EPIGA

On a decomposition of p-adic Coxeter orbits

Alexander B. Ivanov

Abstract. We analyze the geometry of some p-adic Deligne–Lusztig spaces $X_w(b)$, introduced by the author in a paper in Crelle, attached to an unramified reductive group $\mathbf{G}$ over a non-archimedean local field. We prove that when $\mathbf{G}$ is classical, b basic and w Coxeter, $X_w(b)$ decomposes as a disjoint union of translates of a certain integral p-adic Deligne–Lusztig space. Along the way we extend some observations of DeBacker and Reeder on rational conjugacy classes of unramified tori to the case of extended pure inner forms and prove a loop version of the Frobenius-twisted Steinberg cross section.

Alexander B. Ivanov
Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
(new address) Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstrasse 150, 44780 Bochum, Germany
e-mail: a.ivanov@rub.de

arXiv:2109.01424v3 [math.AG] 26 Sep 2023

Contents

1. Introduction

Let $\mathbf{G}$ be an unramified reductive group over a non-archimedean local field k. In [Iva23] p-adic Deligne–Lusztig spaces $X_w(b)$ attached to $\mathbf{G}$ were constructed. This allows one to apply the rich variety of methods from Deligne–Lusztig theory [DL76] to the study of representations of the p-adic group $\mathbf{G}(k)$. Indeed, many results in this direction were obtained recently; *cf.* [CI23, CS17, CO21, DI20] and references therein.

The classical Deligne–Lusztig varieties are smooth schemes of finite type over a finite field. In contrast, the $X_w(b)$ are sheaves for the arc-topology, *cf.* [BM21], on perfect algebras over the residue field of k. Concerning their representability, the situation is slightly delicate: In many cases it is known that the $X_w(b)$ are ind-representable; *cf.* [Iva23, Theorem C]. Also, there are examples where $X_w(b)$ is known to be an ind-scheme which is not a scheme; *cf.* [Iva23, Corollary 10.2]. On the other hand, for $\mathbf{G} = \mathbf{GL}_n$, c Coxeter and b basic, $X_c(b)$ is a scheme; *cf.* [CI23, Proposition 2.6]. Moreover, the proof of this last fact has led to a quite explicit description of (a cover of) $X_c(b)$ in this case, eventually allowing one to compute large parts of its cohomology and relate it to the local Langlands correspondences, which was the main task of [CI23, CI21b].

Motivated by the last sentence, we analyze here the geometry of the spaces $X_c(b)$ and their natural covers $\dot{X}_{\bar{c}}(b)$ when $\mathbf{G}$ is a classical group,[1] c is a (twisted) Coxeter element and b is basic. Under these assumptions our main result, Theorem 1.1, shows that $X_c(b)$ and $\dot{X}_{\bar{c}}(b)$ admit natural decompositions as disjoint unions of translates of certain integral level Deligne–Lusztig spaces of Coxeter type. Those integral level spaces are easier to handle. In particular, they are affine schemes, and it follows that $X_c(b)$ and $\dot{X}_{\bar{c}}(b)$ are schemes, proving [Iva23, Conjecture 1.1] in this case. Our decomposition result, along with the Mackey-type formula shown in [DI20] (as well as techniques from [Lus76, CO21]), can be applied—as in [CI23]—to study the cohomology of $X_c(b)$ and $\dot{X}_{\bar{c}}(b)$ and smooth $\mathbf{G}(k)$-representations therein. Beside our main result, we prove some related results on rational conjugacy classes of tori and a version of Steinberg's cross section; see Section 1.4 below for a summary. Now, we introduce the relevant notation and state our main result, Theorem 1.1.

---

[1] We call an unramified k-group *classical* if each connected component of its Dynkin diagram is of type $A_{n-1}, B_m, C_m, D_m, {}^2A_{n-1}$ or ${}^2D_m$; see Definition 5.1 for more details.

1.1. Setup

Let $\check{k}$ denote the completion of a maximal unramified extension of k, and let $\mathcal{O}_k, \mathcal{O}_{\check{k}}$ be the rings of integers of $k, \check{k}$. Let σ be the Frobenius of $\check{k}/k$. Let $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ be a centralizer of a maximal split k-torus and a k-rational Borel subgroup of the unramified group $\mathbf{G}$. (Note that $\mathbf{T}$ is a k-rational maximal torus of $\mathbf{G}$.) Let $\mathbf{U}$, resp. $\mathbf{U}^-$, be the unipotent radical of $\mathbf{B}$, resp. the opposite Borel subgroup. Let W be the Weyl group of $\mathbf{T}$ and $S \subseteq W$ the set of simple reflections determined by $\mathbf{B}$. Denote by $\mathcal{B}_{\mathbf{G}, \check{k}}$ the Bruhat–Tits building of the adjoint group $\mathbf{G}^{\mathrm{ad}}$ of $\mathbf{G}$ over $\check{k}$ and by $\mathcal{A}_{\mathbf{T}, \check{k}}$ the apartment of $\mathbf{T}$ therein. Denote by σ the action induced by Frobenius on $W, S, \mathcal{B}_{\mathbf{G}, \check{k}}, \mathcal{A}_{\mathbf{T}, \check{k}}$ and $\mathbf{G}(\check{k})$ (as well as its subgroups).

An element $b \in \mathbf{G}(\check{k})$ is called *basic* if its Newton point ν_b factors through the center of $\mathbf{G}$; *cf.* [Kot85]. Twisting σ by $\mathrm{Ad}(b)$, one gets an inner k-form $\mathbf{G}_b$ of $\mathbf{G}$, see Section 2.1.3. Similarly, for $w \in W$, twisting the Frobenius on $\mathbf{T}(\check{k})$ by $\mathrm{Ad}(w)$, one gets an (outer) k-form $\mathbf{T}_w$ of $\mathbf{T}$.

The space $X_w(b)$, introduced in [Iva23], is the intersection of $L\mathcal{O}(w) \subseteq L(\mathbf{G}/\mathbf{B})^2$ with the graph of $b\sigma \colon L(\mathbf{G}/\mathbf{B}) \to L(\mathbf{G}/\mathbf{B})$, where L denotes the loop functor (see Section 2.1.1). Similarly, we have its coverings $\dot{X}_{\bar{w}}(b)$, where $\bar{w}$ lies in $\overline{F}_w / \ker \bar{\kappa}_w$, a certain discrete quotient of the set $F_w(\check{k}) \subseteq \mathbf{G}(\check{k})$ of all lifts of w; *cf.* Sections 3.4 and 4. For more details on $X_w(b)$ and $\dot{X}_{\bar{w}}(b)$, see Section 2.2 and Definition 4.1. The group $\mathbf{G}_b(k)$ acts on $X_w(b)$, and $\mathbf{G}_b(k) \times \mathbf{T}_w(k)$ acts on $\dot{X}_{\bar{w}}(b)$. The natural map $\dot{X}_{\bar{w}}(b) \to X_w(b)$ is $\mathbf{G}_b(k)$-equivariant.

An element $c \in W$ is called *twisted Coxeter*, or simply *Coxeter*, if any of its reduced expressions contains precisely one element from each σ-orbit on S; *cf.* [Spr74, Section 7]. Usually, we denote by w an arbitrary element of W, whereas the letter c is reserved for Coxeter elements.

We refer to Section 2 for more notation and setup.

1.2. Main result

Let $c \in W$ be Coxeter, $\bar{c} \in \overline{F}_c / \ker \bar{\kappa}_c$ a lift of c and $b \in \mathbf{G}(\check{k})$ basic. We prove a decomposition result for $X_c(b)$ and $\dot{X}_{\bar{c}}(b)$. The space $X_c(b)$ is always non-empty, and—after an equivariant isomorphism—one can assume that c is a special Coxeter element (see Definition 5.2) and b is a lift of c. Moreover, in Proposition 6.1, we will see that either $\dot{X}_{\bar{c}}(b) = \varnothing$, or—after an equivariant isomorphism—we may even assume that b lies over $\bar{c}$. In the rest of Section 1.2 we work under these simplifying assumptions (which do not result in any loss of generality).

As b lies over c, the affine transformation $b\sigma$ of $\mathcal{A}_{\mathbf{T}, \check{k}}$ has precisely one fixed point $\mathbf{x}$ (this follows from [Spr74, Lemma 7.4] as c is Coxeter). Let $\mathcal{G}_{\mathbf{x}}$ be the corresponding parahoric $\mathcal{O}_{\check{k}}$-model of $\mathbf{G}$. It descends to an $\mathcal{O}_k$-model $\mathcal{G}_{\mathbf{x}, b}$ of $\mathbf{G}_b$. In particular, the Frobenius $\sigma_b := \mathrm{Ad}(b) \circ \sigma$ of $L\mathbf{G}$ fixes the subgroup $L^+ \mathcal{G}_{\mathbf{x}}$. Here L and L^+ denote the functors of (positive) loops; see Section 2.1.1. We also have the corresponding parahoric model $\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}$ of $\mathbf{G}^{\mathrm{ad}}$. Consider the scheme $\dot{X}_{\bar{c}, b}^{\mathcal{G}_{\mathbf{x}}}$ defined by the cartesian diagram

$$(1.1) \qquad \begin{array}{ccc} \dot{X}_{\bar{c}, b}^{\mathcal{G}_{\mathbf{x}}} & \longrightarrow & L^+ ({}^c \mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}} \\ \downarrow & & \downarrow \\ L^+ \mathcal{G}_{\mathbf{x}} & \longrightarrow & L^+ \mathcal{G}_{\mathbf{x}}, \end{array}$$

where the lower map is $g \mapsto g^{-1} \sigma_b(g)$ and where $({}^c \mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}$ is the closure of ${}^c \mathbf{U} \cap \mathbf{U}^-$ in $\mathcal{G}_{\mathbf{x}}$. We also consider the quotient for the pro-étale (equivalently, arc-)topology,

$$(1.2) \qquad X_{c, b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}} = \dot{X}_{\bar{c}, b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}} / \mathbf{T}_c^{\mathrm{ad}}(k),$$

where $\dot{X}_{\bar{c}, b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}}$ is (1.1) applied to the parahoric model $\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}$ of the adjoint group $\mathbf{G}^{\mathrm{ad}}$. Then $X_{c, b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}}$ and $\dot{X}_{\bar{c}, b}^{\mathcal{G}_{\mathbf{x}}}$ are infinite-dimensional affine schemes; *cf.* Proposition 2.2.

Theorem 1.1. *Let* **G** *be an unramified group of classical type. Let* $c, \bar{c}$ *and* b *be as above. Then there exist a* $\mathbf{G}_b(k)$-*equivariant isomorphism*

$$(1.3) \qquad X_c(b) = \coprod_{\gamma \in \mathbf{G}_b^{\mathrm{ad}}(k)/\mathcal{G}_{\mathbf{x},b}^{\mathrm{ad}}(\mathcal{O}_k)} \gamma X_{c,b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}}$$

and a $\mathbf{G}_b(k) \times \mathbf{T}_c(k)$-*equivariant isomorphism*

$$(1.4) \qquad \dot{X}_{\bar{c}}(b) \cong \coprod_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \dot{X}_{c,b}^{\mathcal{G}_{\mathbf{x}}}.$$

After necessary preparations, Theorem 1.1 will be proven in Sections 6 and 7. Let us outline the proof. By a v-descent argument, (1.3) follows from a particular case of (1.4) for absolutely almost simple adjoint groups. Showing (1.4) in this case presents the main technical difficulty, which is solved in Section 7 by using a special property of the Newton polygons of isocrystals and analyzing each irreducible Dynkin type separately (here the assumption that c is Coxeter becomes crucial). Finally, (1.4) can be deduced in general from (1.3). We expect Theorem 1.1 to hold for all unramified reductive groups **G**.

At this point, let us stress that p-adic Deligne–Lusztig spaces are quite different in nature from affine Deligne–Lusztig varieties (say, at Iwahori level). The latter were introduced by Rapoport in [Rap05]. They are important because their unions appear as special fibers of Rapoport–Zink spaces. Note, for example, that whereas affine Deligne–Lusztig varieties are (perfect) schemes locally of finite type over $\overline{\mathbb{F}}_q$, the p-adic Deligne–Lusztig spaces are essentially never of finite type over $\overline{\mathbb{F}}_q$. On the other hand, there is, remarkably, some relation between these two types of objects. In [CI21a] it was shown that the inverse limit of a certain system of affine Deligne–Lusztig varieties (of increasing level) for the group $\mathbf{GL}_n$ is isomorphic to a p-adic Deligne–Lusztig space $\dot{X}_{\bar{c}}(b)$ for the same group and a Coxeter element c. We expect this result to generalize to other groups, especially in light of the similarity of the decomposition of Coxeter-type spaces $X_c(b)$ provided by Theorem 1.1 and a very similar-looking decomposition result of He–Nie–Yu [HNY22, Theorem 1.1(3)] for affine Deligne–Lusztig varieties attached to elements of the affine Weyl group with finite Coxeter part.

Let us also note that Theorem 1.1 implies [Iva23, Conjecture 1.1] in the following cases.

Corollary 1.2. *If* **G** *is an unramified group of classical type,* $c \in W$ *Coxeter,* $\bar{c} \in \overline{F}_c/\ker \bar{\kappa}_c$ *arbitrary and* b *basic, then* $X_c(b)$ *and* $\dot{X}_{\bar{c}}(b)$ *are disjoint unions of affine schemes.*

1.3. Quasi-split case

If b is σ-conjugate to 1, Theorem 1.1 admits a clearer reformulation. Let $c \in W$ be Coxeter and $\bar{c} \in \overline{F}_c/\ker \bar{\kappa}_c$. If $\dot{X}_{\bar{c}}(1) \neq \varnothing$, then a (*i.e.*, any) lift $\dot{c} \in \mathbf{G}(\breve{k})$ of $\bar{c}$ is σ-conjugate to 1 (by Corollary 4.6). Moreover, choosing $\dot{c}$ appropriately, we may additionally assume that the (unique and automatically hyperspecial) fixed point $\mathbf{x} = \mathbf{x}_{\dot{c}}$ of $\dot{c}\sigma$ on $\mathcal{A}_{\mathbf{T},\breve{k}}$ lies in $\mathcal{A}_{\mathbf{T},k} = \mathcal{A}_{\mathbf{T},\breve{k}}^{\langle\sigma\rangle}$ (*cf.* Section 6.4). In this situation we have the integral level analogues $X_c^{\mathcal{G}_{\mathbf{x}}}(1)$ and $\dot{X}_{\dot{c}}^{\mathcal{G}_{\mathbf{x}}}(1)$ of $X_c(1)$ and $\dot{X}_{\bar{c}}(1)$, which were introduced in [DI20, Definition 4.1.1]; see Section 2.2. In this situation we have the following corollary, proven in Section 6.4.

Corollary 1.3. *With notation as above, there are* **G**(k)- (*resp.* **G**(k) × **T**_c(k)-)*equivariant isomorphisms*

$$X_c(1) \cong \coprod_{\gamma \in \mathbf{G}^{\mathrm{ad}}(k)/\mathcal{G}_{\mathbf{x},1}^{\mathrm{ad}}(\mathcal{O}_k)} \gamma X_c^{\mathcal{G}_{\mathbf{x}}}(1) \quad and \quad \dot{X}_{\bar{c}}(1) \cong \coprod_{\gamma \in \mathbf{G}(k)/\mathcal{G}_{\mathbf{x},1}(\mathcal{O}_k)} \gamma \dot{X}_{\dot{c}}^{\mathcal{G}_{\mathbf{x}}}(1).$$

1.4. Two related results

In the rest of the introduction we briefly discuss two further themes of this article. First, we review in Section 3, following DeBacker and Reeder [DeB06, DR09, Ree11], the parametrization of rational conjugacy

classes of unramified maximal k-tori in pure inner forms of $\mathbf{G}$. Moreover, in Sections 3.3–3.4 we extend their discussion to all extended pure inner forms and give a description of rational conjugacy classes in terms of $\overline{F}_w / \ker \bar{\kappa}_w$, the discrete parameter space for the spaces $\dot{X}_{\dot{w}}(b)$ over $X_w(b)$. In Section 3.5 we investigate in detail the Coxeter case , where (under some mild assumptions on $\mathbf{G}$) it turns out that the set of different covers $\dot{X}_{\dot{c}}(b)$ of $X_c(b)$ for b basic is in canonical bijection with the set of rational conjugacy classes of unramified Coxeter tori in the inner form $\mathbf{G}_b$ of $\mathbf{G}$; see Corollary 4.7. Thus the p-adic Deligne–Lusztig space $X_c(b)$ "can see" the difference between the various rational conjugacy classes of tori whose stable class is given by c.

Secondly, in Section 5 we show a loop version of the twisted Steinberg cross section. Steinberg's original result [Ste65, Proposition 8.9] states roughly that if $\dot{c} \in F_c(\check{k})$ is any lift of a Coxeter element $c \in W$, any ${}^c\mathbf{U} \cap \mathbf{U}$-orbit on ${}^c\mathbf{U}$ for the action given by $g(u) = g^{-1} u \operatorname{Ad}(\dot{c})(g)$ contains precisely one point in ${}^c\mathbf{U} \cap \mathbf{U}^-$ (k can be replaced by any field here). In [HL12, Section 3.14(c)] this is generalized in various respects, in particular to the case where k is replaced by a finite field $\mathbb{F}_q$ and the action is Frobenius-twisted, $i.e.$, given by $g(u) = g^{-1} u \sigma_{\dot{c}}(g)$, where σ is the (geometric) Frobenius over $\mathbb{F}_q$ and $\sigma_{\dot{c}} = \operatorname{Ad}(\dot{c}) \circ \sigma$. This can be used to prove affineness of classical Deligne–Lusztig varieties attached to Coxeter elements (for example, by combining [DL76, Corollary 1.12] and [HL12, Section 4.3]). In our setup, with k non-archimedean local, (to perform a reduction step in the proof of Theorem 1.1) we need the same result for the action of $L({}^c\mathbf{U} \cap \mathbf{U})$ on $L({}^c\mathbf{U})$. A part of the proof in [HL12] can be adapted, but at some point in [HL12] the Ax–Grothendieck theorem[2] is used, which is impossible in our setting (as $L\mathbf{U}$ is not even a scheme). Therefore, we carry through a different argument, at least for classical groups and special Coxeter elements.

Outline

In Section 2 we introduce more notation, prove a fact about integral p-adic Deligne–Lusztig spaces and review the Borovoi fundamental group and σ-conjugacy classes. In Section 3 we review (and extend) some results on rational/stable conjugacy classes of unramified maximal tori of $\mathbf{G}$. In Section 4 we review and extend some results from [Iva23] about the maps $\dot{X}_{\dot{w}}(b) \to X_w(b)$. In Section 5 we prove loop (and integral loop) versions of the twisted Steinberg cross section. This is used in Section 6 to replace $\dot{X}_{\dot{c}}(b)$ (and its integral analogue) by a more tractable object. The main part of the article consists of Sections 6 and 7, where Theorem 1.1 is proven.

Acknowledgements

2. Setup and preliminaries

Recall that $\mathbf{G}$ is an unramified group over the local field k. We use the notation from Section 1.1.

2.1. More notation

Given a group G, $g \in G$ and any subset $H \subseteq G$, we write ${}^g H = gHg^{-1}$. For an abelian group A, we write A_{tors} for its torsion subgroup.

---

[2]It states, for an endomorphism $X \to X$ of a scheme of finite type over an algebraically closed field Ω, that if $X(\Omega) \to X(\Omega)$ is injective, then it is also surjective.

Let p be the residue characteristic of k. For a ring R of characteristic p, denote by Perf_R the category of perfect algebras over R. Denote by ϖ a uniformizer of k and by $\mathbb{F}_q$ (resp. $\overline{\mathbb{F}}_q$) the residue field of k (resp. $\check{k}$).

For R in $\mathrm{Perf}_{\mathbb{F}_q}$, let $\mathbb{W}(R)$ be the unique (up to unique isomorphism) ϖ-adically complete and separated $\mathcal{O}_k$-algebra in which ϖ is not a zero-divisor and which satisfies $\mathbb{W}(R)/\varpi\mathbb{W}(R) = R$. In other words, if $W(R)$ denotes the p-typical Witt-vectors of k, then $\mathbb{W}(R) = W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_k$ if $\mathrm{char}\, k = 0$ and $\mathbb{W}(R) = R[[\varpi]]$ if $\mathrm{char}\, k = p$. Denote by $\phi = \phi_R$ the Frobenius automorphisms of $\mathbb{W}(R)$, $\mathbb{W}(R)[1/\varpi]$, induced by $x \mapsto x^q$ on R (this notation is only used in the proof of Proposition 5.3 and Section 7). We have $\mathbb{W}(\mathbb{F}_q)[1/\varpi] = k$ and $\mathbb{W}(\overline{\mathbb{F}}_q)[1/\varpi] = \check{k}$ (so that $\phi_{\overline{\mathbb{F}}_q} = \sigma$).

Denote by k^{alg} a fixed algebraic closure of k containing the maximal algebraic subextension of $\check{k}/k$. Denote by $\langle\sigma\rangle \cong \hat{\mathbb{Z}}$ the group of continuous automorphisms of $\check{k}/k$.

2.1.1. Loop functors.— For a $\check{k}$-scheme X, let

$$LX \colon \mathrm{Perf}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Sets}, \quad R \longmapsto X\left(\mathbb{W}(R)\left[1/\varpi^{-1}\right]\right)$$

be the loop space of X. If X is affine of finite type over $\check{k}$, then LX is representable by an ind-scheme. For an $\mathcal{O}_{\check{k}}$-scheme $\mathcal{X}$, let

$$L^+\mathcal{X} \colon \mathrm{Perf}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Sets}, \quad R \longmapsto \mathcal{X}(\mathbb{W}(R))$$

be the space of positive loops of $\mathcal{X}$. Moreover, if $n \geq 1$, let

$$L_n^+\mathcal{X} \colon \mathrm{Perf}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Sets}, \quad R \longmapsto \mathcal{X}(\mathbb{W}(R)/\varpi^n\mathbb{W}(R))$$

be the truncated version of $L^+\mathcal{X}$. By [BGA18, Proposition-Definition 6.2], $L_n^+\mathcal{X}$ is a scheme (resp. affine scheme) for arbitrary (resp. affine) $\mathcal{X}$, and if $\mathcal{X}$ is of finite type, then $L^+\mathcal{X}$ is too. Moreover, $L^+\mathcal{X} = \varprojlim_n L_n^+\mathcal{X}$, along affine transition maps (cf. [BGA18, Proposition 8.5]). Thus, if $\mathcal{X}$ is a scheme (resp. affine scheme), then $L^+\mathcal{X}$ is too.

If X is the base change to $\check{k}$ of a k-scheme X_0, then the functor LX carries a natural automorphism, the geometric Frobenius, denoted by σ_{X_0} (or simply by σ if X_0 is clear from context). Similarly, we have the geometric Frobenius $\sigma = \sigma_{\mathcal{X}_0}$ on $L^+\mathcal{X}$ if the $\mathcal{O}_{\check{k}}$-scheme $\mathcal{X}$ is the base change of an $\mathcal{O}_k$-scheme $\mathcal{X}_0$.

2.1.2. Groups.— We write $\mathbf{G}^{\mathrm{ad}}$, resp. $\mathbf{G}^{\mathrm{sc}}$, for the adjoint group, resp. the simply connected cover of the derived group, of $\mathbf{G}$. If $\mathbf{H} \subseteq \mathbf{G}$ is a closed subgroup of $\mathbf{G}$, denote by $\mathbf{H}^{\mathrm{ad}}$, resp. $\mathbf{H}^{\mathrm{sc}}$, its image in $\mathbf{G}^{\mathrm{ad}}$, resp. its preimage in $\mathbf{G}^{\mathrm{sc}}$. We identify the Weyl groups of $\mathbf{G}^{\mathrm{ad}}$ and $\mathbf{G}^{\mathrm{sc}}$ with W. We denote by $\mathbf{Z}$ the center of $\mathbf{G}$.

Write $X_*(\mathbf{T})$, resp. $X^*(\mathbf{T})$, for the free $\mathbb{Z}$-module of cocharacters, resp. characters, of $\mathbf{T}$. Denote by $\Phi \subseteq X^*(\mathbf{T})$, resp. $\Phi^\vee \subseteq X_*(\mathbf{T})$, the set of roots, resp. coroots, of $\mathbf{G}$; denote by Φ^+, resp. Δ, the set of positive, resp. simple, roots, determined by $\mathbf{B}$; denote by $s_\alpha \in W$ the reflection attached to $\alpha \in \Phi$. For $\xi \in X_*(\mathbf{T})$, denote by ϖ^ξ the image in $\mathbf{T}(\check{k})$ of ϖ under ξ.

Write $\mathbf{N}$ for the normalizer of $\mathbf{T}$. We have the reductive $\mathcal{O}_k$-model $\mathcal{T}$ of $\mathbf{T}$, and $\mathcal{T}(\mathcal{O}_{\check{k}})$ is the maximal compact subgroup of $\mathbf{T}(\check{k})$. The quotient $\widetilde{W} = \mathbf{N}(\check{k})/\mathcal{T}(\mathcal{O}_{\check{k}})$ is the *extended affine Weyl group* of $\mathbf{T}$ in $\mathbf{G}$. It is naturally an extension $\widetilde{W} = W \ltimes X_*(\mathbf{T})$, whose splittings (resp. σ-equivariant splittings) $W \to \widetilde{W}$ correspond to hyperspecial points of $\mathcal{A}_{\mathbf{G},\check{k}}$ (resp. of $\mathcal{A}_{\mathbf{G},\check{k}}^{\langle\sigma\rangle}$).

2.1.3. Twisting Frobenius.— For $b \in \mathbf{G}(\check{k})$, we have the k-group $\mathbf{G}_b$, which is defined as the functorial σ-centralizer of b; cf. [RZ96, 1.12] or [Kot97, 3.3, Appendix A]. Then $\mathbf{G}_b(k) = \{g \in \mathbf{G}(\check{k}) \colon g^{-1}b\sigma(g) = b\}$. If b is basic (cf. Section 2.4), then $\mathbf{G}_b$ is an inner form of $\mathbf{G}$, we may identify $\mathbf{G}_b(\check{k}) = \mathbf{G}(\check{k})$ and we denote by $\sigma_b := \mathrm{Ad}(b) \circ \sigma$ the twisted Frobenius on $\mathbf{G}(\check{k})$. We also write σ_b for the corresponding geometric Frobenius on $L\mathbf{G}_b = L\mathbf{G}$.

Suppose that the basic element b lies in $\mathbf{N}(\breve{k})$ and that $\mathbf{x} \in \mathcal{A}_{\mathbf{T},\breve{k}}$ is fixed by the action of $b\sigma$. Let $\mathcal{G}_{\mathbf{x}}$ be the corresponding parahoric model of $\mathbf{G}$ over $\mathcal{O}_{\breve{k}}$. Then σ_b fixes $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_{\breve{k}}) \subseteq \mathbf{G}(\breve{k})$, and $L^+\mathcal{G}_{\mathbf{x}} \subseteq L\mathbf{G}$. Thus $\mathcal{G}_{\mathbf{x}}$ descends to an $\mathcal{O}_k$-model of $\mathbf{G}_b$, which we denote by $\mathcal{G}_{\mathbf{x},b}$. We have $L^+\mathcal{G}_{\mathbf{x},b} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q = L^+\mathcal{G}_{\mathbf{x}}$.

Similarly, for $w \in W$, we have the (outer) $\breve{k}/k$-form $\mathbf{T}_w$ of $\mathbf{T}$, satisfying $\mathbf{T}_w(k) = \{t \in \mathbf{T}(\breve{k}) : \mathrm{Ad}\,w(\sigma(t)) = t\}$. We may identify $\mathbf{T}_w(\breve{k}) = \mathbf{T}(\breve{k})$, and we denote by $\sigma_w := \mathrm{Ad}(w) \circ \sigma$ the twisted Frobenius on this group, as well as on $L\mathbf{T}$.

2.1.4. Topologies.— We work with two topologies on $\mathrm{Perf}_{\mathbb{F}_p}$: the pro-étale topology, *cf.* [BS15], and the arc-topology, *cf.* [BM21]. Recall that a map $S' \to S$ of qcqs schemes is an arc-cover if any immediate specialization in S lifts to S'. By [BM21, Theorem 5.16], the arc-topology on perfect qcqs $\mathbb{F}_p$-schemes coincides with the canonical topology. Thus any perfect qcqs scheme is an arc-sheaf. In particular, for any qcqs scheme $\mathcal{X}/\mathcal{O}_k$, $L^+\mathcal{X}$ is an arc-sheaf. For any quasi-projective scheme X/k, LX is an arc-sheaf by [Iva23, Theorem A].

2.1.5. Locally profinite groups.— As in [Iva23, Section 4], we regard a locally profinite group H as a (representable) functor
$$S \longmapsto C^0(|S|, H)$$
on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. In particular, $\mathbf{G}_b(k)$ and $\mathbf{T}_w(k)$ are such functors. If H is profinite, this functor is even represented by an affine scheme.

2.1.6. Galois cohomology.— For the definition and standard facts about non-abelian cohomology, we refer to [Ser94]. If M is a discrete $\langle\sigma\rangle$-group, where $\langle\sigma\rangle$ is as in Section 2.1, we write $H^1(\sigma, M)$ for the corresponding first cohomology set. We identify a continuous 1-cocycle $a\colon \langle\sigma\rangle \to M$ with $a(\sigma) \in M$ and write $[a(\sigma)] \in H^1(\sigma, M)$ for the corresponding cohomology class.

We will write $H^1(\sigma, \mathbf{G})$, resp. $H^1(\sigma_w, \mathbf{T})$, instead of $H^1(\sigma, \mathbf{G}(\breve{k}))$, resp. $H^1(\sigma, \mathbf{T}_w(\breve{k}))$. By a theorem of Steinberg [Ser94, Theorem III.1'], resp. its extension [BS68, Section 8.6] if char $k > 0$, we have $H^1(\sigma, \mathbf{G}) = H^1(k, \mathbf{G})$ and $H^1(\sigma_w, \mathbf{T}) = H^1(k, \mathbf{T}_w)$ via inflation along $\mathrm{Gal}_k \twoheadrightarrow \langle\sigma\rangle$. We will always use the notation $H^1(\sigma, \mathbf{G})$ and $H^1(\sigma_w, \mathbf{T})$ without further comment.

2.2. Review of some p-adic Deligne–Lusztig spaces

2.2.1. Spaces $X_w(b)$ and $\dot{X}_{\dot{w}}(b)$.— Let $w \in W$, $\dot{w} \in \mathbf{N}(\breve{k})$ be a lift of w and $b \in \mathbf{G}(\breve{k})$. Denote by $\mathcal{O}(w) \subseteq (\mathbf{G}/\mathbf{B})^2$ (resp. $\mathcal{O}(\dot{w}) \subseteq \mathbf{G}/\mathbf{U}$) the $\mathbf{G}$-orbit corresponding to w (resp. $\dot{w}$) under the Bruhat decomposition. The *p-adic Deligne–Lusztig spaces* $X_w(b)$ and $\dot{X}_{\dot{w}}(b)$ were defined in [Iva23, Definition 8.3] by the cartesian diagrams of functors on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$
\begin{array}{ccc}
X_w(b) & \longrightarrow & L\mathcal{O}(w) \\
\downarrow & & \downarrow \\
L(G/B) & \xrightarrow{(\mathrm{id}, b\sigma)} & L(G/B) \times L(G/B)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\dot{X}_{\dot{w}}(b) & \longrightarrow & L\dot{\mathcal{O}}(\dot{w}) \\
\downarrow & & \downarrow \\
L(G/U) & \xrightarrow{(\mathrm{id}, b\sigma)} & L(G/U) \times L(G/U),
\end{array}
$$

where $b\sigma\colon L(G/B) \xrightarrow{\sim} L(G/B)$ and b acts by left multiplication. Then $X_w(b)$ and $\dot{X}_{\dot{w}}(b)$ are sheaves for the arc-topology (see [BM21]) on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$; *cf.* [Iva23, Corollary 8.4]. There is a natural action of $\mathbf{G}_b(k)$ on $X_w(b)$ and of $\mathbf{G}_b(k) \times \mathbf{T}_w(k)$ on $\dot{X}_{\dot{w}}(b)$.

2.2.2. Spaces $\dot{X}_{\dot{c},b}^{\mathcal{G}_{\mathbf{x}}}$ and $X_{c,b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}}$.— These spaces were introduced in (1.1) and (1.2). Here we prove their representability.

Lemma 2.1. *Let $\{H_n\}_{n\geq 1}$ be an inverse system of finite groups, acting compatibly and freely on an inverse system $\{X_n\}_{n\geq 1}$ of pro-étale (resp. arc-)sheaves on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Write $H = \varprojlim_n H_n$ and $X = \varprojlim_n X_n$. The natural map $X/H \to \varprojlim_n X_n/H_n$ is an isomorphism, where the quotients are formed for the pro-étale (resp. arc-)topology.*

Proof. Consider the case of pro-étale topology. The existence of the map is easy. For the injectivity, it is enough to show that the map from the presheaf quotient $R \mapsto X(R)/H(R)$ to $\varprojlim_n X_n/H_n$ is injective. Let $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Suppose $s_1, s_2 \in X(R)$ map to the same element in $(\varprojlim_n X_n/H_n)(R) = \varprojlim_n (X_n/H_n)(R)$. Then for each n, the images $s_{1,n}, s_{2,n}$ of s_1, s_2 in $X_n(R)/H_n(R)$ coincide (as any quotient presheaf is automatically separated). Thus for each $n \geq 1$, $S_n = \{h_n \in H_n(R): h_n s_{1,n} = s_{2,n}\} \neq \varnothing$. This set forms an inverse system. By the freeness of the action, we have $\#S_n = 1$, and it follows that $\varprojlim_n S_n \neq \varnothing$. Then $h \in \varprojlim_n S_n \subseteq H(R)$ satisfies $hs_1 = s_2$. This proves the injectivity. For the surjectivity, let $s_n \in X_n(R)/H_n(R)$ be a compatible collection of elements. For each n, find some pro-étale cover $R \to R_n$ such that $s_n|_{R_n}$ lifts to $\tilde{s}_n \in X_n(R_n)$. As $R \to R' = \varinjlim_n R_n$ is still a pro-étale cover, *cf.* [BS15, proof of Lemma 4.1.8], we may replace each R_n with R' and $\tilde{s}_n$ with $\tilde{s}_n|_{R'}$. Now, we construct an element $\tilde{s} \in X(R')$ mapping to $(s_n)_{n\geq 1}$. Starting with $\tilde{s}_1$ if $n = 1$, and working by induction on n, suppose that $(\tilde{s}_i)_{i=1}^n$ is compatible. As the system $(s_n)_{n\geq 1}$ is compatible, we can find some $h_{n+1} \in H_{n+1}(R')$ such that $h_{n+1}\tilde{s}_{n+1}$ maps to $\tilde{s}_n$ under $X_{n+1}(R') \to X_n(R')$. Then replace $\tilde{s}_{n+1}$ by $h_{n+1}\tilde{s}_{n+1}$, so that $(\tilde{s}_i)_{i=1}^{n+1}$ becomes compatible. We conclude by induction. The argument for the arc-topology is the same, using [BM21, Corollary 2.18] instead of [BS15, Lemma 4.1.8] $\qquad\square$

For $n \geq 1$, let $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b,n}$ be defined by the diagram (1.1) with L^+ replaced by L_n^+ everywhere. Then all $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b,n}$ are affine schemes and $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b} = \varprojlim_n \dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b,n}$. We also have the truncated version of (1.2): let $\mathcal{T}$ be the closure of $\mathbf{T}$ in $\mathcal{G}_\mathbf{x}$ and $\mathcal{T}_c$ the $\mathcal{O}_k$-torus, arising by twisting the Frobenius on $\mathcal{T}$ by c. Then put $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b,n} := \dot{X}^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{\bar{c},b,n}/T_n$, where $T_n := \mathcal{T}_c^{\mathrm{ad}}(\mathcal{O}_k/\varpi^n)$.

Proposition 2.2. *The schemes $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b}$ and $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b}$ are affine and perfect. Also, $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b} = \dot{X}^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{\bar{c},b}/\mathbf{T}_c(k)$ remains unchanged if one takes the quotient with respect to the arc-topology instead of the pro-étale topology. Moreover, $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b} = \varprojlim_n X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b,n}$.*

Proof. The claim for $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b}$ follows from the definition and the affineness of $L^+\mathcal{X}$ for an affine scheme $\mathcal{X}/\mathcal{O}_{\check{k}}$; *cf.* Section 2.1.1. Suppose that $\mathbf{G}$ is adjoint. For $n \geq 1$, let $T_n = \mathcal{T}_c^{\mathrm{ad}}(\mathcal{O}_k/\varpi^n)$ be as above. Note that the finite group T_n acts freely on the affine scheme $\dot{X}^{\mathcal{G}}_{\bar{c},b,n}$ and we have $X^{\mathcal{G}}_{c,b} = (\dot{X}^{\mathcal{G}}_{\bar{c},b}/T)_{\mathrm{proet}} = \varprojlim_n (\dot{X}^{\mathcal{G}}_{\bar{c},b,n}/T_n)_{\mathrm{proet}}$ by Lemma 2.1. By [Sta14, Tag 07S7], the fppf-quotient $(\dot{X}^{\mathcal{G}}_{\bar{c},b,n}/T_n)_{\mathrm{fppf}}$ is a scheme, which is even affine (since $\dot{X}^{\mathcal{G}}_{\bar{c},b,n}$ is), and $\dot{X}^{\mathcal{G}}_{\bar{c},b,n} \to (\dot{X}^{\mathcal{G}}_{\bar{c},b,n}/T_n)_{\mathrm{fppf}}$ is an fppf-torsor. As T_n is finite and acts freely on $\dot{X}^{\mathcal{G}}_{\bar{c},b,n}$, this is even an étale torsor. From this, it follows that $(\dot{X}^{\mathcal{G}}_{\bar{c},b,n}/T_n)_{\mathrm{fppf}} = (\dot{X}^{\mathcal{G}}_{\bar{c},b,n}/T_n)_{\mathrm{ét}}$, the quotient for the étale topology. In particular, the latter is an affine perfect scheme, hence already a pro-étale and even an arc-sheaf. This implies that the sheafification of the quotient for the arc- and pro-étale topologies agree, as well as the fact that $X^{\mathcal{G}}_{c,b}$, being the inverse limit of affine perfect schemes, is itself one. $\qquad\square$

Remark 2.3. The map $g \mapsto g^{-1}\sigma_b(g): L^+\mathcal{G}_\mathbf{x} \to L^+\mathcal{G}_\mathbf{x}$ is pro-étale and—by Lang's theorem applied to all truncations—also surjective. Thus $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b}$ and $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b}$ and their truncations are always non-empty. Also, it follows that $\dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b}$ and $X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b}$ are infinite-dimensional. The truncations are perfections of finite-dimensional schemes. More precisely, as the Lang map is étale, we have

$$\dim \dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b,n} = \dim X^{\mathcal{G}_\mathbf{x}^{\mathrm{ad}}}_{c,b,n} = \dim L_n^+(^c\mathbf{U} \cap \mathbf{U}^-)_\mathbf{x} = \ell(c)n,$$

where $\ell(c)$ denotes the length of the Coxeter element $c \in W$.

Conjecture 2.4. *The scheme $X_{c,b}^{\mathcal{G}_{\mathbf{x}}^{\mathrm{ad}}}$ is an $\mathbb{A}^\infty$-bundle over a classical Deligne–Lusztig variety attached to the reductive quotient of the special fiber of $\mathcal{G}_{\mathbf{x}}$.*

This is easily verified by hand for $\mathbf{G} = \mathbf{GL}_n$, using the explicit description in [CI21a].

2.2.3. Spaces $X_w^{\mathcal{G}}(1)$, $\dot{X}_{\dot{w}}^{\mathcal{G}}(1)$.— These spaces only appear in Sections 1.3 and 6.4. Suppose $\mathbf{x} \in \mathcal{A}_{\mathbf{T},\breve{k}}^{\langle \sigma \rangle}$ is a hyperspecial point, and let $\mathcal{G} = \mathcal{G}_{\mathbf{x}}$ be the corresponding reductive $\mathcal{O}_{\breve{k}}$-model of $\mathbf{G}$. The reductive $\mathcal{O}_k$-group $\mathcal{G}_{\mathbf{x},1}$ is quasi-split, and the closure $\mathbf{B}_{\mathbf{x}}$ of $\mathbf{B}$ in $\mathcal{G}$ is an $\mathcal{O}_k$-rational Borel subgroup. Moreover, $\mathcal{G}$ has a Bruhat decomposition, in the sense that $(\mathcal{G}/\mathbf{B}_{\mathbf{x}})^2 \otimes_{\mathcal{O}_k} \mathcal{O}_{\breve{k}}$ contains locally closed subschemes $\mathcal{O}(w)$ $(w \in W)$, flat over $\mathcal{O}_{\breve{k}}$, which fiberwise induce the usual Bruhat decomposition. If $\mathbf{N}_{\mathbf{x}}, \mathbf{U}_{\mathbf{x}}$ are the closures of $\mathbf{N}, \mathbf{U}$ in $\mathcal{G}$, one similarly has $\mathcal{O}(\dot{w}) \subseteq (\mathcal{G}/\mathbf{U}_{\mathbf{x}})^2 \otimes_{\mathcal{O}_k} \mathcal{O}_{\breve{k}}$ $(\dot{w} \in \mathbf{N}_{\mathbf{x}}(\mathcal{O}_{\breve{k}}))$. Taking $b = 1$, $\dot{w} \in \mathbf{N}_{\mathbf{x}}(\mathcal{O}_{\breve{k}})$ and replacing $\mathbf{G}, \mathbf{B}, \mathbf{U}$ with $\mathcal{G}, \mathbf{B}_{\mathbf{x}}, \mathbf{U}_{\mathbf{x}}$ and L with L^+ everywhere in the diagrams in Section 2.2.1, we obtain integral p-adic Deligne–Lusztig spaces $X_w^{\mathcal{G}}(1)$ and $\dot{X}_{\dot{w}}^{\mathcal{G}}(1)$, as introduced in [DI20, Definition 4.1.1]. They are acted on by $\mathcal{G}_{\mathbf{x},1}(\mathcal{O}_k)$ and $\mathcal{G}_{\mathbf{x},1}(\mathcal{O}_k) \times \mathcal{T}(\mathcal{O}_k)$, respectively.

2.3. Fundamental group and the Kottwitz map

The natural map $\mathbf{T}^{\mathrm{sc}} \to \mathbf{T}$ induces an injection $X_*(\mathbf{T}^{\mathrm{sc}}) \to X_*(\mathbf{T})$, and we identify $X_*(\mathbf{T}^{\mathrm{sc}})$ with its image. The fundamental group of $\mathbf{G}$ in the sense of Borovoi [Bor98] is the group

$$\pi_1(\mathbf{G}) = X_*(\mathbf{T})/X_*(\mathbf{T}^{\mathrm{sc}}).$$

It is independent of the choice of $\mathbf{T}$ and functorial in $\mathbf{G}$, the induced W-action on it is trivial, and it admits a Frobenius action, which we denote by σ. Kottwitz constructed a homomorphism $\tilde{\kappa}_{\mathbf{G}} \colon \mathbf{G}(\breve{k}) \to \pi_1(\mathbf{G})$; see [Kot97, Section 7] and [PR08, Section 2.a.2]. Reformulated, a classical result of Kottwitz [Kot84, Section 6] states that $\tilde{\kappa}_{\mathbf{G}}$ induces an isomorphism $H^1(\sigma, \mathbf{G}) \cong \pi_1(\mathbf{G})_{\langle \sigma \rangle, \mathrm{tors}}$. For tori, we have $\pi_1(-) = X_*(-)$. For any $w \in W$, any k-rational embedding $\mathbf{T}_w \to \mathbf{G}$ induces the map

$$X_*(\mathbf{T})_{\langle \sigma_w \rangle, \mathrm{tors}} = H^1(\sigma_w, \mathbf{T}) \longrightarrow H^1(\sigma, \mathbf{G}) = \pi_1(\mathbf{G})_{\langle \sigma \rangle, \mathrm{tors}},$$

which is obtained from $X_*(\mathbf{T}) \to \pi_1(\mathbf{G})$ by taking σ_w- (resp. σ-)invariants and passing to the torsion subgroup. From the definition of $\pi_1(\mathbf{G})$ we deduce an exact sequence

$$(2.1) \qquad X_*(\mathbf{T}^{\mathrm{sc}})_{\langle \sigma_w \rangle} \xrightarrow{\beta_w} X_*(\mathbf{T})_{\langle \sigma_w \rangle} \longrightarrow \pi_1(\mathbf{G})_{\langle \sigma \rangle} \longrightarrow 0.$$

In particular,

$$(2.2) \qquad \ker\left(H^1(\sigma_w, \mathbf{T}) \longrightarrow H^1(\sigma, \mathbf{G})\right) = \mathrm{im}(\beta_w)_{\mathrm{tors}}$$

Remark 2.5. If $\mathbf{G}$ is adjoint and w Coxeter, then $\beta_w = 0$ and (equivalently) $X_*(\mathbf{T})_{\langle \sigma_w \rangle} \cong \pi_1(\mathbf{G})_{\langle \sigma \rangle}$. For split groups, this follows from [Spr74, Corollary 8.3]. In the non-split case, note that there is some $n \geq 1$ such that $\sigma_w^n = \sigma_{w_{\mathrm{spl}}}$ as automorphisms of $X_*(\mathbf{T})$ and $X_*(\mathbf{T}^{\mathrm{sc}})$, where w_{spl} is some (non-twisted) Coxeter element in W and $\sigma_{w_{\mathrm{spl}}} := \mathrm{Ad}(w_{\mathrm{spl}}) \circ \sigma^n$ is formed with respect to the base change of $\mathbf{G}$ to the unramified extension of k of degree n. Then, as automorphisms of $X_*(\mathbf{T})$ and of $X_*(\mathbf{T}^{\mathrm{sc}})$, $\sigma_{w_{\mathrm{spl}}} - 1$ factors through $\sigma_w - 1$, so that we have a surjection $X_*(\mathbf{T})_{\langle \sigma_w \rangle} \twoheadrightarrow X_*(\mathbf{T})_{\langle \sigma_{w_{\mathrm{spl}}} \rangle}$ and a similar surjection for $X_*(\mathbf{T}^{\mathrm{sc}})$. These surjections form a commutative square with the maps β_w and $\beta_{w_{\mathrm{spl}}}$ (as both are induced by the inclusion $X_*(\mathbf{T}^{\mathrm{sc}}) \to X_*(\mathbf{T})$). As $\beta_{w_{\mathrm{spl}}} = 0$ by the split case, it follows that also $\beta_w = 0$.

By [PR08, Section 5] we may reinterpret $\tilde{\kappa}_{\mathbf{G}}$ as a map of ind-schemes $\tilde{\kappa}_{\mathbf{G}} \colon L\mathbf{G} \to \pi_1(\mathbf{G})$, where we regard $\pi_1(\mathbf{G})$ as a discrete perfect scheme via Section 2.1.5. Passing to σ-coinvariants gives a surjective map

$$(2.3) \qquad \kappa_{\mathbf{G}} \colon L\mathbf{G} \longrightarrow \pi_1(\mathbf{G})_{\langle \sigma \rangle}.$$

Finally, let us note that for $w \in W$, we have the exact sequence

$$(2.4) \qquad 1 \longrightarrow \mathbf{T}_w(k) \longrightarrow L\mathbf{T} \overset{\sigma_w - 1}{\longrightarrow} L\mathbf{T} \overset{\kappa_w}{\longrightarrow} X_*(\mathbf{T})_{\langle \sigma_w \rangle} \longrightarrow 1,$$

where we write κ_w instead of $\kappa_{\mathbf{T}_w}$. Moreover, κ_w factors through the natural projection $\bar{\kappa}_w \colon X_*(\mathbf{T}) \to X_*(\mathbf{T})_{\langle \sigma_w \rangle}$.

2.4. Newton map and σ-conjugacy classes

Let $\mathfrak{f} \in \mathrm{Perf}_{\mathbb{F}_q}$ be an algebraically closed field, so that $L = \mathbb{W}(\mathfrak{f})[1/\varpi]$ is an extension of $\check{k}$. Then L/k is equipped with the Frobenius σ lifting that of $\mathfrak{f}/\mathbb{F}_q$ and $L^\sigma = k$. In [Kot85] the set

$$B(\mathbf{G}) = H^1(\sigma^{\mathbb{Z}}, \mathbf{G}(L)) \cong \mathbf{G}(L)/\mathrm{Ad}_\sigma \mathbf{G}(L)$$

of σ-conjugacy classes of $\mathbf{G}(L)$ in considered, where x, y are σ-conjugate if there is some $g \in \mathbf{G}(L)$ with $g^{-1} x \sigma(g) = y$. Denote the σ-conjugacy class of $b \in \mathbf{G}(L)$ by $[b]$ (or $[b]_{\mathbf{G}}$ if ambiguity is possible). The set $B(\mathbf{G})$ is independent of the choice of $\mathfrak{f}$. To a σ-conjugacy class $[b]$, one can attach the *Newton point* $\nu_b \in (W \backslash X_*(\mathbf{T})_{\mathbb{Q}})^{\langle \sigma \rangle}$. Moreover, the Kottwitz map $\kappa_{\mathbf{G}}(\mathfrak{f})$ from (2.3) factors through a map $\kappa_{\mathbf{G}} \colon B(\mathbf{G}) \twoheadrightarrow \pi(\mathbf{G})_{\langle \sigma \rangle}$. Then the map

$$(2.5) \qquad (\nu, \kappa_{\mathbf{G}}) \colon B(\mathbf{G}) \longrightarrow (W \backslash X_*(\mathbf{T})_{\mathbb{Q}})^{\langle \sigma \rangle} \times \pi_1(\mathbf{G})_{\langle \sigma \rangle}$$

is injective. An element $b \in \mathbf{G}(L)$ is called *basic* if ν_b is central in $\mathbf{G}$. We denote by $B(\mathbf{G})_{\mathrm{bas}} \subseteq B(\mathbf{G})$ the subset of basic classes. There is a canonical injection $H^1(\sigma, \mathbf{G}) \hookrightarrow B(\mathbf{G})_{\mathrm{bas}}$. In [Kot85], the isomorphism $H^1(\sigma, \mathbf{G}) \cong \pi_1(\mathbf{G})_{\langle \sigma \rangle, \mathrm{tors}}$ was extended to the natural commutative diagram

$$(2.6) \qquad \begin{array}{ccc} H^1(\sigma, \mathbf{G}) & \overset{\sim}{\longrightarrow} & \pi_1(\mathbf{G})_{\langle \sigma \rangle, \mathrm{tors}} \\ \Big\downarrow & & \Big\downarrow \\ B(\mathbf{G})_{\mathrm{bas}} & \overset{\sim}{\longrightarrow} & \pi_1(\mathbf{G})_{\langle \sigma \rangle}, \end{array}$$

where the lower map is the restriction of $\kappa_{\mathbf{G}}$ to $B(\mathbf{G})_{\mathrm{bas}}$.

We recall how to compute the Newton point ν_b of $b \in \mathbf{N}(L)$; see *e.g.* [RV14, Remark 2.3(iii)]. Let $\tilde{w} \in \widetilde{W}$ be the image of b under $\mathbf{N}(L) \twoheadrightarrow \widetilde{W}$. There is an integer $d > 0$ such that σ^d acts trivially on $\widetilde{W}$ and such that $\mu = \tilde{w} \sigma(\tilde{w}) \ldots \sigma^{d-1}(\tilde{w}) \in X_*(\mathbf{T})$. Then ν_b is the image of $d^{-1} \mu$ in $W \backslash X_*(\mathbf{T})_{\mathbb{Q}}$. In particular, ν_b only depends on $\tilde{w}$, and it makes sense to define $\nu_{\tilde{w}} = \nu_b$. Moreover, we call $\tilde{w}$ basic if $\nu_{\tilde{w}}$ is central.

Finally, we recall that any σ-conjugacy class has a representative in $\mathbf{N}(\check{k})$ and that the induced surjection $\mathbf{N}(\check{k}) \twoheadrightarrow B(\mathbf{G})$ factors through $\widetilde{W} \twoheadrightarrow B(\mathbf{G})$; *cf.* [GHK+10, Corollary 7.2.2] for split groups and [Hel4, Theorem 3.7] in general. In particular, the Kottwitz homomorphism induces a map $\bar{\kappa}_{\mathbf{G}} \colon \widetilde{W} \to \pi_1(\mathbf{G})_{\langle \sigma \rangle}$.

3. Rational and stable conjugacy classes of unramified tori

Recall that two maximal k-tori $\mathbf{T}_1, \mathbf{T}_2$ in $\mathbf{G}$ are *stably conjugate* if there exists a $g \in \mathbf{G}(k^{\mathrm{alg}})$ such that ${}^g(\mathbf{T}_1(k)) = \mathbf{T}_2(k)$. If two maximal k-tori are rationally conjugate, they are also stably conjugate. Thus any stable conjugacy class breaks up into a (finite) union of rational conjugacy classes. We review from [DeB06, DR09, Ree11] how this happens for unramified tori of pure inner forms of $\mathbf{G}$. Then we investigate the case of *extended* pure inner forms of $\mathbf{G}$. We wish to point out that the description from [DeB06] in terms of the Bruhat–Tits building covers this case, but we are interested in a slightly different description, which is closely related to the spaces $\dot{X}_{\tilde{w}}(b)$.

3.1. Stable conjugacy classes

For $b \in \mathbf{N}(\check{k})$, the twisted Frobenius $\sigma_b = \mathrm{Ad}\, b \circ \sigma$ of $\mathbf{G}(\check{k})$ induces one on W. Now suppose $b \in \mathbf{N}(\check{k})$ is basic. By [DeB06, Lemma 4.3.1], there is a natural injection from the set of stable conjugacy classes of unramified maximal k-tori in $\mathbf{G}_b$ into $H^1(\sigma_b, W)$, which is even a bijection when $\mathbf{G}_b$ is quasi-split (*e.g.*, $[b] = [1]$). Identifying $H^1(\sigma_b, W) \xrightarrow{\sim} H^1(\sigma, W)$ via the natural twisting map $x \mapsto xb$ on cocycles (using [Ser94, Proposition I.35 bis]), we get

$$(3.1) \qquad \{\text{stable conjugacy classes of } k\text{-tori in } \mathbf{G}_b\} \hookrightarrow H^1(\sigma, W),$$

mapping the class of $\mathbf{T}(\subseteq \mathbf{G}_b)$ to the image of b in $H^1(\sigma, W)$.

Definition 3.1. For $b \in \mathbf{N}(\check{k})$ basic and $w \in W$, denote by $\mathfrak{T}(\mathbf{G}_b, w)$ the *stable conjugacy class* of all unramified maximal k-tori in $\mathbf{G}_b$, which corresponds to $[w] \in H^1(\sigma, W)$ under (3.1).

The elements of $\mathfrak{T}(\mathbf{G}_b, w)$ are precisely the images of k-embeddings $\mathbf{T}_w \to \mathbf{G}_b$.

Remark 3.2. We recall the construction from the proof of [DeB06, Lemma 4.3.1]. Any unramified maximal k-torus of $\mathbf{G}_b$ is of the form ${}^g\mathbf{T}$ for some $g \in \mathbf{G}_b(\check{k})$. Moreover, ${}^g\mathbf{T}$ is k-rational if and only if $\sigma_b({}^g\mathbf{T}) = {}^g\mathbf{T}$, which is equivalent to $g^{-1}b\sigma(g) \in \mathbf{N}(\check{k})$. In that case, if $w \in W$ is the image of $g^{-1}b\sigma(g)$, then ${}^g\mathbf{T} \in \mathfrak{T}(\mathbf{G}_b, w)$. Replacing g by $g\dot{v}$ for any lift $\dot{v} \in \mathbf{N}(\check{k})$ of some $v \in W$ has the effect of replacing w by $v^{-1}w\sigma(v)$.

3.2. Rational conjugacy classes for pure inner forms

It is easy to see that the surjection $\mathbf{N}(\check{k}) \twoheadrightarrow \widetilde{W}$ induces an isomorphism[3] $H^1(\sigma, \mathbf{N}(\check{k})) \cong H^1(\sigma, \widetilde{W})$.

The following is a consequence of [Ree11, Proposition 6.1];[4] see also [DR09, Section 2] and [DeB06, Section 4.3]. Suppose that $[b] \in B(\mathbf{G})_{\mathrm{bas}}$ comes from a class $\xi \in H^1(\sigma, \mathbf{G})$ (*cf.* (2.6)), *i.e.*, $\mathbf{G}_b$ is a pure inner form of $\mathbf{G}$. Then rational conjugacy classes of unramified maximal k-tori in $\mathbf{G}_b$ are in bijection with preimages of ξ under the natural map $H^1(\sigma, \widetilde{W}) \xrightarrow{\sim} H^1(\sigma, \mathbf{N}(\check{k})) \to H^1(\sigma, \mathbf{G})$; *cf.* [Ree11, Lemma 6.2] (and use twisting). There are maps

$$(3.2) \qquad H^1(\sigma, W) \xleftarrow{q_0} H^1(\sigma, \widetilde{W}) \xrightarrow{\pi_0} H^1(\sigma, \mathbf{G}).$$

By Section 3.1, given $[w] \in H^1(\sigma, W)$, represented by $w \in W$, and $\xi \in H^1(\sigma, \mathbf{G})$, represented by $b \in \mathbf{N}(\check{k})$, the set $\mathfrak{T}(\mathbf{G}_b, w)/\mathrm{Ad}\,\mathbf{G}_b(k)$ of rational conjugacy classes in $\mathfrak{T}(\mathbf{G}_b, w)$ is in bijection with $\pi_0^{-1}(\xi) \cap q_0^{-1}([w])$. In particular, $[w]$ is in the image of (3.1) if and only if $\pi_0^{-1}(\xi) \cap q_0^{-1}([w]) \neq \varnothing$.

3.3. Rational conjugacy classes for extended pure inner forms

We wish to extend (3.2) to all extended pure inner forms. We have $X_*(\mathbf{Z}) \subseteq X_*(\mathbf{T}) \subseteq \widetilde{W}$. Define

$$\widetilde{Z}^1(\sigma, \widetilde{W}) = \left\{\tilde{w} \in \widetilde{W} \colon \nu_{\tilde{w}} \in X_*(\mathbf{Z})_{\mathbb{Q}}\right\},$$

where $\nu_{\tilde{w}}$ is as in Section 2.4. In other words, $\widetilde{Z}^1(\sigma, \widetilde{W})$ is precisely the set of all basic elements in $\widetilde{W}$ in the sense of Section 2.4. Define an equivalence relation on $\widetilde{Z}^1(\sigma, \widetilde{W})$ by $x \sim y$ if and only if there exists a $g \in \widetilde{W}$ such that $y = g^{-1}x\sigma(g)$. Put

$$\widetilde{H}^1(\sigma, \widetilde{W}) = \widetilde{Z}^1(\sigma, \widetilde{W})/\sim.$$

Note that we have $H^1(\sigma, \widetilde{W}) \subseteq \widetilde{H}^1(\sigma, \widetilde{W})$ and $Z^1(\sigma, \widetilde{W}) \subseteq \widetilde{Z}^1(\sigma, \widetilde{W})$ is precisely the set of those $\tilde{w}$ with $\nu_{\tilde{w}} = 0$. Recalling from (2.6) that $H^1(\sigma, \mathbf{G}) \subseteq B(\mathbf{G})_{\mathrm{bas}}$, we have the following lemma.

---

[3]Indeed, one can use [Ser94, Corollary 2 to Proposition I.39 and Corollary to Proposition I.41] along with the pro-version of Lang's theorem for the group $\mathcal{T}(\mathcal{O}_{\check{k}}) = \ker(\mathbf{N}(\check{k}) \twoheadrightarrow \widetilde{W})$ with varying rational structures given by σ_b ($b \in \mathbf{N}(\check{k})$).

[4]In fact, [Ree11] works with Gal_k-cohomology and *all* maximal k-tori. To deduce our statements, one can run literally the same arguments with $\langle\sigma\rangle$ replacing Gal_k and unramified k-tori replacing all k-tori.

Lemma 3.3. *There is a natural surjection* $\pi \colon \widetilde{H}^1(\sigma, \widetilde{W}) \to B(\mathbf{G})_{\mathrm{bas}}$ *extending the surjection* $\pi_0 \colon H^1(\sigma, \widetilde{W}) \to H^1(\sigma, \mathbf{G})$.

Proof. Suppose $\tilde{w}_1 \sim \tilde{w}_2 \in \widetilde{Z}^1(\sigma, \widetilde{W})$, *i.e.,* $\tilde{w}_2 = g^{-1}\tilde{w}_1\sigma(g)$. Then obviously $\bar{\kappa}_{\mathbf{G}}(\tilde{w}_1) = \bar{\kappa}_{\mathbf{G}}(\tilde{w}_2) \in \pi_1(\mathbf{G})_{\langle\sigma\rangle}$, and an easy computation shows that $\nu_{\tilde{w}_1} = \nu_{\tilde{w}_2}$. Thus some (*i.e.,* any) lifts to $\mathbf{G}(\check{k})$ of $\tilde{w}_1, \tilde{w}_2$ are σ-conjugate. This gives a map $\widetilde{H}^1(\sigma, \widetilde{W}) \to B(\mathbf{G})$ whose image is contained in $B(\mathbf{G})_{\mathrm{bas}}$ as each $\tilde{w} \in \widetilde{Z}^1(\sigma, \widetilde{W})$ is basic. The surjectivity is clear. □

The projection $\widetilde{W} \twoheadrightarrow W$ induces a map $q \colon \widetilde{H}^1(\sigma, \widetilde{W}) \to H^1(\sigma, W)$. Altogether, we have the diagram

$$(3.3) \qquad\qquad H^1(\sigma, W) \xleftarrow{\ q\ } \widetilde{H}^1(\sigma, \widetilde{W}) \xrightarrow{\ \pi\ } B(\mathbf{G})_{\mathrm{bas}}$$

extending (3.2). DeBacker and Reeder's parametrization from Section 3.2 admits the following completely analogous generalization to all extended pure inner forms.

Proposition 3.4. *Given* $[w] \in H^1(\sigma, W)$ *and* $[b] \in B(\mathbf{G})_{\mathrm{bas}}$, *we have a natural bijection* $\Upsilon(\mathbf{G}_b, w)/\operatorname{Ad}\mathbf{G}_b(k) \cong \pi^{-1}([b]) \cap q^{-1}([w])$. *In particular,* $[w]$ *is in the image of* (3.1) *if and only if* $\pi^{-1}([b]) \cap q^{-1}([w]) \neq \varnothing$.

Proof. Any unramified maximal k-torus in $\mathbf{G}_b$ is of the form ${}^g\mathbf{T}$ with $g^{-1}b\sigma(g) \in \mathbf{N}(\check{k})$; see Remark 3.2. Let $\tilde{w}_g \in \widetilde{W}$ be the image of $g^{-1}b\sigma(g)$. Then $\nu_{\tilde{w}_g} = \nu_b$, and hence $\tilde{w}_g \in \widetilde{Z}^1(\sigma, \widetilde{W})$ and $\pi([\tilde{w}_g]) = [b]$. Let ${}^{g_i}\mathbf{T}$ ($i = 1, 2$) be two such tori with corresponding elements $\tilde{w}_i \in \widetilde{Z}^1(\sigma, \widetilde{W})$. Suppose that the ${}^{g_i}\mathbf{T}$ are rationally conjugate in $\mathbf{G}_b$, *i.e.,* there is some $h \in \mathbf{G}_b(k)$ with ${}^{g_1}\mathbf{T} = {}^{hg_2}\mathbf{T}$. Then $g_1^{-1}hg_2 \in \mathbf{N}(\check{k})$, and letting $\tilde{v} \in \widetilde{W}$ be its image, we compute

$$(3.4) \qquad \tilde{v}\tilde{w}_2\sigma(\tilde{v})^{-1} = g_1^{-1}hg_2g_2^{-1}b\sigma(g_2)\sigma(g_2^{-1}h^{-1}g_1) = g_1^{-1}hb\sigma(h^{-1})\sigma(g_1) = g_1^{-1}b\sigma(g_1) = \tilde{w}_1;$$

i.e., $\tilde{w}_1$ and $\tilde{w}_2$ define the same class in $\widetilde{H}^1(\sigma, \widetilde{W})$. Conversely, given $[\tilde{w}] \in \tilde{\pi}^{-1}([b])$, we can lift $\tilde{w}$ to some $\dot{w} \in \mathbf{N}(\check{k})$, which necessarily belongs to $[b]$. If $\dot{w} = g^{-1}b\sigma(g)$, one immediately checks that $\tilde{w} \mapsto {}^g\mathbf{T}$ induces an inverse to the above map. This shows that rational conjugacy classes of unramified tori in $\mathbf{G}_b$ are in bijection with $\pi^{-1}([b])$, establishing the analogue of [Ree11, Lemma 6.2]. The rest of the proof (the analogue of [Ree11, Lemma 6.4]) is done analogously. □

3.4. A description of $\pi^{-1}([b]) \cap q^{-1}([w])$

Fix $w \in W$ and $[b] \in B(\mathbf{G})_{\mathrm{bas}}$. Put

$$F_w \overset{\mathrm{def}}{=\!=} \text{fiber of } \mathbf{N} \to W \text{ over } w \quad \text{and} \quad \overline{F}_w \overset{\mathrm{def}}{=\!=} \text{fiber of } \widetilde{W} \to W \text{ over } w.$$

We regard F_w as a $\check{k}$-scheme and $\overline{F}_w$ as a discrete $\mathbb{F}_q$-scheme (or, sometimes, as a discrete set). Then F_w is a (trivial) $\mathbf{T}_{\check{k}}$-torsor and $\overline{F}_w$ is a (trivial) $X_*(\mathbf{T})$-torsor. The action of $\mathbf{T}_{\check{k}}$ on F_w is given by $t, \tilde{w} \mapsto t\tilde{w}$, and the action of $X_*(\mathbf{T})$ on $\overline{F}_w$ is induced by this. Recall the maps $\kappa_w \colon L\mathbf{T} \to X_*(\mathbf{T})_{\langle\sigma_w\rangle}$ and $\bar{\kappa}_w \colon X_*(\mathbf{T}) \to X_*(\mathbf{T})_{\langle\sigma_w\rangle}$ from Section 2.3.

Lemma 3.5. *The natural surjection* $F_w(\check{k}) \twoheadrightarrow \overline{F}_w$ *induces an isomorphism* $LF_w/\ker\kappa_w \cong \overline{F}_w/\ker\bar{\kappa}_w$. *This discrete set is a* (*trivial*) $X_*(\mathbf{T})_{\langle\sigma_w\rangle}$-*torsor.*

Proof. As LF_w (resp. $\overline{F}_w$) is a trivial $L\mathbf{T}$-torsor (resp. $X_*(\mathbf{T})$-torsor), everything follows from $L\mathbf{T}/\ker\kappa_w = X_*(\mathbf{T})/\ker\bar{\kappa}_w = X_*(\mathbf{T})_{\langle\sigma_w\rangle}$. □

Observe that for $\tilde{w} \in \overline{F}_w$, the values of $\nu_{\tilde{w}}$ and $\bar{\kappa}_{\mathbf{G}}(\tilde{w})$ remain constant if $\tilde{w}$ is multiplied by an element in $\ker\bar{\kappa}_w$. Indeed, for $\bar{\kappa}_w$ this is clear, and for ν this follows from the description in Section 2.4. In particular, $\bar{\kappa}_{\mathbf{G}}|_{\overline{F}_w}$ induces a map

$$(3.5) \qquad\qquad \bar{\kappa}^w \colon \overline{F}_w/\ker\bar{\kappa}_w \longrightarrow \pi_1(\mathbf{G})_{\langle\sigma\rangle}.$$

Moreover, from Section 2.4 we immediately deduce the following lemma.

Lemma 3.6. *The map* (2.5) *factors through an injection* $(\nu, \bar\kappa^w)\colon \overline{F}_w/\ker\bar\kappa_w \to (W\backslash X_*(\mathbf{T})_\mathbb{Q})^{\langle\sigma\rangle} \times \pi_1(\mathbf{G})_{\langle\sigma\rangle}$.

Fix $w \in W$ and $[b] \in B(\mathbf{G})_{\mathrm{bas}}$. Put

$$[b]\cap\overline{F}_w \overset{\mathrm{def}}{=\!=} \mathrm{im}\left([b]\cap F_w(\check k)\hookrightarrow F_w(\check k)\twoheadrightarrow\overline{F}_w\right).$$

Then $[b]\cap\overline{F}_w\subseteq[b]\cap\widetilde{Z}^1(\sigma,\widetilde{W})$, inducing a map $[b]\cap\overline{F}_w\to\pi^{-1}([b])\subseteq\widetilde{H}^1(\sigma,\widetilde{W})$. This obviously factors through a map $[b]\cap\overline{F}_w\to\pi^{-1}([b])\cap q^{-1}([w])$. By Lemma 3.6, $[b]\cap\overline{F}_w$ is stable under the action of $\ker\bar\kappa_w$, and we have the (finite) quotient set $[b]\cap\overline{F}_w/\ker\bar\kappa_w$, consisting of all $\bar w\in\overline{F}_w/\ker\bar\kappa_w$ satisfying $\nu_{\bar w}=\nu_b$ and $\bar\kappa^w(\bar w)=\kappa_{\mathbf{G}}(b)$.

Proposition 3.7.

  (i) *The quotient* $[b]\cap\overline{F}_w/\ker\bar\kappa_w$ *is either empty or a* $\ker(H^1(\sigma_w,\mathbf{T})\to H^1(\sigma,\mathbf{G})) = \mathrm{im}(\beta_w)_{\mathrm{tors}}$*-torsor* *(cf.* (2.2)*). It is empty if and only if* $\Upsilon(\mathbf{G}_b,w)=\varnothing$.

  (ii) *The centralizer* $C_W(w\sigma)\subseteq W$ *of* $w\sigma\in W\rtimes\langle\sigma\rangle$ *acts on* $[b]\cap\overline{F}_w/\ker\bar\kappa_w$ *in a natural way, and there are canonical isomorphisms*

$$\left([b]\cap\overline{F}_w/\ker\bar\kappa_w\right)/C_W(w\sigma)\xrightarrow{\sim}\pi^{-1}([b])\cap q^{-1}([w])\cong\Upsilon(\mathbf{G}_b,w)/\mathrm{Ad}\,\mathbf{G}_b(k).$$

Proof. We have $\Upsilon(\mathbf{G}_b,w)\neq\varnothing\Leftrightarrow\exists g\in\mathbf{G}(\check k)$ such that $g^{-1}b\sigma(g)\in F_w(\check k)\Leftrightarrow[b]\cap\overline{F}_w\neq\varnothing$, where the first equivalence follows from Remark 3.2. To show the first claim of (i), we may assume that there is an element $\tilde w_1\in[b]\cap\overline{F}_w\neq\varnothing$. Recall that $\overline{F}_w$ is a $X_*(\mathbf{T})$-torsor.

Lemma 3.8. *For* $\tau\in X_*(\mathbf{T})$, *we have* $\tau\tilde w_1\in[b]\cap\overline{F}_w\Leftrightarrow\bar\kappa_w(\tau)\in\ker(H^1(\sigma_w,\mathbf{T})\to H^1(\sigma,\mathbf{G}))$.

Proof. We have $\tau\tilde w_1\in[b]\cap\overline{F}_w$ if and only if $\bar\kappa_{\mathbf{G}}(\tau)=0$ (as $\kappa_{\mathbf{G}}$ is a group homomorphism) and $\nu_{\tau\tilde w_1}=\nu_{\tilde w_1}$. Now, $\bar\kappa_{\mathbf{G}}(\tau)=0$ means that the image $\bar\kappa_w(\tau)\in X_*(\mathbf{T})_{\langle\sigma_w\rangle}$ maps to $0\in\pi_1(\mathbf{G})_{\langle\sigma\rangle}$, which is equivalent to $\bar\kappa_w(\tau)\in\mathrm{im}(\beta_w)$, with β_w as in (2.1). Now we use the algorithm to compute ν from Section 2.4. Note that the integer d can be chosen the same for $\tilde w_1$ and $\tau\tilde w_1$. Then let $\mu_{\tilde w_1}$ and $\mu_{\tau\tilde w_1}$ be as described in Section 2.4. Then the equality $\nu_{\tau\tilde w_1}=\nu_{\tilde w_1}$ is equivalent to the existence of some $v\in W$ such that the equality

$$\sum_{i=0}^{d-1}(w\sigma)^i(\tau)=v(\mu_{\tilde w_1})-\mu_{\tilde w_1}$$

holds in $X_*(\mathbf{T})$. But $\tilde w_1$ is basic, so $\mu_{\tilde w_1}$ is central; *i.e.*, the right-hand side is zero. Thus $\tau\tilde w_1\in[b]\cap\overline{F}_w$ if and only if $\bar\kappa_w(\tau)\in\mathrm{im}(\beta_w)$ and $\sum_{i=0}^{d-1}(w\sigma)^i(\tau)=0$. By [DR09, Lemma 2.4.1], the latter condition is equivalent to $\tau\in\bar\kappa_w^{-1}(X_*(\mathbf{T})_{\langle\sigma_w\rangle,\mathrm{tors}})=\bar\kappa_w^{-1}(H^1(\sigma_w,\mathbf{T}))$. $\qquad\square$

By Lemma 3.8, $[b]\cap\overline{F}_w$ (being non-empty) is a torsor under the preimage in $X_*(\mathbf{T})$ of $\ker(H^1(\sigma,\mathbf{T}_w)\to H^1(\sigma,\mathbf{G}))\subseteq X_*(\mathbf{T})_{\langle\sigma_w\rangle}$. This preimage contains $\ker\bar\kappa_w$, and factoring it out, we arrive at the claim. This finishes the proof of (i).

In (ii), it is clear that we have a map $[b]\cap\overline{F}_w\to\pi^{-1}([b])\cap q^{-1}([w])$, and one checks that it factors through $[b]\cap\overline{F}_w/\ker\bar\kappa_w$. Moreover, surjectivity of this map amounts to the tautological fact that any element in $[w]\subseteq W$ is σ-conjugate to w. Next, let us describe the natural action of $C_W(w\sigma)$ on $[b]\cap\overline{F}_w/\ker\bar\kappa_w$. Let $\bar w\in[b]\cap\overline{F}_w/\ker\bar\kappa_w$ with preimage $\tilde w\in[b]\cap\overline{F}_w$. Let $v\in C_W(w\sigma)$, and let $\tilde v\in\widetilde{W}$ be any preimage. Then the action of v sends $\bar w$ to the image of $\tilde v^{-1}\tilde w\sigma(\tilde v)$ in $[b]\cap\overline{F}_w/\ker\bar\kappa_w$. This is independent of the choice of $\tilde v$, as if $\tilde v'$ is another choice, putting $\tau=\tilde v^{-1}\tilde v'\in X_*(\mathbf{T})$ gives the element $\tilde v'^{-1}\tilde w\sigma(\tilde v')=\tau^{-1}\tilde v^{-1}\tilde w\sigma(\tilde v)\sigma(\tau)$, which has the same image in $[b]\cap\overline{F}_w/\ker\bar\kappa_w$. Similarly, one shows that it is independent of the choice of $\tilde w$. Now, $\bar w'\in[b]\cap\overline{F}_w/\ker\bar\kappa_w$ has the same image as $\bar w$ in $\pi^{-1}([b])\cap q^{-1}([w])$ if and only if there is some $\tilde v\in\widetilde{W}$ with $\tilde v^{-1}\tilde w\sigma(\tilde v)=\tilde w'$ for some lift $\tilde w'\in\overline{F}_w$ of $\bar w'$. By the definition of the $C_W(w\sigma)$-action, this is equivalent to the fact that $\bar w$ and $\bar w'$ lie in the same $C_W(w\sigma)$-orbit. Thus $C_W(w\sigma)$-orbits in $[b]\cap\overline{F}_w/\ker\bar\kappa_w$ coincide with fibers of the surjection onto $\pi^{-1}([b])\cap q^{-1}([w])$, and we are done. $\qquad\square$

Remark 3.9. Let us explicate the correspondence in Proposition 3.7(ii). If $\bar{w} \in [b] \cap \overline{F}_w / \ker \bar{\kappa}_w$ represents a class on the left side, $\dot{w} \in [b] \cap F_w(\check{k})$ is a (*i.e.*, any) lift of $\bar{w}$, and $g \in \mathbf{G}(\check{k})$ satisfies $g^{-1} b \sigma(g) = \dot{w}$, then the map $\mathrm{Ad}(g) \circ \iota \colon \mathbf{T}_w \to \mathbf{G}_b$ is k-rational, its image ${}^g\mathbf{T} \subseteq \mathbf{G}_b$ lies in $\Upsilon(\mathbf{G}_b, w)$ and the $\mathbf{G}_b(k)$-conjugacy class of ${}^g\mathbf{T}$ is the image of $\bar{w}$ under the above map.

3.5. The case of Coxeter tori

We keep the setup of Sections 3.3 and 3.4. Furthermore, throughout Section 3.5 we assume that $w = c \in W$ is a *Coxeter element*. Recall $\beta_c \colon X_*(\mathbf{T}^{\mathrm{sc}})_{\langle \sigma_c \rangle} \to X_*(\mathbf{T})_{\langle \sigma_c \rangle}$ from (2.1).

Proposition 3.10. *Let $[b] \in B(\mathbf{G})_{\mathrm{bas}}$ and $c \in W$ be Coxeter. Then the following hold:*

(i) *The image $\mathrm{im}(\beta_c)$ is finite.*
(ii) *We have*
$$[b] \cap \overline{F}_c / \ker \bar{\kappa}_c = (\bar{\kappa}^c)^{-1}(\kappa_{\mathbf{G}}(b)).$$
This is an $\mathrm{im}(\beta_c)$-torsor. In particular, $[b] \cap \overline{F}_c \neq \varnothing$.

Proof. (i) As c Coxeter, $X_*(\mathbf{T}^{\mathrm{sc}})^{\langle \sigma_c \rangle} = 0$; hence $X_*(\mathbf{T}^{\mathrm{sc}})_{\langle \sigma_c \rangle}$ is finite. Hence $\mathrm{im}(\beta_c)$ is finite. For (ii) we first record the following easy and well-known lemma.

Lemma 3.11. *If c is Coxeter, all elements in $\overline{F}_c$ are basic.*

Proof. Let $\tilde{c} \in \overline{F}_c$. If μ is as in the algorithm to compute ν_b from Section 2.4, we see that $\mu \in X_*(\mathbf{T})_{\mathbb{Q}}^{\langle \sigma_c \rangle} \subseteq X_*(\mathbf{Z})_{\mathbb{Q}}$, the last inclusion being true as c is Coxeter. $\qquad\square$

Now note that $\bar{\kappa}_{\mathbf{G}}|_{\overline{F}_c} \colon \overline{F}_c \to \pi_1(\mathbf{G})_{\langle \sigma \rangle}$ is surjective. This holds as $\overline{F}_c$ is a $X_*(\mathbf{T})$-torsor, $\kappa_{\mathbf{G}}$ is a group homomorphism and $X_*(\mathbf{T}) \to \pi_1(\mathbf{G})_{\langle \sigma \rangle}$ is surjective. Taking any preimage b' of $\kappa_{\mathbf{G}}(b)$ in $\overline{F}_c$, Lemma 3.11 shows that b' is basic. Moreover, we know that $\kappa_{\mathbf{G}}$ induces a bijection $B(\mathbf{G})_{\mathrm{bas}} \to \pi_1(\mathbf{G})_{\langle \sigma \rangle}$. It follows that $b' \in [b]$; *i.e.*, $[b] \cap \overline{F}_c \neq \varnothing$. Moreover, as this holds for any b' in the preimage, we even get the equality claimed in (ii). Now, $(\bar{\kappa}^c)^{-1}(\kappa_{\mathbf{G}}(b))$ is a $\ker(X_*(\mathbf{T}_{\langle \sigma_c \rangle} \to \pi_1(\mathbf{G})_{\langle \sigma \rangle}) = \mathrm{im}(\beta_c)$-torsor. $\qquad\square$

Remark 3.12. It follows from Proposition 3.10(ii) that for any $[b] \in B(\mathbf{G})_{\mathrm{bas}}$ and any Coxeter element c, there exists a representative of $[b]$ in $F_c(\check{k})$. This is a converse to Lemma 3.11.

Proposition 3.13. *Let $c \in W$ be Coxeter. Suppose one of the following holds:*

(i) *We have $[b] = 1$.*
(ii) *We have $\mathbf{G} = \prod_{i=1}^s \mathbf{G}_i$, where for each i, $\mathbf{G}_i = \mathrm{Res}_{k_i/k} \mathbf{G}_i'$ for a finite unramified extension k_i/k and an absolutely almost simple k_i-group $\mathbf{G}_i'$.*

Then the $C_W(c\sigma)$-action on $[b] \cap \overline{F}_c / \ker \bar{\kappa}_c$ is trivial, and there is a natural bijection
$$[b] \cap \overline{F}_c / \ker \bar{\kappa}_c \xrightarrow{\sim} \Upsilon(\mathbf{G}_b, c) / \mathrm{Ad}\,\mathbf{G}_b(k).$$

Proof. Choose some σ-equivariant splitting $\iota \colon W \to \widetilde{W}$ (which exists, as $\mathbf{G}$ is unramified), and denote its image by $\dot{W}$. Let $\dot{c} \in \overline{F}_c \cap \dot{W}$. Let $\tau \in X_*(\mathbf{T})$ be such that $\tau \dot{c} \in [b] \cap \overline{F}_c$. Let $v \in C_W(c\sigma)$, and let $\dot{v} = \iota(v)$. The action of v sends the image of $\tau \dot{c}$ in $[b] \cap \overline{F}_c / \ker \bar{\kappa}_c$ to the image in $[b] \cap \overline{F}_c / \ker \bar{\kappa}_c$ of the element $\dot{v}^{-1} \tau \dot{c} \sigma(\dot{v})$. As ι is σ-equivariant, $\dot{v} \in C_{\dot{W}}(\dot{c}\sigma)$, so $\dot{v}^{-1} \tau \dot{c} \sigma(\dot{v}) = \dot{v}^{-1} \tau \dot{v} \dot{c} = \mathrm{Ad}(v)^{-1}(\tau)\dot{c}$; *i.e.*, we must show (replacing v with v^{-1}) that $(\mathrm{Ad}\, v - \mathrm{id})(\tau) = \mathrm{Ad}\, v(\tau) - \tau \in \ker \bar{\kappa}_c$.

In case (i), we have $[1] \cap \overline{F}_c = X_*(\mathbf{T}^{\mathrm{sc}}) \cdot \dot{c}$. Thus, as $\ker \bar{\kappa}_c^{\mathrm{sc}} \subseteq \ker \bar{\kappa}_c$, it suffices to show that we have $(\mathrm{Ad}\, v - \mathrm{id})(X_*(\mathbf{T}^{\mathrm{sc}})) \subseteq \ker \bar{\kappa}_c^{\mathrm{sc}}$. In other words, we are reduced to the case that $\mathbf{G}$ is simply connected, which is covered by (ii).

In case (ii), one is immediately reduced to the case that $\mathbf{G} = \mathrm{Res}_{k'/k} \mathbf{G}'$ with k'/k a finite subextension of $\check{k}/k$ and $\mathbf{G}'$ an absolutely almost simple k'-group. Let $d = [k' : k]$. Write $X_*(\mathbf{T}) = \bigoplus_{i=1}^d X_*(\mathbf{T}')$ and $W = \prod_{i=1}^d W'$, with σ-action given by $(v_1, \ldots, v_d) \mapsto (\sigma'(v_d), v_1, \ldots, v_{d-1})$ on both objects, where σ' is the

$\check{k}/k'$-Frobenius for $\mathbf{G}'$. If $c, c_1 \in W$ are σ-conjugate, the statement claimed in the proposition holds for c if and only if it holds for c_1. Thus we may assume that $c = (c', 1, \ldots, 1)$, where $c' \in W'$ is a Coxeter element in W'. Suppose we know the claim for $\mathbf{G}', c'$; *i.e.*, for any $v' \in C_{W'}(c'\sigma')$, $(\operatorname{Ad} v' - \operatorname{id})(X_*(\mathbf{T}')) \subseteq (\sigma'_{c'} - \operatorname{id})(X_*(\mathbf{T}'))$. We have

$$\ker \bar{\kappa}_c = (\sigma_c - \operatorname{id})(X_*(\mathbf{T})) = \{\sigma'_{c'}(\alpha_d) - \alpha_1, \alpha_1 - \alpha_2, \ldots, \alpha_{r-1} - \alpha_d : \alpha_i \in X_*(\mathbf{T}') \text{ for } 1 \leq i \leq d\}$$

$$= \left\{ \left(\sigma'_{c'}(\alpha_r) - \alpha_d - \sum_{i=1}^{d-1} \beta_i, \beta_1, \ldots, \beta_{d-1} \right) : \beta_i, \alpha_d \in X_*(\mathbf{T}') \text{ for } 1 \leq i \leq d-1 \right\}$$

The inclusion $W' \to W$ into the first factor induces an isomorphism $C_{W'}(c'\sigma') \xrightarrow{\sim} C_W(c\sigma)$, $v' \mapsto (v', \ldots, v') = v$. For $(e_i)_{i=1}^d \in X_*(\mathbf{T})$, we have $(\operatorname{Ad} v - \operatorname{id})((e_i)_i) = ((\operatorname{Ad} v' - \operatorname{id})(e_i))_{i=1}^d$. This lies in $\ker \bar{\kappa}_c$ as, setting $\beta_i := (\operatorname{Ad} v' - \operatorname{id})(e_{i+1})$ for $1 \leq i \leq d-1$, it suffices to find an $\alpha_d \in X_*(\mathbf{T}')$ such that $\sigma'_{c'}(\alpha_d) - \alpha_d = \operatorname{Ad} v'(\sum_{i=1}^d e_i) - \sum_{i=1}^d e_i$, which works by assumption for $\mathbf{G}'$.

It remains to check the claim for an absolutely almost simple group $\mathbf{G}$ over k. Let $d > 0$ be the smallest positive integer such that σ^d acts trivially on the Dynkin diagram of $\mathbf{G}$. By [Spr74, Theorem 7.6(v)], $C_W(c\sigma)$ is the cyclic subgroup of W generated by $c^{\mathrm{spl}} = c\sigma(c) \ldots \sigma^{d-1}(c)$. Thus, it suffices to check that $(\operatorname{Ad} c^{\mathrm{spl}} - \operatorname{id})(X_*(\mathbf{T})) \subseteq \ker \bar{\kappa}_c = \operatorname{im}(\sigma_c - \operatorname{id} : X_*(\mathbf{T}) \to X_*(\mathbf{T}))$. Observe that $\sigma_c(\ker \bar{\kappa}_c) = \ker \bar{\kappa}_c$ (as σ_c is an automorphism and for any $v \in X_*(\mathbf{T})$, $\sigma_c(\sigma_c(v) - v) = \sigma_c(\sigma_c(v)) - \sigma_c(v) \in \ker \bar{\kappa}_c$). Now σ^d acts trivially on $X_*(\mathbf{T})$, so it suffices to check that for all $\tau \in X_*(\mathbf{T})$, $c^{\mathrm{spl}}\sigma^d(\tau) - \tau \in \ker \bar{\kappa}_c$, or equivalently, $\sigma_c^d(v) - v \in \ker \bar{\kappa}_c$. We now prove that $\sigma_c^i(v) - v \in \ker \bar{\kappa}_c$ for all $i > 0$. For $i = 1$, this holds as $\ker \bar{\kappa}_c = \operatorname{im}(\sigma_c - 1)$. For $i > 1$, we have $\sigma_c^i(v) - v = \sigma_c^i(v) - \sigma_c^{i-1}(v) + \sigma_c^{i-1}(v) - v = \sigma_c^{i-1}(\sigma_c(v) - v) + \sigma_c^{i-1}(v) - v$. Now we have $\sigma_c^{i-1}(v) - v \in \ker \bar{\kappa}_c$ by induction on i, and $\sigma_c^{i-1}(\sigma_c(v) - v) \in \sigma_c^{i-1}(\ker \bar{\kappa}_c) = \ker \bar{\kappa}_c$. $\qquad\square$

Proposition 3.13 becomes false without any assumptions on $\mathbf{G}$, b, as the next example shows.

Example 3.14. Suppose $\operatorname{char} k \neq 2$. Let $\mathbf{G}' = \mathbf{SL}_2 \times \mathbf{SL}_2$ and $\mathbf{G} = \mathbf{G}'/\mu_2$, embedded diagonally. Let $\mathbf{T}' \subseteq \mathbf{G}'$ be the product of diagonal tori in $\mathbf{SL}_2$ and $\mathbf{T}$ its image in $\mathbf{G}$. Let $\varepsilon_1 = ((1, -1), (0, 0))$, $\varepsilon_2 = ((0, 0), (1, -1))$ be a $\mathbb{Z}$-basis of $X_*(\mathbf{T}')$. We may identify $X_*(\mathbf{T}) = \{\frac{a_1\varepsilon_1 + a_2\varepsilon_2}{2} \in X_*(\mathbf{T}')_{\mathbb{Q}} : a_1, a_2 \in \mathbb{Z}, a_1 \equiv a_2 \mod 2\}$. Let s be the image of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the Weyl group of the diagonal torus in $\mathbf{SL}_2$, so that $c = (s, s)$ represents a Coxeter element in W. Let $\dot{c}_0 = \left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) \in F_c(\check{k})$ be a hyperspecial lift, and let $\bar{c}_0$ be its image in $\overline{F}_c$. We have $C_W(c) = W \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Note that $\tau_1 = (\varpi^{\frac{1}{2}}, -\varpi^{-\frac{1}{2}}), (\varpi^{\frac{1}{2}}, -\varpi^{-\frac{1}{2}}) \in \mathbf{G}(k)$ by Galois descent. The action of the Frobenius σ on all objects below is trivial, so we ignore it. We have $\pi_1(\mathbf{G}) = X_*(\mathbf{T})/X_*(\mathbf{T}') \cong \mathbb{Z}/2\mathbb{Z}$, the non-trivial element being the class of $\frac{\varepsilon_1 + \varepsilon_2}{2}$. It is represented by the basic element $b_1 = \dot{c}_0\tau_1$. We have $\ker \bar{\kappa}_c = \mathbb{Z} \cdot 2\varepsilon_1 + \mathbb{Z} \cdot (\varepsilon_2 - \varepsilon_1) \subseteq X_*(\mathbf{T})$.

Case $b = 1$.— We have $[1] \cap \overline{F}_c = X_*(\mathbf{T}') \cdot \bar{c}_0$. As $X_*(\mathbf{T}')/\ker \bar{\kappa}_c = \{\bar{0}, \bar{\varepsilon}_1\} \cong \mathbb{Z}/2\mathbb{Z}$, we have $[1] \cap \overline{F}_c/\ker \bar{\kappa}_c = (X_*(\mathbf{T}')/\ker \bar{\kappa}_c) \cdot \bar{c}_0 = \{\bar{0}, \bar{\varepsilon}_1\} \cdot \bar{c}_0$. Now it is immediate to check that the action of $C_W(c)$ from Proposition 3.7 on this set is trivial. For example, $\operatorname{Ad}(s, 1)$ send $\varepsilon_i \in X_*(\mathbf{T}')$ to $(-1)^i \varepsilon_i$ ($i \in 1, 2$), so $\operatorname{Ad}(s, 1)(\varepsilon_i) - \varepsilon_i \in \ker \bar{\kappa}_c$. This is in agreement with Proposition 3.13 and also with the Bruhat–Tits picture: $\mathbf{G}^{\mathrm{ad}} \cong \mathbf{PGL}_2 \times \mathbf{PGL}_2$, so an alcove of $\mathcal{B}_{\mathbf{G},k}$ is a square whose vertices are hyperspecial of type (α, β) with $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$ the parity of the valuation of the determinant of the lattice representing a vertex of $\mathcal{B}_{\mathbf{PGL}_2}$. The action of $\tau_1 \in \mathbf{G}(k)$ identifies the vertices $(0, 0) \leftrightarrow (1, 1)$ and $(0, 1) \leftrightarrow (1, 0)$, so that there are two $\mathbf{G}(k)$-orbits in hyperspecial vertices of a base alcove and hence (by [DeB06]) two rational conjugacy classes of Coxeter tori.

Case $b = b_1$.— We have $[b_1] \cap \overline{F}_c = (X_*(\mathbf{T}) \smallsetminus X_*(\mathbf{T}')) \cdot \bar{c}_0$. As $(X_*(\mathbf{T}) \smallsetminus X_*(\mathbf{T}'))/\ker \bar{\kappa}_c = \{\frac{\bar{\varepsilon}_1 + \bar{\varepsilon}_2}{2}, \frac{\bar{\varepsilon}_1 + 3\bar{\varepsilon}_2}{2}\}$, we have $[b_1] \cap \overline{F}_c/\ker \bar{\kappa}_c = \{\frac{\bar{\varepsilon}_1 + \bar{\varepsilon}_2}{2}, \frac{\bar{\varepsilon}_1 + 3\bar{\varepsilon}_2}{2}\} \cdot \bar{c}_0$. The action of $(s, 1) \in C_W(c)$ identifies the two elements in $[b_1] \cap \overline{F}_c/\ker \bar{\kappa}_c$ as $\frac{\varepsilon_1 + 3\varepsilon_2}{2} - \operatorname{Ad}(s, 1)(\frac{\varepsilon_1 + \varepsilon_2}{2}) = \frac{\varepsilon_1 + 3\varepsilon_2}{2} - \frac{-\varepsilon_1 + \varepsilon_2}{2} = \varepsilon_1 + \varepsilon_2 \in \ker \bar{\kappa}_c$; *i.e.*, the $C_W(c)$-action is non-trivial. This agrees with Bruhat–Tits theory as the adjoint building of $\mathbf{G}_{b_1}$ over k is a point (it is the barycenter of an alcove of $\mathcal{B}_{\mathbf{G},k}$), so that there is just one $\mathbf{G}_{b_1}(k)$-conjugacy class of Coxeter tori.

4. The maps $\dot{X}_{\bar{w}}(b) \to X_w(b)$

Here we review and extend some results from [Iva23, Section 11]. By [Iva23, Lemma 11.5] the space $\dot{X}_{\dot{w}}(b)$ depends—up to a $\mathbf{G}_b(k) \times \mathbf{T}_w(k)$-equivariant isomorphism—only on the image of $\dot{w}$ in the discrete set $LF_w/\ker\kappa_w$ and not on $\dot{w}$. As the concrete choice of $\dot{w}$ never plays an important role, we make the following simplifying definition. Recall from Lemma 3.5 that $LF_w/\ker\kappa_w = \overline{F}_w/\ker\bar{\kappa}_w$.

Definition 4.1. For $b \in \mathbf{G}(\check{k})$ and $\bar{w} \in \overline{F}_w/\ker\bar{\kappa}_w$, we denote by $\dot{X}_{\bar{w}}(b)$ the space $\dot{X}_{\dot{w}}(b)$ for an arbitrary lift $\dot{w} \in F_w(\check{k})$ of $\bar{w}$.

Proposition 4.2 (*cf.* [Iva23, Propositions 11.4 and 11.9]). *Let $w \in W$ and $b \in \mathbf{G}(\check{k})$. There is a canonical map of arc-sheaves*

$$\alpha_{w,b} \colon X_w(b) \longrightarrow \overline{F}_w/\ker\overline{\kappa}_w$$

defining a clopen decomposition $X_w(b) = \coprod_{\bar{w} \in \overline{F}_w/\ker\overline{\kappa}_w} X_w(b)_{\bar{w}}$, where $X_w(b)_{\bar{w}} := \alpha_{w,b}^{-1}(\{\bar{w}\})$. For $\bar{w} \in \overline{F}_w/\ker\bar{\kappa}_w$, $\dot{X}_{\bar{w}}(b) \to X_w(b)_{\bar{w}}$ is a pro-étale $\mathbf{T}_w(k)$-torsor. In particular, $\dot{X}_{\bar{w}}(b) \neq \varnothing$ if and only if $\bar{w} \in \mathrm{im}(\alpha_{w,b})$.

Proof. Combine [Iva23, Propositions 11.4 and 11.9] with Lemma 3.5. $\qquad\square$

We have the following emptyness criterion for $\dot{X}_{\bar{w}}(b)$, *i.e.*, a constraint on the image of $\alpha_{w,b}$. Recall the map $\bar{\kappa}^w \colon \overline{F}_w/\ker\bar{\kappa}_w \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}$ from (3.5).

Proposition 4.3. *Let $b \in \mathbf{G}(\check{k})$ and $\bar{w} \in \overline{F}_w/\ker\bar{\kappa}_w$. If $\dot{X}_{\bar{w}}(b) \neq \varnothing$, then $\kappa_{\mathbf{G}}(b) = \bar{\kappa}^w(\bar{w})$. In particular, $\alpha_{w,b}$ factors through a map $\alpha_{w,b} \colon X_w(b) \to (\bar{\kappa}^w)^{-1}(\kappa_{\mathbf{G}}(b))$.*

Proof. Let $\dot{w} \in F_w(\check{k})$ be a lift of $\bar{w}$. As $\dot{X}_{\dot{w}}(b) \neq \varnothing$, there is some algebraically closed field $\mathfrak{f} \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ such that $\dot{X}_{\dot{w}}(b)(\mathfrak{f}) \neq \varnothing$. Let $\bar{g} \in \dot{X}_{\dot{w}}(b)(\mathfrak{f})$, and let $g \in \mathbf{G}(L)$ be a lift of $\bar{g}$. Then $g^{-1}b\sigma(g) \in \mathbf{U}(L)\dot{w}\mathbf{U}(L)$. We clearly have $\kappa_{\mathbf{G}}(g^{-1}b\sigma(g)) = \kappa_{\mathbf{G}}(b)$. Furthermore, any element of $\mathbf{U}(L)$ can be conjugated to some Iwahori subgroup, and the Kottwitz map is trivial on Iwahori subgroups. Thus $\kappa_{\mathbf{G}}(\mathbf{U}) = 1$. Thus $\kappa_{\mathbf{G}}(\mathbf{U}(L)\dot{w}\mathbf{U}(L)) = \kappa_{\mathbf{G}}(\dot{w})$, and the first claim follows. The last claim follows from the first and Proposition 4.2. $\qquad\square$

Recall the map $\beta_w \colon X_*(\mathbf{T}^{\mathrm{sc}})_{\langle\sigma_w\rangle} \to X_*(\mathbf{T})_{\langle\sigma_w\rangle}$ from (2.1). We have the following diagram with compatible simply transitive actions:

$$(4.1) \qquad \begin{array}{ccccc} (\bar{\kappa}^w)^{-1}(\kappa_{\mathbf{G}}(b)) & \lhook\joinrel\longrightarrow & \overline{F}_w/\ker\bar{\kappa}_w & \overset{\bar{\kappa}^w}{\relbar\joinrel\twoheadrightarrow} & \pi_1(\mathbf{G})_{\langle\sigma\rangle} \\[4pt] \circlearrowright & & \circlearrowright & & \circlearrowright \\[4pt] \mathrm{im}(\beta_w) & \lhook\joinrel\longrightarrow & X_*(\mathbf{T})_{\langle\sigma_w\rangle} & \longrightarrow\joinrel\twoheadrightarrow & \pi_1(\mathbf{G})_{\langle\sigma\rangle}, \end{array}$$

where the actions are induced by the left multiplication action of $X_*(\mathbf{T})$ on $\overline{F}_w$.

Corollary 4.4. *If $\beta_w = 0$, then $\alpha_{w,b}$ factors through a point, i.e., there is at most one $\bar{w} \in \overline{F}_w/\ker\bar{\kappa}_w$ such that $\dot{X}_{\bar{w}}(b) \neq \varnothing$. In this case, $X_w(b) \neq \varnothing$ if and only if $\dot{X}_{\bar{w}}(b) \neq \varnothing$ and $\dot{X}_{\bar{w}}(b) \to X_w(b)$ is a pro-étale $\mathbf{T}_w(k)$-torsor.*

Corollary 4.5. *If $\mathbf{G}$ is of adjoint type and w is Coxeter, the conclusions of Corollary 4.4 hold. Moreover, in this case, the map $\dot{X}_{\bar{w}}(b) \to X_w(b)$ is quasi-compact.*

Proof. The first claim follows from Corollary 4.4 and Remark 2.5. The last claim follows as $\dot{X}_{\bar{w}}(b) \to X_w(b)$ is a torsor under the profinite group $\mathbf{T}_w(k)$. $\qquad\square$

In the case that b is basic and $w = c$ is Coxeter, the situation simplifies as follows.

Corollary 4.6. *Suppose b is basic and c is Coxeter. For any $\bar{c} \in \overline{F}_c/\ker\bar{\kappa}_c$ such that $X_{\bar{c}}(b) \neq \varnothing$, there exists a $b_1 \in [b] \cap F_c(\check{k})$ lying over $\bar{c}$, and there is a $\mathbf{G}_b(k) \times \mathbf{T}_c(k)$-equivariant isomorphism $\dot{X}_{\bar{c}}(b) \cong \dot{X}_{\bar{c}}(b_1)$.*

Proof. By Proposition 4.3, $\dot{X}_{\bar{c}}(b) \neq \varnothing$ forces $\bar{c} \in (\bar{\kappa}^c)^{-1}(\bar{\kappa}_{\mathbf{G}}(b)) = [b] \cap \overline{F}_c / \ker \bar{\kappa}_c$, where the equality holds by Proposition 3.10. This implies the existence of b_1. For any $g \in \mathbf{G}(\check{k})$ such that $g^{-1} b \sigma(g) = b_1$, $x \mapsto gx$ induces an equivariant isomorphism $\dot{X}_{\bar{c}}(b) \cong \dot{X}_{\bar{c}}(b_1)$ by [Iva23, Remark 8.7]. □

Let us also record the following consequence.

Corollary 4.7. *Suppose b is basic and c is Coxeter. Then $\alpha_{c,b}$ factors through a surjection*

$$\alpha_{c,b} \colon X_c(b) \longrightarrow [b] \cap \overline{F}_c / \ker \bar{\kappa}_c.$$

There is a canonical surjection from the set of different non-empty pieces $X_c(b)_{\bar{c}} \subseteq X_c(b)$ to the set $\mathfrak{T}(\mathbf{G}_b, c) / \operatorname{Ad} \mathbf{G}_b(k)$ of rational conjugacy classes of unramified Coxeter tori in $\mathbf{G}_b$.

In particular, if $\mathbf{G}, b$ satisfy the assumptions of Proposition 3.13, then this surjection is a 1-to-1 correspondence.

Proof. The surjectivity of the map $\alpha_{c,b}$ in the corollary follows (for example) from Theorem 1.1. The rest immediately follows from Propositions 3.7, 3.10 and 3.13 □

We finish this section with the following remark on the non-basic case.

Remark 4.8. If $[b]$ is non-basic and c Coxeter, then $[b] \cap \overline{F}_c / \ker \bar{\kappa}_c = \varnothing$ by Lemma 3.11. Hence—although we still have $\operatorname{im}(\beta_c)_{\mathrm{tors}} = \operatorname{im}(\beta_c)$ just as in Proposition 3.10—we cannot conclude that $[b] \cap \overline{F}_c / \ker \kappa_c = (\bar{\kappa}^c)^{-1}(\kappa_{\mathbf{G}}(b))$. In particular, our arguments do not imply that $X_c(b)$ (equivalently, all of its covering spaces $\dot{X}_{\bar{c}}(b)$ with $\bar{c} \in \overline{F}_c / \ker \bar{\kappa}_c$) is empty. In agreement with this, $X_c(b)$ indeed can be non-empty; *cf.* [Iva23, Theorem 13.2].

5. A variant of Steinberg's cross section

By the *Dynkin diagram* $\operatorname{Dyn}(\mathbf{G})$ of the unramified group $\mathbf{G}$, we mean the Dynkin diagram of the split group $\mathbf{G}_{\check{k}}$ endowed with the action induced by the Frobenius σ. There is a unique decomposition into σ-stable subdiagrams $\operatorname{Dyn}(\mathbf{G}) = \bigsqcup_i \Gamma_i$ such that σ cyclically permutes $\pi_0(\Gamma_i)$ for each i. For each i, let $\Gamma_{i,0}$ be a connected component of Γ_i, so that $\Gamma_i = \bigsqcup_{j=1}^{d_i} \sigma^j(\Gamma_{i,0})$. Then $(\Gamma_{i,0}, \sigma^{d_i})$ is a *connected* Dynkin diagram of a group over k_{d_i}, an unramified extension of k of degree d_i.

Definition 5.1. We say that the unramified group $\mathbf{G}$ is *of classical type* if each connected component $(\Gamma_{i,0}, \sigma^{d_i})$ as above of $\operatorname{Dyn}(\mathbf{G})$ is of one of the following types: A_{n-1} $(n \geq 2)$, B_m $(m \geq 2)$, C_m $(m \geq 2)$, D_m $(m \geq 4)$, $^2A_{n-1}$ $(n \geq 3)$ or ^2D_m $(m \geq 4)$.

Although Theorem 1.1 and its corollaries essentially hold for all Coxeter elements (*cf.* the discussion preceding Theorem 1.1), it will be convenient to work with particular ones. Therefore, we introduce the following auxiliary notion.

Definition 5.2. Let $\mathbf{G}$ be of classical type. Let $c \in W$ be a Coxeter element. We call c *special* in the following cases:

- If $\mathbf{G}$ is absolutely almost simple, c is special if it is of the form[5] given in (7.5) (resp. (7.7), (7.9), (7.13), (7.18), (7.26)) if $\mathbf{G}$ is of type A_{n-1} (resp. C_m, B_m, D_m, $^2A_{n-1}$, ^2D_m).
- If $\mathbf{G} = \operatorname{Res}_{k'/k} \mathbf{G}'$ for a finite unramified extension k'/k and an absolutely almost simple k'-group $\mathbf{G}'$, we may identify the Weyl group W of $\mathbf{G}$ with $\prod_{i=1}^{[k':k]} W'$, where W' is the Weyl group of $\mathbf{G}'$ and σ acts cyclically. Then c is special if it is of the form $(c', 1, \ldots, 1)$, where c' is a special Coxeter element of W'.
- If $\mathbf{G}$ is isogeneous to a product $\prod_{i=1}^m \mathbf{G}_i$ of almost simple k-groups $\mathbf{G}_i$, we may identify $W = \prod_{i=1}^m W_i$, where W_i is the Weyl group of $\mathbf{G}_i$. Then $c = (c_i)_{i=1}^m$ is special if all c_i are special.

---

[5]The explicit shape of these elements only becomes important when everything boils down to explicit calculations in Section 7. For the sake of better readability, we postpone the explicit description until the relevant notation is introduced.

5.1. A loop version of Steinberg's cross section

For classical groups and special Coxeter elements, we now prove the existence of certain Steinberg cross sections in our setup. This is a Frobenius-twisted loop version of [Ste65, Proposition 8.9] and [HL12, Theorem 3.6 and Section 3.14], which will be an important ingredient in the proof of Theorem 1.1.

Let $c \in W$ be a Coxeter element. If $b \in \mathbf{G}(\check{k})$ is any lift of c, then $b\sigma$ has a unique fixed point $\mathbf{x} = \mathbf{x}_b$ on $\mathcal{A}_{\mathbf{T},\check{k}}$. Let $\mathcal{G}_{\mathbf{x}}$ denote the corresponding parahoric $\mathcal{O}_{\check{k}}$-model of $\mathbf{G}$. We have the twisted Frobenius $\sigma_b = \mathrm{Ad}(b) \circ \sigma$ on $L\mathbf{G}, L^+\mathcal{G}_{\mathbf{x}}$. Let $({}^c\mathbf{U} \cap \mathbf{U})_{\mathbf{x}}$ be the closure of ${}^c\mathbf{U} \cap \mathbf{U}$ in $\mathcal{G}_{\mathbf{x}}$. Similarly, we have the closures $({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}$ and $({}^c\mathbf{U})_{\mathbf{x}}$ of ${}^c\mathbf{U} \cap \mathbf{U}^-$ and ${}^c\mathbf{U}$ in $\mathcal{G}_{\mathbf{x}}$.

Proposition 5.3. *Let $\mathbf{G}$ be of classical type and $c \in W$ a special Coxeter element. Then*

$$\alpha_b \colon L({}^c\mathbf{U} \cap \mathbf{U}) \times L({}^c\mathbf{U} \cap \mathbf{U}^-) \longrightarrow L({}^c\mathbf{U}), \quad (x, y) \longmapsto x^{-1}y\sigma_b(x)$$

is an isomorphism. The injectivity of α_b holds for arbitrary elements $c \in W$ which are of minimal length in an elliptic σ-conjugacy class (cf. [HL12]).

Furthermore, α_b restricts to an isomorphism

$$\alpha_{b,\mathbf{x}} \colon L^+({}^c\mathbf{U} \cap \mathbf{U})_{\mathbf{x}} \times L^+({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}} \longrightarrow L^+({}^c\mathbf{U})_{\mathbf{x}}.$$

Remark 5.4. It looks very plausible that Proposition 5.3 holds for arbitrary Coxeter elements c. It should be possible to prove this by adapting [HL12, Section 3.5, Theorem 3.6 and Section 3.14] to the present setting. Let us make the relation of Proposition 5.3 with [HL12] more clear. With obvious notation (in particular, c is Coxeter and b any lift of it), [HL12, Theorem 3.6] says that $\mathbf{U} \times (\mathbf{U} \cap {}^b\mathbf{U}^-) \to \mathbf{U}b\mathbf{U}$, $(u, z) \mapsto uz\sigma(u)^{-1}$ is an isomorphism if $\mathbf{G}$ is reductive over a finite field with Frobenius morphism σ (and, in fact, in various other setups). As an easy corollary of this, [HL12, Section 3.14] states that $\alpha'_b \colon (\mathbf{U} \cap {}^{\sigma^{-1}(c)^{-1}}\mathbf{U}) \times (\mathbf{U} \cap {}^c\mathbf{U}^-) \to \mathbf{U}$, $(x, y) \mapsto xy\sigma_b(x)^{-1}$ is an isomorphism in the same setup. Now the finite field version (without the loop functors) of the map α_b of Proposition 5.3 is obtained from α'_b by a series of changes of variables (cf. the first paragraph of the proof of Proposition 5.3), so that α_b is an isomorphism if and only if α'_b is.

Proof of Proposition 5.3. We work with the pro-étale topology on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and regard α_b as a map of pro-étale sheaves. The injectivity of α_b may be derived from [HL12]. First, change the variable by setting $y = \sigma_b(y_1)$. It remains to show that $L(\mathbf{U} \cap {}^c\mathbf{U}) \times L(\mathbf{U} \cap {}^{\sigma^{-1}(c^{-1})}\mathbf{U}^-) \to L({}^c\mathbf{U})$, $x, y_1 \mapsto x^{-1}\sigma_b(y_1 x)$ is injective. Compose this map with the isomorphism $(\cdot)^{-1} \circ \sigma^{-1} \circ \mathrm{Ad}(b) \colon L({}^c\mathbf{U}) \to L\mathbf{U}$. It remains to show that this composition, given by $x, y_1 \mapsto x^{-1}y_1^{-1}\sigma^{-1}(b^{-1}xb)$, is injective. Finally, change the variables by putting $x = x_2^{-1}$ and $y_1 = y_2^{-1}$, and set $\tilde{b} := \sigma^{-1}(b)^{-1}$ with image $\tilde{c}$ in W. We are reduced to showing that $L(\mathbf{U} \cap {}^{\sigma(\tilde{c})^{-1}}\mathbf{U}) \times L(\mathbf{U} \cap {}^{\tilde{c}}\mathbf{U}^-) \to L\mathbf{U}$, $x_2, y_2 \mapsto x_2 y_2 \tilde{b}\sigma^{-1}(x_2)^{-1}\tilde{b}^{-1}$ is injective. This follows from [HL12, Section 3.14(b)] applied to $A = \mathbb{W}(R)[1/\varpi]$ (for any $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$) and $\chi = \phi_R^{-1}$.

We now show the surjectivity of α_b. We may assume that $\mathbf{G}$ is of adjoint type. Then $\mathbf{G} = \prod_{i=1}^r \mathbf{G}_i$ with $\mathbf{G}_i$ almost simple k-groups, and we are immediately reduced to the case that $\mathbf{G}$ is almost simple over k.

Lemma 5.5. *Let k_d/k be an unramified extension of degree $d \geq 1$, and suppose that $\mathbf{G} = \mathrm{Res}_{k_d/k} \mathbf{G}'$ for a k_d-group $\mathbf{G}'$. If the surjectivity claim for α_b of Proposition 5.3 holds for $\mathbf{G}'$, then it holds for $\mathbf{G}$.*

Proof. Identify $\mathbf{G}_{\check{k}} = \prod_{i=1}^d \mathbf{G}'_{\check{k}}$ and similarly for the Weyl groups W, W' and unipotent radicals $\mathbf{U}, \mathbf{U}'$ of Borels of $\mathbf{G}, \mathbf{G}'$, so that the Frobenius σ permutes cyclically the components. As c is special, $c = (c', 1, \ldots, 1)$ for a special Coxeter element $c' \in W'$. We then have ${}^c\mathbf{U} \cap \mathbf{U} = ({}^{c'}\mathbf{U}' \cap \mathbf{U}') \times \mathbf{U}' \times \cdots \times \mathbf{U}'$ and ${}^c\mathbf{U} \cap \mathbf{U}^- = ({}^{c'}\mathbf{U}' \cap \mathbf{U}'^-) \times 1 \times \cdots \times 1$. The lift b of c is then given by $(b', \tau_2, \ldots, \tau_d)$ for a lift b' of c' to $\mathbf{G}'(\check{k})$ and some $\tau_i \in \mathbf{T}'(\check{k})$. Then the map α_b from Proposition 5.3 is given by

$$(x_i)_{i=1}^d, y_1 \longmapsto (x_1^{-1}y_1 \mathrm{Ad}(b')(\sigma'(x_d)), x_2^{-1}\mathrm{Ad}(\tau_2)(x_1), \ldots x_d^{-1}\mathrm{Ad}(\tau_2)(x_{d-1})).$$

From this, surjectivity of α_b for $\mathbf{G}$ immediately reduces to surjectivity of $\alpha_{b'}$ for $\mathbf{G}'$. □

Lemma 5.6. *If $\widetilde{\alpha}_b \colon L(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U}) \times L(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U}^-) \to L\mathbf{U}$ denotes the map $\widetilde{\alpha}_b(x,y) = x^{-1}y\sigma(bxb^{-1})$, then* $\mathrm{Ad}(b^{-1})(\alpha_b(x,y)) = \widetilde{\alpha}_b(\mathrm{Ad}(b^{-1})(x), \mathrm{Ad}(b^{-1})(y))$.

Proof. This follows immediately from a computation. $\qquad\square$

As $\mathbf{G}$ splits over $\breve{k}$, Lemma 5.5 reduces us to the case that $\mathbf{G}$ is absolutely almost simple over k. It suffices to show that the map $\widetilde{\alpha}_b$ from Lemma 5.6 is surjective. Write $A = L(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U})$, $B = L(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U}^-)$, $C = L\mathbf{U}$. For $\star \in \{A, B, C\}$, write $\Phi_\star = \{\alpha \in \Phi \colon L\mathbf{U}_\alpha \subseteq \star\}$, where $\mathbf{U}_\alpha \subseteq \mathbf{G}$ denotes the root subgroup corresponding to α. We call a subset $\Psi \subseteq \Phi$ *closed under addition* if for any $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Psi$.

Lemma 5.7. *There exist a positive integer r and subsets*

$$\Phi_C = \Psi_1 \supseteq \Psi_2 \supseteq \cdots \supseteq \Psi_r = \Phi_B$$

satisfying the following conditions:
   (i) *For all $1 \le i \le r$, Ψ_i and $\Psi_i \setminus \Psi_r$ are closed under addition.*
   (ii) *For all $1 \le i < r$ and all $\alpha, \beta \in \Psi_i$, the implication $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Psi_{i+1}$ holds.*
   (iii) *For all $1 \le i \le r$, $\sigma(c(\Psi_i \setminus \Psi_r)) \subseteq \Psi_i$.*

Proof. We give the desired partitions in Section 7.1.4 (type A_{n-1}), Section 7.2.4 (types B_m, C_m), Section 7.4.9 (type D_m), Section 7.5.7 (type ${}^2A_{n-1}$) and Section 7.6.4 (type ${}^2D_m$) after the relevant notation is introduced. In each case, it is straightforward to check that the conditions of the lemma hold for the given partition; we omit this checking. $\qquad\square$

Lemma 5.8. *Let I be a finite set not containing 0. Let $\lambda \colon I \to I \cup \{0\}$ be a map such that $\lambda^{-1}(i)$ has at most one element if $i \in I$. For any $i \in I \cap \lambda(I)$, let $\varepsilon_i \in \breve{k}^\times$. Consider the map $\delta \colon \prod_{i \in I} L\mathbb{G}_a \to \prod_{i \in I} L\mathbb{G}_a$ given by $\delta((a_i)_{i \in I}) = (b_i)_{i \in I}$, where $b_i = 0$ if $\lambda^{-1}(i) = \varnothing$ and $b_i = \varepsilon_i \phi(a_{\lambda^{-1}(i)})$ otherwise. Then the map $x \mapsto x - \delta(x)$ is surjective for the pro-étale topology.*

Proof. There are some integers $s \ge r \ge 0$ and a disjoint decomposition $I = \bigsqcup_{i=1}^r I_i \,\dot\cup\, \bigsqcup_{i=r+1}^s I_i$ such that for $1 \le i \le r$, λ restricts to a map $I_i \to I_i$ which is a full cycle of length $\#I_i$ (first type), and for $r+1 \le i \le s$, λ restricts to a map $I_i \to I_i \dot\cup \{0\}$, which is, after choosing an isomorphism $I_i \cong \{1, \ldots, m\}$, given by $\lambda(j) = j - 1$ for all $j \in I_i$ (second type). It suffices to show that the restriction of δ to $\prod_{j \in I_i} L\mathbb{G}_a$ is surjective. Thus we may assume that I is one of the I_i. If $I = \{1, \ldots, m\}$ is of the second type, one has the filtration $G^\bullet$ of $G = \prod_{i=1}^m L\mathbb{G}_a$ by closed subschemes (ind-group schemes) $G_j = \prod_{i=1}^j L\mathbb{G}_a$, stable by δ, such that $x \mapsto x - \delta(x)$ induces the identity on the associated graded object $\mathrm{gr}\, G^\bullet$. From this, the result easily follows. If I is of the first type, we may assume $I = \mathbb{Z}/m\mathbb{Z}$ and $\lambda(i) = i + 1$. Replacing ϕ with ϕ^m, one is then immediately reduced to the case that $m = 1$ and $\delta \colon L\mathbb{G}_a \to L\mathbb{G}_a$, $\delta(x) = \varepsilon\phi(x)$ for some $\varepsilon \in \breve{k}^\times$. Then Lemma 5.9 finishes the proof. $\qquad\square$

Lemma 5.9. *Let $\varepsilon \in \breve{k}$. Then the map $\mathrm{id} - \varepsilon\phi \colon L\mathbb{G}_a \to L\mathbb{G}_a$, $x \mapsto x - \varepsilon\phi(x)$ is surjective for the pro-étale topology.*

Proof. If $\varepsilon \notin \mathcal{O}_{\breve{k}}^\times$, then $\mathrm{id} - \varepsilon\phi$ is an isomorphism. Indeed, if $\mathrm{ord}_\varpi(\varepsilon) > 0$, $\sum_{i=0}^\infty (\varepsilon\phi)^i$ is a meaningful endomorphism of $L\mathbb{G}_a$ (as $W(R)$ is ϖ-adically complete for each $R \in \mathrm{Perf}$) and the inverse of $\mathrm{id} - \varepsilon\phi$. If $\mathrm{ord}_\varpi(\varepsilon) < 0$, a similar argument goes through using $\mathrm{id} - \varepsilon\phi = -\varepsilon\phi(\mathrm{id} - (\varepsilon\phi)^{-1})$. Now assume $\varepsilon \in \mathcal{O}_{\breve{k}}^\times$. We have $L\mathbb{G}_a = \varinjlim_{m \to -\infty} \varpi^m L^+\mathbb{G}_a$, and it suffices to show that $\mathrm{id} - \varepsilon\phi$ is surjective on $L^+\mathbb{G}_a = \varprojlim_{r \to \infty} L_r^+\mathbb{G}_a$. This follows from a pro-version of Lang's theorem. $\qquad\square$

We now prove the surjectivity of α_b. For a subset $\Psi \subseteq \Phi^+$ closed under addition, let $\mathbf{U}_\Psi$ denote the unique closed subgroup of $\mathbf{U}$ whose $\breve{k}$-points are generated by $\{\mathbf{U}_\alpha(\breve{k})\}_{\alpha \in \Psi}$. As a $\breve{k}$-scheme, $\mathbf{U}_\Psi$ is isomorphic to $\prod_{\alpha \in \Psi} \mathbf{U}_\alpha$. Write $U_\Psi = L\mathbf{U}_\Psi$. Let $\{\Psi_i\}_{i=1}^r$ be a filtration as in Lemma 5.7. For $1 \le i \le r$, we have the closed subschemes (ind-group schemes) $C_i := U_{\Psi_i}$ of $L\mathbf{U}$, with $C_1 = C$, $C_r = B$. Moreover, condition (ii) from

Lemma 5.7 guarantees that $C_{i+1} \subseteq C_i$ is a closed normal subgroup, with abelian quotient. As $H^1(k, \mathbf{H}) = 0$ for a split unipotent k-group $\mathbf{H}$, we have $C_i/C_{i+1} \cong \prod_{\alpha \in \Psi_i \setminus \Psi_{i+1}} L\mathbf{U}_\alpha$. Moreover, by condition (i), for $1 \le i \le r-1$, C_i has the closed subgroup $A_i = U_{\Psi_i \setminus \Psi_r}$, and condition (ii) implies that A_{i+1} is normal in A_i and that the inclusion $A_i \hookrightarrow C_i$ induces an isomorphism $\gamma_i : A_i/A_{i+1} \xrightarrow{\sim} C_i/C_{i+1}$. Moreover, by condition (iii), the map $\sigma \circ \mathrm{Ad}(b) : A_i \to C_i$ induces a map $\delta_i : A_i/A_{i+1} \to C_i/C_{i+1}$. The composition $\gamma_i^{-1} \delta_i$ is an endomorphism of $A_i/A_{i+1} \cong \prod_{\alpha \in \Psi_i \setminus \Psi_{i+1}} L\mathbf{U}_\alpha$, which is of the form given in Lemma 5.8. From Lemma 5.8 it follows that $a \mapsto a \cdot (\gamma_i^{-1} \delta_i)(a)^{-1}$ is a surjective endomorphism of A_i/A_{i+1}.

Let $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Let $x_i \in C_i(R)$, and let $\bar{x}_i$ be its image in $(C_i/C_{i+1})(R)$. By the above, after replacing R with a pro-étale cover, we may find some $a_i \in A_i(R)$ with image $\bar{a}_i \in (A_i/A_{i+1})(R)$ such that $\bar{a}_i \cdot (\gamma_i^{-1} \delta_i)(\bar{a}_i^{-1}) = \gamma_i^{-1}(\bar{x}_i)$. Applying γ_i, we deduce the equation $\gamma_i(\bar{a}_i)^{-1} \bar{x}_i \delta_i(\bar{a}_i) = 1$ in $(C_i/C_{i+1})(R)$. The left-hand side is precisely the image of $\widetilde{\alpha}_b(a_i, x_i) \in C_i(R)$ in $(C_i/C_{i+1})(R)$. In other words, for each $x_i \in C_i(R)$, we may find (after possibly enlarging R) some $a_i \in A_i$ such that $\widetilde{\alpha}_b(a_i, x_i) \in C_{i+1}$. Starting with an arbitrary $x = x_1 \in C(R)$ and applying this procedure consequently for $i = 1, 2, \ldots, r-1$, we deduce that after possibly enlarging R, there are some $a_1, \ldots, a_{r-1} \in A(R)$ such that for $a = a_{r-1} a_{r-2} \ldots a_1$, we have $\widetilde{\alpha}_b(a, x) \in C_r(R) = B(R)$. (Here we use that $\widetilde{\alpha}_b(b, \widetilde{\alpha}_b(a, x)) = \widetilde{\alpha}_b(ab, x)$.) This finishes the proof that α_b is an isomorphism.

Next we turn to $\alpha_{b,\mathbf{x}}$. Note that it is well defined as for $x \in L^+(^c\mathbf{U} \cap \mathbf{U})_{\mathbf{x}}$, $\sigma_b(x) \in L(^c\mathbf{U}) \cap L^+\mathcal{G}_{\mathbf{x}} = L^+(^c\mathbf{U}_{\mathbf{x}})$, the intersection taken in $L\mathbf{G}$. Being the restriction of α_b, it is injective, as α_b is, and it remains to prove its surjectivity. Applying $\mathrm{Ad}(b^{-1})$, which conjugates $\mathcal{G}_{\mathbf{x}}$ to $\mathcal{G}_{\mathbf{y}}$, where $\mathbf{y} = b^{-1}(\mathbf{x})$, and the analogue of Lemma 5.6, it suffices to show that (with the obvious notation) the map $\widetilde{\alpha}_{b,\mathbf{x}} : L^+(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U})_{\mathbf{y}} \times L^+(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U}^-)_{\mathbf{y}} \to L^+\mathbf{U}_{\mathbf{y}}$, $\widetilde{\alpha}_b(x, y) = x^{-1} y \sigma(bxb^{-1})$ is surjective. By the same reasoning as in Lemma 5.5 (and the paragraph preceding it), we are reduced to the case that $\mathbf{G}$ is absolutely almost simple over k. For $\Psi \subseteq \Phi^+$, let $\mathbf{U}_{\Psi,\mathbf{y}}$ denote the closure of $\mathbf{U}_\Psi$ in $\mathcal{G}_{\mathbf{y}}$. We have $\mathbf{U}_{\Psi,\mathbf{y}} \cong \prod_{\alpha \in \Psi} \mathbf{U}_{\alpha,\mathbf{y}}$. Replacing C_i, B, A_i above with $C_i^+ = L^+\mathbf{U}_{\Psi_i,\mathbf{y}}$, $B^+ = L^+(\mathbf{U} \cap {}^{c^{-1}}\mathbf{U}^-)$, $A_i^+ = L^+\mathbf{U}_{\Psi_i \setminus \Psi_r,\mathbf{y}}$ and using the integral version of Lemma 5.8 (with essentially the same proof), we can run the above inductive argument for the quotients A_i^+/A_{i+1}^+, deducing the surjectivity of $\alpha_{b,\mathbf{x}}$. $\qquad\square$

5.2. A variant of a Deligne–Lusztig space

Throughout this section we work with the arc-topology on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Steinberg's cross section allows us to give a different presentation of the Deligne–Lusztig spaces $X_c(b)$, $\dot{X}_{\dot{c}}(b)$. We have the following general definition.

Definition 5.10. Let $w \in W$, $\dot{w} \in F_w(\check{k})$ and $b \in \mathbf{G}(\check{k})$. Define $\dot{X}_{\dot{w},b}$ and $\tilde{X}_{w,b}$ by the following cartesian diagrams:

$$
\begin{array}{ccc}
\dot{X}_{\dot{w},b} & \longrightarrow & L(^w\mathbf{U} \cap \mathbf{U}^-) \cdot \dot{w}b^{-1} \\
\cap\downarrow & & \cap\downarrow \\
L\mathbf{G} & \xrightarrow{g \mapsto g^{-1}\sigma_b(g)} & L\mathbf{G}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\tilde{X}_{w,b} & \longrightarrow & L(^w\mathbf{U} \cap \mathbf{U}^-) \cdot L\mathbf{T} \\
\cap\downarrow & & \cap\downarrow \\
L\mathbf{G} & \xrightarrow{g \mapsto g^{-1}\sigma_b(g)} & L\mathbf{G},
\end{array}
$$

and let $X_{w,b} := \tilde{X}_{w,b}/L\mathbf{T}$.

Remark 5.11. The map $g \mapsto g^{-1}\sigma_b(g) : L\mathbf{G} \to L\mathbf{G}$ factors through $\kappa_{\mathbf{G}}^{-1}(\kappa_{\mathbf{G}}(b)) \subseteq L\mathbf{G}$. Thus the map $\tilde{X}_{c,b} \to (^c\mathbf{U} \cap \mathbf{U}^-) \cdot L\mathbf{T}$ factors through $(^c\mathbf{U} \cap \mathbf{U}^-) \cdot \kappa_{\mathbf{G}}|_{L\mathbf{T}}^{-1}(\kappa_{\mathbf{G}}(b))$. So, in the upper right entry of the right diagram of Definition 5.10, we could have written $(^c\mathbf{U} \cap \mathbf{U}^-) \cdot \kappa_{\mathbf{G}}|_{L\mathbf{T}}^{-1}(\kappa_{\mathbf{G}}(b))$ without changing the definition.

The group $\mathbf{G}_b(k) \times \mathbf{T}_w(k)$ acts on $\dot{X}_{\dot{w},b}$ by $g, t : x \mapsto gxt$, and $\mathbf{G}_b(k)$ acts on $X_{w,b}$ by $g : x \mapsto gx$. We only use these spaces in the case that $w = c$ is a (special) Coxeter element.

Proposition 5.12. *Let $b \in \mathbf{G}(\check{k})$. Let $c \in W$ be a Coxeter element and $\dot{c} \in F_c(\check{k})$. Whenever the conclusion of Proposition 5.3 holds (e.g., $\mathbf{G}$ classical, c special Coxeter), the following hold:*

(i) *There is a $\mathbf{G}_b(k) \times \mathbf{T}_c(k)$-equivariant isomorphism $\dot{X}_{\dot{c}}(b) \cong \dot{X}_{\dot{c},b}$.*

(ii) *If $b \in F_c(\check{k})$, then there is a $\mathbf{G}_b(k)$-equivariant isomorphism $X_c(b) \cong X_{c,b}$.*

Proof. (i) Consider the sheafification $\dot{X}'$ of the functor on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$X_1 \colon R \longmapsto \left\{ g \in L\mathbf{G}(R) \colon g^{-1}\sigma_b(g) \in L({}^c\mathbf{U})(R)\dot{c}b^{-1} = \dot{c}L\mathbf{U}(R)b^{-1} \right\} / L(\mathbf{U} \cap {}^c\mathbf{U})(R).$$

By [Iva23, Proposition 12.1] $\dot{X}_{\dot{c}}(b)$ and $\dot{X}'$ are equivariantly isomorphic. It remains to show that $\dot{X}' \cong \dot{X}_{\dot{c},b}$ equivariantly. Note that $\dot{X}_{\dot{c},b}$ is the sheafification of

$$X_2 \colon R \longmapsto \left\{ g \in L\mathbf{G}(R) \colon g^{-1}\sigma_b(g) \in L({}^c\mathbf{U} \cap \mathbf{U}^-)(R) \cdot \dot{c}b^{-1} \right\}.$$

There is an obvious equivariant map $X_2 \to X_1$, of which we claim that it is an isomorphism. This follows once we check that

$$L({}^c\mathbf{U} \cap \mathbf{U}) \times L({}^c\mathbf{U} \cap \mathbf{U}^-) \cdot \dot{c}b^{-1} \longrightarrow \dot{c}L\mathbf{U}b^{-1} = L({}^c\mathbf{U}) \cdot \dot{c}b^{-1}, \quad x, y' \longmapsto x^{-1}y'\sigma_b(x)$$

is an isomorphism. Substituting $y' = y\dot{c}b^{-1}$, with y ranging in $L({}^c\mathbf{U} \cap \mathbf{U}^-)$, this map becomes $(x, y) \mapsto \left(x^{-1}y\dot{c}\sigma(x)\dot{c}^{-1} \right) \cdot \dot{c}b^{-1}$, which is an isomorphism by Proposition 5.3. The proof of (ii) is completely analogous. $\square$

Proposition 5.13. *Suppose $\mathbf{G}$ is of adjoint type. Let $c \in W$ be as in Proposition 5.12 and $b \in F_c(\check{k})$. For any $\dot{c} \in F_c(\check{k})$ satisfying $\kappa_{\mathbf{G}}(b) = \kappa_{\mathbf{G}}(\dot{c})$, there are $\mathbf{G}_b(k)$-equivariant isomorphisms $X_c(b) \cong X_{c,b} \cong \dot{X}_{\dot{c},b}/\mathbf{T}_c(k)$.*

Proof. By Proposition 5.12, it suffices to show that $\dot{X}_{\dot{c},b}/\mathbf{T}_c(k) \cong X_{c,b}$. Writing

$$\tau := \dot{c}^{-1}b \in (L\mathbf{T} \cap \kappa_{\mathbf{G}}^{-1}(\kappa_{\mathbf{G}}(b)))(\overline{\mathbb{F}}_q),$$

these sheaves are the sheafifications of the quotient presheaves

$$Y_1 \colon R \longmapsto \{ g \in L\mathbf{G}(R) \colon g^{-1}\sigma_b(g) \in L(\mathbf{U}^- \cap {}^c\mathbf{U})(R) \cdot \tau \} / \mathbf{T}_c(k)(R) \qquad \text{and}$$

$$Y_2 \colon R \longmapsto \{ g \in L\mathbf{G}(R) \colon g^{-1}\sigma_b(g) \in L(\mathbf{U}^- \cap {}^c\mathbf{U})(R) \cdot L\mathbf{T}(R) \} / L\mathbf{T}(R),$$

respectively. We have an obvious map $Y_1 \to Y_2$, and we claim its sheafification is an isomorphism. First, the map between presheaves itself is injective (which already implies that its sheafification is injective, as quotient presheaves are separated). Indeed, let $g, h \in Y_1(R)$. Then $\sigma_b(g) = gu_g\tau$, $\sigma_b(h) = hu_h\tau$ for some $u_g, u_h \in L(\mathbf{U}^- \cap {}^c\mathbf{U})(R)$. Suppose their images in $Y_2(R)$ agree, *i.e.*, there is some $t \in L\mathbf{T}(R)$ such that $gt = h$. Then $\sigma_b(g)\sigma_b(t) = \sigma_b(h)$, and hence $gu_g\tau\sigma_b(t) = hu_h\tau$; *i.e.*,

$$\sigma_c(t) = \sigma_b(t) = \tau^{-1}u_g^{-1}tu_h\tau$$

(the first equality holds as $b \in F_c(\check{k})$). Thus, $\sigma_c(t) = tu$ for some $u \in L(\mathbf{U}^- \cap {}^c\mathbf{U})(R)$, which implies $u = 1$, and hence $t \in (L\mathbf{T})^{\sigma_c}(R) = \mathbf{T}_c(k)(R)$.

Secondly, let $g \in L\mathbf{G}(R)$ satisfy $\sigma_b(g) = gu_g\tau'$ for some $u \in ({}^c\mathbf{U} \cap \mathbf{U}^-)(R)$ and $\tau' \in L\mathbf{T}(R)$. By Remark 5.11, $\kappa_{\mathbf{G}}(\tau') = \kappa_{\mathbf{G}}(b)$. Changing g to gt for some $t \in L\mathbf{T}(R)$ has no effect on the image of g in $Y_2(R)$ but amounts to replacing τ' with $\tau't^{-1}\sigma(t)$. Thus, to show that $Y_1 \to Y_2$ is surjective when passing to sheafification, it remains to check that the σ_c-twisted action morphism

$$L\mathbf{T} \times (L\mathbf{T} \cap \kappa_{\mathbf{G}}^{-1}(\kappa_{\mathbf{G}}(b))) \longrightarrow L\mathbf{T} \cap \kappa_{\mathbf{G}}^{-1}(\kappa_{\mathbf{G}}(b)), \quad t, \tau' \longmapsto t^{-1}\tau'\sigma_c(t)$$

is surjective for the arc-topology. In fact, it is even surjective for the pro-étale topology. Indeed, the space $L\mathbf{T} \cap \kappa_{\mathbf{G}}^{-1}(\kappa_{\mathbf{G}}(b))$ is fibered over $(\bar{\kappa}_{\mathbf{G}}|_{X_*(\mathbf{T})})^{-1}(\kappa_{\mathbf{G}}(b))$, all fibers being isomorphic to $L^+\mathcal{T}$. Using pro-Lang for the σ_c-stable connected subscheme $L^+\mathcal{T} \subseteq L\mathbf{T}$, we are reduced to showing that the map of discrete abelian groups $X_*(\mathbf{T}) \times (\bar{\kappa}_{\mathbf{G}}|_{X_*(\mathbf{T})})^{-1}(\kappa_{\mathbf{G}}(b)) \to (\bar{\kappa}_{\mathbf{G}}|_{X_*(\mathbf{T})})^{-1}(\bar{\kappa}_{\mathbf{G}}(b))$ given by $\mu, \chi \mapsto \chi + \sigma_c(\mu) - \mu$ is surjective, *i.e.*, that the fibers of $\bar{\kappa}_{\mathbf{G}}|_{X_*(\mathbf{T})}$ are torsors under $\mathrm{im}(\sigma_c - \mathrm{id} \colon X_*(\mathbf{T}) \to X_*(\mathbf{T})) = \ker \bar{\kappa}_c$. This holds by Remark 2.5 as $\mathbf{G}$ is adjoint and c Coxeter. $\square$

6. Decomposition into integral p-adic spaces

In this section we begin with the proof of Theorem 1.1. Before that, we show that this theorem is indeed sufficiently general to describe *all* spaces $X_c(b)$ and $\dot{X}_{\bar{c}}(b)$ for an unramified classical group $\mathbf{G}$, c Coxeter and b basic.

Proposition 6.1. *Let $b \in \mathbf{G}(\check{k})$ be basic, $c \in W$ Coxeter and $\bar{c} \in \overline{F}_c / \ker \bar{\kappa}_c$. Suppose $\dot{X}_{\bar{c}}(b) \neq \varnothing$. Then there exist*
- *a special Coxeter element $c_1 \in W$ (cf. Definition 5.2),*
- *a lift $\bar{c}_1 \in \overline{F}_{c_1} / \ker \bar{\kappa}_{c_1}$ and*
- *an element $b_1 \in [b] \cap F_c(\check{k})$ which lies over $\bar{c}_1$*

such that $X_c(b) \cong X_{c_1}(b_1)$ and $\dot{X}_{\bar{c}}(b) \cong \dot{X}_{\bar{c}_1}(b_1)$, $\mathbf{G}_b(k)$- (resp. $\mathbf{G}_b(k) \times \mathbf{T}_c(k)$-)equivariantly.

Proof. By [Iva23, Corollary 8.18] there exists a special Coxeter element c_1 such that $X_c(b) \cong X_{c_1}(b)$ equivariantly. Moreover, we may find some $\bar{c}_1$ over c_1 such that $\dot{X}_{\bar{c}}(b)$ is equivariantly isomorphic to $\dot{X}_{\bar{c}_1}(b)$; cf. [Iva23, Lemma 8.16]. By Corollary 4.6 we may pick $b_1 \in [b] \cap F_{c_1}(\check{k})$ lying over $\bar{c}_1$. $\qquad\square$

The rest of the article is devoted to the proof of Theorem 1.1. Thus we start with
- a special Coxeter element $c \in W$ and $\bar{c} \in \overline{F}_c / \ker \bar{\kappa}_c$,
- a basic element b which lies in $F_c(\check{k})$ and lifts $\bar{c}$.

In Section 6.1 we reduce (1.3) to the case that $\mathbf{G}$ is absolutely almost simple and adjoint. In Section 6.2 we then prove (1.3) by a v-descent argument, assuming the technical Proposition 6.3 (which essentially is (1.4) for absolutely almost simple adjoint groups, cf. Corollary 6.4). Then it remains to deduce (1.4) from (1.3). This is done in Section 6.3. Finally, in Section 7 we prove the remaining Proposition 6.3, which is the most technical part of the proof. We also point out that specialness of c is only used in the proof of Proposition 6.3 and nowhere else in the proof of Theorem 1.1.

6.1. Reduction of (1.3) to the adjoint absolutely almost simple case

First, note that both sides of the equality claimed in the theorem only depend on $\mathbf{G}^{\mathrm{ad}}$ (for the left-hand side, use that $\mathbf{G}/\mathbf{B} = \mathbf{G}^{\mathrm{ad}}/\mathbf{B}^{\mathrm{ad}}$). Hence we may assume that $\mathbf{G}$ is of adjoint type. Then $\mathbf{G}$ decomposes as a direct product of almost simple k-groups, and both sides of the claimed isomorphism (1.3) do so accordingly. Thus we are reduced to the case that $\mathbf{G}$ is of adjoint type and almost simple.

If $\mathbf{G}$ is adjoint and almost simple, then $\mathbf{G} = \mathrm{Res}_{k'/k}\mathbf{G}'$ for some finite unramified extension k'/k and some absolutely almost simple k'-group $\mathbf{G}'$. As $B(\mathbf{G}) = B(\mathbf{G}')$, cf. [Kot85, 1.10], (which allows us to change b appropriately inside its σ-conjugacy class), Lemma 6.2 reduces us to the case that $\mathbf{G}$ is adjoint absolutely almost simple.

We state and prove Lemma 6.2 in bigger generality than just for adjoint absolutely almost simple groups and Coxeter elements. Let us fix notation first. Let k'/k be a finite unramified extension of degree d, let $\mathbf{H}'$ be an unramified reductive group over k', with maximal quasi-split torus $\mathbf{T}'$. Let $\mathbf{H} = \mathrm{Res}_{k'/k}\mathbf{H}'$. Let W' and $W \cong \prod_{i=1}^{d} W'$ be the Weyl groups of $(\mathbf{H}', \mathbf{T}')$ and $(\mathbf{H}, \mathrm{Res}_{k'/k}\mathbf{T})$. Identifying the buildings of $\mathbf{H}$ over k and of $\mathbf{H}'$ over k', any point $\mathbf{x} \in \mathcal{A}_{\mathbf{T},\check{k}}$ defines a parahoric model $\mathcal{H}'_{\mathbf{x}}$ of $\mathbf{H}'$, and we have $\mathcal{H}_{\mathbf{x}} = \mathrm{Res}_{\mathcal{O}_{k'}/\mathcal{O}_k}\mathcal{H}'_{\mathbf{x}}$; cf. [HR20, Proposition 4.7]. Note that for definition (1.1) to work, we only need that the affine transformation $b\sigma$ fixes $\mathbf{x}$ (assumptions on c to be Coxeter and on b to be basic are redundant), so part (ii) of the following lemma makes sense.

Lemma 6.2. *Let $w' \in W'$, $b' \in \mathbf{H}'(\check{k})$, and let $w = (w', 1, \ldots, 1) \in W$, $b = (b', 1, \ldots, 1) \in \mathbf{H}(\check{k})$.*
- (i) *We have $X_w^{\mathbf{H}}(b) \cong X_{w'}^{\mathbf{H}'}(b')$.*
- (ii) *If $b' \in F_{w'}(\check{k})$ is such that $b'\sigma'$ fixes $\mathbf{x}$, then $\dot{X}_{\bar{w},b}^{\mathcal{G}_x} \cong \dot{X}_{\bar{w}',b'}^{\mathcal{G}'_x}$ and $X_{w,b}^{\mathcal{G}_x} \cong X_{w',b'}^{\mathcal{G}'_x}$, where $\bar{w}$ (resp. $\bar{w}'$) is the image of b (resp. b') in $\overline{F}_w / \ker \bar{\kappa}_w$ (resp. $\overline{F}_{w'} / \ker \bar{\kappa}_{w'}$),*

These natural isomorphisms are compatible with the natural identifications $\mathbf{H}(k) = \mathbf{H}'(k')$ and $\mathcal{H}(\mathcal{O}_k) = \mathcal{H}'(\mathcal{O}_{k'})$.

Proof. (i) Let $\mathbf{B}'$ be a k'-rational Borel subgroup of $\mathbf{H}'$ which contains $\mathbf{T}'$, and let $\mathbf{B} = \mathrm{Res}_{k'/k} \mathbf{B}'$ be the corresponding Borel subgroup of $\mathbf{H}$. As $\breve{k}$-groups, we have $\mathbf{H} \cong \prod_{i=1}^d \mathbf{H}'$, with Frobenius permuting the components cyclically. As $L(\cdot)$ commutes with products, we have $L(\mathbf{H}/\mathbf{B}) = \prod_{i=1}^d L(\mathbf{H}'/\mathbf{B}')$. Let $g = (g_1, \ldots, g_d) \in L(\mathbf{G}/\mathbf{B})$. Then $g \in X_w(b)$ if and only if the relative position of $(g_1, \ldots, g_d)$ and $(\sigma'(g_d), g_1, \ldots, g_{d-1})$ is $(w', 1, \ldots, 1)$, where σ' is the geometric Frobenius on $L(\mathbf{G}'/\mathbf{B}')$. This hold if and only if $g_1 = g_2 = \cdots = g_d$ and $g_1 \in X_{w'}^{\mathbf{H}'}(b)$. Thus $g \mapsto g_1$ defines the claimed isomorphism.

(ii) Just as above, we have $\mathcal{H}_{\mathbf{x}} \cong \prod_{i=1}^d \mathcal{H}'_{\mathbf{x}}$. The same holds after applying $L^+(\cdot)$. Moreover, ${}^w\mathbf{U} \cap \mathbf{U}^- = ({}^{w'}\mathbf{U}' \cap \mathbf{U}'^-, 1, \ldots, 1)$. Thus, $g = (g_1, \ldots, g_d) \in L^+\mathcal{H}_{\mathbf{x}}$ lies in $\dot{X}_{w,b}^{\mathcal{H}_{\mathbf{x}}}$ if and only if

$$g^{-1} \sigma_b(g) = (g_1^{-1} \mathrm{Ad}(b')(\sigma'(g_d)), g_2^{-1} g_1, \ldots, g_d^{-1} g_{d-1})) \in (L^+({}^{w'}\mathbf{U}' \cap \mathbf{U}'^-), 1, \ldots, 1),$$

which is the case if and only if $g_1 \in \dot{X}_{w,b}^{\mathcal{H}'_{\mathbf{x}}}$ and the g_i $(i = 2, \ldots, d)$ are determined by appropriate formulas in terms of g_1. Thus $g \mapsto g_1$ defines the required isomorphism. The last isomorphism follows from the first. $\qquad\square$

Altogether, in the proof of (1.3) we may assume $\mathbf{G}$ to be absolutely almost simple and adjoint.

6.2. Disjoint decomposition

Here we reduce (1.3) to Proposition 6.3. Using Proposition 5.12 we may replace the left-hand side in (1.3), resp. (1.4), by $X_{c,b}$, resp. $\dot{X}_{b,b}$ (as b lies over $\bar{c}$). Consider the following (auxiliary) subsheaf of $\dot{X}_{b,b}$:

$$
\begin{array}{ccc}
\dot{X}_{b,b,\mathcal{O}} & \longrightarrow & L^+({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}} \\
\cap\downarrow & & \cap\downarrow \\
L\mathbf{G} & \xrightarrow{g \mapsto g^{-1}\sigma_b(g)} & L\mathbf{G}.
\end{array}
$$

We then have the following result, whose proof occupies Section 7.

Proposition 6.3. *Suppose $\mathbf{G}$ is absolutely almost simple and adjoint. We have $\dot{X}_{b,b} = \dot{X}_{b,b,\mathcal{O}}$.*

Corollary 6.4. *Suppose $\mathbf{G}$ is absolutely almost simple and adjoint. We have a $\mathbf{G}_b(k) \times \mathbf{T}_c(k)$-equivariant isomorphism*

$$\dot{X}_{b,b} = \bigsqcup_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \dot{X}_{\bar{c},b}^{\mathcal{G}_{\mathbf{x}}},$$

where $\dot{X}_{\bar{c},b}^{\mathcal{G}_{\mathbf{x}}}$ is as in (1.1).

Proof. Consider the composed map

(6.1) $$\dot{X}_{b,b} \hookrightarrow L\mathbf{G} \longrightarrow L\mathbf{G}/L^+\mathcal{G}_{\mathbf{x}},$$

where the second map is the natural projection. By Proposition 6.3, for any $R \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and any $g \in \dot{X}_{b,b}(R)$, we have $\sigma_b(g) \in gL^+({}^c\mathbf{U} \cap \mathbf{U})_{\mathbf{x}}(R) \subseteq gL^+\mathcal{G}_{\mathbf{x}}(R)$. Now $L^+\mathcal{G}_{\mathbf{x}} \subseteq L\mathbf{G}$ is a σ_b-stable subsheaf; thus σ_b acts on $L\mathbf{G}/L^+\mathcal{G}_{\mathbf{x}}$. It follows that (6.1) factors through $(L\mathbf{G}/L^+\mathcal{G}_{\mathbf{x}})^{\sigma_b} = \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$, which is a discrete set. The preimage of $1 \cdot \mathcal{G}_{\mathbf{x}}(\mathcal{O}_k)$ is precisely $\dot{X}_{\bar{c},b}^{\mathcal{G}_{\mathbf{x}}}$. $\qquad\square$

We now prove (1.3) for $\mathbf{G}$ as in Corollary 6.4 (this suffices by Section 6.1). We need to descend the disjoint union decomposition of Corollary 6.4 to $X_{c,b}$. Recall that by [Ryd10, Corollary 2.9], a quasi-compact morphism of schemes is universally subtrusive if and only if it satisfies the condition in [BS17, Definition 2.1], *i.e.*, is a v-cover. We need the following version of [Ryd10, Theorem 4.1].

Lemma 6.5. *Let $S' \to S$ be a quasi-compact universally subtrusive map between schemes. Let $E \subseteq S$ be a subset, $E' = f^{-1}(E)$. Then E is closed (resp. open) if and only if E' is closed (resp. open).*

Proof. It suffices to prove the claim for closed sets. The "only if" part is clear. For the other direction, it suffices by [Ryd10, Corollary 1.5] to show that E is closed in the constructible topology and stable under specializations if the same holds for E'. As f is quasi-compact, f is closed in the constructible topology; *cf.* [Ryd10, Proposition 1.2]. Hence $E = f(E')$ is closed in the constructible topology. Let $x_0 \rightsquigarrow x_1$ be a specialization relation in S with $x_0 \in E$. As f is universally subtrusive, $x_0 \rightsquigarrow x_1$ lifts to a specialization relation $y_0 \rightsquigarrow y_1$. Then $y_0 \in E'$ and as E' is closed, also $y_1 \in E'$. Hence $x_1 = f(y_1) \in f(E') = E$. □

The space $X_{c,b}$ is an ind-scheme by Proposition 5.12 and [Iva23, Theorem C]. Thus for any map $T \to X_{c,b}$ with T a quasi-compact scheme, the fiber product $T \times_{X_{c,b}} \dot{X}_{b,b}$ with the scheme $\dot{X}_{b,b}$ is again a scheme. Moreover, the map $T \times_{X_{c,b}} \dot{X}_{b,b} \to T$ is quasi-compact and a v-cover by Corollary 4.5. Hence this map is universally subtrusive, *cf.* [Ryd10, Corollary 2.9], and by Lemma 6.5 and Corollary 6.4, T admits a disjoint union decomposition $T = \coprod_{\gamma \in G_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} T_\gamma$ which is functorial in T. Doing this for all maps $T \to X_{c,b}$ from a quasi-compact scheme T into $X_{c,b}$, we deduce a disjoint union decomposition of $X_{c,b}$ indexed by the same set. The subscheme (ind-scheme) corresponding to $\gamma = 1$ is the quotient of $\dot{X}^{\mathcal{G}_{\mathbf{x}}}_{\tilde{c},b}$ by $\mathbf{T}_c(k)$, which equals $X^{\mathcal{G}_{\mathbf{x}}}_{c,b}$ by definition. This proves (1.3).

6.3. Proof of (1.4)

Using Proposition 5.12 we may replace the left-hand side in (1.4) by $\dot{X}_{b,b}$ (as b lies over $\bar{c}$). If $\mathbf{G}$ is of adjoint type, (1.4) follows from Corollary 6.4 and the discussion in Section 6.1. We now proceed in two steps.

6.3.1. Special case: The center Z of G is an unramified induced torus.—

Lemma 6.6. *The natural map $p \colon \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k) \to \mathbf{G}^{\mathrm{ad}}_b(k)/\mathcal{G}^{\mathrm{ad}}_{\mathbf{x},b}(\mathcal{O}_k)$ is surjective, and its fibers are $X_*(\mathbf{Z})^{\langle\sigma\rangle}$-torsors.*

Proof. As $\mathbf{Z}$ is an induced torus, $H^1(\sigma, \mathbf{Z}) = 0$. Hence $\mathbf{Z}(k) \hookrightarrow \mathbf{G}_b(k) \twoheadrightarrow \mathbf{G}^{\mathrm{ad}}_b(k)$ is exact. Let $\mathcal{Z}$ denote the reductive $\mathcal{O}_k$-model of $\mathbf{Z}$. Using $H^1(\sigma, \mathcal{Z}(\mathcal{O}_{\check{k}})) = 0$ (by the pro-version of Lang's theorem), we deduce the exact sequence $\mathcal{Z}(\mathcal{O}_k) \hookrightarrow \mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k) \twoheadrightarrow \mathcal{G}^{\mathrm{ad}}_{\mathbf{x},b}(\mathcal{O}_k)$. Factoring this sequence out from the first one, and using $\mathbf{Z}(k)/\mathcal{Z}(\mathcal{O}_k) \cong X_*(\mathbf{Z})^{\langle\sigma\rangle}$ (which again follows from $H^1(\sigma, \mathcal{Z}(\mathcal{O}_{\check{k}})) = 0$), we obtain the lemma. □

Let $\pi \colon L\mathbf{G} \to L\mathbf{G}^{\mathrm{ad}}$ be the natural map. It is surjective as $\pi_1(\mathbf{G}) \to \pi_1(\mathbf{G}^{\mathrm{ad}})$ is. For $g \in \mathbf{G}$, write $\bar{g}$ for $\pi(g)$. For any $\tau \in X_*(\mathbf{Z})$, choose a lift $\tilde{\tau} \in \mathbf{Z}(\check{k})$ subject to the condition that $\tilde{\tau} \in \mathbf{Z}(k)$ if $z \in X_*(\mathbf{Z})^{\langle\sigma\rangle}$. We have $\pi^{-1}(L^+\mathcal{G}^{\mathrm{ad}}_{\mathbf{x}}) = \coprod_{\tau \in X_*(\mathbf{Z})} \tilde{\tau} L^+ \mathcal{G}_{\mathbf{x}}$ (indeed, $L^+\mathcal{G}_{\mathbf{x}} \subseteq \pi^{-1}(L^+\mathcal{G}^{\mathrm{ad}}_{\mathbf{x}})$, and the quotient is the discrete scheme $X_*(\mathbf{Z})$). Consequently, if $\bar{\gamma} \in \mathbf{G}^{\mathrm{ad}}_b(k)/\mathcal{G}^{\mathrm{ad}}_{\mathbf{x},b}(\mathcal{O}_k)$ with a fixed lift $\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$, then

$$(6.2) \qquad \pi^{-1}(\bar{\gamma}\dot{X}^{\mathcal{G}^{\mathrm{ad}}_{\mathbf{x}}}_{b,b}) = \coprod_{\tau \in X_*(\mathbf{Z})} \gamma\tilde{\tau}Y, \quad \text{where } Y \overset{\text{def}}{=} \left\{ g \in L^+\mathcal{G}_{\mathbf{x}} \colon \bar{g}^{-1}\sigma_b(\bar{g}) \in L^+\left({}^c\mathbf{U}^{\mathrm{ad}} \cap \mathbf{U}^{\mathrm{ad}-}\right)_{\mathbf{x}} \right\}.$$

Lemma 6.7. *If $\tau \in X_*(\mathbf{Z}) \setminus X_*(\mathbf{Z})^{\langle\sigma\rangle}$, then $\dot{X}_{b,b} \cap \gamma\tilde{\tau}Y = \varnothing$.*

Proof. It suffices to prove that $(\dot{X}_{b,b} \cap \gamma\tilde{\tau}Y)(\mathfrak{f}) = \varnothing$ for any algebraically closed field $\mathfrak{f} \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. Let $g \in L^+\mathcal{G}_{\mathbf{x}}(\mathfrak{f})$ be such that $\gamma\tilde{\tau}g \in (\dot{X}_{b,b} \cap \gamma\tilde{\tau}Y)(\mathfrak{f})$. Then $g^{-1}\sigma_b(g) \in \pi^{-1}(L^+({}^c\mathbf{U}^{\mathrm{ad}} \cap \mathbf{U}^{\mathrm{ad}-})_{\mathbf{x}})(\mathfrak{f}) = (L^+\mathcal{Z})(\mathfrak{f}) \cdot L^+({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}(\mathfrak{f})$, where $\mathcal{Z}$ is the reductive $\mathcal{O}_k$-model of $\mathbf{Z}$. Write $g^{-1}\sigma_b(g) = \zeta u$ with $\zeta \in (L^+\mathcal{Z})(\mathfrak{f})$ and $u \in L^+({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}(\mathfrak{f})$. Then

$$(\gamma\tilde{\tau}g)^{-1}\sigma_b(\gamma\tilde{\tau}g) = \tilde{\tau}^{-1}\sigma(\tilde{\tau})g^{-1}\sigma_b(g) = \tilde{\tau}^{-1}\sigma(\tilde{\tau})\zeta u,$$

where we have $\sigma_b(\tilde{\tau}) = \sigma(\tilde{\tau})$ as $\tilde{\tau}$ is central. This cannot lie in $L^+({}^c\mathbf{U} \cap \mathbf{U}^-)(\mathfrak{f})$ as the image of $\zeta \in L^+\mathcal{Z}(\mathfrak{f}) \subseteq L\mathbf{Z}(\mathfrak{f})$ in $X_*(\mathbf{Z})$ is 0 but the image of $\tilde{\tau}^{-1}\sigma(\tilde{\tau})$ in $X_*(\mathbf{Z})$ is non-zero by assumption. □

As $\dot{X}_{b,b} \subseteq \pi^{-1}(\dot{X}_{b,b}^{\mathrm{ad}})$, the disjoint decomposition for $\dot{X}_{b,b}^{\mathrm{ad}}$ (*cf.* beginning of Section 6.3) implies, together with (6.2) and Lemmas 6.6 and 6.7, that $\dot{X}_{b,b}$ is the disjoint union of all $\gamma \dot{X}_{b,b}^{\mathcal{G}_{\mathbf{x}}}$, where γ varies through $\mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$, which is precisely (1.4).

6.3.2. General case.— An arbitrary $\mathbf{G}$ admits a derived embedding $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ such that $\widetilde{\mathbf{G}}$ is unramified reductive and the center of $\widetilde{\mathbf{G}}$ is an unramified induced torus. (This was pointed out to us by M. Borovoi; *cf.* [Bor21].) Now the result follows from Section 6.3.1 and the next proposition.

Proposition 6.8. *Let* $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ *be a derived embedding. If* (1.4) *holds for* $\widetilde{\mathbf{G}}$, *then it holds for* $\mathbf{G}$.

Proof. We identify $\mathbf{G}$ with a subgroup of $\widetilde{\mathbf{G}}$. Knowing $\dot{X}_{b,b}^{\widetilde{\mathbf{G}}} = \bigsqcup_{\gamma \in \widetilde{\mathbf{G}}_b(k)/\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}}$ as subsheaves of $L\widetilde{\mathbf{G}}$, we have to show that $\dot{X}_{b,b} = \bigsqcup_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \dot{X}_{\bar{c},b}^{\mathcal{G}_{\mathbf{x}}}$.

Lemma 6.9. *Let* $\mathfrak{f} \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ *be an algebraically closed field. Let* $L := \mathbb{W}(\mathfrak{f})[1/\varpi]$. *We have* $\mathbf{G}(L) \cap \widetilde{\mathcal{G}}_{\mathbf{x}}(\mathcal{O}_L) = \mathcal{G}_{\mathbf{x}}(\mathcal{O}_L)$.

Proof. By [HR08, Proposition 3] we have $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_L) = \mathrm{Fix}_{\mathbf{G}(L)}(\mathbf{x}) \cap \ker \tilde{\kappa}_{\mathbf{G}}$ and $\widetilde{\mathcal{G}}_{\mathbf{x}}(\mathcal{O}_L) = \mathrm{Fix}_{\widetilde{\mathbf{G}}(L)}(\mathbf{x}) \cap \ker \tilde{\kappa}_{\widetilde{\mathbf{G}}}$, where $\mathrm{Fix}(\mathbf{x})$ denotes the stabilizer of the vertex $\mathbf{x}$ in the respective group (we identify the adjoint groups of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$). As $\pi_1(\mathbf{G}) \to \pi_1(\widetilde{\mathbf{G}})$ is injective, *cf.* [Bor98, Lemma 1.5], we have $\ker \tilde{\kappa}_{\widetilde{\mathbf{G}}} \cap \mathbf{G}(L) = \ker \tilde{\kappa}_{\mathbf{G}}$. As the actions on the adjoint building are compatible with the inclusion $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$, we have $\mathrm{Fix}_{\widetilde{\mathbf{G}}(L)}(\mathbf{x}) \cap \mathbf{G}(L) = \mathrm{Fix}_{\mathbf{G}(L)}(\mathbf{x})$. Putting all these together, we deduce $\widetilde{\mathcal{G}}_{\mathbf{x}}(\mathcal{O}_L) \cap \mathbf{G}(L) = \mathcal{G}_{\mathbf{x}}(\mathcal{O}_L)$. $\square$

Taking σ_b-invariants in Lemma 6.9, we deduce that $\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k) \cap \mathbf{G}_b(k) = \mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$. In particular, the natural map $\mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k) \to \widetilde{\mathbf{G}}_b(k)/\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)$ is injective. Pick some $\bar{\gamma} \in \widetilde{\mathbf{G}}_b(k)/\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)$, represented by $\gamma \in \widetilde{\mathbf{G}}_b(k)$.

Lemma 6.10. *If* $\gamma \dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}} \cap L\mathbf{G} \neq \varnothing$, *then* $\bar{\gamma} \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$.

Proof. By assumption, the sheaf $\gamma \dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}} \cap L\mathbf{G}$ admits a geometric point. Let $\mathfrak{f} \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ be algebraically closed, such that there exists some $x \in \dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}}(\mathfrak{f})$ with $\gamma x \in L\mathbf{G}(\mathfrak{f})$. Write $L = \mathbb{W}(\mathfrak{f})[1/\varpi]$. By the definition of $\dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}}$, we have

$$x^{-1}\sigma_b(x) \in ({}^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}(\mathcal{O}_L) \subseteq \mathcal{G}_{\mathbf{x}}(\mathcal{O}_L)$$

(here, we identify $\mathbf{U} \subseteq \mathbf{G}$ with the unipotent radical of a Borel of $\widetilde{\mathbf{G}}$). As $H^1(\sigma_b, L^+\mathcal{G}_{\mathbf{x}}) = 0$, we find some $\zeta \in \mathcal{G}_{\mathbf{x}}(\mathcal{O}_L)$ such that $x^{-1}\sigma_b(x) = \zeta^{-1}\sigma_b(\zeta)$. In particular, $x\zeta^{-1} \in \widetilde{\mathcal{G}}_{\mathbf{x}}(\mathcal{O}_L)^{\sigma_b} = \widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)$. From this and $\gamma \in \widetilde{\mathbf{G}}_b(k)$, it follows that $\gamma x\zeta^{-1} \in \widetilde{\mathbf{G}}_b(k)$. Moreover, $\gamma x\zeta^{-1} \in \mathbf{G}(L)$ as $\gamma x \in \mathbf{G}(L)$ and $\zeta \in \mathcal{G}_{\mathbf{x}}(\mathcal{O}_L) \subseteq \mathbf{G}(L)$. Combining these observations, we deduce $\gamma x\zeta^{-1} \in \mathbf{G}(L)^{\sigma_b} = \mathbf{G}_b(k)$. Finally, $x\zeta^{-1} \in \widetilde{\mathcal{G}}_{\mathbf{x}}(\mathcal{O}_L)$; thus γ and $\gamma x\zeta^{-1}$ represent the same class in $\mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)$ as $\mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k) \to \widetilde{\mathbf{G}}_b(k)/\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)$ is injective. The lemma is proved. $\square$

Next we claim that

(6.3) $$\dot{X}_{\bar{c},b}^{\widetilde{\mathcal{G}}_{\mathbf{x}}} \cap L\mathbf{G} = \dot{X}_{\bar{c},b}^{\mathcal{G}_{\mathbf{x}}}.$$

Lemma 6.11. *Let* X *be an ind-(reduced scheme) over* $\overline{\mathbb{F}}_q$ *and let* Y, Z *be two closed subfunctors, which are themselves ind-(reduced schemes). If* $Y(\mathfrak{f}) = Z(\mathfrak{f})$ *for any algebraically closed field* $\mathfrak{f}/\overline{\mathbb{F}}_q$, *then* $Y = Z$.

Proof. Write $X = \varinjlim_i X_i$ for qcqs reduced schemes X_i and an index set I. Then $Y \cap X_i, Z \cap X_i$ are closed reduced subschemes of X_i, which agree on geometric points. Thus $Y \cap X_i = Z \cap X_i$ for each i, and hence also $Y = Z$ as the X_n exhaust X. $\square$

By combining Lemma 6.11 with Lemma 6.9 we see that $L\mathbf{G} \cap L^+\widetilde{\mathcal{G}}_\mathbf{x} = L^+\mathcal{G}_\mathbf{x}$ (note that any ind-(perfect scheme) is automatically an ind-(reduced scheme)). As $\dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \subseteq L^+\widetilde{\mathcal{G}}_\mathbf{x}$ by definition, we deduce $\dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L\mathbf{G} = \dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L^+\mathcal{G}_\mathbf{x}$. Now, $\dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L^+\mathcal{G}_\mathbf{x} = \dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b}$ is immediate, proving (6.3).

Finally, observe that $\dot{X}_{b,b} = \dot{X}^{\widetilde{\mathbf{G}}}_{b,b} \cap L\mathbf{G}$. Now we compute

$$\dot{X}_{b,b} = \dot{X}^{\widetilde{\mathbf{G}}}_{b,b} \cap L\mathbf{G} = \bigsqcup_{\gamma \in \widetilde{\mathbf{G}}_b(k)/\widetilde{\mathcal{G}}_{\mathbf{x},b}(\mathcal{O}_k)} \left(\gamma \dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L\mathbf{G} \right) = \bigsqcup_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \left(\gamma \dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L\mathbf{G} \right)$$

$$= \bigsqcup_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \left(\dot{X}^{\widetilde{\mathcal{G}}_\mathbf{x}}_{\bar{c},b} \cap L\mathbf{G} \right) = \bigsqcup_{\gamma \in \mathbf{G}_b(k)/\mathcal{G}_{\mathbf{x},b}(\mathcal{O}_k)} \gamma \dot{X}^{\mathcal{G}_\mathbf{x}}_{\bar{c},b},$$

where the second equality is by the assumption of the proposition, the third is by Lemma 6.10 and the fifth holds by (6.3). This proves the proposition. □

6.4. Quasi-split case

We work in the setup of Section 1.3. In particular, we have the Coxeter element c, and we have $\bar{c} \in [1] \cap \overline{F}_c/\ker \bar{\kappa}_c$ (if $\bar{c} \in \overline{F}_c/\ker \bar{\kappa}_c \setminus [1]$, then $\dot{X}_{\bar{c}}(1) = \varnothing$ by Corollary 4.6). First, we justify that a choice of $\dot{c}$ as claimed in Section 1.3 is indeed possible. It is well known (in the quasi-split case) that $\Upsilon(\mathbf{G}, c)/\operatorname{Ad}\mathbf{G}(k)$ is in natural bijection with $\mathbf{G}(k)$-orbits in the set of hyperspecial vertices of $\mathcal{B}_{\mathbf{G},k} = \mathcal{B}^{\langle \sigma \rangle}_{\mathbf{G},\check{k}}$ (cf. [DeB06] or [DI20, Lemma 2.6.2]). Let $[\mathbf{T}_{\bar{c}}] \in \Upsilon(\mathbf{G}, c)/\operatorname{Ad}\mathbf{G}(k)$ be the class corresponding to $\bar{c}$ under the surjection $[1] \cap \overline{F}_c/\ker \bar{\kappa}_c \twoheadrightarrow \Upsilon(\mathbf{G}, c)/\operatorname{Ad}\mathbf{G}(k)$ from Proposition 3.7. Then let$\mathbf{x}$ be any hyperspecial vertex in $\mathcal{B}_{\mathbf{G},k}$ corresponding to $[\mathbf{T}_{\bar{c}}]$ under the above bijection. As $\mathbf{x}$ is hyperspecial, c admits a lift $\dot{c} \in F_c(\check{k}) \cap \mathcal{G}_\mathbf{x}(\mathcal{O}_{\check{k}})$. As c is Coxeter, the affine transformation $\dot{c}\sigma$ of $\mathcal{A}_{\mathbf{T},\check{k}}$ admits a unique fixed point, which necessarily equals $\mathbf{x}$. Comparing the two parametrizations of $\Upsilon(\mathbf{G}, c)/\operatorname{Ad}\mathbf{G}(k)$ above, we see that the image of $\dot{c}$ equals the image of $\bar{c}$ in $([1] \cap \overline{F}_c/\ker \bar{\kappa}_c)/C_W(c\sigma)$. Replacing $\dot{c}$ with $\gamma \dot{c}$ for an appropriate $\gamma \in C_W(c\sigma)$, we have that $\dot{c}$ lifts $\bar{c}$.

Now we prove Corollary 1.3. We only prove the first isomorphism (the second is completely analogous). We may assume that $\mathbf{G}$ is adjoint. By the pro-version of Lang's theorem for $\mathcal{G}_\mathbf{x}$, we may find an $h \in \mathcal{G}_\mathbf{x}(\mathcal{O}_{\check{k}})$ such that $h^{-1}\dot{c}\sigma(h) = 1$. By [Iva23, Remark 8.7], $g \mapsto hg\colon L\mathbf{G} \to L\mathbf{G}$ induces an isomorphism $X_c(\dot{c}) \xrightarrow{\sim} X_c(1)$ which is equivariant with respect to $\operatorname{Ad}(h)\colon \mathbf{G}(k) \xrightarrow{\sim} \mathbf{G}_{\dot{c}}(k)$. Moreover, $\operatorname{Ad}(h)$ restricts to an isomorphism $\mathcal{G}_{\mathbf{x},1}(\mathcal{O}_{\check{k}}) \xrightarrow{\sim} \mathcal{G}_{\mathbf{x},\dot{c}}(\mathcal{O}_{\check{k}})$, thus inducing $\mathbf{G}(k)/\mathcal{G}_{\mathbf{x},1}(\mathcal{O}_{\check{k}}) \xrightarrow{\sim} \mathbf{G}_{\dot{c}}(k)/\mathcal{G}_{\mathbf{x},\dot{c}}(\mathcal{O}_{\check{k}})$. Exploiting Theorem 1.1 for $X_c(\dot{c})$, we are reduced to showing that $g \mapsto hg\colon L^+\mathcal{G}_\mathbf{x} \to L^+\mathcal{G}_\mathbf{x}$ induces an isomorphism $X^{\mathcal{G}_\mathbf{x}}_{c,\dot{c}} \xrightarrow{\sim} X^{\mathcal{G}_\mathbf{x}}_c(1)$. We compute (quotients are taken in the category of arc-sheaves)

$$X^{\mathcal{G}^{\mathrm{ad}}_\mathbf{x}}_{c,\dot{c}} = \{g \in L^+\mathcal{G}_\mathbf{x}\colon g^{-1}\sigma_{\dot{c}}(g) \in L^+(^c\mathbf{U} \cap \mathbf{U}^-)_\mathbf{x}\}/\mathbf{T}_c(k)$$

$$\cong \{g \in L^+\mathcal{G}_\mathbf{x}\colon g^{-1}\sigma_{\dot{c}}(g) \in L^+(^c\mathbf{U} \cap \mathbf{U}^-)_\mathbf{x} \cdot L^+\mathcal{T}\}/L^+\mathcal{T}$$

$$\cong \{g \in L^+\mathcal{G}_\mathbf{x}\colon g^{-1}\sigma(g) \in L^+(^c\mathbf{U} \cap \mathbf{U}^-)_\mathbf{x} \cdot L^+\mathcal{T} \cdot \dot{c}\}/L^+\mathcal{T}$$

$$\cong \{g \in L^+\mathcal{G}_\mathbf{x}\colon g^{-1}\sigma(g) \in \dot{c} \cdot L^+\mathbf{B}_\mathbf{x}\}/L^+(^c\mathbf{U} \cap \mathbf{U})_\mathbf{x}L^+\mathcal{T}$$

$$\cong \{g \in L^+(\mathcal{G}_\mathbf{x}/\mathbf{B}_\mathbf{x})\colon g^{-1}\sigma(g) \in L^+\mathbf{B}_\mathbf{x} \cdot \dot{c} \cdot L^+\mathbf{B}_\mathbf{x}\}/L^+\mathbf{B}_\mathbf{x} \cong X^{\mathcal{G}_\mathbf{x}}_c(1),$$

where the third isomorphism is $g \mapsto hg$ and in the fourth isomorphism, we exploit the last statement of Proposition 5.3.

7. Estimates via Newton polygons

In this section we prove Proposition 6.3 for an absolutely almost simple classical group $\mathbf{G}^{\mathrm{ad}}$ of adjoint type (so, compared to Section 6.2, we change our notation, and the group which was denoted by $\mathbf{G}$ there is $\mathbf{G}^{\mathrm{ad}}$ here). It will be more convenient to work with a certain central extension $p\colon \mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$, whose kernel $\mathbf{Z} = \ker(p)$ is an unramified torus. (Which central extension we take is specified in each particular case of our case-by-case study below; see Sections 7.1.1, 7.2.1, 7.3.1, 7.4.1, 7.5.1, 7.6.1.) Then we have the exact sequence

$$(7.1) \qquad X_*(\mathbf{Z})_{\langle\sigma\rangle} \longrightarrow \pi_1(\mathbf{G})_{\langle\sigma\rangle} \longrightarrow \pi_1(\mathbf{G}^{\mathrm{ad}})_{\langle\sigma\rangle} \longrightarrow 0.$$

Remark 7.1. In our case-by-case study below, the left map will be injective, except in type $^2A_{n-1}$ with n even. (This is clear in the split cases A_{n-1}, B_m, C_m, D_m; see Sections 7.5.2 and 7.6.2 for the cases $^2A_{n-1}$ (n odd) and ^2D_m.) In type $^2A_{n-1}$ with n even, the left map will be the zero map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ (see Section 7.5.2).

We let $\bar{b} \in F_c^{\mathrm{ad}}(\check{k}) \subseteq \mathbf{G}^{\mathrm{ad}}(\check{k})$ be a basic element lying over a Coxeter element $c \in W$, so that we have to show that $X_{\bar{b},\bar{b}}^{\mathrm{ad}} = X_{\bar{b},\bar{b},\mathcal{O}}^{\mathrm{ad}}$ within $L\mathbf{G}^{\mathrm{ad}}$. It suffices to show that the Lang map $g \mapsto g^{-1}\sigma_{\bar{b}}(g)\colon X_{\bar{b},\bar{b}}^{\mathrm{ad}} \to L(^c\mathbf{U} \cap \mathbf{U})$ factors through the closed subscheme $L^+(^c\mathbf{U} \cap \mathbf{U})_{\mathbf{x}}$. As all rings in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ are reduced, it suffices to do so on geometric points. Let $\mathfrak{f}$ denote a fixed algebraically closed field in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and put $L := \mathbb{W}(\mathfrak{f})[1/\varpi]$.

As $\pi_1(\mathbf{G})_{\langle\sigma\rangle} \to \pi_1(\mathbf{G}^{\mathrm{ad}})_{\langle\sigma\rangle}$ is surjective, $\bar{b}$ lifts to a basic element $b \in F_c(\check{k}) \subseteq \mathbf{G}(\check{k})$. For any element $\bar{\zeta} \in \ker\big(X_*(\mathbf{Z})_{\langle\sigma\rangle} \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}\big)$, fix a representative $\zeta \in X_*(\mathbf{Z})$ of $\bar{\zeta}$. For each such ζ, we have the commutative diagram

$$(7.2)$$

$$
\begin{array}{ccc}
\dot{X}_{b\varpi^\zeta,b} & \xrightarrow{\;\;q_{b,\zeta}\;\;} & \dot{X}_{\bar{b},\bar{b}}^{\mathrm{ad}} \\[4pt]
\big\uparrow & & \big\uparrow \\[4pt]
\dot{X}_{b\varpi^\zeta,b,\mathcal{O}} & \longrightarrow & \dot{X}_{\bar{b},\bar{b},\mathcal{O}}^{\mathrm{ad}},
\end{array}
$$

where the objects on the left (resp. right) are contained in $L\mathbf{G}$ (resp. $L\mathbf{G}^{\mathrm{ad}}$), and where

$$\dot{X}_{b\varpi^\zeta,b,\mathcal{O}} \overset{\mathrm{def}}{=} \text{preimage of } L^+(^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}\varpi^\zeta \text{ under } g \mapsto g^{-1}\sigma_b(g)\colon L\mathbf{G} \to L\mathbf{G}$$

is a subsheaf of $\dot{X}_{b\varpi^\zeta,b}$.

Lemma 7.2. *We have*

$$\bigcup_{\bar{\zeta}\in\ker\big(X_*(\mathbf{Z})_{\langle\sigma\rangle}\to\pi_1(\mathbf{G})_{\langle\sigma\rangle}\big)} q_{b,\zeta}(\dot{X}_{b\varpi^\zeta,b}(\mathfrak{f})) = \dot{X}_{\bar{b},\bar{b}}(\mathfrak{f}).$$

Proof. Let $\bar{g} \in \dot{X}_{\bar{b},\bar{b}}^{\mathrm{ad}}(\mathfrak{f})$. The map $L\mathbf{G}(\mathfrak{f}) \to L\mathbf{G}^{\mathrm{ad}}(\mathfrak{f})$ is surjective (as $H^1(L, \mathbf{Z}) = 0$). Thus we may lift $\bar{g}$ to some $g \in L\mathbf{G}(\mathfrak{f})$. Then $g^{-1}\sigma_b(g) = uz$ with $u \in (^c\mathbf{U} \cap \mathbf{U}^-)(L)$ and $z \in \mathbf{Z}(L)$. On the other hand, it is clear that under $\kappa_{\mathbf{G}}\colon \mathbf{G}(L) \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}$, the element $g^{-1}\sigma_b(g)$ maps to 0. As also $\kappa_{\mathbf{G}}((^c\mathbf{U} \cap \mathbf{U}^-)(L)) = 0$, we deduce that $\kappa_{\mathbf{G}}(z) = 0$. Thus the image $\bar{\zeta}$ of z in $X_*(\mathbf{Z})_{\langle\sigma\rangle}$ lies in $\ker\big(X_*(\mathbf{Z})_{\langle\sigma\rangle} \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}\big)$. The exactness of (2.4) implies that there exists some $\tau \in \mathbf{Z}(L)$ such that $z = \tau^{-1}\sigma(\tau)\varpi^\zeta$. Replacing g with $g\tau^{-1}$ and recalling that $\mathbf{Z}$ is central in $\mathbf{G}$, we get $g^{-1}\sigma_b(g) \in (^c\mathbf{U} \cap \mathbf{U}^-)(L)\varpi^\zeta$, *i.e.*, $g \in \dot{X}_{b\varpi^\zeta,b}(\mathfrak{f})$. $\qquad\square$

To show that the right arrow in (7.2) is surjective, it suffices (using Lemma 7.2) to show that the left vertical arrow is surjective on $\mathfrak{f}$-points for all $\bar{\zeta}$. Thus, finally, we are reduced to showing that the inclusion $\dot{X}_{b\varpi^\zeta,b,\mathcal{O}}(\mathfrak{f}) \hookrightarrow \dot{X}_{b\varpi^\zeta,b}(\mathfrak{f})$ is surjective, *i.e.*, that

$$(7.3) \qquad \forall g \in \dot{X}_{b\varpi^\zeta,b}(\mathfrak{f}), \quad \text{we have} \quad g^{-1}\sigma_b(g) \in L^+(^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}}(\mathfrak{f})\varpi^\zeta;$$

we will do this in the following subsections case by case. As mentioned in Remark 7.1, we have $\#\ker\left(X_*(\mathbf{Z})_{\langle\sigma\rangle} \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}\right) = 1$, and it will suffice to prove only the surjectivity of $\dot{X}_{b,b,\mathcal{O}}(\mathfrak{f}) \hookrightarrow \dot{X}_{b,b}(\mathfrak{f})$, with the only exception (treated in Section 7.5.6) in the case ${}^2A_{n-1}$ with n even, when $\#\ker\left(X_*(\mathbf{Z})_{\langle\sigma\rangle} \to \pi_1(\mathbf{G})_{\langle\sigma\rangle}\right) = 2$. We will use the following well-known property of isocrystals.

Lemma 7.3. *Let (V, φ) be an isocrystal over $\mathfrak{f}$ which is isoclinic of slope λ and dimension $n \geq 1$. Let v be a* cyclic vector *of (V, φ), that is, an element $v \in V$ such that the set $\{\varphi^i(v)\}_{i=0}^{n-1}$ is a basis of V. Write $\varphi^n(v) = \sum_{i=0}^{n-1} A_i \varphi^i(v)$ with $A_i \in L$. Then $\mathrm{ord}_{\varpi}(A_i) \geq (n-i)\lambda$.*

Proof. The Newton polygon of (V, φ) is the straight line segment connecting the points $(0,0)$ and $(n, \lambda n)$ in the plane. It coincides with the Newton polygon of the polynomial $\sum_{i=0}^{n} A_i X^i \in L[X]$ with $A_n = 1$, which is by definition the convex hull of the points $(i, \mathrm{ord}_{\varpi}(A_{n-i}))$; *cf.* [Bea12, Sections 3.1 and 3.2] or [Ked04, Proposition 5.7] . $\qquad\square$

Finally, note that if $g \in \dot{X}_{b,b}(\mathfrak{f})$, then $g^{-1}\sigma_b(g) = y \in ({}^c\mathbf{U} \cap \mathbf{U}^-)(L)$, *i.e.*, we have

$$(7.4) \qquad\qquad\qquad\qquad b\sigma(g) = gyb.$$

We will now proceed case by case. Below we will denote the fixed point $\mathbf{x}$ of $b\sigma$ by $\mathbf{x}_b$ (to make the notation non-ambiguous). Also recall the notation from Section 2, especially Section 2.1.2.

7.1. Type A_{n-1}

This case is treated in [CI21a, CI23]. We include it for the sake of completeness.

7.1.1. — Let $n \geq 2$, and let V_0 be an n-dimensional k-vector space with an (ordered) basis $\{e_1, \dots, e_n\}$. Whenever convenient, identify V_0 with k^n via this basis. Let $\mathbf{G} = \mathbf{GL}_n(V_0)$.

7.1.2. — Let $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ be the diagonal torus and the upper triangular Borel subgroup. Then $X_*(\mathbf{T}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}\beta_i$, with $\beta_i \colon t \mapsto (1^{\oplus i-1}, t, 1^{\oplus n-i})$. Then $X_*(\mathbf{T}^{\mathrm{sc}}) = \mathbb{Z}\Phi^\vee = \bigoplus_{i=1}^{n-1} \mathbb{Z} \cdot (\beta_{i+1} - \beta_i)$, $X_*(\mathbf{Z}) = \mathbb{Z} \cdot \sum_{i=1}^{n} \beta_i$. Then $\pi_1(\mathbf{G}^{\mathrm{ad}}) = X_*(\mathbf{T})/\mathbb{Z}\Phi^\vee + X_*(\mathbf{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, the isomorphism given by sending 1 to the class $\bar{\beta}_i$ of β_i for any i.

We have $\Phi = \{\alpha_{i-j} \colon 1 \leq i < j \leq n\}$, where $\alpha_{i-j} \colon (t_\lambda)_{\lambda=1}^{n} \mapsto t_i t_j^{-1}$ and $\Delta = \{\alpha_{i-(i+1)} \colon 1 \leq i \leq n-1\}$. We have the Coxeter element

$$(7.5) \qquad\qquad\qquad c = s_{\alpha_{1-2}} s_{\alpha_{2-3}} \dots s_{\alpha_{(n-1)-n}} = (1, 2, \dots, n).$$

For $0 \leq \kappa < n$, consider the lift $\dot{c}_\kappa$ of c to $\mathbf{G}(\check{k})$ given by $e_i \mapsto e_{i+1}$ for $1 \leq i < n$ and $e_n \mapsto \varpi^\kappa e_1$. Then $\dot{c}_\kappa$ maps to $\kappa\bar{\beta}_1 \in \pi_1(\mathbf{G}^{\mathrm{ad}})$, and $\mathbf{G}_{\dot{c}_\kappa}$ run through all inner forms of $\mathbf{G}$.

We identify $\mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \check{k}}$ with $X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{R}}$ by sending $0 \in X_*(\mathbf{T})_{\mathbb{R}}$ to the σ-stable hyperspecial point corresponding to $\bigoplus_{i=1}^{n} \mathcal{O}_{\check{k}} e_i \subseteq V_0 \otimes_k \check{k}$.

The group ${}^c\mathbf{U} \cap \mathbf{U}^-$ is generated by the root subgroups attached to α_{i-1} ($1 < i \leq n$). An element $y \in ({}^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is given by $e_1 \mapsto e_1 + \sum_{i=2}^{n} a_{i-1} e_i$; $e_i \mapsto e_i$ for $2 \leq i \leq n$.

7.1.3. — Let $b = \dot{c}_\kappa$ with $0 \leq \kappa < n$. We compute $\mathbf{x}_b = \left(-\frac{(i-1)\kappa}{n}\right)_{i=1}^{n}$. Let $g \in \dot{X}_{b,b}(\mathfrak{f})$. Then (7.4) holds with y as described in Section 7.1.2. We have to show that $y \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, or equivalently (see *e.g.* [MP94, Section 2.5])

$$(7.6) \qquad\qquad \mathrm{ord}_{\varpi}(a_i) \geq \langle \alpha_{(i+1)-1}, \mathbf{x}_b \rangle = -\frac{\kappa i}{n} \qquad \forall 1 \leq i \leq n-1.$$

Consider the ϕ-linear isocrystal $(V, \varphi) = (V_0 \otimes_k L, b\sigma)$ over $\mathfrak{f}$. It is isoclinic of slope $\frac{\kappa}{n}$. For $1 \leq i \leq n$, let $g_i = g(e_i)$. Then $\sigma(g)(e_i) = \sigma(g_i)$ for all i. Thus, (7.4) implies $\varphi(g_i) = gyb(e_i)$. Using this equation for all $i \neq n$ and writing $v := g_1$, we deduce $g_i = \varphi^{i-1}(v)$ for $1 \leq i \leq n-1$. In particular, $\{\varphi^i(v)\}_{i=0}^{n-1}$ is an L-basis

of V. Using $\varphi(g_i) = gyb(e_i)$ for $i = n$ and inserting the values for g_i, we arrive at the equation

$$\varphi^n(v) = \varpi^\kappa v + \sum_{i=1}^{n-1} \varpi^\kappa a_i \varphi^i(v).$$

Now Lemma 7.3 shows that $\mathrm{ord}_\varpi(a_i) \geq \frac{\kappa}{n}(n-i) - \kappa = -\frac{\kappa i}{n}$. This gives (7.6), and we are done.

7.1.4. Proof of Lemma 5.7 for type A_{n-1}.— The action of c on Φ^+ is as follows: $c(\alpha_{i-j}) = \alpha_{(i+1)-(j+1)}$ for $1 \leq i < j < n$; $c(\alpha_{i-n}) \in \Phi^-$ for $1 \leq i < n$. For $1 \leq i \leq r = n-1$, put $\Psi_i = \{\alpha_{i'-j'} \in \Phi^+ : j' \geq i+1\}$.

7.2. Type C_m

7.2.1.— Let $m \geq 2$. Let V_0 be a $2m$-dimensional k-vector space with an (ordered) basis $\{e_1, \ldots, e_m, e_{-m}, \ldots e_{-1}\}$. Whenever convenient, we will identify V_0 with k^{2m} via this basis. Let $\langle \cdot, \cdot \rangle$ be the alternating bilinear form on V_0 determined by $\langle e_{\pm i}, e_{\pm j} \rangle = 0$ if $i \neq j$ and $\langle e_i, e_{-i} \rangle = 1$ for $1 \leq i \leq m$. Let $\mathbf{G}$ be the closed k-subgroup of $\mathbf{GL}(V_0)$ of elements preserving $\langle \cdot, \cdot \rangle$ up to a scalar; *i.e.*,

$$\mathbf{G}(R) = \{g \in \mathbf{GL}(V_0)(R) : \langle gu, gv \rangle = \lambda(g)\langle u, v \rangle \text{ for all } u, v \in V_0 \otimes_k R\},$$

where $\lambda(g) \in R^\times$ is a unit only depending on g. The group $\mathbf{G}$ is the *group of symplectic similitudes*, and the character $\lambda \colon \mathbf{G} \to \mathbb{G}_m$ is called the *similitude character*.

7.2.2.— Let $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ be the diagonal torus and the Borel subgroup consisting of upper triangular matrices in $\mathbf{G}$. Then $\mathbf{T} = \{(t_0 t_1, \ldots, t_0 t_m, t_m^{-1}, \ldots, t_1^{-1}) : t_i \in \mathbb{G}_m\}$ and $X_*(\mathbf{T}) = \bigoplus_{i=0}^m \mathbb{Z}\varepsilon_i$, where

$$\varepsilon_0 \colon t \longmapsto (t^{\oplus m}, 1^{\oplus m}) \quad \text{and} \quad \varepsilon_i \colon t \longmapsto (1^{\oplus i-1}, t, 1^{\oplus 2(m-i-1)}, t^{-1}, 1^{\oplus i-1}) \text{ for } 1 \leq i \leq m$$

are cocharacters of $\mathbf{T}$. We have $X_*(\mathbf{T}^{\mathrm{ad}}) = X_*(\mathbf{T})/\mathbb{Z} \cdot (2\varepsilon_0 - \sum_{i=1}^m \varepsilon_i)$ and $X_*(\mathbf{T}^{\mathrm{sc}}) = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$. In particular, $\pi_1(\mathbf{G}^{\mathrm{ad}}) = X_*(\mathbf{T}^{\mathrm{ad}})/X_*(\mathbf{T}^{\mathrm{sc}}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the class of ε_0.

We have $\Phi = \{\alpha_{\pm i \pm j} \colon (t_i)_{i=1}^m \mapsto t_i^{\pm 1} t_j^{\pm 1} : 1 \leq i < j \leq m\} \cup \{\alpha_{\pm 2 \cdot i} \colon (t_i)_{i=1}^m \mapsto t_i^{\pm 2} : 1 \leq i \leq m\}$ and $\Delta = \{\alpha_{i-(i+1)} \colon 1 \leq i \leq m-1\} \cup \{\alpha_{2 \cdot m}\}$. We have the Coxeter element

(7.7) $$c = s_{\alpha_{1-2}} \cdot \cdots \cdot s_{\alpha_{(m-1)-m}} s_{\alpha_{2m}} = (1, 2, \ldots, m-1, m, 2m, 2m-1, \ldots, m+1).$$

We consider two lifts $\dot{c}_0, \dot{c}_1$ of c to $\mathbf{G}(\check{k})$. The first is given by $\dot{c}_0(e_{\pm i}) = e_{\pm(i+1)}$ for $1 \leq i \leq m-1$, $\dot{c}_0(e_{\pm m}) = e_{\mp 1}$. The second is $\dot{c}_1 = \dot{c}_0 \varepsilon_0(\varpi)$. Then $\dot{c}_0, \dot{c}_1$ are basic, and the image of $\dot{c}_0$, resp. $\dot{c}_1$, in $\mathbf{G}^{\mathrm{ad}}(\check{k})$ represents $0 \in \pi_1(\mathbf{G}^{\mathrm{ad}})$, resp. $\bar{\varepsilon}_0 \in \pi_1(\mathbf{G}^{\mathrm{ad}})$. Thus the pairs $(\mathbf{G}, \sigma_{\dot{c}_i})$ $(i = 1, 2)$ represent the two inner forms of $\mathbf{G}$.

We identify $\mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \check{k}}$ with $X_*(\mathbf{T}^{\mathrm{ad}})_\mathbb{R}$ by sending $0 \in X_*(\mathbf{T})_\mathbb{R}$ to the σ-stable hyperspecial point ("the origin") corresponding to the self-dual lattice $\bigoplus_{i=1}^m (\mathcal{O}_{\check{k}} e_i \oplus \mathcal{O}_{\check{k}} e_{-i}) \subseteq V_0 \otimes_k \check{k}$.

The group $^c\mathbf{U} \cap \mathbf{U}^-$ is generated by root subgroups attached to α_{i-1} $(2 \leq i \leq m)$ and $\alpha_{-2 \cdot 1}$. Explicitly, any element $y \in (^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is an R-linear map in $\mathbf{GL}(V_0)(R)$ given by $e_1 \mapsto e_1 + \sum_{i=2}^m a_{i-1} e_i + a_m e_{-m}$; $e_i \mapsto e_i$ for $2 \leq i \leq m$ and $i = -1$; $e_{-i} \mapsto e_{-i} - a_{i-1} e_{-1}$ for all $2 \leq i \leq m$.

7.2.3.— Let $b = \dot{c}_\kappa$ with $\kappa \in \{0, 1\}$. We compute that $\mathbf{x}_b = 0$ if $\kappa = 0$ and that $\mathbf{x}_b = \sum_{i=1}^m \lambda_i \varepsilon_i$ with $\lambda_i = -\frac{m}{4} + \frac{i-1}{2}$ for $1 \leq i \leq m$ if $\kappa = 1$. Let $g \in \dot{X}_{b,b}(\mathfrak{f})$. Then (7.4) holds with y as described in Section 7.2.2, depending on some $a_1, \ldots, a_m \in L$. We have to show that $y \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, which is (as in Section 7.1.3) equivalent to

(7.8) $$\mathrm{ord}_\varpi(a_i) \geq \frac{\kappa i}{2} \quad \forall 1 \leq i \leq m.$$

Consider the ϕ-linear isocrystal $(V, \varphi) = (V_0 \otimes_k L, b\sigma)$ over $\mathfrak{f}$. It is isoclinic of slope $\frac{\kappa}{2}$. For $i \in \{\pm 1, \ldots, \pm m\}$, let $g_i := g(e_i)$. Then $\sigma(g)(e_i) = \sigma(g_i)$ for all i. Hence (7.4) implies $\varphi(g_i) = gyb(e_i)$. Using this equation for

all $i \neq m+1$ and writing $v := g_1$, we deduce that $g_i = \varpi^{\kappa(1-i)}\varphi^{i-1}(v)$ for $2 \leq i \leq m$, $g_{-1} = \varpi^{-\kappa m}\varphi^m(v)$ and

$$g_{-i} = \varpi^{-\kappa m}\varphi^{m+i-1}(v) + \sum_{j=1}^{i-1}\varpi^{-\kappa m}\phi^{i-1-j}(a_j)\varphi^{m+i+1-j}(v) \quad \text{for } 2 \leq i \leq m.$$

In particular, $\{\varphi^i(v)\}_{i=0}^{2m-1}$ is a basis of V; *i.e.*, v is a cyclic vector for V. Finally, applying $\varphi(g_i) = gyb(e_i)$ for $i = m+1$ and inserting the above formulas for the g_i, we deduce

$$\varphi^{2m}(v) = \varpi^{\kappa m}v + \sum_{i=1}^{m}\varpi^{\kappa(m-i)}a_i\varphi^i(v) - \sum_{m+1}^{2m}\phi^{i-m}(a_{2m-i})\varphi^i(v).$$

Now Lemma 7.3 shows that $\mathrm{ord}_\varpi(a_i) \geq 0$ for all $1 \leq i \leq m$ if $j = 0$ and that $\mathrm{ord}_\varpi(\varpi^{m-i}a_i) \geq \frac{1}{2}(2m-i)$, *i.e.*, that $\mathrm{ord}_\varpi(a_i) \geq \frac{i}{2}$ for all $1 \leq i \leq m$, if $j = 1$. This shows (7.8), and we are done.

7.2.4. Proof of Lemma 5.6 for types C_m and B_m.— The action of c on Φ^+ is as follows: for $1 \leq i < j < m$, $c(\alpha_{i\pm j}) = \alpha_{(i+1)\pm(j+1)}$; for $1 \leq i < m$, $c(\alpha_{i-m}) = \alpha_{1+(i+1)}$; for $1 \leq i \leq m-1$, $c(\alpha_{i+m}) \in \Phi^-$; for $1 \leq i \leq m-1$, $c(\alpha_{2\cdot i}) = \alpha_{2\cdot(i+1)}$; $c(\alpha_{2\cdot m}) \in \Phi^-$.

We set $r = m+1$; $\Psi_{m+1} = \{\alpha_{i+m}\}_{i=1}^{m-1} \cup \{\alpha_{2\cdot m}\}$; $\Psi_m = \{\alpha_{i+j}: 1 \leq i < j \leq m\} \cup \{\alpha_{2\cdot i}\}_{i=1}^{m}$; and for $1 \leq i_0 \leq m-1$, $\Psi_{i_0} = \Psi_m \cup \{\alpha_{i-j}: 1 \leq i < j \leq m \text{ and } i_0 < j\}$. Replacing all roots with the corresponding coroots, we obtain by duality the desired partition for type B_m (under the induced isomorphism of Weyl groups, the special Coxeter element for type C_m maps to the one in Section 7.3.2).

7.3. Type B_m

7.3.1. — Let $m \geq 2$. Let V_0 be a $(2m+1)$-dimensional k-vector space with an (ordered) basis $\{e_1, \ldots, e_m, e_{m+1}, e_{-m}, \ldots e_{-1}\}$. Whenever convenient, identify V_0 with k^{2m+1} via this basis. Let $Q: V_0 \to k$ be the split quadratic form given by $Q\left(\sum_{i=1}^{m}(a_ie_i + a_{-i}e_i) + a_{m+1}e_{m+1}\right) = \sum_{i=1}^{m}a_ia_{-i} + a_{m+1}^2$. Let $\mathbf{G} = \mathbf{SO}(V_0, Q)$ be the split odd orthogonal group attached to (V_0, Q); *i.e.*,

$$\mathbf{G}(R) = \{g \in \mathbf{GL}(V_0)(R): \langle gu, gv \rangle = \langle u, v \rangle \text{ for all } u, v \in V_0 \otimes_k R, \det(g) = 1\}.$$

It is of adjoint type. (The computations below work independently of the characteristic and the residue characteristic of k.)

7.3.2. — Let $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ be the diagonal torus and the Borel subgroup consisting of upper triangular matrices in $\mathbf{G}$. Then $\mathbf{T} = \{(t_1, \ldots, t_m, 1, t_m^{-1}, \ldots, t_1^{-1}): t_i \in \mathbb{G}_m\}$ and $X_*(\mathbf{T}) = \bigoplus_{i=1}^{m}\mathbb{Z}\varepsilon_i$, where

$$\varepsilon_i: t \longmapsto (1^{\oplus i-1}, t, 1^{\oplus 2(m-i-1)}, t^{-1}, 1^{\oplus i-1}) \text{ for } 1 \leq i \leq m$$

are cocharacters of $\mathbf{T}$. We have $X_*(\mathbf{T}^{sc}) = \{\sum_{i=1}^{m}a_i\varepsilon_i \in X_*(\mathbf{T}): \sum_i a_i \text{ even}\}$. Thus $\pi_1(\mathbf{G}) = X_*(\mathbf{T})/X_*(\mathbf{T}^{sc}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the class of ε_i for any i.

We have $\Phi = \{\alpha_{\pm i \pm j}: (t_\lambda)_{\lambda=1}^{m} \mapsto t_i^{\pm 1}t_j^{\pm 1}: 1 \leq i < j \leq m\} \cup \{\alpha_i: (t_\lambda)_{\lambda=1}^{m} \mapsto t_i: 1 \leq i \leq m\}$ and $\Delta = \{\alpha_{i-(i+1)}: 1 \leq i \leq m-1\} \cup \{\alpha_m\}$. We have the Coxeter element

$$(7.9) \qquad c = s_{\alpha_{1-2}} \cdot \cdots \cdot s_{\alpha_{(m-1)-m}}s_{\alpha_m} = (1, 2, \ldots, m, 2m+1, 2m, \ldots, m+2).$$

We consider two lifts $\dot{c}_0, \dot{c}_1$ of c to $\mathbf{G}(\check{k})$. The first is given by $\dot{c}_0(e_{\pm i}) = e_{\pm(i+1)}$ for $1 \leq i \leq m-1$, $\dot{c}_0(e_{\pm m}) = e_{\mp 1}$ and $\dot{c}_0(e_{m+1}) = -e_{m+1}$. The second is given by $\dot{c}_1(e_{\pm i}) = e_{\pm(i+1)}$ for $1 \leq i \leq m-1$, $\dot{c}_1(e_{\pm m}) = \varpi^{\pm 1}e_{\mp 1}$ and $\dot{c}_1(e_{m+1}) = -e_{m+1}$. Then $\dot{c}_0, \dot{c}_1$ are basic, and the image of $\dot{c}_0$, resp. $\dot{c}_1$, in $\mathbf{G}^{ad}(\check{k})$ represents $0 \in \pi_1(\mathbf{G})$, resp. $\bar{\varepsilon}_1 \in \pi_1(\mathbf{G})$. Thus the $\mathbf{G}_{\dot{c}_i}$ $(i = 1, 2)$ are the two inner forms of $\mathbf{G}$.

We identify $\mathcal{A}_{\mathbf{T},\check{k}}$ with $X_*(\mathbf{T})_\mathbb{R}$ by sending $0 \in X_*(\mathbf{T})_\mathbb{R}$ to the σ-stable hyperspecial point ("the origin") corresponding to the self-dual lattice $\bigoplus_{i=1}^{m}\left(\mathcal{O}_{\check{k}}e_i \oplus \mathcal{O}_{\check{k}}e_{-i}\right) \oplus \mathcal{O}_{\check{k}}e_{m+1} \subseteq V_0 \otimes_k \check{k}$.

The group ${}^c\mathbf{U} \cap \mathbf{U}^-$ is generated by root subgroups attached to α_{i-1} ($2 \le i \le m$) and α_{-1}. Explicitly, any element $y \in ({}^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is an R-linear map in $\mathbf{GL}(V_0)(R)$ given by $e_1 \mapsto e_1 + \sum_{i=2}^m a_{i-1} e_i + a_m e_{m+1} - a_m^2 e_{-1}$; $e_i \mapsto e_i$ for $2 \le i \le m$ and $i = -1$; $e_{m+1} \mapsto e_{m+1} - 2a_m e_{-1}$; $e_{-i} \mapsto e_{-i} - a_{i-1} e_{-1}$ for $2 \le i \le m$.

7.3.3. — Let $b = \dot{c}_\kappa$ with $\kappa \in \{0, 1\}$. We have $\mathbf{x}_b = 0$ if $\kappa = 0$ and $\mathbf{x}_b = -\frac{1}{2} \sum_{i=1}^m \varepsilon_i$ if $\kappa = 1$. Let $g \in \dot{X}_{b,b}(\mathfrak{f})$. Then (7.4) holds with y as described in Section 7.3.2, depending on some $a_1, \ldots, a_m \in L$. We have to show that $y \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, which is (as in Section 7.1.3) equivalent to

$$(7.10) \qquad \operatorname{ord}_\omega(a_i) \ge 0 \quad \forall 1 \le i \le m-1 \quad \text{and} \quad \operatorname{ord}_\omega(a_m) \ge \frac{\kappa}{2}.$$

Consider the ϕ-linear isocrystal $(V, \varphi) = (V_0 \otimes_k L, b\sigma)$ over $\mathfrak{f}$. It is isoclinic of slope 0. For $i \in \{\pm 1, \ldots, \pm m, m+1\}$, let $g_i := g(e_i)$. Then (7.4) implies that $\varphi(g_i) = gyb(e_i)$ for all i. Using this equation for all $i \ne m+1, m+2$ and writing $v := g_1$ and $u := g_{m+1}$, we see that $g_i = \varphi^{i-1}(v)$ for $2 \le i \le m$, $g_{-1} = \omega^{-\kappa} \varphi^m(v)$ and

$$g_{-i} = \omega^{-\kappa} \left(\varphi^{m+i-1}(v) + \sum_{j=1}^{i-1} \phi^{i-1-j}(a_j) \varphi^{m+i-1-j}(v) \right) \quad \text{for } 2 \le i \le m.$$

In particular, it follows that $\{\varphi^i(v)\}_{i=0}^{2m-1} \cup \{u\}$ are linearly independent over L. Now, evaluating the equation $\varphi(g_i) = gyb(e_i)$ for $i = m+1$ and $i = m+2$ gives

$$(7.11) \qquad 2a_m \varphi^m(v) = u + \varphi(u),$$

$$(7.12) \qquad \varphi^{2m}(v) = v + \sum_{i=1}^{m-1} a_i \varphi^i(v) - a_m^2 \omega^{-\kappa} \varphi^m(v) - \sum_{i=m+1}^{2m} \phi^{i-m}(a_{2m-i}) \varphi^i(v) + a_m u.$$

First suppose $a_m = 0$. Then (7.12) together with the linear independence of $\{\varphi^i(v)\}_{i=0}^{2m-1}$ shows that v spans a $2m$-dimensional sub-isocrystal of V and is a cyclic vector in it. Then Lemma 7.3 and (7.12) show that $\operatorname{ord}_\omega(a_i) \ge 0$ for $1 \le i \le m-1$.

Now suppose $a_m \ne 0$. Then $a_m \in L^\times$, and $\lambda := \frac{a_m}{\phi(a_m)} \in L^\times$ satisfies $\operatorname{ord}_\omega(\lambda) = 0$. Apply $\lambda \varphi(\cdot)$ to (7.12), and add the result to (7.12); then use (7.11) to eliminate u:

$$\lambda \varphi^{2m+1}(v) = v + (a_1 + \lambda)\varphi(v) + \sum_{i=2}^{m-1} (a_i + \lambda \phi(a_{i-1}))\varphi^i(v) + * \cdot \varphi^m(v)$$

$$+ \left(\phi(a_{m-1}) - \lambda \phi(a_m)^2 \omega^{-\kappa} \right) \varphi^{m+1}(v) + \sum_{i=m+2}^{2m} * \cdot \varphi^i(v),$$

where the $*$ denote some unspecified elements of L. Moreover, v is a cyclic vector for V. Indeed, $\{\varphi^i(v)\}_{i=0}^{2m-1} \cup \{u\}$ is a basis of V, and hence it follows from (7.12) and $a_m \ne 0$, that $\{\varphi^i(v)\}_{i=0}^{2m}$ also is. Using the cyclicity of v, the last equation, the fact that $\operatorname{ord}_\omega(\lambda) = 0$ and Lemma 7.3, we deduce by induction on i that $\operatorname{ord}_\omega(a_i) \ge 0$ for $i = 1, \ldots, m$ and that, moreover, $\operatorname{ord}_\omega(a_m) \ge \frac{1}{2}$ if $\kappa = 1$. This agrees with what we had to show in (7.10).

7.4. Type D_m

7.4.1. — Let $m \ge 4$. Let V_0 be a $2m$-dimensional k-vector space with an (ordered) basis $\{e_1, \ldots, e_m, e_{-m}, \ldots e_{-1}\}$. Whenever convenient, we use this basis to identify V_0 with k^{2m}. Consider the quadratic form $Q\left(\sum_{i=1}^m (a_i e_i + a_{-i} e_{-i}) \right) = \sum_{i=1}^m a_i a_{-i}$ on V_0. We have the orthogonal group $\mathbf{O}(V_0, Q) \subseteq \mathbf{GL}(V_0)$. Via the Clifford algebra attached to Q, we have the surjective Dickson morphism $D_Q : \mathbf{O}(V_0, Q) \to \mathbb{Z}/2\mathbb{Z}$;

cf. [Con14, Proposition C.2.2]. Its kernel is the special orthogonal group $\mathbf{SO}(V_0, Q)$ of Q. Finally, let $\mathbf{G} = \mathbf{GSO}(V_0, Q)$ be the closed subgroup of $\mathbf{GL}(V_0)$ of elements which up to a scalar lie in $\mathbf{SO}(V_0, Q)$.[6]

7.4.2. — Let $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ be the diagonal torus and the Borel subgroup consisting of upper triangular matrices in $\mathbf{G}$. Then $\mathbf{T} = \{(t_0 t_1, \ldots, t_0 t_m, t_m^{-1}, \ldots, t_1^{-1}) \colon t_i \in \mathbb{G}_m\}$ and $X_*(\mathbf{T}) = \bigoplus_{i=0}^m \mathbb{Z}\varepsilon_i$, where

$$\varepsilon_0 \colon t \longmapsto (t^{\oplus m}, 1^{\oplus m}) \quad \text{and} \quad \varepsilon_i \colon t \longmapsto (1^{\oplus i-1}, t, 1^{\oplus 2(m-i)}, t^{-1}, 1^{\oplus i-1}) \text{ for } 1 \le i \le m$$

are cocharacters of $\mathbf{T}$. We have $X_*(\mathbf{T}^{\mathrm{sc}}) = \mathbb{Z}\Phi^\vee = \{\sum_{i=1}^m \lambda_i \varepsilon_i \colon \sum_i \lambda_i \text{ even}\}$, $X_*(\mathbf{T}^{\mathrm{der}}) = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ and $X_*(\mathbf{Z}) = \mathbb{Z}(2\varepsilon_0 - \sum_{i=1}^m \varepsilon_i)$. We have $\pi_1(\mathbf{G}^{\mathrm{ad}}) = X_*(\mathbf{T})/\mathbb{Z}\Phi^\vee + X_*(\mathbf{Z})$. If m is even, $\pi_1(\mathbf{G}^{\mathrm{ad}}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the classes of ε_0 and ε_i for an arbitrary $1 \le i \le m$. If m is odd, $\pi_1(\mathbf{G}^{\mathrm{ad}}) \cong \mathbb{Z}/4\mathbb{Z}$, generated by the class of ε_0. In any case, $\pi_1(\mathbf{G}^{\mathrm{ad}})$ has the distinguished subgroup $\pi_1(\mathbf{G}^{\mathrm{der}}) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the class of some (*i.e.*, any) ε_i $(1 \le i \le m)$.

We have $\Phi = \{\alpha_{\pm i \pm j} \colon (t_\lambda)_{\lambda=1}^m \mapsto t_i^{\pm 1} t_j^{\pm 1} \colon 1 \le i < j \le m\}$ and $\Delta = \{\alpha_{i-(i+1)} \colon 1 \le i \le m-1\} \cup \{\alpha_{(m-1)+m}\}$. We have the Coxeter element

$$(7.13) \qquad c = s_{\alpha_{1-2}} \cdot \cdots \cdot s_{\alpha_{(m-1)-m}} s_{\alpha_{(m-1)+m}} = (1, 2, \ldots, m-2, m-1, 2m, 2m-1, \ldots, m+1)(m, m+1).$$

We consider three lifts $\dot{c}_0, \dot{c}_1, \dot{c}_2$ of c to $\mathbf{G}(\breve{k})$. For $\kappa \in \{0, 1\}$, $\dot{c}_\kappa$ is given by $\dot{c}_\kappa(e_{\pm i}) = e_{\pm(i+1)}$ for $1 \le i \le m-2$, $\dot{c}_\kappa(e_{\pm(m-1)}) = e_{\pm 1}$, $\dot{c}_\kappa(e_{\pm m}) = \varpi^{\mp \kappa} e_{\mp m}$. The third is given by $\dot{c}_2 = \dot{c}_0 \varpi^{\varepsilon_0}$. Then $\tilde{\kappa}_{\mathbf{G}}(\dot{c}_0) = 0$, $\tilde{\kappa}_{\mathbf{G}}(\dot{c}_1)$ is the non-trivial class in $\pi_1(\mathbf{G}^{\mathrm{der}})$ and $\tilde{\kappa}_{\mathbf{G}}(\dot{c}_2) = \bar{\varepsilon}_0 \in \pi_1(\mathbf{G}) \smallsetminus \pi_1(\mathbf{G}^{\mathrm{der}})$. Thus the $\mathbf{G}_{\dot{c}_\kappa}$ $(\kappa \in \{0, 1, 2\})$ are three of the four inner forms of $\mathbf{G}$. The fourth inner form is conjugate to $(\mathbf{G}, \dot{c}_2)$ by the outer automorphism of $\mathbf{G}$ induced by an inner automorphism of the orthogonal group $\mathbf{GO}(V_0)$; hence we need not to consider it separately.

We identify $\mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \breve{k}}$ with $X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{R}}$ by sending $0 \in X_*(\mathbf{T})_{\mathbb{R}}$ to the σ-stable hyperspecial point ("the origin") corresponding to the self-dual lattice $\bigoplus_{i=1}^m (\mathcal{O}_{\breve{k}} e_i \oplus \mathcal{O}_{\breve{k}} e_{-i}) \subseteq V_0 \otimes_k \breve{k}$.

The group $^c\mathbf{U} \cap \mathbf{U}^-$ is generated by root subgroups attached to α_{i-1} $(2 \le i \le m)$ and α_{-1-m}. Explicitly, any element $n \in (^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is an R-linear map in $\mathbf{GL}(V_0)(R)$ given by $e_1 \mapsto e_1 + \sum_{i=2}^m a_{i-1} e_i + a_m e_{-m} - a_{m-1} a_m e_{-1}$; $e_i \mapsto e_i$ for $2 \le i \le m-1$ and $i = -1$; $e_m \mapsto e_m - a_m e_{-1}$; $e_{-i} \mapsto e_{-i} - a_{i-1} e_{-1}$ for all $2 \le i \le m$.

7.4.3. — Let $b = \dot{c}_\kappa$ with $\kappa \in \{0, 1, 2\}$. We compute $\mathbf{x}_b = 0$ if $\kappa = 0$; $\mathbf{x}_b = -\frac{1}{2}\varepsilon_m$ if $\kappa = 1$; $\mathbf{x}_b = -\frac{1}{4}\varepsilon_m + \sum_{i=1}^{m-1}(-\frac{1}{4} + \frac{1}{2}(i-1))\varepsilon_i$ if $\kappa = 2$. Let $g \in \dot{X}_{b,b}(\mathbb{f})$. Then (7.4) holds, with n as described in Section 7.4.2, depending on some $a_1, \ldots, a_m \in L$. We have to show that $y \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, which (as in Section 7.1.3) is equivalent to

$$(7.14) \qquad \begin{aligned} &\mathrm{ord}_\varpi(a_i) \ge 0 \text{ for } 1 \le i \le m && \text{if } \kappa = 0, \\ &\mathrm{ord}_\varpi(a_i) \ge 0 \text{ for } 1 \le i \le m-1 \text{ and } \mathrm{ord}_\varpi(a_m) \ge 1 && \text{if } \kappa = 1, \\ &\mathrm{ord}_\varpi(a_i) \ge \frac{i}{2} \text{ for } 1 \le i \le m-2, \mathrm{ord}_\varpi(a_{m-1}) \ge \frac{m-2}{4} \text{ and } \mathrm{ord}_\varpi(a_m) \ge \frac{m}{4} && \text{if } \kappa = 2. \end{aligned}$$

Consider the ϕ-linear isocrystal $(V, \varphi) = (V_0 \otimes_k L, b\sigma)$ over $\mathbb{f}$. It is isoclinic of slope 0 if $\kappa \in \{0, 1\}$ and slope $\frac{1}{2}$ if $\kappa = 2$. For $i \in \{\pm 1, \ldots, \pm m\}$, put $g_i := \varphi(e_i)$. Then (7.4) implies that $\varphi(g_i) = gyb(e_i)$ for all i. Using this equation for all $i \ne m+1, m+2$ and writing $v := g_1$ and $u := g_m$, we compute that $g_i = \varpi^{A_i} \varphi^{i-1}(v)$ for $2 \le i \le m$, $g_{-1} = \varpi^{A_{-1}} \varphi^m(v)$, $g_{-m} = \varpi^{A_{-m}} \varphi(u) + \varpi^{A'_{-m}} a_{m-1} \varphi^{m-1}(v)$ and

$$g_{-i} = \varpi^{A_{-i}}\left(\varphi^{m+i-2}(v) + \sum_{d=m-1}^{m-3+i} \phi^{d-m+1}(a_{m-d+i-2})\varphi^d(v)\right) \quad \text{for } 2 \le i \le m-1,$$

---

[6]If $\mathrm{char}\, k \ne 2$, then the Dickson morphism is not necessary to describe $\mathbf{G}$. For any k-algebra R, we have

$$\mathbf{G}(R) = \left\{g \in \mathbf{GL}(V_0)(R) \colon Q(gv) = \lambda(g)Q(v) \text{ for all } u, v \in V_0 \otimes_k R \text{ and } \frac{\det(g)}{\lambda(g)^m} = 1\right\},$$

where $\lambda(g) \in R^\times$ is a unit only depending on g.

where $A_i = A'_{-m} = 0$ for all $i \in \{\pm 1, \ldots, \pm m\}$ if $\kappa = 0$; $A_i = A'_{-m} = 0$ for all $i \neq -m$ and $A_{-m} = -1$ if $\kappa = 1$; $A_i = 1 - i$ for $2 \leq i \leq m - 1$, $A_{-1} = A'_{-m} = A_{-i} = 1 - m$ for $2 \leq i \leq m - 1$ and $A_{-m} = -1$ if $\kappa = 2$.

Lemma 7.4. *The vectors $\{\varphi^i(v)\}_{i=0}^{2m-3}$, $u, \varphi(u)$ form a basis of V.*

Proof. From the formulas for the $g_{\pm i}$ above, it follows that the vectors are linearly independent, hence form a basis by dimension reasons. $\qquad\square$

Now, evaluating the equation $\varphi(g_i) = gyb(e_i)$ for $i = m + 1$ and $i = m + 2$ gives the formulas

$$(7.15) \qquad \varphi^{2m-2}(v) = \varpi^{B_0} v + \sum_{i=1}^{m-2} \varpi^{B_i} a_i \varphi^i(v) - \sum_{i=m}^{2m-3} \phi^{i-m+1}(a_{2m-2-i}) \varphi^i(v) + \varpi^{B'_0} a_{m-1} u + \varpi^{B'_1} a_m \varphi(u),$$

$$(7.16) \qquad \varphi^2(u) = \varpi^{C_0} u - \varpi^{C_1} \phi(a_{m-1}) \varphi^m(v) - \varpi^{C_2} a_m \varphi^{m-1}(v),$$

where all B_i, B'_i, C_i are 0 if $\kappa = 0$; $B'_1 = -1$, $C_1 = 1$ and all other B_i, B'_i, C_i are 0 if $\kappa = 1$; $B_i = m - 1 - i$ for $0 \leq i \leq m - 2$, $B'_0 = B'_1 = m - 2$, $C_0 = 1$ and $C_1 = C_2 = 2 - m$ if $\kappa = 2$. Put

$$\mu := \varpi^D a_m \phi(a_m) - a_{m-1} \phi(a_{m-1}) \in L,$$

where $D = 0$ if $\kappa = 0$; $D = -2$ if $\kappa = 1$; $D = 1$ if $\kappa = 2$. We have five cases, which we consider separately in the following subsections.

7.4.4. Case $a_m \neq 0$, $a_{m-1} \neq 0$, $\mu \neq 0$.— Suppose $a_m a_{m-1} \mu \neq 0$. Apply $\frac{a_m}{\phi(a_{m-1})} \varphi(\cdot)$ to (7.15), and subtract (7.15) from the resulting equation. This gives an expression of $\frac{a_m}{\phi(a_{m-1})} \varphi^{2m-1}(v)$ as a linear combination of $\{\varphi^i(v)\}_{i=0}^{2m-2}$, $\varphi^2(u)$ and u. Substitute (7.16) into this expression to eliminate $\varphi^2(u)$. The coefficient of u in the resulting equation is $\mu \neq 0$, and it gives an expression of u as a linear combination of $\{\varphi^i(v)\}_{i=0}^{2m-1}$:

$$\varpi^E u = \varpi^{E_0} \frac{\phi(a_{m-1})}{\mu} v - \frac{\varpi^{E_1} a_m - \varpi^{E'_1} \phi(a_{m-1}) a_1}{\mu} \varphi(v) - \sum_{i=2}^{m-2} \frac{\varpi^{E_i} a_m \phi(a_{i-1}) - \varpi^{E'_i} \phi(a_{m-1}) a_i}{\mu} \varphi^i(v)$$

$$(7.17) \qquad - \frac{\varpi^{E_{m-1}} a_m \phi(a_{m-2}) - \varpi^{E'_{m-1}} a_m^2 \phi(a_m)}{\mu} \varphi^{m-1}(v)$$

$$- \frac{\phi(a_{m-1}) \phi(a_{m-2}) - \varpi^{E'_m} \phi(a_{m-1}) a_m \phi(a_m)))}{\mu} \varphi^m(v) + \sum_{i=m+1}^{2m-1} * \cdot \varphi^i(v) + \varpi^{E_{2m-1}} \frac{a_m}{\mu} \varphi^{2m-1}(v),$$

where the $* \in L$ denote unspecified coefficients; all $E, E_i, E'_i = 0$ if $\kappa = 0$; $E = E_0 = E'_i = 0$ ($1 \leq i \leq m - 2$), $E'_m = E_{2m-1} = E_i = -1$ ($1 \leq i \leq m - 1$), $E'_{m-1} = -2$ if $\kappa = 1$; $E = m - 2$, $E_0 = m - 1$, $E_i = m - i$ and $E'_i = m - 1 - i$ for $1 \leq i \leq m - 1$, $E'_m = E_{2m-1} = 0$ if $\kappa = 2$.

As $a_m \neq 0$, the coefficient of $\varphi^{2m-1}(v)$ in (7.17) is non-zero. Using this observation, as well as the version of it after applying $\varphi(\cdot)$, shows, together with Lemma 7.4, that v is a cyclic vector for V. Now insert (7.17) into (7.15) to eliminate u there. This gives (after multiplication with an element in $L^\times$) an expression

$$\frac{\mu}{\phi(\mu)} \varphi^{2m}(v) = \sum_{i=0}^{2m-1} \beta_i \varphi^i(v)$$

with $\beta_i \in L$ satisfying $\mathrm{ord}_\varpi(\beta_i) \geq (2m - i) \cdot \lambda$, where λ is the slope of (V, φ), by Lemma 7.3. Let $\gamma := -\frac{a_{m-1}}{\phi(a_m)} + \frac{\phi^2(a_{m-1})\mu}{\phi(a_m)\phi(\mu)}$.

Lemma 7.5. *We have $\mathrm{ord}_\varpi(\gamma) \geq 0$ if $\kappa = 0$, $\mathrm{ord}_\varpi(\gamma) \geq -1$ if $\kappa = 1$ and $\mathrm{ord}_\varpi(\gamma) \geq 1$ if $\kappa = 2$.*

Proof. One checks that $\gamma = \varpi^D \cdot \frac{\phi^2(a_{m-1})a_m - \phi^2(a_m)a_{m-1}}{\phi(\mu)}$. First assume $\kappa = 0$. If $\mathrm{ord}_\varpi(a_{m-1}) > \mathrm{ord}_\varpi(a_m)$, then $\mathrm{ord}_\varpi(\phi^2(a_{m-1})a_m - \phi^2(a_m)a_{m-1}) \geq \mathrm{ord}_\varpi(a_{m-1}) + \mathrm{ord}_\varpi(a_m) \geq 2\,\mathrm{ord}_\varpi(a_m) = \mathrm{ord}_\varpi(\phi(\mu))$, and we are done as $D = 0$ in this case. One concludes similarly in the case $\mathrm{ord}_\varpi(a_m) < \mathrm{ord}_\varpi(a_{m-1})$. Finally, suppose

$\mathrm{ord}_\varpi(a_{m-1}) = \mathrm{ord}_\varpi(a_m) =: \alpha$. Write $x = \varpi^\alpha a_m$, $y = \varpi^\alpha a_{m-1}$, so that $x, y \in \mathcal{O}_L^\times$. Write $A := x\phi(x) - y\phi(y)$, $B := x\phi^2(y) - y\phi^2(x) \in \mathcal{O}_L$. It suffices to show that $\mathrm{ord}_\varpi(B) \geq \mathrm{ord}_\varpi(A)$. We now compute in $\mathcal{O}_L / \left(\varpi^{\mathrm{ord}_\varpi(A)}\right)$, where we surely have $A = 0$, i.e., $x\phi(x) = y\phi(y)$, or equivalently $\phi\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)^{-1}$. Exploiting this equation twice, we see $\phi^2\left(\frac{x}{y}\right) = \phi\left(\left(\frac{x}{y}\right)^{-1}\right) = \frac{x}{y}$, which shows that $\phi^2(x)y = x\phi^2(y)$ in $\mathcal{O}_L / \left(\varpi^{\mathrm{ord}_\varpi(A)}\right)$, i.e. $B = 0$ in $\mathcal{O}_L / \left(\varpi^{\mathrm{ord}_\varpi(A)}\right)$, i.e., $\mathrm{ord}_\varpi(B) \geq \mathrm{ord}_\varpi(A)$.

Suppose $\kappa = 1$. Write $a_m = \varpi a_m'$. Then $\gamma = \varpi^{-1} \cdot \frac{\phi^2(a_{m-1})a_m' - \phi^2(a_m')a_{m-1}}{\phi(a_m'\phi(a_m') - a_{m-1}\phi(a_{m-1}))}$, where the fraction is just γ from case $\kappa = 0$ for a_{m-1} and a_m', and we are done by applying the above.

Finally, suppose $\kappa = 2$. Then $\mu = \varpi a_m \phi(a_m) - a_{m-1}\phi(a_{m-1})$, and the claim is immediately checked in both cases, $\mathrm{ord}_\varpi(a_m) < \mathrm{ord}_\varpi(a_{m-1})$ and the opposite. $\qquad\square$

Now we proceed case by case.

Case $b = \dot{c}_0$.— The slope is $\lambda = 0$; i.e., $\mathrm{ord}_\varpi(\beta_i) \geq 0$ for all i. One computes $\beta_1 = a_1 + \gamma$ and $\beta_2 = a_2 + \phi(a_1)\gamma - \frac{\mu}{\phi(\mu)}$. As $\mathrm{ord}_\varpi\left(\frac{\mu}{\phi(\mu)}\right) = 0$, Lemma 7.5 implies $\mathrm{ord}_\varpi(a_i) \geq 0$ for $i = 1, 2$. Using this as the first step, and computing $\beta_i = a_i - \frac{\mu\phi^2(a_{i-2})}{\phi(\mu)} + \phi(a_{i-1})\gamma$ for $3 \leq i \leq m-2$, we conclude by induction (using Lemma 7.5) that $\mathrm{ord}_\varpi(a_i) \geq 0$ for all $1 \leq i \leq m-2$. Next, we have $\beta_{m-1} = a_{m-1}a_m - \frac{\mu\phi^2(a_{m-3})}{\phi(\mu)} + \phi(a_{m-2})\gamma$, which (using the above) implies that $\mathrm{ord}_\varpi(a_{m-1}a_m) \geq 0$. Thus, if $\mathrm{ord}_\varpi(a_{m-1}) = \mathrm{ord}_\varpi(a_m)$, we deduce that this number is at least 0, and we are done according to (7.14). Finally, suppose $\mathrm{ord}_\varpi(a_{m-1}) \neq \mathrm{ord}_\varpi(a_m)$. We have $\beta_m = -\left(1 + \frac{\mu}{\phi(\mu)}\right)\phi(a_{m-2}) + a_{m-1}\phi(a_{m-1}) + \frac{\mu}{\phi(\mu)}\phi(a_m)\phi^2(a_m)$, which shows (as $\mathrm{ord}_\varpi(\beta_m) \geq 0$) that $\mathrm{ord}_\varpi\left(a_{m-1}\phi(a_{m-1}) + \frac{\mu}{\phi(\mu)}\phi(a_m)\phi^2(a_m)\right) \geq 0$. As the two summands have different valuations by assumption, both valuations must be non-negative, which implies $\mathrm{ord}_\varpi(a_{m-1}), \mathrm{ord}_\varpi(a_m) \geq 0$. We are done by (7.14).

Case $b = \dot{c}_1$.— The slope is again $\lambda = 0$, so that $\beta_i \geq 0$ for all i. We have $\beta_1 = a_1 + \varpi\gamma$, $\beta_2 = a_2 - \frac{\mu}{\phi(\mu)} + \varpi\phi(a_1)\gamma$ and $\beta_i = a_i - \frac{\mu\phi^2(a_{i-2})}{\phi(\mu)} + \varpi\phi(a_{i-1})\gamma$ for $3 \leq i \leq m-2$. As in the case $b = \dot{c}_0$, we deduce $\mathrm{ord}_\varpi(a_i) \geq 0$ for $1 \leq i \leq m-2$. We have $\beta_{m-1} = a_{m-1}a_m + \varpi\phi(a_{m-2})\gamma$. Using Lemma 7.5, we deduce that if $\mathrm{ord}_\varpi(a_{m-1}) = \mathrm{ord}_\varpi(a_m) - 1$, then this number must be non-negative, and we are done according to (7.14). Assume $\mathrm{ord}_\varpi(a_{m-1}) \neq \mathrm{ord}_\varpi(a_m) - 1$. We compute $\beta_m = -\left(1 + \frac{\mu}{\phi(\mu)}\right)\phi(a_{m-2}) + \varpi a_{m-1}\phi(a_{m-1}) + \varpi^{-1}\frac{\mu}{\phi(\mu)}\phi(a_m)\phi^2(a_m)$, and it follows that $\mathrm{ord}_\varpi(a_{m-1}), \mathrm{ord}_\varpi(a_m) - 1 \geq 0$.

Case $b = \dot{c}_2$.— The slope is $\lambda = \frac{1}{2}$, so that for any i, $\mathrm{ord}_\varpi(\beta_i) \geq m - \frac{i}{2}$, or equivalently $\mathrm{ord}_\varpi(\beta_i) \geq m - \lfloor\frac{i}{2}\rfloor$. We have $\beta_1 = \varpi^{m-1}(a_1 + \gamma)$, $\beta_2 = \varpi^{m-2}(a_2 + \gamma\phi(a_1) - \varpi\frac{\mu}{\phi(\mu)})$, $\beta_i = \varpi^{m-i}(a_i + \gamma\phi(a_{i-1}) - \varpi\frac{\mu}{\phi(\mu)}\phi^2(a_{i-2}))$ for $3 \leq i \leq m-2$, $\beta_{m-1} = a_{m-1}a_m + \varpi\gamma\phi(a_{m-2}) - \varpi^2\frac{\mu}{\phi(\mu)}\phi^2(a_{m-3})$ and $\beta_m = -\varpi(1 + \frac{\mu}{\phi(\mu)})\phi(a_{m-2}) + a_{m-1}\phi(a_{m-1}) + \frac{\mu}{\phi(\mu)}\phi(a_m)\phi^2(a_m)$. As in the previous two cases, we deduce that (7.14) holds.

In the following cases, we do not give all details as the computations are similar to (and easier than) those in Section 7.4.4.

7.4.5. Case $a_{m-1} \neq 0$, $a_m = 0$.— In this case (7.16) gives $\varphi^m(v) = \frac{1}{\phi(a_{m-1})}(u - \varphi^2(u))$. By Lemma 7.4 this implies that u is a cyclic vector for V. Apply $\varphi^m(\cdot)$ to (7.15), and insert the above expression of $\varphi^m(v)$ in terms of u to eliminate v. This gives an expression $\varphi^{2m}(u) = \sum_{i=0}^{2m-1} \beta_i\varphi^i(u)$. Making all β_i explicit for $1 \leq i \leq m$ and using Lemma 7.3, one immediately deduces (7.14).

7.4.6. Case $a_{m-1} = 0$, $a_m \neq 0$.— This case is completely analogous to the one in Section 7.4.5 (one has $\varphi^{m-1}(v) = \frac{1}{a_{m-1}}(u - \varphi^2(u))$, from which the cyclicity of u and an explicit formula $\varphi^{2m}(u) = \sum_{i=0}^{2m-1} \beta_i\varphi^i(u)$ follow).

7.4.7. Case $a_{m-1} = 0$, $a_m = 0$.— By Lemma 7.4 v is a cyclic vector for the $(2m-2)$-dimensional subisocrystal with L-basis $\{\varphi^i(v)\}_{i=0}^{2m-3}$. The result follows easily from Lemma 7.3 by looking at the coefficients of (7.15).

7.4.8. Case $a_{m-1} \neq 0$, $a_m \neq 0$, $\mu = 0$.— (Note that $\kappa \neq 2$ in this case.) As $a_{m-1} \neq 0$, (7.15) along with Lemma 7.4 show that the vectors $\{\varphi^i(v)\}_{i=0}^{2m-2}$ are linearly independent. Further, following the same steps as in the first lines of Section 7.4.4 and exploiting $\mu = 0$, we arrive at an expression $\frac{a_m}{\phi(a_{m-1})}\varphi^{2m-1}(v) = \sum_{i=0}^{2m-2} \beta_i \varphi^i(v)$, thus showing that v is a cyclic vector for the $(2m-1)$-dimensional subisocrystal generated by it. Proceeding as in the previous cases, one verifies (7.14). This finishes the proof in type D_m.

7.4.9. Proof of Lemma 5.6 for type D_m.— The action of c on Φ^+ is as follows: for $1 \leq i < j < m-1$, $c(\alpha_{i\pm j}) = \alpha_{(i+1)\pm(j+1)}$; for $1 \leq i \leq m-2$, $c(\alpha_{i-(m-1)}) = \alpha_{1+(i+1)}$; for $1 \leq i \leq m-2$, $c(\alpha_{i\pm m}) = \alpha_{(i+1)\mp m}$; for $1 \leq i \leq m-2$, $c(\alpha_{i+(m-1)}) \in \Phi^-$, $c(\alpha_{(m-1)\pm m}) \in \Phi^-$.

Set $r = m+1$; $\Psi_{m+1} = \{\alpha_{i+(m-1)}\}_{i=1}^{m-2} \cup \{\alpha_{(m-1)-m}, \alpha_{(m-1)+m}\}$; $\Psi_m = \{\alpha_{i+j} : 1 \leq i < j \leq m\} \cup \{\alpha_{(m-1)-m}, \alpha_{(m-1)+m}\}$; $\Psi_{m-1} = \Psi_m \cup \{\alpha_{i-m}\}_{i=1}^{m-2} \cup \{\alpha_{i+m}\}_{i=1}^{m-2}$; for $1 \leq i_0 \leq m-2$, $\Psi_{i_0} = \Psi_{m-1} \cup \{\alpha_{i-j} : 1 \leq i < j \leq m-1 \text{ and } j > i_0\}$.

7.5. Type $^2A_{n-1}$

7.5.1. — Let $n \geq 2$. Let $m = \lfloor \frac{n}{2} \rfloor$, so that $n = 2m+1$ if n odd and $n = 2m$ if n is even. Let V_0, e_i be as in Section 7.1.1. Let $\mathbf{G}$ be the unitary group attached to the standard hermitian form on $V_0 \otimes_k k_2$, relative to the unramified extension k_2/k of degree 2. We may identify $\mathbf{G} \otimes_k k_2 = \mathbf{GL}_n(V_0 \otimes_k k_2)$, so that for any $\mathbb{F}_q$-algebra R_0 with $\widetilde{R} = \mathrm{W}(R_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)[1/\varpi]$, the Frobenius σ on $\mathbf{G}(\widetilde{R}) = \mathbf{GL}_n(V_0)(\widetilde{R})$ is given by

$$g \longmapsto \sigma(g) = J\sigma_0(g)^{-1,T}J^{-1},$$

where σ_0 is the Frobenius on $\mathbf{GL}(V_0)(\widetilde{R})$ corresponding to the natural k-structure on $\mathbf{GL}(V_0)$, $(\cdot)^T$ is transposition and $J \in \mathbf{GL}_n(V_0)(k)$ is given by $e_i \mapsto e_{n+1-i}$ ($1 \leq i \leq n$).

7.5.2. — Let $\mathbf{T}, \mathbf{B}, \beta_i, \alpha_{i-j}$ be as in Section 7.1.2. Note that $\mathbf{T}, \mathbf{B}$ are both k-rational (with respect to σ). Moreover, the action of σ on $\pi_1(\mathbf{G}^{\mathrm{ad}})$ is by multiplication with -1. Thus, $\pi_1(\mathbf{G}^{\mathrm{ad}})_{\langle\sigma\rangle} = 1$ if n is odd, and $\pi_1(\mathbf{G}^{\mathrm{ad}})_{\langle\sigma\rangle} \cong \mathbb{Z}/2\mathbb{Z}$, with the non-trivial element represented by $\bar{\beta}_i$ for any i, when n is even. We have an isomorphism $\mathbb{Z} \xrightarrow{\sim} \pi_1(\mathbf{G})$, $1 \mapsto$ class of β_1, under which $X_*(\mathbf{Z}) \subseteq \pi_1(\mathbf{G})$ corresponds to $n\mathbb{Z}$. The action of σ corresponds to multiplication by -1. Justifying Remark 7.1 in this case, we deduce from this that the induced map $X_*(\mathbf{Z})_{\langle\sigma\rangle} \cong \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbf{G})_{\langle\sigma\rangle}$ is an isomorphism when n odd and is 0 when n even.

The sets Φ and Δ are the same as in Section 7.1.2. The action of σ on Δ is given by $\sigma: \alpha_{i-(i+1)} \mapsto \alpha_{(n-i)-(n-i+1)}$. We have the Coxeter element

$$(7.18) \qquad\qquad c = s_{\alpha_{1-2}} \cdots \cdots s_{\alpha_{m-(m+1)}} = (1, 2, \ldots, m, m+1).$$

We consider the lift $\dot{c}_0$ of c to $\mathbf{G}(\check{k})$, given by $\dot{c}_0(e_i) = e_{i+1}$ for $1 \leq i \leq m$, $\dot{c}_0(e_{m+1}) = e_1$ and $\dot{c}_0(e_i) = e_i$ for the remaining i. If $n = 2m$ is even, we also consider the lift $\dot{c}_1$ given by $\dot{c}_1(e_i) = \dot{c}_0(e_i)$ for $i \neq m+1$, $\dot{c}_1(e_{m+1}) = \varpi e_1$. Then $\dot{c}_0, \dot{c}_1$ are basic, and if n is even, $\bar{\kappa}_{\mathbf{G}}(\dot{c}_0) = 0$, $\bar{\kappa}_{\mathbf{G}}(\dot{c}_1) \neq 0$ in $\pi_1(\mathbf{G})_{\langle\sigma\rangle}$. Thus, if $n = 2m+1$ is odd, the quasi-split group $\mathbf{G}$ (represented by the pair $(\mathbf{G}, \dot{c}_0)$) has no non-trivial inner forms; if $n = 2m$ is even, there are two inner forms, represented by the pairs $(\mathbf{G}, \dot{c}_\kappa)$ with $\kappa \in \{0, 1\}$.

We identify $\mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \check{k}}$ with $X_*(\mathbf{T}^{\mathrm{ad}})_{\mathbb{R}}$ by sending 0 to the σ-stable hyperspecial point ("the origin") corresponding to the standard lattice $\bigoplus_{i=1}^n \mathcal{O}_{\check{k}} e_i$.

The group $^c\mathbf{U} \cap \mathbf{U}^-$ is generated by the root subgroups attached to α_{i-1} ($2 \leq i \leq m+1$). For a $\check{k}$-algebra R, any element $y \in (^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is an element of $\mathbf{GL}(V_0)(R)$, determined by $e_1 \mapsto e_1 + \sum_{\lambda=1}^m a_\lambda e_{\lambda+1}$, $e_i \mapsto e_i$ ($2 \leq i \leq n$) for some $a_1, \ldots, a_m \in R$.

7.5.3. — Let $b = \dot{c}_\kappa$ with $\kappa = 0$ if n odd and $\kappa \in \{0, 1\}$ if n even. We compute that $\mathbf{x}_b = 0$ if $\kappa = 0$ and that $\mathbf{x}_b$ is the image of $\sum_{\lambda=1}^m \frac{1}{2}\beta_\lambda - \sum_{\lambda=m+2}^n \frac{1}{2}\beta_\lambda$ under $X_*(\mathbf{T})_\mathbb{R} \twoheadrightarrow X_*(\mathbf{T}^{\mathrm{ad}})_\mathbb{R} \cong \mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \check{k}}$ if $\kappa = 1$ (and $n = 2m$ even). Let $g \in \dot{X}_{b,b}(\mathfrak{f})$. Then (7.4) holds with y as described in Section 7.5.2, depending on some $a_1, \ldots, a_m \in L$. We have to show that $y \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, which is (as in Section 7.1.3) equivalent to

$$\text{(7.19)} \qquad \mathrm{ord}_\varpi(a_i) \geq 0 \quad \forall\, 1 \leq i \leq m$$

(independently of what κ is).

Let $c^{\mathrm{spl}} = c\sigma(c)$, and let $\dot{c}_\kappa^{\mathrm{spl}} = \dot{c}_\kappa \sigma(\dot{c}_\kappa)$. We write b^{spl} for $\dot{c}^{\mathrm{spl}}$ if $b = \dot{c}_\kappa$. We have the automorphism $b\sigma \colon g \mapsto \dot{b}\sigma(g)$ of $L\mathbf{G}$ (regarded as a sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$). As $\sigma^2 = \sigma_0^2$ (recall the identification in Section 7.5.1), we have

$$\text{(7.20)} \qquad (b\sigma)^2 = b^{\mathrm{spl}}\sigma^2 = b^{\mathrm{spl}}\sigma_0^2.$$

Let $V = V_0 \otimes_k L$, and denote by $\sigma_0 \colon V \to V$ the Frobenius automorphism $v \otimes \lambda \mapsto v \otimes \phi(\lambda)$. Then $\varphi := b^{\mathrm{spl}}\sigma_0^2 \colon V \to V$ is a ϕ^2-linear automorphism of V. Moreover, (V, φ) is an isocrystal over $\mathfrak{f}$ (relative to $\mathbb{F}_{q^2}$) of slope 0.

For $1 \leq i \leq n$, let $g_i := g(e_i)$. Then $\sigma_0^2(g)(e_i) = \sigma_0^2(g_i)$. Using (7.20), (7.4) applied twice gives the equation

$$\text{(7.21)} \qquad b^{\mathrm{spl}}\sigma_0^2(g) = g \cdot \left(y \cdot {}^b\sigma(y)\right) \cdot b^{\mathrm{spl}}.$$

Applying both sides of this to $e_i \in V$ and using the above, for $1 \leq i \leq n$, we get

$$\text{(7.22)} \qquad \varphi(g_i) = g \cdot \left(y \cdot {}^b\sigma(y)\right) \cdot b^{\mathrm{spl}}(e_i).$$

7.5.4. — Assume that $n = 2m + 1$ is odd (and $b = \dot{c}_0$). Then

$$c^{\mathrm{spl}} = (1, 2, \ldots, m, m+1, 2m+1, 2m, \ldots, m+3, m+2),$$

b^{spl} is its lift with all entries 1 or 0, and $y \cdot {}^b\sigma(y)$ is given by $e_1 \mapsto e_1 + \sum_{\lambda=1}^m a_i e_{i+1} - \phi(a_m)e_{2m+1}$; $e_i \mapsto e_i$ for $2 \leq i \leq m+1$ and $i = 2m+1$; $e_i \mapsto e_i - \phi(a_{2m+1-i})e_{2m+1}$ for $m+2 \leq i \leq 2m$. Put $v := g_1$. Applying (7.22) for all $i \neq m+2$, we deduce that $g_i = \varphi^{i-1}(v)$ for $1 \leq i \leq m+1$, $g_i = \varphi^{3m+2-i}(v) + \sum_{\lambda=1}^{2m+1-i} \phi^{1+2(2m-i-\lambda+1)}(a_\lambda)\phi^{3m+2-i-\lambda}(v)$ for $m+2 \leq i \leq 2m+1$. As $g \in \mathbf{GL}(V_0)(\check{k})$, the set $\{g_i\}_{i=1}^{2m+1}$ is a basis of V. This easily implies that the set $\{\varphi^i(v)\}_{i=0}^{2m}$ is a basis of V; *i.e.*, v is a cyclic vector for (V, φ). Now, applying (7.22) with $i = m+2$ and inserting the values of g_i, we deduce

$$\varphi^{2m+1}(v) = v + \sum_{i=1}^m a_i \varphi^i(v) - \phi(a_m)\varphi^{m+1}(v) - \sum_{i=m+2}^{2m} \phi^{1+2(i-m-1)}(a_{2m-i+1})\varphi^i(v).$$

Applying Lemma 7.3, we deduce (7.19).

7.5.5. — Assume that $n = 2m$ is even. Then

$$c^{\mathrm{spl}} = (1, 2, \ldots, m-1, m, 2m, 2m-1, \ldots, m+3, m+2),$$

c_0^{spl} is its lift with all entries 1 or 0, and c_1^{spl} is given by $c_1^{\mathrm{spl}}(e_m) = \varpi^{-1}e_{2m}$, $c_1^{\mathrm{spl}}(e_{m+2}) = \varpi e_1$, $c_1^{\mathrm{spl}}(e_i) = c_0^{\mathrm{spl}}(e_i)$ for all $i \neq m, m+2$. Next, $y \cdot {}^b\sigma(y)$ is given by $e_1 \mapsto e_1 + \sum_{\lambda=1}^m a_i e_{i+1} - \varpi^{-\kappa}\phi(a_{m-1})e_{2m}$; $e_i \mapsto e_i$ for $2 \leq i \leq m$ and $i = 2m$; $e_{m+1} \mapsto e_{m+1} - \phi(a_m)e_{2m}$; $e_i \mapsto e_i - \phi(a_{2m-i})e_{2m}$ for $m+2 \leq i \leq 2m-1$. Put $v := g_1$ and $u := g_{m+1}$. Applying (7.22) for all $i \neq m+1, m+2$, we deduce $g_i = \varphi^{i-1}(v)$ for $1 \leq i \leq m$; $g_i = \varpi^\kappa \varpi^{3m-i}(v) + \varpi^\kappa \sum_{j=1}^{2m-i} \phi^{1+2(2m-i-j)}(a_j)\varphi^{3m-i-j}(v)$ for $m+2 \leq i \leq 2m$. As in the previous cases, from this description we deduce the following.

Lemma 7.6. *The set $\{\varphi^i(v)\}_{i=0}^{2m-2} \cup \{u\}$ is a basis of V.*

Inserting the above values for g_i into (7.22) for $i = m+1$ and $i = m+2$, we deduce

$$(7.23) \qquad 0 = u - \varphi(u) - \varpi^\kappa \phi(a_m) \varphi^m(v),$$

$$(7.24) \qquad \varphi^{2m-1}(v) = v + \sum_{i=1}^{m-1} a_i \varphi^i(v) - \phi(a_{m-1}) \varphi^m(v) - \sum_{i=m+1}^{2m-2} \phi^{1+2(i-m)}(a_{2m-1-i}) \varphi^i(v) + a_m u.$$

If $a_m = 0$, Lemma 7.6 shows that v is a cyclic vector for the $(2m-1)$-dimensional isocrystal generated by it. Then (7.24) and Lemma 7.3 finish the proof by showing formulas (7.19).

We may now assume $a_m \neq 0$. Dividing (7.24) by a_m, we obtain an expression of u as a linear combination of $\{\varphi^i(v)\}_{i=0}^{2m-1}$, with $\varphi^{2m-1}(v)$ appearing with non-zero coefficient. Then Lemma 7.6 shows that v is a cyclic vector for V. Moreover, inserting this expression of u into (7.23) and multiplying with a_m, we eliminate u and obtain an expression of the form $\varphi^{2m}(v) = \sum_{i=0}^{2m-1} \gamma_i \varphi^i(v)$ with fixed $\gamma_i \in L$. By Lemma 7.3 we have $\operatorname{ord}_\varpi(\gamma_i) \geq 0$ for all i. We then compute that $\gamma_{2m-1} = 1 - \frac{a_m}{\phi^2(a_m)} \phi^{1+2(m-1)}(a_1)$, $\gamma_i = \phi^{1+2(i-m)}(a_{2m-1-i}) - \frac{a_m}{\phi^2(a_m)} \phi^{1+2(i-m)} a_{2m-i}$ for $m+1 \leq i \leq 2m-2$ and $\gamma_m = -\varpi^\kappa a_m \phi(a_m) + \phi(a_{m-1}) - \frac{a_m}{\phi^2(a_m)} \phi^2(a_{m-1})$. Using induction, it follows from these equations that $\operatorname{ord}_\varpi(a_i) \geq 0$ for $1 \leq i \leq m$. This shows (7.19).

7.5.6. — Suppose that $n = 2m$ is even and $b = \dot{c}_\kappa$ ($\kappa \in \{0, 1\}$) as in Section 7.5.3. We have the central cocharacter $\zeta = \sum_{i=1}^n \beta_i \in X_*(\mathbf{Z})$, whose image in $X_*(\mathbf{Z})_{\langle \sigma \rangle} \cong \mathbb{Z}/2\mathbb{Z}$ is non-trivial but which maps to $0 \in \pi_1(\mathbf{G})_{\langle \sigma \rangle}$. To finish the proof in the $^2A_{n-1}$-case (*cf.* the discussion after Lemma 7.2), we have to show that for all $g \in X_{b\varpi^\zeta, b}(\mathbb{f})$, we have $g^{-1} \sigma_b(g) \in (^c\mathbf{U} \cap \mathbf{U}^-)_{\mathbf{x}_b}(\mathcal{O}_L) \varpi^\zeta$. As in (7.4), we can write

$$(7.25) \qquad b\sigma(g) = gy\varpi^\zeta b$$

with $y \in (^c\mathbf{U} \cap \mathbf{U}^-)(L)$ depending on $a_1, \ldots, a_m \in L$ as described in Section 7.5.2. We thus have to show that (7.19) holds for these a_i. However, doing the same computation as in Section 7.5.3 (with b^{spl} and other notation as there), we see that

$$b^{\mathrm{spl}} \sigma_0^2(g) = (b\sigma)^2(g) = gy\varpi^{\zeta + \sigma(\zeta) b} \sigma(y) b^{\mathrm{spl}}$$

(where we used that ϖ^ζ and $\varpi^{\sigma(\zeta)}$ are central). As $\zeta + \sigma(\zeta) = 0$, we arrive simply at (7.21) and may proceed as there (and in Section 7.5.5) to deduce (7.19).

7.5.7. Proof of Lemma 5.6 for type $^2A_{n-1}$. — The action of c on Φ^+ is as follows: for $1 \leq i < j \leq m$, $c(\alpha_{i-j}) = \alpha_{(i+1)-(j+1)}$; for $1 \leq i \leq m$ and $m+2 \leq j \leq n$, $c(\alpha_{i-j}) = \alpha_{(i+1)-j}$ and $c(\alpha_{(m+1)-j}) = \alpha_{1-j}$; for $m+2 \leq i < j \leq n$, $c(\alpha_{i-j}) = \alpha_{i-j}$; for $1 \leq j \leq m$, $c(\alpha_{(m+1)-j}) \in \Phi^-$. The action of σ on positive roots is given by $\sigma(\alpha_{i-j}) = \alpha_{(n+1-j)-(n+1-i)}$.

We set $r = m+2$; $\Psi_{m+2} = \{\alpha_{(m+1)-j}\}_{j=1}^m$; $\Psi_{m+1} = \Psi_{m+2} \cup \{\alpha_{i-j} : 1 \leq i \leq m \text{ and } m+2 \leq j \leq n\}$; for $m \geq i_0 \geq 2$, $\Psi_{i_0} = \Psi_{i_0+1} \cup \{\alpha_{i-i_0}\}_{i=1}^{i_0-1} \cup \{\alpha_{(n-i_0)-j}\}_{j=n-i_0+1}^n$; $\Psi_1 = \Phi^+$.

7.6. Type ^2D_m

7.6.1. — Let $m \geq 4$, V_0, Q be as in Section 7.4.1. Consider the closed subgroup $\mathbf{GSO}(V_0, Q) \subseteq \mathbf{GL}(V_0)$, which was denoted by $\mathbf{G}$ in Section 7.4.1. It is k-split, and its derived group is of type D_m. First, suppose $\operatorname{char} k \neq 2$. Fix some $\Delta \in \mathcal{O}_k^\times$ which is not a square in k, and fix a square root $\sqrt{\Delta} \in \mathcal{O}_{\bar{k}}^\times$. Let $h_0 \in \mathbf{GL}_{2m}(V_0)(k)$ be given by $h_0(e_i) = e_i$ for all $i \neq \pm m$ and $h_0(e_{\pm m}) = (-2\Delta)^{\pm 1} e_{\mp m}$. Now suppose $\operatorname{char} k = 2$. Then let $h_0 \in \mathbf{GL}_{2m}(V_0)(k)$ be given by $h_0(e_i) = e_i$ for all $i \neq \pm m$ and $h_0(e_{\pm m}) = e_{\mp m}$. In this section we let $\mathbf{G}$ be the k-group $\mathbf{GSO}(V_0, Q)_{h_0}$ in the sense of Section 2.1.3. In particular, for any $\mathbb{F}_q$-algebra R_0 with $\widetilde{R} = \mathbb{W}(R_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)[1/\varpi]$, the geometric Frobenius on $\mathbf{G}(\widetilde{R}) = \mathbf{GSO}(V_0, Q)(\widetilde{R}) \subseteq \mathbf{GL}(V_0)(\widetilde{R})$ is given by

$$g \longmapsto \sigma(g) = h_0 \sigma_0(g) h_0^{-1},$$

where σ_0 is the Frobenius on $\mathbf{GL}(V_0)(\widetilde{R})$ corresponding to the natural k-structure on $\mathbf{GL}(V_0)$.

Lemma 7.7. *The derived group of the k-group $\mathbf{G}$ is of type ^2D_m.*

Proof. First suppose $\operatorname{char} k \neq 2$. Let $Q' : V_0 \to k$ be the quadratic form $Q'\left(\sum_{i=1}^{m}(a_i e_i + a_{-i} e_{-i})\right) = \sum_{i=1}^{m-1} a_i a_{-i} + a_m^2 - a_{-m}^2 \Delta$. It is non-split of Witt index $m-1$; hence $\mathbf{SO}(V_0, Q')$ is of type ^2D_m. It suffices to construct a k-isomorphism $\mathbf{GSO}(V_0, Q') \cong \mathbf{G}$. Let $h_1 \in \mathbf{GL}(V_0)(\check{k})$ be the element given by $h_1(e_{\pm i}) = e_{\pm i}$ for $1 \leq i \leq m-1$ and $h_1(e_m) = \frac{1}{2\sqrt{\Delta}} e_m + \frac{1}{2} e_{-m}$, $h_1(e_{-m}) = \sqrt{\Delta} e_m - \Delta e_{-m}$. Then $h_1^T \sigma_0(h_1)^{-1,T} = h_0$. If Q (resp. Q') denotes the matrix representing Q (resp. Q') with respect to the fixed basis of V_0 (so that $Q(v) = v^T Q v$ for all $v \in V$), one immediately checks that $h_1 Q h_1^T = Q'$. This means that $v \mapsto h_1^T v : (V, Q') \to (V, Q)$ is a $\check{k}$-isometry (*i.e.*, $Q'(v) = Q(h_1^T v)$) and that $\operatorname{Int}(h_1^{T,-1}) : \mathbf{GL}(V_0)_{\check{k}} \to \mathbf{GL}(V_0)_{\check{k}}$ restricts to a $\check{k}$-rational isomorphism $\operatorname{Int}(h_1^{T,-1}) : \mathbf{GSO}(V_0, Q)_{\check{k}} \to \mathbf{GSO}(V_0, Q')_{\check{k}}$. (Indeed, if $g \in \mathbf{GSO}(V_0, Q)$, then $g^t Q g = Q$ and one computes using $h_1 Q h_1^T = Q'$ that $\operatorname{Int}(h_1^{T,-1})(g)^T Q' \operatorname{Int}(h_1^{T,-1})(g) = Q'$.) For $\widetilde{R}$ as above, this isomorphism transfers the geometric Frobenius σ_0 of $\mathbf{GSO}(V_0, Q')(\widetilde{R})$ to the Frobenius σ given by

$$\sigma(g) = \operatorname{Int}(h_1^{T,-1})\left(\sigma_0(\operatorname{Int}(h_1^{T,-1})(g))\right) = \operatorname{Int}(h_0)(\sigma_0(g)).$$

Now suppose $\operatorname{char} k = 2$. Let $Q' : V_0 \to k$ be the quadratic form $Q'\left(\sum_{i=1}^{m}(a_i e_i + a_{-i} e_{-i})\right) = \sum_{i=1}^{m-1} a_i a_{-i} + a_m^2 + a_m a_{-m} + \Delta a_{-m}^2$, where $\Delta \in \mathcal{O}_k^\times$ is such that $X^2 + X + \Delta \in k[X]$ is irreducible. It is non-split of Witt index $m-1$; hence $\mathbf{SO}(V_0, Q')$ is of type ^2D_m. Let $\lambda \in \mathcal{O}_{\check{k}}^\times$ be a root of $X^2 + X + \Delta$. Let $h_1 \in \mathbf{GL}(V_0)(\check{k})$ be given by $h_1(e_i) = e_i$ for $i \neq \pm m$, $h_1(e_m) = e_m + \lambda e_{-m}$, $h_1(e_{-m}) = e_m + \lambda^{-1} \Delta e_{-m}$. Then h_1 induces a $\check{k}$-isometry between (V, Q') and (V, Q). We may conclude as above, computing that $h_1^T \sigma_0(h_1)^{-1,T} = h_0$. $\square$

7.6.2. — Via the identification $\mathbf{G} \otimes_k \check{k} \cong \mathbf{GSO}(V_0, Q)_{\check{k}}$, we can use $\mathbf{T}, \mathbf{B}$, $\varepsilon_i \in X_*(\mathbf{T})$ ($0 \leq i \leq m$) from Section 7.4.2. The induced action of σ on $X_*(\mathbf{T})$ is given by $\sigma(\varepsilon_i) = \varepsilon_i$ for $1 \leq i \leq m-1$, $\sigma(\varepsilon_m) = -\varepsilon_m$, $\sigma(\varepsilon_0) = \varepsilon_0 - \varepsilon_m$. It follows that $\pi_1(\mathbf{G}^{\mathrm{ad}})_{\langle \sigma \rangle} \cong \mathbb{Z}/2\mathbb{Z}$, generated by the class of ε_0.

We have isomorphisms $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \pi_1(\mathbf{G})$, $(1,0) \mapsto \varepsilon_0$, $(0,1) \mapsto \bar{\varepsilon}_1$ and $\mathbb{Z} \xrightarrow{\sim} X_*(\mathbf{Z})$, $1 \mapsto 2\varepsilon_0 - \sum_{i=1}^{m} \varepsilon_i$. The inclusion $X_*(\mathbf{Z}) \to \pi_1(\mathbf{G})$ corresponds to the map $1 \mapsto (2,1)$ if m is odd and to $1 \mapsto (2,0)$ if m is even. Moreover, σ acts on $X_*(\mathbf{Z})$ as the identity, and the action on $\pi_1(\mathbf{G})$ is given by $(a, b) \mapsto (0, (a \bmod 2) + b)$. Justifying Remark 7.1 in this case, one easily deduces that the induced map $X_*(\mathbf{Z})_{\langle \sigma \rangle} \cong \mathbb{Z} \to \mathbb{Z} \cong \pi_1(\mathbf{G})_{\langle \sigma \rangle}$ is multiplication by 2, hence injective.

We have the roots $\alpha_{\pm i \pm j}$ of $\mathbf{T}$ in $\mathbf{G}$ as in Section 7.4.2. The σ-action on the simple roots fixes all $\alpha_{i-(i+1)}$ ($1 \leq i \leq m-2$) and interchanges $\alpha_{(m-1)-m}$ and $\alpha_{(m-1)+m}$. Thus we obtain a Coxeter element

$$(7.26) \qquad c = s_{\alpha_{1-2}} \cdot \dots \cdot s_{\alpha_{(m-1)-m}} = (1, 2, \dots, m-1, m)(2m, 2m-1, \dots, m+2, m+1).$$

We consider two lifts $\dot{c}_0, \dot{c}_1$ of c to $\mathbf{G}(\check{k})$. For $\kappa \in \{0, 1\}$, $\dot{c}_\kappa$ is given by $\dot{c}_\kappa(e_i) = \varpi^\kappa e_{i+1}$ for $1 \leq i \leq m-1$; $\dot{c}_\kappa(e_m) = \varpi^\kappa e_1$; $\dot{c}_\kappa(e_{-i}) = e_{-(i+1)}$ for $1 \leq i \leq m-1$; $\dot{c}_\kappa(e_{-m}) = e_{-1}$. Then $\kappa_{\mathbf{G}}(\dot{c}_0) = 0$, $\kappa_{\mathbf{G}}(\dot{c}_1) = \bar{\varepsilon}_0$ in $\pi_1(\mathbf{G})_{\langle \sigma \rangle}$. Thus the $\mathbf{G}_{\dot{c}_j}$ ($\kappa \in \{0, 1\}$) are the two inner forms of $\mathbf{G}$.

We use the $\mathcal{A}_{\mathbf{T}^{\mathrm{ad}}, \check{k}}$ as in Section 7.4.2. The group $^c\mathbf{U} \cap \mathbf{U}^-$ is generated by root subgroups attached to α_{i-1} ($2 \leq i \leq m$). Explicitly, any element $y \in (^c\mathbf{U} \cap \mathbf{U}^-)(R)$ is an R-linear map in $\mathbf{GL}(V_0)(R)$ given by $e_1 \mapsto e_1 + \sum_{i=2}^{m} a_{i-1} e_i$; $e_i \mapsto e_i$ for $2 \leq i \leq m$ and $i = -1$; $e_{-i} \mapsto e_{-i} - a_{i-1} e_{-1}$ for all $2 \leq i \leq m$.

7.6.3. — Let $b = \dot{c}_\kappa$ with $\kappa \in \{0, 1\}$. We compute $\mathbf{x}_b = 0$ if $\kappa = 0$ and $\mathbf{x}_b = \sum_{i=1}^{m} (-\frac{m}{4} + \frac{i}{2}) \varepsilon_i$ if $\kappa = 1$. Let $g \in \dot{X}_{b,b}(\mathfrak{f})$. Then (7.4) holds, with n as described in Section 7.6.2, depending on some $a_1, \dots, a_{m-1} \in L$. We have to show that $n \in \mathcal{G}_{\mathbf{x}_b}(\mathcal{O}_L)$, which is (as in Section 7.1.3) equivalent to

$$(7.27) \qquad \operatorname{ord}_\varpi(a_i) \geq \frac{\kappa i}{2} \quad \text{for } 1 \leq i \leq m-1.$$

Let $V = V_0 \otimes_k L$. Let σ_0 be the ϕ-linear automorphism $x \otimes \lambda \mapsto x \otimes \phi(\lambda)$ of V. Consider the ϕ-linear automorphism $\varphi = b h_0 \sigma_0$ of V. Then (V, φ) is an isocrystal over $\mathfrak{f}$ of slope $\frac{\kappa}{2}$. Equation (7.4) gives $b h_0 \sigma_0(g) = g y b h_0$. Applying this to e_i for $1 \leq \pm i \leq m$, we get $\varphi(g_i) = g y b h_0(e_i)$, where we write

$g_i = g(e_i)$ (and where we use $\sigma_0(g)(e_i) = \sigma_0(g_i)$). Writing $v := g_1$, we deduce $g_i = \varpi^{1-i}\varphi^{i-1}(v)$ and $g_{-i} = -(2\Delta)^{-1}\varpi^{\kappa(1-m)}\left(\varphi^{m-1+i}(v) + \sum_{j=1}^{i-1}\phi^{i-j-1}(a_j)\varphi^{m+i-j-1}(v)\right)$ for $1 \le i \le m$. Inserting this into the equation attached to $\varphi(g_{m+1}) = gybh_0(e_{m+1})$ gives

$$\varphi^{2m}(v) = \varpi^{\kappa m}v + \sum_{i=1}^{m-1}\varpi^{\kappa(m-i)}a_i\varphi^i(v) - \sum_{i=m+1}^{2m-1}\phi^{i-m}(a_{2m-i})\varphi^i(v).$$

It is easy to see that v is cyclic for V. Using Lemma 7.3, (7.27) follows.

7.6.4. Proof of Lemma 5.6 for type ^2D_m. — The action of c on Φ^+ is as follows: for $1 \le i < j < m$, $c(\alpha_{i\pm j}) = \alpha_{(i+1)\pm(j+1)}$; for $1 \le i \le m-1$, $c(\alpha_{i+m}) = \alpha_{1+(i+1)}$; for $1 \le i \le m-1$, $c(\alpha_{i-m}) \in \Phi^-$. Moreover, $\sigma(\alpha_{i\pm j}) = \alpha_{i\pm j}$ if $j \ne m$ and $\sigma(\alpha_{i\pm m}) = \alpha_{i\mp m}$.

Set $r = m+1$; $\Psi_{m+1} = \{\alpha_{i-m}\}_{i=1}^{m-1}$; $\Psi_m = \Psi_{m+1} \cup \{\alpha_{i+j} : 1 \le i < j < m\}$; $\Psi_{m-1} = \Psi_m \cup \{\alpha_{i+m}\}_{i=1}^{m-1}$; for $1 \le i_0 \le m-2$, $\Psi_{i_0} = \Psi_{i_0+1} \cup \{\alpha_{i-(i_0+1)} : 1 \le i < i_0+1\}$.

References

[Bea12]   E. T. Beazley, *Maximal Newton polygons via the quantum Bruhat graph*, 24th International Conference on Formal Power Series and Algebraic Combinatorics (Nagoya, Japan, 2012), pp. 899–910, Discrete Math. Theor. Comput. Sci. Proc., Nancy, 2012.

[BGA18]   A. Bertapelle and C. Gonzàles-Avilés, *The Greenberg functor revisited*, Eur. J. Math. **4** (2018), 1340–1389.

[BM21]   B. Bhatt and A. Mathew, *The arc-topology*, Duke Math. J. **170** (2021), no. 9, 1899–1988.

[BS15]   B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201.

[BS17]   ———, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math. **209** (2017), no. 2, 329–423.

[BS68]   A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups II*, Tôhoku Math. J. (2) **20** (1968), 443–497.

[Bor98]   M. Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626.

[Bor21]   ——— (https://mathoverflow.net/users/4149/mikhail-borovoi), *Regular embeddings of a reductive groups with induced center*, MathOverflow, https://mathoverflow.net/q/397229 (version: 2021-07-11).

[CI21a]   C. Chan and A. B. Ivanov, *Affine Deligne–Lusztig varieties at infinite level*, Math. Ann. **380** (2021), 1801–1890.

[CI21b]   ———, *The Drinfeld stratification for* GL_n, Selecta Math. (N.S.) **27** (2021), no. 3, Paper No. 50.

[CI23]   ———, *On loop Deligne–Lusztig varieties of Coxeter type for GL_n*, Camb. J. Math. **11** (2023), no. 2, 441–505.

[CO21]   C. Chan and M. Oi, *Geometric L-packets of Howe-unramified toral supercuspidal representations*, preprint arXiv:2105.06341 (2021). To appear in J. Eur. Math. Soc.

[CS17]   Z. Chen and A. Stasinski, *The algebraisation of higher Deligne–Lusztig representations*, Selecta Math. (N.S.) **23** (2017), no. 4, 2907–2926.

[Con14]   B. Conrad, *Reductive group schemes*, in: *Autour des Schémas en Groupes Vol. I*, Panor. Synthèses, 42/43, Soc. Math. France, Paris, 2014.

[DeB06]   S. DeBacker, *Parameterizing Conjugacy Classes of Maximal Unramified Tori via Bruhat–Tits Theory*, Michigan Math. J. **54** (2006), no. 1, 157–178.

[DR09]   S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901.

[DL76]   P. Deligne and G. Lusztig, *Representations of Reductive Groups Over Finite Fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DI20]   O. Dudas and A. Ivanov, *Orthogonality relations for deep level Deligne–Lusztig schemes of Coxeter type*, preprint arXiv:2010.15489 (2020).

[GHK+10]   U. Görtz, T. J. Haines, R. E. Kottwitz and D. C. Reuman, *Affine Deligne–Lusztig varieties in affine flag varieties*, Compos. Math. **146** (2010), no. 5, 1339–1382.

[HR08]   T. J. Haines and M. Rapoport, *Appendix: On parahoric subgroups*, Adv. in Math. **219** (2008), no. 1, 188–198. Appendix to [PR08].

[HR20]   T. J. Haines and T. Richarz, *The test function conjecture for local models of Weil-restricted groups*, Compos. Math. **156** (2020), no. 7, 1348–1404.

[He14]   X. He, *Geometric and homological properties of affine Deligne-Lusztig varieties*, Ann. of Math. (2) **179** (2014), no. 1, 367–404.

[HL12]   X. He and G. Lusztig, *A generalization of Steinberg's cross-section*, J. Amer. Math. Soc. **25** (2012), no. 3, 739–757.

[HNY22]   X. He, S. Nie and Q. Yu, *Affine Deligne–Lusztig varieties with finite Coxeter parts*, preprint arXiv:2208.14058 (2022).

[Iva23]   A. Ivanov, *Arc-descent for the perfect loop functor and p-adic Deligne–Lusztig spaces*, J. reine angew. Math. (Crelle) **794** (2023), 1–54.

[Ked04]   K. Kedlaya, *A p-adic local monodromy theorem*, Ann. of Math. (2) **160** (2004), no. 1, 93–184.

[Kot84]   R. Kottwitz, *Stable trace formula: Cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650.

[Kot85]   ——, *Isocrystals with additional structure*, Compos. Math. **56** (1985), no. 2, 201–220.

[Kot97]   ——, *Isocrystals with additional structure. II*, Compos. Math. **109** (1997), no. 3, 255–339.

[Lus76]   G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **38** (1976), 101–159.

[MP94]   A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[PR08]   G. Pappas and M. Rapoport, *Twisted loop groups and their affine flag varieties* (with an appendix by T. J. Haines and M. Rapoport), Adv. Math. **219** (2008), no. 1, 118–198.

[Rap05]   M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, Astérisque **298** (2005), 271–318.

[RV14]   M. Rapoport and E. Viehmann, *Towards a theory of local Shimura varieties*, Münster J. Math. **7** (2014), no. 1, 273–326.

[RZ96]   M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Ann. of Math. Stud., vol 141, Princeton Univ. Press, Princeton, NJ, 1996.

[Ree11]   M. Reeder, *Elliptic centralizers in Weyl groups and their coinvariant representations*, Represent. Theory **15** (2011), 63–111.

[Ryd10]   D. Rydh, *Submersions and effective descent of étale morphisms*, Bull. Soc. Math. France **138** (2010), no. 2, 181–230.

[Ser94]   J.-P. Serre, *Cohomologie Galoisienne*, 5th ed., Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, 1994.

[Spr74]   T. A. Springer, *Regular elements in reflection groups*, Invent. Math. **25** (1974), 159–198.

[Sta14]   The Stacks Project Authors, *The Stacks Project*, https://stacks.math.columbia.edu, 2014.

[Ste65]   R. Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 49—80.