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Remarks on the geometry of the variety of planes of a cubic fivefold

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Dedicated to Claire Voisin on the occasion of her 60th birthday

Abstract. This note presents some properties of the variety of planes $F_2(X) \subset G(3,7)$ of a cubic 5-fold $X \subset \mathbb{P}^6$. A cotangent bundle exact sequence is first derived from the remark made by Iliev and Manivel that $F_2(X)$ sits as a Lagrangian subvariety of the variety of lines of a cubic 4-fold, which is a hyperplane section of X. Using the sequence, the Gauss map of $F_2(X)$ is then proven to be an embedding. The last section is devoted to the relation between the variety of osculating planes of a cubic 4-fold and the variety of planes of the associated cyclic cubic 5-fold.

Keywords. Cubic hypersurfaces, varieties of linear subspaces

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Contents

1. Introduction

To understand the topology and the geometry of smooth complex hypersurfaces $X \subset \mathbb{P}(V^*) \simeq \mathbb{P}^{n+1}$, various auxiliary manifolds have been introduced in the past century, of which the intermediate Jacobian

$$J^n(X) := (H^{k-1,k+2}(X) \oplus \cdots \oplus H^{0,n})/H^n(X,\mathbb{Z})_{/\text{torsion}}$$

when $n = 2k+1$ is odd is one of the most widely known since the seminal work of Clemens–Griffiths ([CG72]) on the cubic 3-fold.

Cubic 5-folds are classically (*cf.* [Gri69]) known to be the only hypersurfaces of dimension greater than 3 for which the intermediate Jacobian, which is in general just a (polarised) complex torus, is a (non-trivial) principally polarised abelian variety.

Another interesting series of varieties classically associated to X are the varieties $F_m(X) \subset G(m+1, V)$ of m-planes contained in X.

Starting from Collino ([Col86]), some properties of the variety of planes $F_2(X) \subset G(3, V)$ of a cubic 5-fold X have been studied in connection with the 21-dimensional intermediate Jacobian $J^5(X)$. In *loc. cit.*, the following is proven.

Theorem 1.1. *For a general cubic $X \subset \mathbb{P}(V^*) \simeq \mathbb{P}^6$, $F_2(X)$ is a smooth irreducible surface, and the Abel–Jacobi map of the family of planes $\Phi_{\mathcal{P}}\colon F_2(X) \to J^5(X)$ is an immersion; i.e., the associate tangent map is injective and induces an isomorphism of abelian varieties*

$$\phi_{\mathcal{P}}\colon \operatorname{Alb}(F_2(X)) \xrightarrow{\sim} J^5(X),$$

where $\mathcal{P} \in \mathrm{CH}^5(F_2(X) \times X)$ is the universal plane over $F_2(X)$. Equivalently, $q_*p^*\colon H^3(F_2(X),\mathbb{Z})_{/\text{torsion}} \to H^5(X,\mathbb{Z})$ is an isomorphism of Hodge structures, where the maps are defined by

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ F_2(X). & & \end{array}$$

In the present note, we investigate some additional properties of $F_2(X)$.

In the first section, we establish the following cotangent bundle exact sequence.

Theorem 1.2. *Let $X \subset \mathbb{P}(V^*)$ be a smooth cubic 5-fold for which $F_2(X)$ is a smooth irreducible surface. Then the cotangent bundle $\Omega_{F_2(X)}$ fits in the exact sequence*

$$0 \longrightarrow \mathcal{Q}_3^*|_{F_2(X)} \longrightarrow \operatorname{Sym}^2\mathcal{E}_3|_{F_2(X)} \longrightarrow \Omega_{F_2(X)} \longrightarrow 0, \tag{1.1}$$

where the tautological rank 3 quotient bundle $\mathcal{E}_3$ and the other bundle appear in the exact sequence

$$0 \longrightarrow \mathcal{Q}_3 \longrightarrow V^* \otimes \mathcal{O}_{G(3,V)} \longrightarrow \mathcal{E}_3 \longrightarrow 0 \tag{1.2}$$

and the first map (of (1.1)) is the contraction with an equation $\mathrm{eq}_X \in \mathrm{Sym}^3 V^*$ *defining* X*, i.e. for any* $[P] \in F_2(X)$, $v \mapsto \mathrm{eq}_X(v,\cdot,\cdot)|_P$.

Classically associated to the Albanese map $\mathrm{alb}_{F_2}\colon F_2(X) \to \mathrm{Alb}(F_2(X))$ of $F_2(X)$, there is the Gauss map

$$\begin{aligned}\mathcal{G}\colon\ \mathrm{alb}_{F_2}(F_2(X)) &\dashrightarrow G\left(2, T_{\mathrm{Alb}(F_2(X)),0}\right)\\ t &\longmapsto T_{\mathrm{alb}_{F_2}(F_2(X))-t,0}\end{aligned}$$

where $\mathrm{alb}_{F_2}(F_2(X)) - t$ designates the translation of $\mathrm{alb}_{F_2}(F_2(X)) \subset \mathrm{Alb}(F_2(X))$ by $-t \in \mathrm{Alb}(F_2(X))$. The map $\mathcal{G}$ is defined on the smooth locus of $\mathrm{alb}_{F_2}(F_2(X))$.

In the second section of the note, we prove the following.

Theorem 1.3. *The Albanese map is an embedding. In particular, the Gauss map is defined everywhere. Moreover,* $\mathcal{G}$ *is an embedding, and its composition with the Plücker embedding*

$$G\left(2,_{\mathrm{Alb}(F_2(X)),0}\right) \simeq G\left(2, H^0\left(\Omega_{F_2}\right)^*\right) \subset \mathbb{P}\left(\bigwedge^2 H^0\left(\Omega_{F_2(X)}\right)^*\right)$$

is the composition of the degree 3 Veronese of the natural embedding $F_2(X) \subset G(3,V) \subset \mathbb{P}(\bigwedge^3 V^*)$ *followed by a linear projection.*

The last section is concerned with some properties of the variety of osculating planes of a cubic 4-fold, namely

$$F_0(Z) := \{[P] \in G(3,H),\ \exists \ell \subset P \text{ line s.t. } P \cap Z = \ell \text{ (set-theoretically)}\}, \tag{1.3}$$

where $Z \subset \mathbb{P}(H^*) \simeq \mathbb{P}^5$ is a smooth cubic 4-fold containing no plane.

This variety admits a natural projection to the variety of lines $F_1(Z)$ of Z whose image (under that projection) has been studied, for example, in [GK21]. The interest of the authors there for the variety $F_0(Z)$ stems from its image in $F_1(Z)$ being the fixed locus of the Voisin self-map of $F_1(Z)$ (see [Voi04]), a map that plays an important role in the understanding of algebraic cycles on the hyper-Kähler 4-fold $F_1(Z)$ (see for example [SV16]).

In [GK21], it is proven that for Z general, $F_0(Z)$ is a smooth irreducible surface, and some of its invariants are computed.

We compute some more invariants of $F_0(Z)$ using its link with the variety of planes $F_2(X_Z)$ of the associated cyclic cubic 5-fold: to a smooth cubic 4-fold $Z = \{\mathrm{eq}_Z = 0\} \subset \mathbb{P}^5$, one can associate the cubic 5-fold $X_Z = \{X_6^3 + \mathrm{eq}_Z(X_0, \ldots, X_5)\}$ which (by linear projection) is the degree 3 cyclic cover of $\mathbb{P}^5$ ramified over Z.

Theorem 1.4. *For* Z *general,* $F_0(Z)$ *is a smooth irreducible surface, and*

(1) $F_2(X_Z)$ *is a degree 3 étale cover of* $F_0(Z)$*,*
(2) $b_1(F_0(Z)) = 0$, $h^2(\mathcal{O}_{F_0(Z)}) = 1070$, $h^1(\Omega_{F_0(Z)}) = 2207$,
(3) $\mathrm{Im}(F_0(Z) \to F_1(Z))$ *is a (non-normal) Lagrangian surface of* $F_1(Z)$.

Remark 1.5. As mentioned by the referee and Frank Gounelas, in [GK21], it is proven that $[\mathrm{Im}(F_0(Z) \to F_1(Z))] = 21[F_1(Z \cap H)]$ in $\mathrm{CH}_2(F_1(Z))$, where $Z \cap H$ is a cubic 3-fold obtained as a general hyperplane section, which implies that $[\mathrm{Im}(F_0(Z) \to F_1(Z))]$ is Lagrangian (see [Huy23, Lemma 6.4.5], for example).

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2. Cotangent bundle exact sequence

Let $X \subset \mathbb{P}(V^*) \simeq \mathbb{P}^6$ be a smooth cubic 5-fold. Its variety of planes $F_2(X) \subset G(3, V)$ is the zero locus of the section of $\operatorname{Sym}^3 \mathcal{E}_3$ (where $\mathcal{E}_3$ is defined by (1.2)) induced by an equation $\mathrm{eq}_X \in H^0(\mathcal{O}_{\mathbb{P}^6}(3))$ of X.

Let us gather some basic properties of $F_2(X)$ before proving Theorem 1.2.

It is proven in [Col86, Proposition 1.8] that $F_2(X)$ is connected for any X, so that by Bertini-type theorems, for X general, $F_2(X)$ is a smooth irreducible surface.

As such an $F_2(X)$ is cut out of $G(3, V)$ by a regular section of the rank 10 vector bundle $\operatorname{Sym}^3 \mathcal{E}_3$, the Koszul resolution says that the structure sheaf $\mathcal{O}_{F_2(X)}$ is quasi-isomorphic to the complex

$$0 \longrightarrow \wedge^{10} \operatorname{Sym}^3 \mathcal{E}_3^* \longrightarrow \wedge^9 \operatorname{Sym}^3 \mathcal{E}_3^* \longrightarrow \cdots \longrightarrow \operatorname{Sym}^3 \mathcal{E}_3^* \longrightarrow \mathcal{O}_{G(3,V)} \longrightarrow 0, \tag{2.1}$$

where the differentials are given by the section of $\operatorname{Sym}^3 \mathcal{E}_3$. By the adjunction formula,

$$K_{F_2(X)} \simeq K_{G(3,V)} \otimes \det(\operatorname{Sym}^3 \mathcal{E}_3|_{F_2(X)}) \simeq \mathcal{O}_{G(3,V)}(3)|_{F_2(X)} := \mathcal{O}_{F_2(X)}(3).$$

Theorem 1.1 (see also Theorem 3.1 below) implies that $h^{1,0}(F_2(X)) = h^0(\Omega_{F_2(X)}) = h^{2,3}(X) = 21$, and we can use software to compute the other Hodge numbers (see also [Gam]). We use the package Schubert2 of Macaulay2:

(1) The Koszul resolution of $\mathcal{O}_{F_2(X)}$ gives $\chi(\mathcal{O}_{F_2(X)}) = \sum_{i=0}^{10} (-1)^i \chi(\wedge^i \operatorname{Sym}^3 \mathcal{E}_3^*)$. We can get the result $\chi(\mathcal{O}_{F_2(X)}) = 3213$ using the following code:

```
loadPackage "Schubert2"
G=flagBundle{4,3}
(Q,E)= bundles G
F=symmetricPower(3,dual(E))
chi(exteriorPower(0,F))-chi(exteriorPower(1,F))+chi(exteriorPower(2,F))
-chi(exteriorPower(3,F))+chi(exteriorPower(4,F))-chi(exteriorPower(5,F))
+chi(exteriorPower(6,F))-chi(exteriorPower(7,F))+chi(exteriorPower(8,F))
-chi(exteriorPower(9,F))+chi(exteriorPower(10,F))
```

Then we get $h^{0,2}(F_2(X)) = \chi(\mathcal{O}_{F_2(X)}) - 1 + h^{0,1}(F_2(X)) = 3233$.

(2) Next, Noether's formula reads $\chi_{\mathrm{top}}(F_2(X)) = 12\chi(\mathcal{O}_{F_2(X)}) - \int_{F_2(X)} c_1(K_{F_2(X)})^2$, and as

$$\begin{aligned}\int_{F_2(X)} c_1\left(K_{F_2(X)}\right)^2 &= \int_{F_2(X)} c_1\left(\mathcal{O}_{G(3,V)}(3)|_{F_2(X)}\right)^2 \\ &= \int_{G(3,V)} [F_2(X)] \cdot c_1\left(\mathcal{O}_{G(3,V)}(3)\right)^2 \\ &= 9 \int_{G(3,V)} c_{10}\left(\operatorname{Sym}^3 \mathcal{E}_3\right) \cdot c_1\left(\mathcal{O}_{G(3,V)}(1)\right)^2,\end{aligned}$$

the number $\int_{F_2(X)} c_1(K_{F_2(X)})^2 = 3^2 \times 2835 = 25515$ can be obtained using the code

```
loadPackage "Schubert2"
G=flagBundle{4,3}
```

```
(Q,E)= bundles G
F=symmetricPower(3,E)
cycle=chern(1,exteriorPower(3,E))*chern(1,exteriorPower(3,E))*chern(10,F)
integral cycle
```

Then we get $b_2(F_2(X)) = \chi_{\mathrm{top}}(F_2(X))-2+2b_1(F_2(X)) = 13041-2+4\times 21 = 13123$ and $h^{1,1}(F_2(X)) = b_2(F_2(X)) - 2h^{0,2}(F_2(X)) = 6657$.

Associated to X, there is also its variety of lines $F_1(X) \subset G(2,V)$. It is a smooth Fano variety of dimension 6 which is cut out by a regular section of $\mathrm{Sym}^3\mathcal{E}_2$, where $\mathcal{E}_2$ is the tautological rank 2 quotient bundle appearing in an exact sequence

$$0 \longrightarrow \mathcal{Q}_2 \longrightarrow V^* \otimes \mathcal{O}_{G(2,V)} \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

Let us examine the relation between the two auxiliary varieties by introducing the flag variety

$$\begin{array}{ccc} \mathrm{Fl}(2,3,V) & \xrightarrow{\ e\ } & \mathrm{Gr}(2,V) \\ {\scriptstyle t}\big\downarrow & & \\ \mathrm{Gr}(3,V), & & \end{array}$$

where $t\colon \mathrm{Fl}(2,3,V) \simeq \mathbb{P}(\wedge^2\mathcal{E}_3) \to \mathrm{Gr}(3,V)$ and $e\colon \mathrm{Fl}(2,3,V) \simeq \mathbb{P}(\mathcal{Q}_2) \to \mathrm{Gr}(2,V)$. For the tautological quotient line bundles, we have $\mathcal{O}_t(1) \simeq e^*\mathcal{O}_{\mathrm{Gr}(2,V)}(1)$ and $\mathcal{O}_e(1) \simeq t^*\mathcal{O}_{\mathrm{Gr}(3,V)}(1) \otimes e^*\mathcal{O}_{\mathrm{Gr}(2,V)}(-1)$.

On $\mathrm{Fl}(2,3,V)$, the relation between the two tautological bundles is given by the exact sequence

$$0 \longrightarrow e^*\mathcal{O}_{G(2,V)}(-1) \otimes t^*\mathcal{O}_{G(3,V)}(1) \longrightarrow t^*\mathcal{E}_3 \longrightarrow e^*\mathcal{E}_2 \longrightarrow 0. \tag{2.2}$$

We can restrict the flag bundle to get

$$\begin{array}{ccc} \mathbb{P}_{F_2} := \mathbb{P}\left(\wedge^2\mathcal{E}_3|_{F_2(X)}\right) & \xrightarrow{\ e_{F_2}\ } & F_1(X) \\ {\scriptstyle t_{F_2}}\big\downarrow & & \\ F_2(X). & & \end{array}$$

We have the following property.

Proposition 2.1. *The tangent map Te_{F_2} of e_{F_2} is injective; i.e., e_{F_2} is an immersion. Moreover, the "normal bundle" $N_{\mathbb{P}_{F_2}/F_1(X)} := e_{F_2}^*T_{F_1(X)}/T_{\mathbb{P}_{F_2}}$ of $\mathbb{P}_{F_2}$ admits the following description:*

$$0 \longrightarrow t_{F_2}^*(\mathcal{Q}_3^*|_{F_2(X)}) \otimes \mathcal{O}_e(1) \longrightarrow t_{F_2}^*\mathrm{Sym}^2\mathcal{E}_3 \otimes \mathcal{O}_e(1) \longrightarrow N_{\mathbb{P}_{F_2}/F_1(X)} \longrightarrow 0. \tag{2.3}$$

Proof. (1) Let us first prove that e_{F_2} is an immersion. Let us recall the natural isomorphism between the two presentations of the tangent space of $\mathrm{Fl}(2,3,V)$: looking at t, we can write

$$T_{\mathrm{Fl}(2,3,V),([\ell],[P])} \simeq \mathrm{Hom}(\langle P\rangle, V/\langle P\rangle) \oplus \mathrm{Hom}(\langle \ell\rangle, \langle P\rangle/\langle \ell\rangle),$$

and looking at e, we have

$$T_{\mathrm{Fl}(2,3,V),([\ell],[P])} \simeq \mathrm{Hom}(\langle \ell\rangle, V/\langle \ell\rangle) \oplus \mathrm{Hom}(\langle P\rangle/\langle \ell\rangle, V/\langle P\rangle),$$

where we denote by $\langle K\rangle \subset V$ the linear subspace whose projectivisation is $K \subset \mathbb{P}(V^*)$. For a given decomposition $\langle P\rangle \simeq \langle \ell\rangle \oplus \langle P\rangle/\langle \ell\rangle$, the isomorphism takes the following form:

$$\begin{array}{rcl} \mathrm{Hom}(\langle P\rangle, V/\langle P\rangle) \oplus \mathrm{Hom}(\langle \ell\rangle, \langle P\rangle/\langle \ell\rangle) & \longrightarrow & \mathrm{Hom}(\langle \ell\rangle, V/\langle \ell\rangle) \oplus \mathrm{Hom}(\langle P\rangle/\langle \ell\rangle, V/\langle P\rangle). \\ (f,\ g) & \longmapsto & \left(f|_{\langle \ell\rangle} + g,\ f|_{\langle P\rangle/\langle \ell\rangle}\right) \end{array}$$

Notice that, by definition, we have $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \{0\}$, so that in proving that $T_{([\ell],[P])}e_{F_2}$ is injective, we can examine the two components separately.

Now we have the exact sequence

$$0 \longrightarrow N_{\ell/P} \longrightarrow N_{\ell/X} \longrightarrow N_{P/X}|_{\ell} \longrightarrow 0,$$

from which we get

$$0 \longrightarrow \underbrace{H^0(\mathcal{O}_\ell(1))}_{\simeq\langle\ell\rangle^*} \longrightarrow H^0(N_{\ell/X}) \longrightarrow H^0(N_{P/X}|_\ell) \longrightarrow 0 = H^1(\mathcal{O}_\ell(1)), \tag{2.4}$$

and we have $T_{F_1(X),[\ell]} \simeq H^0(N_{\ell/X})$.

A linear form on P defining ℓ is given by any generator of $(\langle P\rangle/\langle\ell\rangle)^* \subset \langle P\rangle^*$, so that

$$T_{\mathbb{P}(\wedge^2\mathcal{E}_3|_{F_2(X)}),([\ell],[P])} \simeq \underbrace{T_{F_2(X),[P]}}_{\simeq H^0(N_{P/X})} \oplus \underbrace{\langle P\rangle^*/(\langle P\rangle/\langle\ell\rangle)^*}_{\simeq\langle\ell\rangle^*}.$$

The second summand is readily seen to inject into $T_{F_1(X),\langle\ell\rangle}$ by (2.4).

Next, we have the exact sequence

$$0 \longrightarrow N_{P/X}(-1) \longrightarrow N_{P/X} \longrightarrow N_{P/X}|_\ell \longrightarrow 0,$$

which gives rise to

$$0 \longrightarrow H^0\left(N_{P/X}(-1)\right) \longrightarrow H^0\left(N_{P/X}\right) \longrightarrow H^0\left(N_{P/X}|_\ell\right) \longrightarrow H^1\left(N_{P/X}(-1)\right) \longrightarrow H^1\left(N_{P/X}\right). \tag{2.5}$$

To prove that $T_{([\ell],[P])}e_{F_2}$ is injective, it is thus sufficient to prove that $H^0(N_{P/X}(-1)) = 0$.

Consider the exact sequence

$$0 \longrightarrow N_{P/X} \longrightarrow \underbrace{N_{P/\mathbb{P}^6}}_{\simeq(V/\langle P\rangle)\otimes\mathcal{O}_P(1)} \xrightarrow{\alpha} \underbrace{N_{X/\mathbb{P}^6}|_P}_{\simeq\mathcal{O}_P(3)} \longrightarrow 0. \tag{2.6}$$

Up to a projective transformation, we can assume $P = \{X_0 = \cdots = X_3 = 0\}$, so that eq_X has the following form:

$$X_0Q_0 + X_1Q_1 + X_2Q_2 + X_3Q_3 + \sum_{i=4}^{6} X_iD_i(X_0,X_1,X_2,X_3) + R(X_0,X_1,X_2,X_3) \tag{2.7}$$

where R is a homogeneous cubic polynomial, the D_i, $4 \le i \le 6$, are homogeneous quadratic polynomials in the variables $(X_k)_{k\le 3}$ and the Q_i, $0 \le i \le 3$, are homogeneous quadratic polynomials in $(X_i)_{4\le i\le 6}$. With this notation, X is smooth along P if and only if $\mathrm{Span}((Q_i|_P)_{i=0,\dots,3})$ is base-point-free. We recall the following result found in [Col86, Proposition 1.2 and Corollary 1.4].

Proposition 2.2. *For X smooth along P, the following properties are equivalent:*

(1) *The variety $F_2(X)$ is smooth at $[P]$.*
(2) *The set $(Q_0,\dots,Q_3)$ is linearly independent.*
(3) *The map $H^0(\alpha)\colon H^0(N_{P/\mathbb{P}^6}) \simeq (V/\langle P\rangle)\otimes H^0(\mathcal{O}_P(1)) \to H^0(N_{X/\mathbb{P}^6}|_P) \simeq H^0(\mathcal{O}_P(3))$, $(L_0,\dots,L_3) \mapsto \sum_i L_iQ_i$ is surjective.*

Now tensoring (2.6) by $\mathcal{O}_P(-1)$, we get the long exact sequence

$$0 \longrightarrow H^0(N_{P/X}(-1)) \longrightarrow V/\langle P\rangle \xrightarrow{H^0(\alpha(-1))} H^0(\mathcal{O}_P(2)) \longrightarrow H^1(N_{P/X}(-1)) \longrightarrow 0 = H^1(\mathcal{O}_P)^{\oplus 4}. \tag{2.8}$$

The map $H^0(\alpha(-1))$ is given by the quadrics $(Q_0,\dots,Q_3)$. As $F_2(X)$ is smooth by assumption, the latter are linearly independent; thus $H^0(\alpha(-1))$ is injective; *i.e.*, we have $H^0(N_{P/X}(-1)) = 0$. In particular, $H^0(N_{P/X}) \subset H^0(N_{P/X}|_\ell)$; hence, looking at (2.6) and (2.4), we see that $T_{([\ell],[P])}e_{F_2}$ is injective.

(2) We want now to establish the exact sequence (2.3). Pulling back the natural exact sequence of locally free sheaves, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\mathbb{P}_{F_2}} & \longrightarrow & T_{\mathrm{Fl}(2,3,V)}|_{\mathbb{P}_{F_2}} & \longrightarrow & (t^*\operatorname{Sym}^3\mathcal{E}_3)|_{\mathbb{P}_{F_2}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Te_{F_2}} & & \downarrow{\scriptstyle Te|_{\mathbb{P}_{F_2}}} & & \downarrow{\scriptstyle \overline{Te|_{\mathbb{P}_{F_2}}}} & & \\
0 & \longrightarrow & e_{F_2}^* T_{F_1(X)} & \longrightarrow & e_{F_2}^* T_{\mathrm{Gr}(2,V)}|_{F_1(X)} & \longrightarrow & e_{F_2}^*\operatorname{Sym}^3\mathcal{E}_2|_{F_1(X)} & \longrightarrow & 0,
\end{array}$$

which by the snake lemma yields

$$0 \longrightarrow \operatorname{Ker}\left(Te|_{\mathbb{P}_{F_2}}\right) \longrightarrow \operatorname{Ker}\left(\overline{Te|_{\mathbb{P}_{F_2}}}\right) \longrightarrow \operatorname{coker}\left(Te_{F_2}\right) \longrightarrow 0.$$

By the definition of the normal bundle, we get $\operatorname{coker}(Te_{F_2}) \simeq N_{\mathbb{P}_{F_2}/F_1(X)}$. The restriction of the exact sequence of locally free sheaves

$$0 \longrightarrow T_{\mathrm{Fl}(2,3,V)/\mathrm{Gr}(2,7)} \longrightarrow T_{\mathrm{Fl}(2,3,V)} \longrightarrow e^*T_{\mathrm{Gr}(2,V)} \longrightarrow 0$$

still being exact, we get $\ker(Te|_{\mathbb{P}_{F_2}}) \simeq T_{\mathrm{Fl}(2,3,V)/\mathrm{Gr}(2,V)}|_{\mathbb{P}_{F_2}}$. The relative tangent bundle appears in the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathrm{Fl}(2,3,V)} \longrightarrow e^*V/\mathcal{E}_2^* \otimes \mathcal{O}_e(1) \longrightarrow T_{\mathrm{Fl}(2,3,V)/\mathrm{Gr}(2,V)} \longrightarrow 0.$$

The sequence (2.2) also yields

$$0 \longrightarrow t^*\mathcal{O}_{\mathrm{Gr}(3,V)}(-1) \otimes e^*\mathcal{O}_{\mathrm{Gr}(2,V)}(1) \longrightarrow V/\mathcal{E}_2^* \longrightarrow V/\mathcal{E}_3^* \longrightarrow 0,$$

from which, after twisting that last sequence by $\mathcal{O}_e(1)$, we get $T_{\mathrm{Fl}(2,3,V)/\mathrm{Gr}(2,V)}|_{\mathbb{P}_{F_2}} \simeq t_{F_2}^* V/\mathcal{E}_3^* \otimes \mathcal{O}_e(1)$.

Next, taking the symmetric power of (2.2) we get the exact sequence

$$0 \longrightarrow e^*\mathcal{O}_{\mathrm{Gr}(2,V)}(-1) \otimes t^*\mathcal{O}_{\mathrm{Gr}(3,V)}(1) \otimes t^*\operatorname{Sym}^2\mathcal{E}_3 \longrightarrow t^*\operatorname{Sym}^3\mathcal{E}_3 \longrightarrow e^*\operatorname{Sym}^3\mathcal{E}_2 \longrightarrow 0,$$

so that $\ker(\overline{Te|_{\mathbb{P}_{F_2}}}) \simeq (e^*\mathcal{O}_{\mathrm{Gr}(2,V)}(-1) \otimes t^*\mathcal{O}_{\mathrm{Gr}(3,V)}(1) \otimes t^*\operatorname{Sym}^2\mathcal{E}_3)|_{\mathbb{P}_{F_2}}$. Putting everything together, we get the desired exact sequence. □

For any plane $P_0 \subset X$, looking for example at the associated quadric bundle

$$\begin{array}{ccc}
\widetilde{X_{P_0}} & \hookrightarrow & \mathbb{P}(\mathcal{E}_4) \\
 & \searrow{\scriptstyle \tilde{\gamma}} & \downarrow{\scriptstyle \gamma} \\
 & & B,
\end{array}$$

where $B \simeq \{[\Pi] \in G(4,V),\ P_0 \subset \Pi\} \simeq \mathbb{P}^3$, $\mathcal{E}_4 \simeq \langle P\rangle^* \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ and $\widetilde{X_{P_0}} \in |\mathcal{O}_\gamma(2) \otimes \gamma^*\mathcal{O}_{\mathbb{P}^3}(1)|$, we see that the locus of quadrics of rank at most 2 has codimension (at most) $\binom{4-2+1}{2} = 3$. Moreover, by the Harris–Tu formula ([HT84, Theorem 1 and Theorem 10]), there are (at least) $2\begin{vmatrix} c_2(\mathcal{E}_4\otimes L) & c_3(\mathcal{E}_4\otimes L) \\ c_0(\mathcal{E}_4\otimes L) & c_1(\mathcal{E}_4\otimes L \end{vmatrix} = 31$ of these quadrics (where L has to be thought of as a formal square root of $\mathcal{O}_{\mathbb{P}^3}(1)$).

In particular, the locus $\Gamma = \{([\ell],[P]) \in \mathbb{P}_{F_2},\ \exists [P'] \neq [P],\ ([\ell],[P']) \in \mathbb{P}_{F_2}\}$ has codimension 2 in $\mathbb{P}_{F_2}$ (above the general plane $[P] \in F_2(X)$, there are finitely many lines that belong to another planes $P' \subset X$).

To any hyperplane $H \subset \mathbb{P}(V^*)$ such that $Y := X \cap H$ is a smooth cubic 4-fold containing no plane, we can attach the morphism $j_H \colon F_2(X) \to F_1(Y)$ defined by $[P] \mapsto [P \cap H]$.

The subvariety $F_1(Y) \subset F_1(X)$ is the zero locus of the regular section of $\mathcal{E}_2|_{F_1(X)}$ induced by the equation of $H \subset \mathbb{P}(V^*)$. For any such Y (containing no plane), $e^{-1}(F_1(Y))$ is obviously a section Z_H of $\mathbb{P}_{F_2} \to F_2(X)$, $[P] \mapsto ([P \cap H],[P])$. The smooth surface $Z_H \simeq F_2(X)$ is thus the zero locus of a regular section of $e_{F_2}^*\mathcal{E}_2|_{F_1(X)}$. By Bertini-type theorems, for H general, $Z_H \cap \Gamma$ is 0-dimensional.

As a result, as noticed in [IM08, Proposition 7] (the published version corrects the preprint, in which it is wrongly claimed that j_H is an embedding, as underlined in [Huy23]), $j_H \colon Z_H \simeq F_2(X) \to F_1(Y)$ is isomorphic to its image outside a 0-dimensional subset of $F_2(X)$.

The following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{Z_H} & \longrightarrow & T_{\mathbb{P}_{F_2}}|_{Z_H} & \longrightarrow & N_{Z_H/\mathbb{P}_{F_2}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (e_{F_2}^* T_{F_1(Y)})|_{Z_H} & \longrightarrow & (e_{F_2}^* T_{F_1(X)})|_{Z_H} & \longrightarrow & (e_{F_2}^* N_{F_1(Y)/F_1(X)})|_{Z_H} & \longrightarrow & 0. \end{array}$$

As $Z_H \subset \mathbb{P}_{F_2}$ is the zero locus of a regular section of $e_{F_2}^* \mathcal{E}_2|_{F_1(X)}$, we have $N_{Z_H/\mathbb{P}_{F_2}} \simeq (e_{F_2}^* \mathcal{E}_2|_{F_1(X)})|_{Z_H}$, so that the last vertical arrow in the diagram is an isomorphism. As the second vertical arrow is injective by Proposition 2.1, the first is injective as well. So the snake lemma gives $(e_{F_2}^* T_{F_1(Y)})|_{Z_H}/T_{Z_H} \simeq N_{\mathbb{P}_{F_2}/F_1(X)}|_{Z_H}$.

According to [IM08, Proposition 4], $\mathrm{Im}(j_H)$ is a (non-normal) Lagrangian surface of the hyper-Kähler manifold $F_1(Y)$. In particular, outside a codimension 2 subset of $F_2(X)$, we have

$$\Omega_{Z_H} \simeq \left(e_{F_2}^* T_{F_1(Y)}\right)_{Z_H} / T_{Z_H}.$$

As both sheaves are locally free, the isomorphism holds globally; *i.e.*,

$$\Omega_{F_2(X)} \simeq N_{\mathbb{P}_{F_2}/F_1(X)}|_{Z_H}. \tag{2.9}$$

We can now prove Theorem 1.2

Proof of Theorem 1.2. Looking at (2.9) and (2.3), we see that we only have to check that $\mathcal{O}_e(1)|_{Z_H} \simeq \mathcal{O}_{Z_H}$.

For a (general) hyperplane $H \subset \mathbb{P}(V^*)$, we have a rational map $\varphi\colon \operatorname{Gr}(3,V) \dashrightarrow \operatorname{Gr}(2,\langle H\rangle)$, $P \mapsto P \cap H$ whose indeterminacy locus is $\operatorname{Gr}(3,\langle H\rangle)$. The morphism $j_H\colon F_2(X) \simeq Z_H \to F_1(Y)$ is the restriction of the map φ to $F_2(X)$. To get the result, we will show more generally that $\varphi^*\mathcal{O}_{\operatorname{Gr}(2,\langle H\rangle)}(-1) \otimes \mathcal{O}_{\operatorname{Gr}(3,V)}(1)$ restricts to the trivial line bundle on the open set where φ is defined, *i.e.*, on $\operatorname{Gr}(3,V)\backslash \operatorname{Gr}(3,\langle H\rangle)$.

The subvariety $\operatorname{Gr}(3,\langle H\rangle) \subset \operatorname{Gr}(3,V)$ is the zero locus of a regular section of $\mathcal{E}_3$, so that $N_{\operatorname{Gr}(3,\langle H\rangle)/\operatorname{Gr}(3,V)} \simeq \mathcal{E}_3|_{\operatorname{Gr}(3,\langle H\rangle)}$. After blowing up this locus, we get

$$\begin{array}{ccccc} E_\tau & \overset{j}{\hookrightarrow} & \widetilde{\operatorname{Gr}(3,V)} & & \\ \downarrow & & \downarrow \tau & \searrow \widetilde{\varphi} & \\ \operatorname{Gr}(3,\langle H\rangle) & \overset{i}{\hookrightarrow} & \operatorname{Gr}(3,V) & \overset{\varphi}{\dashrightarrow} & \operatorname{Gr}(2,\langle H\rangle), \end{array}$$

where the exceptional divisor E_τ is isomorphic to $\mathbb{P}(\mathcal{E}_3^*) \simeq \mathbb{P}(\wedge^2\mathcal{E}_3 \otimes \det(\mathcal{E}_3)^{-1})$. So E_τ is isomorphic to the flag variety $\operatorname{Fl}(2,3,\langle H\rangle)$, and $\widetilde{\varphi} \circ j$ correspond to the projection on the Grassmannian of lines; hence

$$\mathcal{O}_{E_\tau}(1) \simeq j^*\widetilde{\varphi}^*\mathcal{O}_{\operatorname{Gr}(2,\langle H\rangle)}(1) \otimes \tau_{E_\tau}^* i^*\mathcal{O}_{\operatorname{Gr}(3,V)}(-1) \quad \text{in } \operatorname{Pic}(E_\tau).$$

As the restriction $\operatorname{Pic}(\operatorname{Gr}(3,V)) \to \operatorname{Pic}(\operatorname{Gr}(3,\langle H\rangle))$ is an isomorphism, so is $\operatorname{Pic}(\widetilde{\operatorname{Gr}(3,V)}) \to \operatorname{Pic}(E_\tau)$; thus

$$\mathcal{O}_{\widetilde{\operatorname{Gr}(3,V)}}(-E) \simeq \widetilde{\varphi}^*\mathcal{O}_{\operatorname{Gr}(2,\langle H\rangle)}(1) \otimes \tau^*\mathcal{O}_{\operatorname{Gr}(3,V)}(-1) \quad \text{in } \operatorname{Pic}\left(\widetilde{\operatorname{Gr}(3,V)}\right).$$

Now pushing forward by τ the short exact sequence defining E, we get

$$\tau_*\widetilde{\varphi}^*\mathcal{O}_{\operatorname{Gr}(2,\langle H\rangle)}(1) \otimes \mathcal{O}_{\operatorname{Gr}(3,V)}(-1) \simeq \tau_*\mathcal{O}_{\widetilde{\operatorname{Gr}(3,V)}}(-E) \simeq \mathcal{I}_{\operatorname{Gr}(3,\langle H\rangle)/\operatorname{Gr}(3,V)},$$

which is indeed trivial on $\operatorname{Gr}(3,V)\backslash \operatorname{Gr}(3,\langle H\rangle)$. □

3. Gauss map of $F_2(X)$

Let $X \subset \mathbb{P}(V^*) \simeq \mathbb{P}^6$ be a smooth cubic hypersurface such that $F_2(X)$ is a smooth (irreducible) surface. We begin this section with the following.

Theorem 3.1. *The following sequence is exact:*

$$0 \longrightarrow H^1\left(\mathcal{O}_{F_2(X)}\right) \longrightarrow \operatorname{Sym}^2 V \otimes \det(V) \xrightarrow{\varphi_{\mathrm{eq}_X} \otimes \mathrm{id}_{\det(V)}} V^* \otimes \det(V) \longrightarrow 0, \tag{3.1}$$

where φ_{eq_X} is defined to be $e_i + e_j \mapsto \mathrm{eq}_X(e_i, e_j, \cdot)$.

Moreover, we have an inclusion $\bigwedge^2 H^1(\mathcal{O}_{F_2(X)}) \subset H^2(\mathcal{O}_{F_2(X)})$, which by Hodge symmetry yields $\bigwedge^2 H^0(\Omega_{F_2(X)}) \subset H^0(K_{F_2(X)})$.

Proof. As $\mathcal{O}_{F_2(X)}$ admits the Koszul resolution (2.1), to understand the cohomology groups $H^i(\mathcal{O}_{F_2(X)})$, we can use the spectral sequence

$$E_1^{p,q} = H^q\left(G(3,V), \wedge^{-p} \operatorname{Sym}^3 \mathcal{E}_3^*\right) \Longrightarrow H^{p+q}\left(\mathcal{O}_{F_2(X)}\right).$$

As a reminder, we borrow from [Jia12] (see also [Spa03]) the following elementary presentation of the Borel–Weil–Bott theorem for a $G(3,W)$ with $\dim(W) = d$.

For any vector space L of dimension f and any decreasing sequence of integers $a = (a_1, \dots, a_f)$, there is an irreducible $GL(L)$-representation (Weyl module) denoted by $\Gamma^{(a_1,\dots,a_f)}L$.

To two decreasing sequences $a = (a_1, \dots, a_{d-e})$ and $b = (b_1, \dots, b_e)$, we can associate the sequence

$$(\phi_1, \dots, \phi_d) = \phi(a,b) := (a_1 - 1, a_2 - 2, \dots, a_{d-e} - (d-e), b_1 - (d-e+1), \dots, b_e - d).$$

We measure how far $\phi(a,b)$ is from being decreasing by introducing $i(a,b) := \#\{\alpha < \beta,\ \phi_\alpha > \phi_\beta\}$.

Finally, let us denote by $\phi(a,b)^+ = (\phi_1^+, \dots, \phi_d^+)$ a re-ordering of $\phi(a,b)$ to make it non-increasing and set $\psi(a,b) := (\phi_1^+ + 1, \dots, \phi_d^+ + d)$.

The Borel–Weil–Bott theorem reads as follows.

Theorem 3.2. *We have*

1. $H^q(G(3,W), \Gamma^a \mathcal{Q}_3^* \otimes \Gamma^b \mathcal{E}_3^*) = 0$ *for* $q \neq i(a,b)$,
2. $H^{i(a,b)}(G(3,W), \Gamma^a \mathcal{Q}_3^* \otimes \Gamma^b \mathcal{E}_3^*) = \Gamma^{\psi(a,b)} W$,

where $\mathcal{Q}_3$ and $\mathcal{E}_3$ are defined by (1.2) and $\Gamma^{\psi(a,b)}W = 0$ if $\psi(a,b)$ is not decreasing.

Now, we want to apply this theorem to compute the $E_1^{p,q}$ of the spectral sequence. Using Sage with the code

```
R=WeylCharacterRing("A2")
V=R(1,0,0)
for k in range(11): print k, V.symmetric_power(3).exterior_power(k)
```

we get the decompositions into irreducible modules of $\wedge^k \operatorname{Sym}^3 \mathcal{E}_3^*$. Then by the Borel–Weil–Bott theorem, we have

$$\begin{aligned}
&(0) & \oplus_i^{12} H^i\left(\mathcal{O}_{G(3,V)}\right) &= \oplus_i H^i\left(\Gamma^{(0,\dots,0)}\mathcal{Q}_3^* \otimes \Gamma^{(0,0,0)}\mathcal{E}_3^*\right) \\
&& &= H^0\left(\mathcal{O}_{G(3,V)}\right) = \Gamma^{(0,\dots,0)} V \simeq \mathbb{C}, \\
&(1) & \oplus_i^{12} H^i\left(\operatorname{Sym}^3 \mathcal{E}_3^*\right) &= \oplus_i^{12} H^i\left(\Gamma^{(3,0,0)}\mathcal{E}_3^*\right) = 0, \\
&(2) & \oplus_i H^i\left(\wedge^2 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &= \oplus_i H^i\left(\Gamma^{(3,3,0)}\mathcal{E}_3^* \oplus \Gamma^{(5,1,0)}\mathcal{E}_3^*\right) \\
&& &= H^4\left(\Gamma^{(5,1,0)}\mathcal{E}_3^*\right) = \Gamma^{(1,\dots,1,0)} V \simeq \wedge^6 V, \\
&(3) & \oplus_i H^i\left(\wedge^3 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &= \oplus_i H^i\left(\Gamma^{(3,3,3)}\mathcal{E}_3^* \oplus \Gamma^{(5,3,1)}\mathcal{E}_3^* \oplus \Gamma^{(6,3,0)}\mathcal{E}_3^* \oplus \Gamma^{(7,1,1)}\mathcal{E}_3^*\right) \\
&& &= H^4\left(\Gamma^{(7,1,1)}\mathcal{E}_3^*\right) = \Gamma^{(3,1,\dots,1)} V \simeq \operatorname{Sym}^2 V \otimes \det(V), \\
&(4) & \oplus_i H^i\left(\wedge^4 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &= \oplus_i H^i\left(\Gamma^{(6,3,3)}\mathcal{E}_3^* \oplus \Gamma^{(6,4,2)}\mathcal{E}_3^* \oplus \Gamma^{(6,6,0)}\mathcal{E}_3^* \oplus \Gamma^{(7,4,1)}\mathcal{E}_3^* \oplus \Gamma^{(8,3,1)}\mathcal{E}_3^*\right)
\end{aligned}$$

$$
\begin{aligned}
&= H^8\left(\Gamma^{(6,6,0)}\mathcal{E}_3^*\right) = \Gamma^{(2,\dots,2,0)}V \\
&\simeq \operatorname{Sym}^2 V^* \otimes \det(V)^{\otimes 2},
\end{aligned}
$$

(5)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^5 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\Big(\Gamma^{(6,6,3)}\mathcal{E}_3^* \oplus \Gamma^{(7,4,4)}\mathcal{E}_3^* \oplus \Gamma^{(7,6,2)}\mathcal{E}_3^* \oplus \Gamma^{(8,4,3)}\mathcal{E}_3^* \oplus \Gamma^{(8,6,1)}\mathcal{E}_3^* \\
&\qquad \oplus \Gamma^{(9,4,2)}\mathcal{E}_3^*\Big) \\
&= H^8\left(\Gamma^{(7,6,2)}\mathcal{E}_3^* \oplus \Gamma^{(8,6,1)}\mathcal{E}_3^*\right) \\
&= \Gamma^{(3,2,\dots,2)}V \oplus \Gamma^{(4,2\dots,2,1)}V \\
&\simeq \left(\operatorname{Sym}^2 V \otimes V^*\right) \otimes \det(V)^{\otimes 2},
\end{aligned}
$$

(6)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^6 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(7,7,4)}\mathcal{E}_3^* \oplus \Gamma^{(8,6,4)}\mathcal{E}_3^* \oplus \Gamma^{(9,6,3)}\mathcal{E}_3^* \oplus \Gamma^{(9,7,2)}\mathcal{E}_3^* \oplus \Gamma^{(10,4,4)}\mathcal{E}_3^*\right) \\
&= H^8\left(\Gamma^{(9,7,2)}\mathcal{E}_3^*\right) \\
&\simeq \Gamma^{(5,3,2\dots,2)}V \simeq \left(\wedge^2 \operatorname{Sym}^2 V\right) \otimes \det(V)^{\otimes 2},
\end{aligned}
$$

(7)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^7 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(7,7,7)}\mathcal{E}_3^* \oplus \Gamma^{(9,7,5)}\mathcal{E}_3^* \oplus \Gamma^{(9,9,3)}\mathcal{E}_3^* \oplus \Gamma^{(10,7,4)}\mathcal{E}_3^*\right) \\
&= H^{12}\left(\Gamma^{(7,7,7)}\mathcal{E}_3^*\right) \simeq \Gamma^{(3,\dots,3)}V \simeq \det(V)^{\otimes 3},
\end{aligned}
$$

(8)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^8 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(10,7,7)}\mathcal{E}_3^* \oplus \Gamma^{(10,9,5)}\mathcal{E}_3^*\right) \\
&= H^{12}\left(\Gamma^{(10,7,7)}\mathcal{E}_3^*\right) = \Gamma^{(6,3,\dots,3)}V \simeq \operatorname{Sym}^3 V \otimes \det(V)^{\otimes 3},
\end{aligned}
$$

(9)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^9 \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(10,10,7)}\mathcal{E}_3^*\right) \\
&= H^{12}\left(\Gamma^{(10,10,7)}\mathcal{E}_3^*\right) \simeq \Gamma^{(6,6,3,\dots,3)}V,
\end{aligned}
$$

(10)
$$
\begin{aligned}
\oplus_i H^i\left(\wedge^{10} \operatorname{Sym}^3 \mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(10,10,10)}\mathcal{E}_3^*\right) \\
&= H^{12}\left(\Gamma^{(10,10,10)}\mathcal{E}_3^*\right) \simeq \Gamma^{(6,6,6,3\dots,3)}V.
\end{aligned}
$$

To understand $H^1(\mathcal{O}_{F_2(X)})$, we have to examine the $E_\infty^{-i,i+1}$ for $i = 0,\dots,10$. As $E_1^{-i,i+1} = 0$ for any $i \neq 3$, we get $E_\infty^{-i,i+1} = 0$ for $i \neq 3$.

On the other hand, for $r \geq 2$, $E_r^{-3,4}$ is defined as the (middle) cohomology of

$$
E_{r-1}^{-(2+r),2+r} \xrightarrow{d_{r-1}} E_{r-1}^{-3,4} \xrightarrow{d_{r-1}} E_{r-1}^{-4+r,6-r}.
$$

From the above computations, we see that $E_1^{-i,i} = 0$ for $i \geq 3$, so that $E_r^{-i,i} = 0$ for any $i \geq 3$ and $r \geq 1$.

So we get $E_2^{-3,4} = \operatorname{Ker}(d_1 \colon E_1^{-3,4} \to E_1^{-2,4})$.

As $E_1^{-1,3} = 0$, we have $E_2^{-1,3} = 0$, so that $E_3^{-3,4} \simeq E_2^{-3,4}$.

As $E_1^{0,2} = 0$, we have $E_3^{0,2} = 0$, so that $E_4^{-3,4} \simeq E_2^{-3,4}$.

As $E_1^{a,b} = 0$ for any $a > 0$, we get $E_\infty^{-3,4} \simeq E_2^{-3,4}$; *i.e.*, the following sequence is exact:

$$
0 \longrightarrow H^1\left(\mathcal{O}_{F_2(X)}\right) \longrightarrow E_1^{-3,4} \xrightarrow{d_1^{-3,4}} E_1^{-2,4}.
$$

Now, $d_1^{-3,4}$ is given by contracting with the section defined by eq_X, so that, choosing a basis $(e_0,\dots,e_6)$ of V, we have

$$
\begin{aligned}
d_1^{-3,4} \colon \operatorname{Sym}^2 V \otimes \det(V) &\longrightarrow \wedge^6 V \simeq V^* \otimes \det(V). \\
(e_i + e_j) \otimes (e_0 \wedge \cdots \wedge e_6) &\longmapsto \sum_k \mathrm{eq}_X(e_i, e_j, e_k)\widehat{e_k} = \mathrm{eq}_X(e_i, e_j, \cdot) \otimes (e_0 \wedge \cdots \wedge e_6)
\end{aligned}
$$

If this map is not surjective, we can choose the basis so that $e_0^* \otimes (e_0 \wedge \cdots \wedge e_6) \notin \operatorname{Im}(d_1^{-3,4})$. Then we get $\mathrm{eq}_X(e_i, e_j, e_0) = 0$ for any i, j, which means that the cubic hypersurface X is a cone with vertex $[e_0]$.

So for a smooth cubic, $d_1^{-3,4}$ is surjective, so (3.1) is exact.

Before tackling the case of $H^2(\mathcal{O}_{F_2(X)})$, we notice that the exterior square of (3.1) gives the following exact sequence:

$$(3.2)\quad 0 \longrightarrow \wedge^2 H^1(\mathcal{O}_{F_2(X)}) \longrightarrow (\wedge^2 \operatorname{Sym}^2 V) \otimes \det(V)^{\otimes 2} \xrightarrow{\varphi_{\mathrm{eq}_X} \otimes \mathrm{id}_{\operatorname{Sym}^2 V \otimes \det(V)}} \operatorname{Sym}^2 V \otimes V^* \otimes \det(V)^{\otimes 2} \xrightarrow{\varphi_{\mathrm{eq}_X} \otimes \mathrm{id}_{V^* \otimes \det(V)}} \operatorname{Sym}^2 V^* \otimes \det(V)^{\otimes 2} \longrightarrow 0.$$

To understand $H^2(\mathcal{O}_{F_2(X)})$, we have to examine the $E_\infty^{-i,i+2}$ for $i = 0, \ldots, 10$. As $E_1^{-i,i+2} = 0$ for $i \neq 2, 6, 10$, we have $E_\infty^{-i,i+2} = 0$ for $i \neq 2, 6, 10$.

Analysis of $E_\infty^{-2,4}$.— As $E_1^{-1,4} = 0$, $E_2^{-2,4}$ is the cokernel of $d_1^{-3,4}$, which has just been proven to be surjective when X is smooth. So $E_2^{-2,4} = 0$, from which we get $E_\infty^{-2,4} = 0$.

Analysis of $E_\infty^{-6,8}$.— Each $E_r^{-6,8}$ is the middle cohomology of

$$E_{r-1}^{-(5+r),6+r} \xrightarrow{d_{r-1}} E_{r-1}^{-6,8} \xrightarrow{d_{r-1}} E_{r-1}^{-7+r,10-r}.$$

From the above computations of the cohomology groups, we see that $E_1^{-(5+r),6+r} = 0$ for any $r \geq 2$, so $E_{r-1}^{-(5+r),6+r} = 0$ for any $r \geq 2$.

So $E_2^{-6,8} = \operatorname{Ker}(d_1^{-6,8} \colon E_1^{-6,8} \to E_1^{-5,8})$.

We see that $E_1^{-7+r,10-r} = 0$ for any $r \geq 3$, so that $E_{r-1}^{-7+r,10-r} = 0$ for any $r \geq 3$. As a result, we get $E_\infty^{-6,8} = E_2^{-6,8}$.

From (3.2), we get that $\operatorname{Coker}(d_1^{-6,8} \colon E_1^{-6,8} \to E_1^{-5,8}) \simeq \operatorname{Sym}^2 V^* \otimes \det(V)^{\otimes 2}$ and $E_\infty^{-6,8} = \operatorname{Ker}(d_1^{-6,8} \colon E_1^{-6,8} \to E_1^{-5,8}) \simeq \wedge^2 H^1(\mathcal{O}_{F_2(X)})$.

Now, the spectral sequence computes the graded pieces of a filtration

$$0 = F^1 \subset F^0 \subset \cdots \subset F^{-10} \subset F^{-11} = H^2\left(\mathcal{O}_{F_2(X)}\right),$$

and we have seen ($E_\infty^{-2,4} = 0$) that all the graded pieces are trivial, but $\mathrm{Gr}^F_{-6} \simeq E_\infty^{-6,8}$ and (*a priori*) $\mathrm{Gr}^F_{-10} \simeq E_\infty^{-10,12}$. As a result, we get $\wedge^2 H^1(\mathcal{O}_{F_2(X)}) \simeq E_\infty^{-6,8} = F^{-6} = \cdots = F^{-9} \subset F^{10} \subset H^2(\mathcal{O}_{F_2(X)})$, proving the inclusion. □

Moreover, we have the following proposition.

Proposition 3.3. *We have* $H^0(\mathcal{Q}_3|^*_{F_2(X)}) \simeq H^0(\mathcal{Q}_3^*) \simeq V$ *and* $H^0(\operatorname{Sym}^2 \mathcal{E}_3|_{F_2(X)}) \simeq H^0(\operatorname{Sym}^2 \mathcal{E}_3) \simeq \operatorname{Sym}^2 V^*$, *and the following sequence is exact:*

$$(3.3)\quad 0 \longrightarrow H^0\left(\mathcal{Q}_3^*|_{F_2(X)}\right) \longrightarrow H^0\left(\operatorname{Sym}^2 \mathcal{E}_3|_{F_2(X)}\right) \longrightarrow H^0\left(\Omega_{F_2(X)}\right) \longrightarrow 0,$$

where the first map is given by $v \mapsto \mathrm{eq}_X(v, \cdot, \cdot)$.

Proof. To understand $H^0(\mathcal{Q}_3^*|_{F_2(X)})$, we use again the Koszul resolution (2.1) tensored by $\mathcal{Q}_3^*$. We have the spectral sequence

$$E_1^{p,q} = H^q\left(G(3,V), \mathcal{Q}_3^* \otimes \wedge^{-p} \operatorname{Sym}^3 \mathcal{E}_3^*\right) \Longrightarrow H^{p+q}\left(\mathcal{Q}_3^*|_{F_2(X)}\right).$$

We again use the Borel–Weil–Bott theorem 3.2 to compute the cohomology groups on $G(3, V)$. The decompositions of the $\wedge^i \operatorname{Sym} \mathcal{E}_3^*$'s into irreducible modules have already been obtained in Theorem 3.1. So we get

$$(0)\quad \oplus_i H^i\left(\mathcal{Q}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)} \mathcal{Q}_3^*\right)$$

$$= H^0\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^*\right) = V,$$

$$(1)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(3,0,0)}\mathcal{E}_3^*\right) = 0,$$

$$(2)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^2\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(3,3,0)}\mathcal{E}_3^* \oplus \Gamma^{(5,1,0)}\mathcal{E}_3^*\right)\right) = 0,$$

$$(3)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^3\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(3,3,3)}\mathcal{E}_3^* \oplus \Gamma^{(5,3,1)}\mathcal{E}_3^* \oplus \Gamma^{(6,3,0)}\mathcal{E}_3^* \oplus \Gamma^{(7,1,1)}\mathcal{E}_3^*\right)\right)$$
$$= H^4\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(7,1,1)}\mathcal{E}_3^*\right) \simeq \Gamma^{(3,2,1,\ldots,1)}V,$$

$$(4)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^4\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(6,3,3)}\mathcal{E}_3^* \oplus \Gamma^{(6,4,2)}\mathcal{E}_3^* \oplus \Gamma^{(6,6,0)}\mathcal{E}_3^* \oplus \Gamma^{(7,4,1)}\mathcal{E}_3^* \oplus \Gamma^{(8,3,1)}\mathcal{E}_3^*\right)\right)$$
$$= 0,$$

$$(5)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^5\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(6,6,3)}\mathcal{E}_3^* \oplus \Gamma^{(7,4,4)}\mathcal{E}_3^* \oplus \Gamma^{(7,6,2)}\mathcal{E}_3^* \oplus \Gamma^{(8,4,3)}\mathcal{E}_3^* \oplus \Gamma^{(8,6,1)}\mathcal{E}_3^* \oplus \Gamma^{(9,4,2)}\mathcal{E}_3^*\right)\right)$$
$$= 0,$$

$$(6)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^6\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(7,7,4)}\mathcal{E}_3^* \oplus \Gamma^{(8,6,4)}\mathcal{E}_3^* \oplus \Gamma^{(9,6,3)}\mathcal{E}_3^* \oplus \Gamma^{(9,7,2)}\mathcal{E}_3^* \oplus \Gamma^{(10,4,4)}\mathcal{E}_3^*\right)\right)$$
$$= H^8\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(9,7,2)}\mathcal{E}_3^*\right)$$
$$\simeq \Gamma^{(5,3,3,2,\ldots,2)}V,$$

$$(7)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^7\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(7,7,7)}\mathcal{E}_3^* \oplus \Gamma^{(9,7,5)}\mathcal{E}_3^* \oplus \Gamma^{(9,9,3)}\mathcal{E}_3^* \oplus \Gamma^{(10,7,4)}\mathcal{E}_3^*\right)\right)$$
$$= 0,$$

$$(8)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^8\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \left(\Gamma^{(10,7,7)}\mathcal{E}_3^* \oplus \Gamma^{(10,9,5)}\mathcal{E}_3^*\right)\right)$$
$$= 0,$$

$$(9)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^9\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(10,10,7)}\mathcal{E}_3^*\right) = 0,$$

$$(10)\qquad \oplus_i H^i\left(\mathcal{Q}_3^* \otimes \wedge^{10}\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(10,10,10)}\mathcal{E}_3^*\right)$$
$$= H^{12}\left(\Gamma^{(1,0,0,0)}\mathcal{Q}_3^* \otimes \Gamma^{(10,10,10)}\mathcal{E}_3^*\right) \simeq \Gamma^{(6,6,6,4,3,3,3)}V.$$

The graded pieces of the filtration on $H^0(\mathcal{Q}_3^*|_{F_2(X)})$ are given by $E_\infty^{-i,i}$, $i = 0,\ldots,10$. From the above calculations, we see that $E_1^{-i,i} = 0$ for any $i \geq 1$; thus $E_\infty^{-i,i} = 0$ for any $i \geq 1$.

On the other hand, $E_1^{0,0} = H^0(\mathcal{Q}_3^*) = V$, and as $E_r^{a,b} = 0$ for any $a > 0$, we have $E_r^{0,0} = \mathrm{Coker}(d_{r-1}\colon E_{r-1}^{-(r-1),r-2} E_{r-1}^{0,0})$ for any $r \geq 2$. But the above calculations give $E_1^{-r,r-1} = 0$ for $r \geq 0$, so that $E_r^{-r,r-1} = 0$ for any $r \geq 1$. Thus $E_\infty^{0,0} = E_1^{0,0}$, proving that $H^0(\mathcal{Q}_3^*|_{F_2(X)}) \simeq H^0(\mathcal{Q}_3^*) \simeq V$.

Now, let us examine $H^0(\mathrm{Sym}^2\,\mathcal{E}_3|_{F_2(X)})$ using the spectral sequence

$$E_1^{p,q} = H^q\left(\mathrm{Sym}^2\,\mathcal{E}_3 \otimes \wedge^{-p}\,\mathrm{Sym}^3\,\mathcal{E}_3^*\right) \Longrightarrow H^{p+q}\left(\mathrm{Sym}^2\,\mathcal{E}_3|_{F_2(X)}\right).$$

Using Sage with the code

```
R=WeylCharacterRing("A2")
V=R(1,0,0)
```

```
W=R(0,0,-1)
for k in range(11): print k,
↪  W.symmetric_power(2)*V.symmetric_power(3).exterior_power(k)
```

and the Borel–Weil–Bott theorem 3.2, we get

$$\begin{aligned}
&(0) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\right) &\simeq \oplus_i H^i\left(\Gamma^{(0,0,-2)}\mathcal{E}_3^*\right)\\
&&&= H^0\left(\Gamma^{(0,0,-2)}\mathcal{E}_3^*\right)\simeq \Gamma^{(0,\dots,0,-2)}V\simeq \operatorname{Sym}^2 V^*,\\
&(1) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\otimes\operatorname{Sym}^3\mathcal{E}_3^*\right) &\simeq \oplus_i H^i\left(\Gamma^{(1,0,0)}\mathcal{E}_3^*\oplus\Gamma^{(2,0,-1)}\mathcal{E}_3^*\oplus\Gamma^{(3,0,-2)}\mathcal{E}_3^*\right)=0,\\
&(2) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\otimes\wedge^2\operatorname{Sym}^3\mathcal{E}_3^*\right) &\simeq \oplus_i H^i\Big(\left(\Gamma^{(3,1,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(3,2,-1)}\mathcal{E}_3^*\oplus\Gamma^{(3,3,-2)}\mathcal{E}_3^*\\
&&&\qquad\oplus\Gamma^{(4,0,0)}\mathcal{E}_3^*\oplus\Gamma^{(4,1,-1)}\mathcal{E}_3^*\oplus\Gamma^{(5,1,-2)}\mathcal{E}_3^*\oplus\Gamma^{(5,0,-1)}\mathcal{E}_3^*\Big)\\
&&&= H^4\left(\Gamma^{(5,1,-2)}\mathcal{E}_3^*\oplus\Gamma^{(5,0,-1)}\mathcal{E}_3^*\right)\\
&&&\simeq \Gamma^{(1,\dots,1,-2)}V\oplus\Gamma^{(1,\dots,1,0,-1)}V,\\
&(3) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\otimes\wedge^3\operatorname{Sym}^3\mathcal{E}_3^*\right) &\simeq \oplus_i H^i\Big(\left(\Gamma^{(3,3,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(4,2,1)}\mathcal{E}_3^*\oplus\left(\Gamma^{(4,3,0)}\mathcal{E}_3^*\right)^{\oplus 2}\\
&&&\qquad\oplus\left(\Gamma^{(5,1,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(5,2,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(5,3,-1)}\mathcal{E}_3^*\right)^{\oplus 2}\\
&&&\qquad\oplus\left(\Gamma^{(6,1,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(6,2,-1)}\mathcal{E}_3^*\oplus\Gamma^{(6,3,-2)}\mathcal{E}_3^*\\
&&&\qquad\oplus\Gamma^{(7,1,-1)}\mathcal{E}_3^*\Big)\\
&&&= H^4\Big(\left(\Gamma^{(5,1,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(6,1,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(7,1,-1)}\mathcal{E}_3^*\Big)\\
&&&\simeq \det(V)^{\oplus 2}\oplus\left(\Gamma^{(2,1,\dots,1,0)}V\right)^{\oplus 2}\oplus\Gamma^{(3,1,\dots,1,-1)}V,\\
&(4) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\otimes\wedge^4\operatorname{Sym}^3\mathcal{E}_3^*\right) &\simeq \oplus_i H^i\Big(\Gamma^{(4,3,3)}\mathcal{E}_3^*\oplus\Gamma^{(4,4,2)}\mathcal{E}_3^*\oplus\left(\Gamma^{(5,3,2)}\mathcal{E}_3^*\right)^{\oplus 2}\\
&&&\qquad\oplus\left(\Gamma^{(5,4,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(6,2,2)}\mathcal{E}_3^*\oplus\left(\Gamma^{(6,3,1)}\mathcal{E}_3^*\right)^{\oplus 4}\\
&&&\qquad\oplus\left(\Gamma^{(6,4,0)}\mathcal{E}_3^*\right)^{\oplus 3}\oplus\Gamma^{(6,5,-1)}\mathcal{E}_3^*\oplus\Gamma^{(6,6,-2)}\mathcal{E}_3^*\\
&&&\qquad\oplus\left(\Gamma^{(7,2,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(7,3,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(7,4,-1)}\mathcal{E}_3^*\\
&&&\qquad\oplus\Gamma^{(8,1,1)}\mathcal{E}_3^*\oplus\Gamma^{(8,2,0)}\mathcal{E}_3^*\oplus\Gamma^{(8,3,-1)}\mathcal{E}_3^*\Big)\\
&&&= \underbrace{H^4\left(\Gamma^{(8,1,1)}\mathcal{E}_3^*\right)}_{\simeq\operatorname{Sym}^3 V\otimes\det(V)}\oplus\underbrace{H^8\left(\Gamma^{(6,6,-2)}\mathcal{E}_3^*\right)}_{\simeq\Gamma^{(2,\dots,2,-2)}V},\\
&(5) & \oplus_i H^i\left(\operatorname{Sym}^2\mathcal{E}_3\otimes\wedge^5\operatorname{Sym}^3\mathcal{E}_3^*\right) &\simeq \oplus_i H^i\Big(\Gamma^{(5,4,4)}\mathcal{E}_3^*\oplus\left(\Gamma^{(6,4,3)}\mathcal{E}_3^*\right)^{\oplus 3}\oplus\left(\Gamma^{(6,5,2)}\mathcal{E}_3^*\right)^{\oplus 2}\\
&&&\qquad\oplus\left(\Gamma^{(6,6,1)}\mathcal{E}_3^*\right)^{\oplus 3}\oplus\Gamma^{(7,3,3)}\mathcal{E}_3^*\oplus\left(\Gamma^{(7,4,2)}\mathcal{E}_3^*\right)^{\oplus 4}\\
&&&\qquad\oplus\left(\Gamma^{(7,5,1)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(7,6,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\left(\Gamma^{(8,3,2)}\mathcal{E}_3^*\right)^{\oplus 2}\\
&&&\qquad\oplus\left(\Gamma^{(8,4,1)}\mathcal{E}_3^*\right)^{\oplus 3}\oplus\Gamma^{(8,5,0)}\mathcal{E}_3^*\oplus\Gamma^{(8,6,-1)}\mathcal{E}_3^*\\
&&&\qquad\oplus\Gamma^{(9,2,2)}\mathcal{E}_3^*\oplus\Gamma^{(9,3,1)}\mathcal{E}_3^*\oplus\Gamma^{(9,4,0)}\mathcal{E}_3^*\Big)\\
&&&= H^8\Big(\left(\Gamma^{(6,6,1)}\mathcal{E}_3^*\right)^{\oplus 3}\oplus\left(\Gamma^{(7,6,0)}\mathcal{E}_3^*\right)^{\oplus 2}\oplus\Gamma^{(8,6,-1)}\mathcal{E}_3^*\Big)
\end{aligned}$$

$$\simeq \left(\Gamma^{(2,\dots,2,1)}V\right)^{\oplus 3} \oplus \left(\Gamma^{(3,2,\dots,2,0)}V\right)^{\oplus 2} \oplus \Gamma^{(4,2,\dots,2,-1)}V,$$

$$(6) \qquad \oplus_i H^i\left(\mathrm{Sym}^2\mathcal{E}_3 \otimes \wedge^6 \mathrm{Sym}^3\mathcal{E}_3^*\right) \simeq \oplus_i H^i\Big(\Gamma^{(6,6,4)}\mathcal{E}_3^* \oplus \left(\Gamma^{(7,5,4)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(7,6,3)}\mathcal{E}_3^*\right)^{\oplus 3}$$
$$\oplus \left(\Gamma^{(7,7,2)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(8,4,4)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(8,5,3)}\mathcal{E}_3^*\right)^{\oplus 2}$$
$$\oplus \left(\Gamma^{(8,6,2)}\mathcal{E}_3^*\right)^{\oplus 3} \oplus \Gamma^{(8,7,1)}\mathcal{E}_3^* \oplus \left(\Gamma^{(9,4,3)}\mathcal{E}_3^*\right)^{\oplus 2}$$
$$\oplus \left(\Gamma^{(9,5,2)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(9,6,1)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \Gamma^{(9,7,0)}\mathcal{E}_3^*$$
$$\oplus \Gamma^{(10,4,2)}\mathcal{E}_3^*\Big)$$
$$= H^8\Big(\left(\Gamma^{(7,7,2)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(8,6,2)}\mathcal{E}_3^*\right)^{\oplus 3} \oplus \Gamma^{(8,7,1)}\mathcal{E}_3^*$$
$$\oplus \left(\Gamma^{(9,6,1)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \Gamma^{(9,7,0)}\mathcal{E}_3^*\Big)$$
$$\simeq \left(\Gamma^{(3,3,2,\dots,2)}V\right)^{\oplus 2} \oplus \left(\Gamma^{(4,2,\dots,2)}V\right)^{\oplus 3} \oplus \Gamma^{(4,3,2,\dots,2,1)}V$$
$$\oplus \left(\Gamma^{(5,2,\dots,2,1)}V\right)^{\oplus 2} \oplus \Gamma^{(5,3,2\dots,2,0)}V,$$

$$(7) \qquad \oplus_i H^i\left(\mathrm{Sym}^2\mathcal{E}_3 \otimes \wedge^7 \mathrm{Sym}^3\mathcal{E}_3^*\right) \simeq \oplus_i H^i\Big(\left(\Gamma^{(7,7,5)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \Gamma^{(8,6,5)}\mathcal{E}_3^* \oplus \left(\Gamma^{(8,7,4)}\mathcal{E}_3^*\right)^{\oplus 2}$$
$$\oplus \Gamma^{(9,5,5)}\mathcal{E}_3^* \oplus \left(\Gamma^{(9,6,4)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \left(\Gamma^{(9,7,3)}\mathcal{E}_3^*\right)^{\oplus 3}$$
$$\oplus \Gamma^{(9,8,2)}\mathcal{E}_3^* \oplus \Gamma^{(9,9,1)}\mathcal{E}_3^* \oplus \Gamma^{(10,5,4)}\mathcal{E}_3^*$$
$$\oplus \Gamma^{(10,6,3)}\mathcal{E}_3^* \oplus \Gamma^{(10,7,2)}\mathcal{E}_3^*\Big)$$
$$= H^8\left(\Gamma^{(9,8,2)}\mathcal{E}_3^* \oplus \Gamma^{(9,9,1)}\mathcal{E}_3^* \oplus \Gamma^{(10,7,2)}\mathcal{E}_3^*\right)$$
$$\simeq \Gamma^{(5,4,2\dots,2)}V \oplus \Gamma^{(5,5,2,\dots,2,1)}V \oplus \Gamma^{(6,3,2,\dots,2)}V,$$

$$(8) \qquad \oplus_i H^i\left(\mathrm{Sym}^2\mathcal{E}_3 \otimes \wedge^8 \mathrm{Sym}^3\mathcal{E}_3^*\right) \simeq \oplus_i H^i\Big(\Gamma^{(8,7,7)}\mathcal{E}_3^* \oplus \Gamma^{(9,7,6)}\mathcal{E}_3^* \oplus \Gamma^{(9,8,5)}\mathcal{E}_3^*$$
$$\oplus \Gamma^{(9,9,4)}\mathcal{E}_3^* \oplus \left(\Gamma^{(10,7,5)}\mathcal{E}_3^*\right)^{\oplus 2} \oplus \Gamma^{(10,8,4)}\mathcal{E}_3^*$$
$$\oplus \Gamma^{(10,9,3)}\mathcal{E}_3^*\Big)$$
$$= H^{12}\left(\Gamma^{(8,7,7)}\mathcal{E}_3^*\right) \simeq \Gamma^{(4,3,\dots,3)}V,$$

$$(9) \qquad \oplus_i H^i\left(\mathrm{Sym}^2\mathcal{E}_3 \otimes \wedge^9 \mathrm{Sym}^3\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(10,8,7)}\mathcal{E}_3^* \oplus \Gamma^{(10,9,6)}\mathcal{E}_3^* \oplus \Gamma^{(10,10,5)}\mathcal{E}_3^*\right)$$
$$= H^{12}\left(\Gamma^{(10,8,7)}\mathcal{E}_3^*\right) \simeq \Gamma^{(6,4,3,\dots,3)}V,$$

$$(10) \qquad \oplus_i H^i\left(\mathrm{Sym}^2\mathcal{E}_3 \otimes \wedge^{10} \mathrm{Sym}^3\mathcal{E}_3^*\right) \simeq \oplus_i H^i\left(\Gamma^{(10,10,8)}\mathcal{E}_3^*\right)$$
$$= H^{12}\left(\Gamma^{(10,10,8)}\mathcal{E}_3^*\right) \simeq \Gamma^{(6,6,4,3,\dots,3)}V.$$

The graded pieces of the filtration on $H^0(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)})$ are given by the $E_\infty^{-i,i}$. We have $E_\infty^{-i,i} = 0$ for any $i \neq 0,4$ since $E_1^{-i,i} = 0$ for $i \neq 0,4$.

As $E_r^{a,b} = 0$ for any $a > 0$ and $E_r^{-r,r-1} = 0$ (because $E_1^{-r,r-1} = 0$) for any $r \geq 1$, we have $E_\infty^{0,0} = E_1^{0,0}$.

In particular, $H^0(\mathrm{Sym}^2\mathcal{E}_3) \simeq E_\infty^{0,0} \subset H^0(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)})$. As $h^0(\mathrm{Sym}^2\mathcal{E}_3) = \dim(\mathrm{Sym}^2 V^*) = 28$, we have $h^0(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)}) \geq 28$. By Hodge symmetry, $h^0(\Omega_{F_2(X)}) = h^1(\mathcal{O}_{F_2(X)}) = 21$ (see Theorem 3.1). So the exactness of the sequence

$$0 \longrightarrow H^0\left(\mathcal{Q}_3^*|_{F_2(X)}\right) \longrightarrow H^0\left(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)}\right) \longrightarrow H^0\left(\Omega_{F_2(X)}\right)$$

implies $H^0(\mathrm{Sym}^2\mathcal{E}_3) = H^0(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)})$ and the surjectivity of the last map. □

According to Theorem 3.1, $\bigwedge^2 H^0(\Omega_{F_2(X)}) \subset H^0(K_{F_2(X)})$. As $K_{F_2(X)} \simeq \mathcal{O}_{G(3,V)}(3)|_{F_2(X)}$, the map $\rho\colon F_2(X) \dashrightarrow |\bigwedge^2 H^0(\Omega_{F_2(X)})|$ is the composition of the degree 3 Veronese of the natural embedding $F_2(X) \subset G(3,V)$ followed by a linear projection. Moreover, we have the following.

Lemma 3.4.

(1) *The canonical bundle $K_{F_2(X)}$ is generated by the sections in $\bigwedge^2 H^0(\Omega_{F_2(X)}) \subset H^0(K_{F_2(X)})$. In particular, $|\bigwedge^2 H^0(\Omega_{F_2(X)})|$ is base-point-free.*

(2) *For any $[P] \in F_2(X)$, the following sequence is exact:*

$$0 \longrightarrow \mathcal{K}_{[P]} \longrightarrow H^0\left(\Omega_{F_2(X)}\right) \xrightarrow{\mathrm{ev}([P])} \Omega_{F_2(X),[P]} \longrightarrow 0,$$

where $\mathcal{K}_{[P]} = \{Q \in H^0(\mathcal{O}_{\mathbb{P}^6}(2)),\ P \subset \{Q=0\}\}/\mathrm{Span}((\mathrm{eq}_X(x,\cdot,\cdot))_{x\in\langle P\rangle})$.

Proof. (1) As $\mathcal{E}_3|_{F_2(X)}$ is globally generated (as a restriction of $\mathcal{E}_3$, which is globally generated, by (1.2)), $\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)}$ is also globally generated. The same holds for $\mathcal{Q}_3^*|_{F_2(X)}$ (by (1.2)). So applying the evaluation to (3.3), we get the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & H^0\left(\mathcal{Q}_3|^*_{F_2(X)}\right)\otimes\mathcal{O}_{F_2(X)} & \to & H^0\left(\mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)}\right)\otimes\mathcal{O}_{F_2(X)} & \to & H^0\left(\Omega_{F_2(X)}\right)\otimes\mathcal{O}_{F_2(X)} & \to 0 \\
& \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_2} & & \downarrow{\scriptstyle \mathrm{ev}_3} & \\
0 \longrightarrow & \mathcal{Q}_3^*|_{F_2(X)} & \longrightarrow & \mathrm{Sym}^2\mathcal{E}_3|_{F_2(X)} & \longrightarrow & \Omega_{F_2(X)} & \longrightarrow 0,
\end{array}$$

where the bottom row is (1.1). As ev_2 is surjective, we get that ev_3 is also surjective; *i.e.*, $\Omega_{F_2(X)}$ is globally generated. Then taking the exterior square of ev_3, we get that $\wedge^2\mathrm{ev}_3$ is surjective:

$$\bigwedge^2 H^0\left(\Omega_{F_2(X)}\right)\otimes\mathcal{O}_{F_2(X)} \xrightarrow{\wedge^2\mathrm{ev}_3} \wedge^2\Omega_{F_2(X)}.$$

Now a base point of $|\bigwedge^2 H^0(\Omega_{F_2(X)})|$ would be a point where $\wedge^2\mathrm{ev}_3$ fails to be surjective. So $|\bigwedge^2 H^0(\Omega_{F_2(X)})|$ is base-point-free.

(2) As $H^0(\mathcal{Q}_3^*|_{F_2(X)}) \simeq H^0(\mathcal{Q}_3^*) \simeq V$ by Proposition 3.3, (1.2) yields $\ker(\mathrm{ev}_1) \simeq \mathcal{E}_3^*|_{F_2(X)}$, so the snake lemma gives the exact sequence. □

Now, let us come back to the Gauss map of $F_2(X)$, that we have defined to be

$$\begin{aligned}
\mathcal{G}\colon \mathrm{alb}_{F_2}(F_2(X)) &\dashrightarrow G\left(2, T_{\mathrm{Alb}(F_2(X)),0}\right),\\
t &\longmapsto T_{\mathrm{alb}_{F_2}(F_2(X))-t,0}
\end{aligned}$$

where $\mathrm{alb}_{F_2}(F_2(X)) - t$ is the translation of $\mathrm{alb}_{F_2}(F_2(X)) \subset \mathrm{Alb}(F_2(X))$ by $-t \in \mathrm{Alb}(F_2(X))$. It is defined on the smooth locus of $\mathrm{alb}_{F_2}(F_2(X))$.

According to [Col86, Section (III)], $T\,\mathrm{alb}_{F_2}$ is injective. So the indeterminacies of $\mathcal{G}$ are resolved by the pre-composition with alb_{F_2}, *i.e.*,

$$\begin{aligned}
F_2(X) &\longrightarrow G\left(2, T_{\mathrm{Alb}(F_2(X)),0}\right)\\
t &\longmapsto T_{-\mathrm{alb}_{F_2}(t)}\mathrm{Translate}(-\mathrm{alb}_{F_2}(t))\left(T_t\,\mathrm{alb}_{F_2}\left(T_{F_2(X),t}\right)\right).
\end{aligned}$$

We have the Plücker embedding

$$G\left(2, T_{\mathrm{Alb}(F_2(X)),0}\right) \simeq G\left(2, H^0\left(\Omega_{F_2(X)}\right)^*\right) \subset \mathbb{P}\left(\bigwedge^2 H^0\left(\Omega_{F_2(X)}\right)^*\right)$$

and the commutative diagram

$$\begin{array}{ccc} F_2(X) & \xrightarrow{\mathrm{alb}_{F_2}} & \mathrm{alb}_{F_2}(F_2(X)) \\ & & \downarrow \mathcal{G} \\ \downarrow \rho & & G\left(2, H^0(\Omega_{F_2(X)})^*\right) \\ & & \cap \downarrow \\ \left|\bigwedge^2 H^0\left(\Omega_{F_2(X)}\right)\right| & \xrightarrow{\cong} & \mathbb{P}\left(\bigwedge^2 H^0\left(\Omega_{F_2(X)}\right)^*\right). \end{array}$$

The following proposition completes the proof of Theorem 1.3.

Proposition 3.5. *The morphism ρ is an embedding, which implies that* alb_{F_2} *is an isomorphism unto its image and $\mathcal{G}$ is an embedding.*

Proof. Let us denote by J_X the Jacobian ideal of X, *i.e.*, the ideal of the polynomial ring generated by $\left(\frac{\partial \mathrm{eq}_X}{\partial X_i}\right)_{i=0,\dots,6}$ and by $J_{X,2}$ its homogeneous part of degree 2. By Proposition 2.2, for any $[P] \in F_2(X)$, $\dim(J_{X,2}|_P) = 4$, so that $\dim(J_X \cap \{Q \in H^0(\mathcal{O}_{\mathbb{P}^6}(2)),\ P \subset \{Q = 0\}\}) = 3$. We have the following.

Lemma 3.6.

(1) *For $[P] \in G(3, V)$, the codimension of $L_P^2 := \{Q \in H^0(\mathcal{O}_{\mathbb{P}^6}(2)),\ P \subset \{Q = 0\}\}$ in $H^0(\mathcal{O}_{\mathbb{P}^6}(2))$ is* 6*. For $[P] \neq [P'] \in G(3, V)$, the codimension of $L_{P,P'}^2 := \{Q \in H^0(\mathcal{O}_{\mathbb{P}^6}(2)),\ P, P' \subset \{Q = 0\}\}$ in L_P^2 is*
   - (a) 6 *if $P \cap P' = \emptyset$,*
   - (b) 5 *if $P \cap P' = \{\mathrm{pt}\}$,*
   - (c) 3 *if $P \cap P' = \{\mathrm{line}\}$.*

(2) *For $[P] \neq [P'] \in F_2(X)$ such that $P \cap P' = \{\mathrm{line}\}$, we have* $\dim(J_X \cap L_{P,P'}^2) \geq 1$*, and if X is general, we even have* $\dim(J_X \cap L_{P,P'}^2) \geq 2$*. So $L_P^2/(J_X \cap L_P^2) + L_{P,P'}^2 \subsetneq L_P^2$, and for X general,* $\dim(L_P^2/(J_X \cap L_P^2) + L_{P,P'}^2) \geq 2$.

Proof. (1) This follows from a direct calculation.

(2) Up to a projective transformation, we can assume $P = \{X_0 = \cdots = X_3 = 0\}$ and $P' = \{X_0 = X_1 = X_2 = X_4 = 0\}$. Then eq_X is of the form (2.7) with the additional conditions $Q_3(0, X_5, X_6) = 0$, $D_5(0, 0, 0, X_3) = 0$, $D_6(0, 0, 0, X_3) = 0$, $R(0, 0, 0, X_3) = 0$.

By definition, the quadrics of the Jacobian ideal are $\frac{\partial \mathrm{eq}_X}{\partial X_i}$, and according to Proposition 2.2, $\left(\frac{\partial \mathrm{eq}_X}{\partial X_i}|_P\right)_{i=0,\dots,3}$ are linearly independent, so that

$$J_X \cap L_P^2 = \mathrm{Span}\left(\left(\frac{\partial \mathrm{eq}_X}{\partial X_i}|_P\right)_{i=4,5,6}\right).$$

For $i \in \{4, 5, 6\}$,

$$\frac{\partial \mathrm{eq}_X}{\partial X_i} = X_0 \frac{\partial Q_0}{\partial X_i} + X_1 \frac{\partial Q_1}{\partial X_i} + X_2 \frac{\partial Q_2}{\partial X_i} + X_3 \frac{\partial Q_3}{\partial X_i} + D_i$$

which, when restricted to P', gives $\frac{\partial \mathrm{eq}_X}{\partial X_i}|_{P'} = X_3 \frac{\partial Q_3}{\partial X_i}(0, X_5, X_6) + D_i(0, 0, 0, X_3)$. But since $Q_3(0, X_5, X_6) = 0$, we have $\frac{\partial Q_3}{\partial X_i}(0, X_5, X_6) = 0$ for $i = 5, 6$, so that $\frac{\partial \mathrm{eq}_X}{\partial X_5}|_{P'} = 0 = \frac{\partial \mathrm{eq}_X}{\partial X_6}|_{P'}$, *i.e.*, $\frac{\partial \mathrm{eq}_X}{\partial X_5}, \frac{\partial \mathrm{eq}_X}{\partial X_6} \in L_{P,P'}^2 \cap J_X$. For X general, those two quadric polynomials are independent.

We have $\dim(J_X \cap L_P^2 + L_{P,P'}^2) = \dim(J_X \cap L_P^2) + \dim(L_{P,P'}^2) - \dim(J_X \cap L_{P,P'}^2)$, which, by the first item of the lemma, yields the result. □

According to Lemma 3.6, for $[P] \neq [P'] \in F_2(X)$, we can always find a quadric $Q \in H^0(\mathcal{O}_{\mathbb{P}^6}(2))$ such that $0 \neq \overline{Q} \in L_P^2/(J_X \cap L_P^2 + L_{P,P'}^2)$; in particular, $Q|_P = 0$ but $Q|_{P'} \neq 0$. Pick another $Q' \in H^0(\mathcal{O}_{\mathbb{P}^6}(2)) \backslash (L_P^2 \cup L_{P'}^2)$

(*i.e.*, $Q'|_P \neq 0$, $Q'|_{P'} \neq 0$) such that $Q'|_{P'}$ is independent of $Q|_{P'}$ and Q and Q' are independent modulo $J_{X,2}$ ($\dim(H^0(\mathcal{O}_{\mathbb{P}^6}(2))/(J_{X,2} \oplus \mathbb{C}[Q])) = 5$).

By Proposition 3.3, such quadrics give rise to 1-forms on $F_2(X)$. Then $Q \wedge Q' \in \bigwedge^2 H^0(\Omega_{F_2(X)})$ vanishes at $[P]$ but not at $[P']$; *i.e.*, $|\bigwedge^2 H^0(\Omega_{F_2(X)})|$ separates points.

Now, given a $[P] \in F_2(X)$, we recall that

$$T_{[P]}F_2(X) = \{u \in \mathrm{Hom}(\langle P\rangle, V/\langle P\rangle),\ \mathrm{eq}_X(x, x, u(x)) = 0\ \forall x \in \langle P\rangle\}$$

(the first order of $\mathrm{eq}_X(x + u(x), x + u(x), x + u(x))$ is 0 for all $x \in \langle P\rangle$).

Let $Q \in L_P^2$ be such that $0 \neq \overline{Q} \in H^0(\mathcal{O}_{\mathbb{P}^6}(2))/J_{X,2}$ and $T_{[P]}F_2(Q) \cap T_{[P]}F_2(X) = \{0\}$. Pick a non-zero $\overline{Q}' \in H^0(\mathcal{O}_{\mathbb{P}^6}(2))/J_{X,2}$ such that $Q'|_P \neq 0$; then $Q \wedge Q' \in \bigwedge^2 H^0(\Omega_{F_2(X)})$ and $(Q \wedge Q')|_P = 0$.

Moreover, given a $u \in T_{[P]}F_2(X)$, we have $d_{[P]}Q(u) \wedge Q'|_P + Q|_P \wedge d_{[P]}Q'(u) = d_{[P]}Q(u) \wedge Q'|_P$, where $d_{[P]}Q(u)$ is the quadratic form $x \mapsto \mathrm{eq}_Q(x, u(x))$ and is non-trivial since $T_{[P]}F_2(Q) \cap T_{[P]}F_2(X) = \{0\}$. Then for Q generic (containing P and such that $T_{[P]}F_2(Q) \cap T_{[P]}F_2(X) = \{0\}$), $d_{[P]}Q(u)$ is linearly independent of $Q'|_P$, so that $Q \wedge Q'$ does vanish along the tangent vector u. So $|\bigwedge^2 H^0(\Omega_{F_2(X)})|$ separates tangent directions. □

4. Variety of osculating planes of a cubic 4-fold

In (1.3), we have previously introduced, for a smooth cubic 4-fold containing no plane $Z \subset \mathbb{P}(H^*) \simeq \mathbb{P}^5$, the variety of osculating planes $F_0(Z) := \{[P] \in G(3, H),\ \exists \ell \subset P \text{ line s.t. } P \cap Z = \ell \text{ (set-theoretically)}\}$.

The variety $F_0(Z)$ lives naturally in $\mathrm{Fl}(2, 3, H)$, *i.e.*,

$$F_0(Z) = \{([\ell], [P]) \in \mathrm{Fl}(2, 3, H),\ P \cap Z = \ell \text{ (set-theoretically)}\},$$

and from the exact sequence (2.2):

$$0 \longrightarrow e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1) \longrightarrow t^*\mathcal{E}_3 \longrightarrow e^*\mathcal{E}_2 \longrightarrow 0,$$

we see that $e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1)$ is, for $([\ell], [P]) \in \mathrm{Fl}(2, 3, H)$, the bundle of equations of $\ell \subset P$. As a result, $F_0(Z)$ is the zero locus on $\mathrm{Fl}(2, 3, H)$ of a section of the rank 9 vector bundle $\mathcal{F}$ defined by the exact sequence

$$0 \longrightarrow e^*\mathcal{O}_{G(2,H)}(-3) \otimes t^*\mathcal{O}_{G(3,H)}(3) \longrightarrow t^* \mathrm{Sym}^3 \mathcal{E}_3 \longrightarrow \mathcal{F} \longrightarrow 0. \tag{4.1}$$

In particular (since $\mathcal{F}$ is globally generated by the sections induced by $H^0(t^* \mathrm{Sym}^3 \mathcal{E}_3)$), by Bertini-type theorems, for Z general, $F_0(Z)$ is a smooth surface with $K_{F_0(Z)} \simeq (t^*\mathcal{O}_{G(3,H)}(3))|_{F_0(Z)}$. Its link to the surface of planes of a cubic 5-fold is the following.

Proposition 4.1. *Denoting by $X_Z = \{X_6^3 - \mathrm{eq}_Z(X_0, \ldots, X_5) = 0\}$ the cyclic cubic 5-fold associated to Z, the linear projection with center $p_0 := [0 : \cdots : 0 : 1]$ induces a degree 3 étale cover $\pi\colon F_2(X_Z) \to F_0(Z)$ given by the torsion line bundle $(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)}$.*

In particular, when $F_0(Z)$ is smooth, $F_2(X_Z)$ and $F_0(Z)$ are smooth and irreducible.

Proof. (1) The point p_0 does not belong to X_Z. In particular, any $[P] \in F_2(X_Z)$ is sent by $\pi_{p_0}\colon \mathbb{P}(V^*) \dashrightarrow \mathbb{P}(H^*)$ to a plane in $\mathbb{P}(H^*)$, where $V = H \oplus \mathbb{C} \cdot p_0$. The restriction of π_{p_0} (also denoted by π_{p_0}) to X is a degree 3 cyclic cover of $\mathbb{P}^5$ ramified over Z. Let us denote by $\tau\colon [a_0 : \cdots : a_6] \mapsto [a_0 : \cdots : a_5 : \xi a_6]$, with ξ a primitive third root of 1, the cover automorphism.

For any $[P] \in F_2(X_Z)$, $\pi_{p_0}\colon \pi_{p_0}^{-1}(\pi_{p_0}(P)) \to \pi_{p_0}(P)$ is a degree 3 cyclic cover ramified over the cubic curve $\pi_{p_0}(P) \cap Z$. It contains the three sections $P, \tau(P), \tau^2(P)$, which in turn all contain (set-theoretically) the ramification curve $\pi_{p_0}(P) \cap Z$, so it is a line; *i.e.*, $([\{\pi_{p_0}(P) \cap Z\}_{\mathrm{red}}], [\pi_{p_0}(P)]) \in F_0(Z)$.

Conversely, for any $([\ell], [P]) \in F_0(Z)$, $\pi_{p_0}^{-1}|_{X_Z}(P) \to P$ is a degree 3 cyclic cover ramified over $\{\ell\}^3$, so it consists of three surfaces isomorphic each to P, *i.e.*, three planes. To make it even more explicit, if $P = \{X_0 = X_1 = X_2 = 0\}$ and $\ell = \{X_0 = X_1 = X_2 = X_3 = 0\}$, then $\pi_{p_0}^{-1}|_{X_Z}(P)$ is defined in $\pi_{p_0}^{-1}(P) \simeq \mathrm{Span}(P, p_0) \simeq \mathbb{P}^3$ by $X_6^3 - aX_3^3$ for some $a \neq 0$ (since Z contains no plane), and we have $X_6^3 - aX_3^3 = (X_6 - bX_3)(X_6 - b'X_3)(X_6 - b''X_3)$, where b, b', b'' are the distinct roots of $y^3 = a$. So $\pi\colon F_2(X_Z) \to F_0(Z)$ is étale of degree 3.

(2) The equation eq_Z defines a section $\sigma_{\mathrm{eq}_Z} \in H^0(t^*\mathrm{Sym}^3\,\mathcal{E}_3) \simeq H^0(\mathrm{Sym}^3\,\mathcal{E}_3)$ and by projection in (4.1) a section $\overline{\sigma_{\mathrm{eq}_Z}}$ of $\mathcal{F}$ whose zero locus is $F_0(Z)$. Restricting (4.1) to $F_0(Z)$, we see that σ_{eq_Z} induces a section of $(e^*\mathcal{O}_{G(2,H)}(-3) \otimes t^*\mathcal{O}_{G(3,H)}(3))|_{F_0(Z)}$ which vanishes nowhere since Z contains no plane. Thus

$$\left(e^*\mathcal{O}_{G(2,H)}(-3) \otimes t^*\mathcal{O}_{G(3,H)}(3)\right)|_{F_0(Z)} \simeq \mathcal{O}_{F_0(Z)}.$$

Now if $(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)} \simeq \mathcal{O}_{F_0(Z)}$, since $(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)}$ is the bundle of equation of $\ell_x \subset P_x$ for any $x = ([\ell_x],[P_x]) \in F_0(Z)$, for any nowhere-vanishing section s of $(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)}$, we would be able to define three distinct sections of $\pi\colon F_2(X_Z) \to F_0(Z)$, namely (symbolically) $[x \mapsto \{X_6 - \xi^k s(x)\}_{\mathrm{Span}(P_x,p_0)}]$, $k = 0,1,2$. But according to [Col86, Proposition 1.8], $F_2(X)$ is connected for any X. Hence we have a contradiction. So $(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)}$ is a non-trivial 3-torsion line bundle.

Moreover, we readily see that for any $[P] \in F_2(X_Z)$, $X_6|_P \neq 0$ is an equation of the line $P \cap \mathbb{P}(H^*)$; *i.e.*, $\pi^*(e^*\mathcal{O}_{G(2,H)}(-1) \otimes t^*\mathcal{O}_{G(3,H)}(1))|_{F_0(Z)}$ has a nowhere-vanishing section, hence is trivial.

(3) When $F_0(Z)$ is smooth, since π is étale, $F_2(X_Z)$ is also smooth. As $F_2(X_Z)$ is connected (by [Col86, Proposition 1.8]), $F_2(X_Z)$ is irreducible, and $\pi(F_2(X_Z)) = F_0(Z)$ is also irreducible. □

Remark 4.2. That $F_0(Z)$ is smooth and irreducible, for Z general, is proven in [GK21, Lemma 4.3] without reference to $F_2(X_Z)$.

In [GK21], the interest for the image $e(F_0(Z)) \subset F_1(Z)$ stems from $e(F_0(Z))$ being the fixed locus of a rational self-map of the hyper-Kähler 4-fold $F_1(Z)$ defined by Voisin (*cf.* [Voi04]).

Proposition 4.3. *For Z general, the tangent map of $e_{F_0} := e|_{F_0(Z)}\colon F_0(Z) \to F_1(Z)$ is injective, and e_{F_0} is the normalisation of $e_{F_0}(F_0(Z))$ and is an isomorphism unto its image outside a finite subset of $F_0(Z)$.*

Moreover, $e_{F_0}(F_0(Z))$ is a (non-normal) Lagrangian surface of the hyper-Kähler 4-fold $F_1(Z)$.

Proof. (1) That e_{F_0} is injective outside a finite number of points follows from a simple dimension count: let us introduce $I := \{(([\ell],[P]),[Z]) \in \mathrm{Fl}(2,3,H) \times |\mathcal{O}_{\mathbb{P}^5}(3)|,\ \ell \subset Z \text{ and } Z \cap P = \ell \text{ set-theoretically}\}$ and $I_2 := \{(([\ell],[P_1],[P_2]),[Z]) \in \mathbb{P}(\mathcal{Q}_2) \times_{G(2,H)} \mathbb{P}(\mathcal{Q}_2) \backslash \Delta_{\mathbb{P}(\mathcal{Q}_2)} \times |\mathcal{O}_{\mathbb{P}^5}(3)|,\ \ell \subset Z \text{ and } Z \cap P_i = \ell,\ i = 1,2 \text{ set-theoretically}\}$. As $\mathrm{Fl}(2,3,H)$ and $\mathbb{P}(\mathcal{Q}_2) \times_{G(2,H)} \mathbb{P}(\mathcal{Q}_2) \backslash \Delta_{\mathbb{P}(\mathcal{Q}_2)}$ are homogeneous, the fibers of $p\colon I \to \mathrm{Fl}(2,3,H)$ (resp. $p_2\colon I_2 \to \mathbb{P}(\mathcal{Q}_2) \times_{G(2,H)} \mathbb{P}(\mathcal{Q}_2) \backslash \Delta_{\mathbb{P}(\mathcal{Q}_2)}$) are isomorphic to each other and are sub-linear systems of $|\mathcal{O}_{\mathbb{P}^5}(3)|$.

Notice that, since $F_0(Z)$ is a surface for Z general, we know that $\dim(I) = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)| + 2$.

Let us analyse the fiber of p_2. To do so, we can assume $\ell = \{X_2 = \cdots = X_5 = 0\}$, $P_1 = \{X_3 = X_4 = X_5 = 0\}$ and $P_2 = \{X_2 = X_4 = X_5 = 0\}$. Then the condition $Z \cap P_1 = \ell$ implies that eq_Z is of the form

$$\mathrm{eq}_Z = \alpha X_2^3 + X_3Q_3 + X_4Q_4 + X_5Q_5 + \sum_{i=0}^{2} X_iD_i(X_3,X_4,X_5) + R(X_3,X_4,X_5), \tag{4.2}$$

where the $Q_i(X_0,X_1,X_2)$ are quadratic forms in X_0,X_1,X_2, the D_i are quadratic forms in X_3,X_4,X_5 and R is a cubic form in X_3,X_4,X_5. Notice that this is the general form of a member of the fiber $p^{-1}([\ell],[P_1])$, in particular, $\dim(p^{-1}([\ell],[P_1])) = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)|) + 2 - \dim(\mathrm{Fl}(2,3,H)) = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)|) - 9$.

The additional condition $Z \cap P_2 = \ell$ implies that $Q_3(X_0,X_1,0) = 0$, $D_0(X_3,0,0) = 0$, $D_1(X_3,0,0) = 0$, which gives $3+1+1 = 5$ constraints. So $\dim(p_2^{-1}(([\ell],[P_1],[P_2]))) = \dim(p^{-1}([\ell],[P_1])) - 5 = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)|) - 14$, hence $\dim(I_2) = \dim(p_2^{-1}(([\ell],[P_1],[P_2]))) + 2\times 3 + \dim(G(2,H)) = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)|)$. As a result, the general fiber of $I_2 \to |\mathcal{O}_{\mathbb{P}^5}(3)|$ is finite. In other words, for $[Z] \in |\mathcal{O}_{\mathbb{P}^5}(3)|$ general, there are only finitely many $\ell \subset Z$

such that there are at least two planes $P_1, P_2 \subset \mathbb{P}^5$ such that $Z \cap P_i = \ell$, $i = 1, 2$, *i.e.*, there is a finite set $\gamma \subset F_0(Z)$ such that $e|_{F_0} \colon F_0(Z)\backslash\gamma \to F_1(Z)$ is a bijection unto its image.

(2) Let us give a description of $T_{F_0(Z),([\ell],[P])}$. We recall that the two projective bundle structures on $\mathrm{Fl}(2,3,H)$ given by $e\colon \mathrm{Fl}(2,3,H) \simeq \mathbb{P}(\mathcal{Q}_2) \to G(2,H)$ and $t\colon \mathrm{Fl}(2,3,H) \simeq \mathbb{P}(\wedge^2\mathcal{E}_3) \to G(3,H)$ yield the following descriptions of the tangent bundle:

$$T_{\mathrm{Fl}(2,3,H),([\ell],[P])} \simeq \mathrm{Hom}(\langle\ell\rangle, H/\langle\ell\rangle) \oplus \mathrm{Hom}(\langle P\rangle/\langle\ell\rangle, H/\langle P\rangle)$$

and

$$T_{\mathrm{Fl}(2,3,H),([\ell],[P])} \simeq \mathrm{Hom}(\langle P\rangle, H/\langle P\rangle) \oplus \mathrm{Hom}(\langle\ell\rangle, \langle P\rangle/\langle\ell\rangle).$$

The isomorphism between the two takes the form

$$\mathrm{Hom}(\langle\ell\rangle, H/\langle\ell\rangle) \oplus \mathrm{Hom}(\langle P\rangle/\langle\ell\rangle, H/\langle P\rangle) \longrightarrow \mathrm{Hom}(\langle P\rangle, H/\langle P\rangle) \oplus \mathrm{Hom}(\langle\ell\rangle, \langle P\rangle/\langle\ell\rangle),$$
$$(\varphi,\psi) \longmapsto \left(\varphi_\perp + \psi, \varphi_\|\right)$$

where $\varphi = (\varphi_\|, \varphi_\perp)$ is the decomposition corresponding to the choice of a decomposition $H/\langle\ell\rangle \simeq \langle P\rangle/\langle\ell\rangle \oplus H/\langle P\rangle$ coming from a decomposition $\langle P\rangle \simeq \langle\ell\rangle \oplus \langle P\rangle/\langle\ell\rangle$.

Around $([\ell],[P]) \in F_0(Z)$, the points of $\mathrm{Fl}(2,3,H)$ are of the form $([(\mathrm{id}_{\langle\ell\rangle} + \varphi)(\langle\ell\rangle)], [(\mathrm{id}_{\langle P\rangle} + \varphi_\perp + \psi)(\langle P\rangle)])$. Let us choose an equation $\lambda \in \langle P\rangle^*$ (a generator of $(\langle P\rangle/\langle\ell\rangle)^*$) of $\ell \subset P$ such that $\mathrm{eq}_Z(x,x,x) = \lambda(x)^3$ for any $x \in \langle P\rangle$.

The first-order deformation of this equation to an equation of $(\mathrm{id}_{\langle\ell\rangle} + \varphi)(\langle\ell\rangle) \subset (\mathrm{id}_{\langle P\rangle} + \varphi_\perp + \psi)(\langle P\rangle)$ is given by $\lambda - \varphi^*(\lambda)$, so that the point associated to (φ,ψ) belongs to $F_0(Z)$ if and only if

$$\mathrm{eq}_Z(x + \varphi_\perp(x) + \psi(x), x + \varphi_\perp(x) + \psi(x), x + \varphi_\perp(x) + \psi(x)) = (1 + c(\varphi,\psi))(\lambda(x) - \varphi^*(\lambda)(x))^3 \quad \forall x \in \langle P\rangle$$

for some term $c(\varphi,\psi) = O(\varphi,\psi)$ constant on $\langle P\rangle$. So at the first order, we get

$$\mathrm{eq}_Z(x, x, \varphi_\perp(x) + \psi(x)) = -\lambda(x)^2\varphi^*(\lambda)(x) + \frac{1}{3}c(\varphi,\psi)\lambda(x)^3 \quad \forall x \in \langle P\rangle. \tag{4.3}$$

The differential of the projection $e_{F_0(Z)}\colon F_0(Z) \to F_1(Z)$ is simply given by $(\varphi,\psi) \mapsto \varphi$.

Let us introduce

$$J := \{(([\ell],[P]),[Z]) \in \mathrm{Fl}(2,3,H) \times |\mathcal{O}_{\mathbb{P}^5}(3)|,\ \ell \subset Z,\ Z \cap P = \ell \text{ and } T_{([\ell],[P])}e|_{F_0} \text{is not injective}\}$$

and analyse the fibers of $p_J\colon J \to \mathrm{Fl}(2,3,H)$, which are isomorphic to each other by the homogeneity of $\mathrm{Fl}(2,3,H)$.

So we can assume $\ell = \{X_2 = \cdots = X_5 = 0\}$ and $P = \{X_3 = \cdots = X_5 = 0\}$, so that eq_Z is of the form (4.2) with $Q_i = a_iX_0^2 + b_iX_1^2 + c_iX_2^2 + d_iX_0X_1 + e_iX_0X_2 + f_iX_1X_2$, $i = 3,4,5$, for some $a_i, \dots, f_i$. We recall that for $\varphi = \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \\ u_5 & v_5 \end{pmatrix} \in \mathrm{Hom}(\langle\ell\rangle, H/\langle\ell\rangle)$ and $\psi = \begin{pmatrix} w_3 \\ w_4 \\ w_5 \end{pmatrix} \in \mathrm{Hom}(\langle P\rangle/\langle\ell\rangle, H/\langle P\rangle)$, the associated subspaces are

$$\ell_{(\varphi,\psi)} = [\lambda, \mu, \lambda u_2 + \mu v_2, \dots, \lambda u_5 + \mu v_5], \quad [\lambda,\mu] \in \mathbb{P}^1,$$

$$P_{(\varphi,\psi)} = [\lambda, \mu, \nu, \lambda u_3 + \mu v_3 + \nu w_3, \lambda u_4 + \mu v_4 + \nu w_4, \lambda u_5 + \mu v_5 + \nu w_5], \quad [\lambda,\mu,\nu] \in \mathbb{P}^2.$$

Now, if $(0,\psi) \in T_{F_0(Z),([\ell],[P])}$, we have at the first order

$$\begin{aligned}
\mathrm{eq}_Z|_{P_{(0,\psi)}} &= \alpha\nu^3 + \sum_{i=3}^{5} \nu w_i(a_i\lambda^2 + b_i\mu^2 + c_i\nu^2 + d_i\lambda\mu + e_i\lambda\nu + f_i\mu\nu) + O\left((\varphi,\psi)^2\right) \\
&= (\alpha + c_3w_3 + c_4w_4 + c_5w_5)\,\nu^3 + [(e_3w_3 + e_4w_4 + e_5w_5)\,\lambda + (f_3w_3 + f_4w_4 + f_5w_5)\,\mu]\,\nu^2 \\
&\quad + (a_3w_3 + a_4w_4 + a_5w_5)\,\lambda^2\nu + (b_3w_3 + b_4w_4 + b_5w_5)\,\mu^2 \\
&\quad + (d_3w_3 + d_4w_4 + d_5w_5)\,\lambda\mu\nu + O\left((\varphi,\psi)^2\right)
\end{aligned}$$

so that looking at (4.3), we see that $(0,\psi) \in T_{F_0(Z),([\ell],[P])}$ if and only if

$$\operatorname{rank}\begin{pmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ d_3 & d_4 & d_5 \\ e_3 & e_4 & e_5 \\ f_3 & f_4 & f_5 \end{pmatrix} \leq 2,$$

which defines a subset of codimension $(3-2)(5-2)=3$.

So $J \subset I$ has codimension 3. As $\dim(I) = \dim(|\mathcal{O}_{\mathbb{P}^5}(3)|)+2$, J does not dominate $|\mathcal{O}_{\mathbb{P}^5}(3)|$; *i.e.*, for the general Z, e_{F_0} is an immersion.

(3) Let us prove that $e_{F_0}(F_0(Z))$ is a Lagrangian surface of $F_1(Z)$. In [IM08], the following explicit description of the symplectic form $\mathbb{C}\cdot\Omega = H^{2,0}(F_1(Z))$ is given: let us introduce the following quadratic form on $\wedge^2 T_{F_1(Z),[\ell]}$ with values in $\mathrm{Hom}((\wedge^2\langle\ell\rangle)^{\otimes 2}, \wedge^4(H/\langle\ell\rangle)$:

$$\begin{aligned} K(u\wedge v, u'\wedge v') &= u(x)\wedge u'(y)\wedge v(x)\wedge v'(y) - u(y)\wedge u'(y)\wedge v(x)\wedge v'(x) \\ &\quad + u(y)\wedge u'(x)\wedge v(y)\wedge v'(x) - u(x)\wedge u'(x)\wedge v(y)\wedge v'(y), \end{aligned}$$

where (x,y) is a basis of $\langle\ell\rangle$. Let us also introduce the following skew-symmetric form:

$$\begin{aligned} \omega\colon\ \wedge^2 T_{F_1(Z),[\ell]} &\longrightarrow (\wedge^2\langle\ell\rangle)^{\otimes 3} \\ u\wedge v &\longmapsto \mathrm{eq}_Z(x,x,u(y))\,\mathrm{eq}_Z(y,y,v(x)) - \mathrm{eq}_Z(x,x,v(y))\,\mathrm{eq}_Z(y,y,u(x)) \\ &\qquad + 2\,\mathrm{eq}_Z(x,y,u(y))\,\mathrm{eq}_Z(x,x,v(y)) - 2\,\mathrm{eq}_Z(x,x,u(y))\,\mathrm{eq}_Z(x,y,v(y)) \\ &\qquad + 2\,\mathrm{eq}_Z(y,y,u(x))\,\mathrm{eq}_Z(x,y,v(x)) - 2\,\mathrm{eq}_Z(x,y,u(x))\,\mathrm{eq}_Z(y,y,v(x)). \end{aligned}$$

According to [IM08, Theorem 1], for $u,v \in T_{F_1(Z),[\ell]}$,

$$K(u\wedge v, u\wedge v) = w(u\wedge v)\Omega_{[\ell]}(u,v).$$

As for a general point $([\ell],[P]) \in F_0(Z)$, $\ell\subset Z$ is of the first type; *i.e.*, in reference to the above presentation (4.2) for $\ell = \{X_2=\cdots=X_5=0\}$, $P=\{X_3=X_4=X_5=0\}$, $\begin{vmatrix} a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \\ a_5 & b_5 & d_5 \end{vmatrix} \neq 0$, it is sufficient to prove the vanishing of $\Omega_{[\ell]}(\mathrm{Im}(T_{([\ell],[P])}e_{F_0}), \mathrm{Im}(T_{([\ell],[P])}e_{F_0}))$ for such a line. So we can assume $\alpha=1$ and

$$\begin{aligned} Q_3 &= X_0^2 + e_3X_0X_2 + f_3X_1X_2 + c_3X_2^2, \\ Q_4 &= X_0X_1 + e_4X_0X_2 + f_4X_1X_2 + c_4X_2^2, \\ Q_5 &= X_1^2 + e_5X_0X_2 + f_5X_1X_2 + c_5X_2^2. \end{aligned}$$

Then as above, for $\varphi = \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \\ u_5 & v_5 \end{pmatrix} \in \mathrm{Hom}(\langle\ell\rangle, H/\langle\ell\rangle)$ and $\psi = \begin{pmatrix} w_3 \\ w_4 \\ w_5 \end{pmatrix} \in \mathrm{Hom}(\langle P\rangle/\langle\ell\rangle, H/\langle P\rangle)$, we have

$$\begin{aligned} \mathrm{eq}_Z|_{P_{(\varphi,\psi)}} &= \nu^3 + \sum_{i=3}^{5}(\lambda u_i + \mu v_i + \nu w_3)Q_i + O((\varphi,\psi)^2) \\ &= (1+c_3w_3+c_4w_4+c_5w_5)\nu^3 + (c_3u_3+e_3w_3+c_4u_4+e_4w_4+c_5u_5+e_5w_5)\lambda\nu^2 \\ &\quad + (c_3v_3+f_3w_3+c_4v_4+f_4w_4+c_5v_5+b_5w_5)\mu\nu^2 \\ &\quad + (w_3+e_3u_3+e_4u_4+e_5u_5)\lambda^2\nu + (w_5+f_3v_3+f_4v_4+f_5v_5)\mu\nu^2 \\ &\quad + (w_4+f_3u_3+e_3v_3+f_4u_4+e_4v_4+f_5u_5+e_5v_5)\lambda\mu\nu \\ &\quad + u_3\lambda^3 + v_5\mu^2 + (v_4+u_5)\lambda\mu^2 + (v_3+u_4)\lambda^2\mu + O\Big((\varphi,\psi)^2\Big), \end{aligned}$$

so that the description (4.3) of $T_{F_0(Z),([\ell],[P])}$ yields

$$\begin{cases} c_3u_3+e_3w_3+c_4u_4+e_4w_4+c_5u_5+e_5w_5=-u_2\\ c_3v_3+f_3w_3+c_4v_4+f_4w_4+c_5v_5+b_5w_5=-v_2\\ w_3+e_3u_3+e_4u_4+e_5u_5=0\\ w_5+f_3v_3+f_4v_4+f_5v_5=0\\ w_4+f_3u_3+e_3v_3+f_4u_4+e_4v_4+f_5u_5+e_5v_5=0\\ v_4=-u_5;\ v_3=-u_4\ u_3=0\ v_5=0. \end{cases}$$

The seven last equations yield $w_3=-(e_4u_4+e_5u_5)$, $w_4=(e_3-f_4)u_4+(e_4-f_5)u_5$, $w_5=f_3u_4+f_4u_5$. Thus the first two give a system

$$\begin{cases} \alpha u_4+\beta u_5=-u_2,\\ -\delta u_4-\alpha u_5=-v_2, \end{cases}$$

where $\alpha=c_4-e_4f_4+e_5f_3$, $\beta=c_5-e_3e_5+e_4^2-e_4f_5+e_5f_4$ and $\delta=e_3f_4-f_4^2-e_4f_3+f_3f_5-c_3$. In particular, the determinant $\Delta=-\alpha^2-\beta\delta$ of the 2×2 system is non-zero for a general choice of the (e_i,f_i,c_i) and $u_4=\frac{1}{\Delta}(\alpha u_2+\beta v_2)$, $u_5=\frac{1}{\Delta}(\delta u_2-\alpha v_2)$. So a basis of $T_{F_1(Z),([\ell],[P])}$ is given by $((u_2=1,v_2=0)$ and $(u_2=0,v_2=1))$:

$$\begin{aligned} \varphi_{u_2}\colon \epsilon_0 &\longmapsto \epsilon_2+\frac{\alpha}{\Delta}\epsilon_4+\frac{\delta}{\Delta}\epsilon_5,\\ \epsilon_1 &\longmapsto -\frac{\alpha}{\Delta}\epsilon_3-\frac{\delta}{\Delta}\epsilon_4 \end{aligned}$$

and

$$\begin{aligned} \varphi_{v_2}\colon \epsilon_0 &\longmapsto \frac{\beta}{\Delta}\epsilon_4-\frac{\alpha}{\Delta}\epsilon_5,\\ \epsilon_1 &\longmapsto \epsilon_2-\frac{\beta}{\Delta}\epsilon_3+\frac{\alpha}{\Delta}\epsilon_4, \end{aligned}$$

where $(\epsilon_0,\dots,\epsilon_5)$ is the (dual) basis associated to the choice of the coordinates X_i. Then we readily compute

$$K(\varphi_{u_2}\wedge\varphi_{v_2})=\begin{vmatrix} 1 & 0 & 0 & 1\\ 0 & -\frac{\alpha}{\Delta} & 0 & -\frac{\beta}{\Delta}\\ \frac{\alpha}{\Delta} & -\frac{\delta}{\Delta} & \frac{\beta}{\Delta} & \frac{\alpha}{\Delta}\\ \frac{\delta}{\Delta} & 0 & -\frac{\alpha}{\Delta} & 0 \end{vmatrix}=0$$

and $\omega(\varphi_{u_2}\wedge\varphi_{v_2})=\frac{5}{\Delta}\neq 0$, hence $\Omega_{[\ell]}(\varphi_{u_2},\varphi_{v_2})=0$. □

Remark 4.4. In [GK21], it is also proven that $F_0(Z)\to e(F_0(Z))$ is the normalisation and that $e(F_0(Z))$ has 3780 non-normal isolated singularities.

As for Z general, e_{F_0} is an immersion, $N_{F_0(Z)/F_1(Z)}:=e_{F_0}^*T_{F_1(Z)}/T_{F_0(Z)}$ is locally free. Moreover, since e_{F_0} is, outside a codimension 2 subset of $F_0(Z)$, an isomorphism unto its image and that image is a Lagrangian subvariety of $F_1(Z)$, we get (outside a codimension 2 subset, thus globally) an isomorphism

$$\Omega_{F_0(Z)}\simeq N_{F_0(Z)/F_1(Z)}.$$

Notice that $F_0(Z)$ naturally lives in $\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})\subset \mathrm{Fl}(2,3,H)$. We have the following.

Lemma 4.5. *The following sequence is exact:*

$$0\longrightarrow e_{F_1}^*\mathcal{O}_{F_1(Z)}(-3)\otimes t_{F_1}^*\mathcal{O}_{G(3,H)}(3)\longrightarrow e_{F_1}^*\mathcal{O}_{F_1}(-1)\otimes t_{F_1}^*(\mathrm{Sym}^2\mathcal{E}_3\otimes\mathcal{O}_{G(3,H)}(1))|_{F_1(Z)}\longrightarrow N_{F_0(Z)/\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})}\longrightarrow 0,$$

where $e_{F_1}\colon \mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})\to F_1(Z)$ *and* $t_{F_1}\colon \mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})\to G(3,H)$.

Proof. We have seen that $F_0(Z) \subset \mathrm{Fl}(2,3,H)$ is the zero locus of a section of $\mathcal{F}$ appearing in the sequence (4.1). Taking the symmetric power of (2.2), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & e^*\mathcal{O}_{G(2,H)}(-3)\otimes t^*\mathcal{O}_{G(3,H)}(3) & \longrightarrow & e^*\mathcal{O}_{G(2,H)}(-3)\otimes t^*\mathcal{O}_{G(3,H)}(3) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & e^*\mathcal{O}_{G(2,H)}(-1)\otimes t^*(\mathrm{Sym}^2\mathcal{E}_3\otimes\mathcal{O}_{G(3,H)}(1)) & \longrightarrow & t^*\mathrm{Sym}^3\mathcal{E}_3 & \longrightarrow & e^*\mathrm{Sym}^2\mathcal{E}_2 & \rightarrow & 0. \end{array}$$

The projection to $e^*\mathrm{Sym}^2\mathcal{E}_2$ of the section $\sigma_{\mathrm{eq}_Z} \in H^0(t^*\mathrm{Sym}^3 E_3)$ induced by eq_Z vanishes on $F_1(Z)$ by the definition of $F_1(Z)$. So it induces a section of

$$e_{F_1}^*\mathcal{O}_{F_1(Z)}(-1)\otimes t_{F_1}^*(\mathrm{Sym}^2\mathcal{E}_3\otimes\mathcal{O}_{G(3,H)}(1)) \simeq (e^*\mathcal{O}_{G(2,H)}(-1)\otimes t^*(\mathrm{Sym}^2\mathcal{E}_3\otimes\mathcal{O}_{G(3,H)}(1)))|_{\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})}.$$

Now the snake lemma in the above diagram gives the result. □

The snake lemma in the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \rightarrow & T_{F_0(Z)} & \rightarrow & T_{\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})}|_{F_0(Z)} & \rightarrow & N_{F_0(Z)/\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})} & \rightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & T_{F_0(Z)} & \longrightarrow & e_{F_0}^* T_{F_1(Z)} & \longrightarrow & N_{F_0(Z)/F_1(Z)} & \longrightarrow & 0 \end{array}$$

and the description of the relative tangent bundle of e_{F_1} give the following.

Proposition 4.6. *The following sequence is exact:*

$$0 \longrightarrow \mathcal{O}_{F_0} \longrightarrow e_{F_0}^*(\mathcal{Q}_2|_{F_1(Z)}\otimes\mathcal{O}_{F_1(Z)}(-1))\otimes t_{F_0}^*(\mathcal{O}_{G(3,H)}(1))|_{F_0} \longrightarrow N_{F_0(Z)/\mathbb{P}(\mathcal{Q}_2|_{F_1(Z)})} \longrightarrow \Omega_{F_0(Z)} \longrightarrow 0.$$

We finish this section by computing the Hodge numbers of $F_0(Z)$.

Proposition 4.7. *We have $H^1(F_0(Z),\mathbb{Z}) = 0$ for any Z for which $F_0(Z)$ is smooth.*

Proof. For the universal variety of planes $r_{\mathrm{univ}}\colon \mathcal{F}_2(\mathcal{X}) \to |\mathcal{O}_{\mathbb{P}^6}(3)|$, $R^3 r_{\mathrm{univ},*}\mathbb{Q}$ is a local system over the open subset $\{[X] \in |\mathcal{O}_{\mathbb{P}^6}(3)|,\ F_2(X) \text{ is smooth}\}$ which, by Proposition 4.1, contains an open subset of the locus of cyclic cubic 5-folds.

As a consequence, the Abel–Jacobi isomorphism $q_*p^*\colon H^3(F_2(X),\mathbb{Q}) \xrightarrow{\sim} H^5(X,\mathbb{Q})$ given by the result of Collino (Theorem 1.1) for general X extends to the case of the general cyclic cubic 5-fold.

But, as noticed in the proof of Proposition 4.1, for any $[P] \in F_0(Z)$, the associated cycle $q(p^{-1}(\pi^{-1}([P])))$ on X_Z is the complete intersection cycle $\mathrm{Span}(P,p_0)\cap X_Z$, which belongs to a family of cycles parametrised by a rational variety, namely $\{[\Pi] \in G(4,V),\ p_0 \in \Pi\} \simeq G(3,H)$. Now, as an abelian variety contains no rational curve, the Abel–Jacobi map $\Phi\colon G(3,H) \to J^5(X_Z)$, $[P] \mapsto [\mathrm{Span}(P,p_0)\cap X_Z] - [\mathrm{Span}(P_0,p_0)\cap X_Z]$ ($[P_0]$ being a reference point) is constant. Hence the restriction $\Phi_{(\pi_*,\mathrm{id}_{X_Z})\mathbb{P}(\mathcal{E}_3)}\colon F_0(Z) \to J^5(X_Z)$ of Φ to the sub-family $(\pi_*,\mathrm{id}_{X_Z})\mathbb{P}(\mathcal{E}_3) \subset F_0(Z)\times X_Z$ (of planes P such that $\mathrm{Span}(P,p_0)\cap X_Z$ consists of three planes) is constant; *i.e.*, $q_*p^*\pi^*\colon H^3(F_0(Z),\mathbb{Z}) \to H^5(X_Z,\mathbb{Z})$ is trivial.

As π is étale, $\pi^*\colon H^3(F_0(Z),\mathbb{Q}) \to H^3(F_2(X_Z),\mathbb{Q})$ is injective, so that the trivial map $q_*p^*\pi^*$ is the composition of a injective map followed by an isomorphism. □

We can then compute the rest of the Hodge numbers:

(1) Again using the package Schubert2 of Macaulay2, we can use the Koszul resolution of $\mathcal{O}_{F_0(Z)}$ by $\wedge^i\mathcal{F}^*$ (where $\mathcal{F}$ is defined by (4.1)) to compute $\chi(\mathcal{O}_{F_0(Z)}) = 1071$ with the following code:

```
loadPackage "Schubert2"
G=flagBundle{3,3}
(Q,E)=bundles G
wE=exteriorPower(2,E)
P=projectiveBundle' wE
p=P.StructureMap
```

```
pl=exteriorPower(3,E)
pol=p^*pl**dual(OO_P(1))
F=p^*symmetricPower(3,E)-symmetricPower(3,pol)
chi(exteriorPower(0,dual(F)))-chi(exteriorPower(1,dual(F)))
+chi(exteriorPower(2,dual(F)))-chi(exteriorPower(3,dual(F)))
+chi(exteriorPower(4,dual(F)))-chi(exteriorPower(5,dual(F)))
+chi(exteriorPower(6,dual(F)))-chi(exteriorPower(7,dual(F)))
+chi(exteriorPower(8,dual(F)))-chi(exteriorPower(9,dual(F)))
```

so we get $h^2(\mathcal{O}_{F_0(Z)}) = 1070$.

(2) Then as π is étale of degree 3, we get $\chi_{\mathrm{top}}(F_0(Z)) = \frac{1}{3}\chi_{\mathrm{top}}(F_2(X_Z)) = 4347$. So $h^{1,1}(F_0(Z)) = 2207$.

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