any $m\in\mathbb{N}$, $|-mK_S|$ has one element mE.

Let H be an ample Cartier divisor on S and $X=\mathbb{P}_S(E)$, where $E := \mathcal{O}_S + \mathcal{O}_S(H)$. Denote by $\pi\colon X\to S$ the natural morphism. We claim $-K_X$ is big. In fact, since

$$\omega_{X/S} = \wedge^2\mathcal{O}_{\mathbb{P}(E)}(-2),$$

we have

$$\begin{aligned} H^0(\mathcal{O}_X(-mK_X)) &= H^0(S,\pi_*(\mathcal{O}_X(-mK_X)))\\ &= H^0(S,\mathrm{Sym}^{2m}(E)\otimes(\wedge^2E)^{\otimes -m}\otimes\omega_S^{\otimes m})\\ &= H^0\left(S,\left(\bigoplus_{i=0}^{2m}\mathcal{O}_S(iH)\right)\otimes\mathcal{O}_S(-mH-mK_S)\right)\\ &= H^0\left(S,\bigoplus_{i=0}^{m}\mathcal{O}_S(iH-mK_S)\right),\end{aligned}$$

and since $-K_S\sim E$ is nef, we have

$$\begin{aligned}\mathrm{vol}_X(-K_X) &= 6\int_0^1 \frac{1}{2}(tH-K_S)^2 = 3\int_0^1 (t^2H^2-2tH(-K_S))\\ &= H^2+3H\cdot(-K_S)>0.\end{aligned}$$

However, the algebra $\bigoplus_{i\le m} H^0(iH-mK_S)$ is not finitely generated, since

$$\sum_{1\le j\le m-1} H^0(\mathcal{O}_S(-jK_S))\otimes H^0(\mathcal{O}_S(H-(m-j)K_S))\longrightarrow H^0(\mathcal{O}_S(H-mK_S))$$

is not surjective for any m. Thus we need generators from $H^0(\mathcal{O}_S(H-mK_S))$ for every m.

By Theorem 1.1, we know $\delta(X)<1$. Here we give a direct verification of this. We denote by $Y\subseteq X$ the section given by

$$E=\mathcal{O}_S\oplus\mathcal{O}_S(H)\longrightarrow\mathcal{O}_S.$$

Then similarly to before, we have

$$H^0(\mathcal{O}_X(-mK_X-m_0Y))=H^0\left(S,\left(\bigoplus_{i=m_0}^{2m}\mathcal{O}_S(iH)\right)\otimes\mathcal{O}_S(-mH-mK_S)\right),$$

where we follow the convention that if $m_0>2m$, then the direct sum is 0. Hence a direct calculation implies

$$\operatorname{vol}(-K_X-tY)\ =\begin{cases} H^2+3H\cdot(-K_S) & \text{if } t\le 1,\\ (2-t)((t^2-t+1)H^2+3tH\cdot(-K_S)) & \text{if } 1\le t\le 2.\end{cases}$$

By an elementary calculation,

$$S_X(Y)=\frac{\frac{7}{4}H^2+5H\cdot(-K_S)}{H^2+3H\cdot(-K_S)}>\frac{5}{3}>1=A_X(Y),$$

which implies $\delta(X)<\frac{3}{5}$.

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