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Coholomogy class of complex approximable algebras

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Abstract. In his work on arithmetic Fujita approximation, Huayi Chen introduces the notion of an approximable graded algebra, which he uses to prove a Fujita-type theorem in the arithmetic setting, and asks if any such algebra is the graded ring of a big line bundle on a projective variety. This was proved to be false by the author in a previous work. A subsequent paper showed that, whilst the approximable algebra is not necessarily a subalgebra of the algebra of graded sections of a big line bundle, it is a full-dimensional subalgebra of the algebra of sections of an infinite Weil divisor. This paper also proved that over $\mathbb{C}$, if an infinite Weil divisor has a finite numerical class, then its section ring is approximable, and the same is true for any full-dimensional subalgebra.

This note proves that the converse is true: any approximable algebra over $\mathbb{C}$ is associated to an infinite Weil divisor whose numerical class converges.

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Contents

1. Introduction

The Fujita approximation theorem, [Fuj94], is an important result in algebraic geometry. It states that whilst the section ring associated to a big line bundle L on an algebraic variety X

$$R(L) \stackrel{\text{def}}{=} \oplus_m H^0(mL, X)$$

is typically not a finitely generated algebra, it can be approximated arbitrarily well by finitely generated algebras. More precisely, we have the following.

Theorem 1.1 (Fujita). *Let X be an algebraic variety, and let L be a big line bundle on X. For any $\epsilon > 0$ there exist a birational modification*

$$\pi : \hat{X} \longrightarrow X$$

and a decomposition of $\mathbb{Q}$-divisors $\pi^*(L) = A + E$ such that

- *A is ample and E is effective,*
- $\mathrm{vol}(A) \geq (1 - \epsilon)\mathrm{vol}(L)$.

In [LM09], Lazarsfeld and Mustaţă use the Newton–Okounkov body associated to A to give a simple proof of Fujita approximation.

In [Che10], Chen uses Lazarsfeld and Mustaţă's work on Fujita approximation to prove a Fujita-type approximation theorem in the arithmetic setting. In the course of this work, he defines the notion of approximable graded algebras, which are exactly those algebras for which a Fujita-type approximation theorem holds.

Definition 1.2. An integral graded algebra $\mathbf{B} = \oplus_m B_m$ with $B_0 = k$ a field is approximable if and only if the following conditions are satisfied:

(1) All the graded pieces B_m are finite-dimensional over k.
(2) For all sufficiently large m the space B_m is non-empty.
(3) For any ϵ there exists a p_0 such that for all $p \geq p_0$ we have that

$$\liminf_{n \to \infty} \frac{\dim(\mathrm{Im}(S^n B_p \to B_{np}))}{\dim(B_{np})} > (1 - \epsilon).$$

In his paper [Che10], Chen asks whether any graded approximable algebra is in fact a subalgebra of the algebra of sections of a big line bundle. A counter-example to this is given in [Mac17a], where the graded approximable algebra is equal to the section ring of an *infinite* divisor.[1] The subsequent paper [Mac17b] shows that any approximable algebra is indeed a subalgebra of the section ring of such a divisor. Moreover, it is established that if X is a smooth complex algebraic variety of dimension d and $D = \sum_{i=1}^{\infty} a_i D_i$ is an

[1] Infinite in this context means an infinite formal sum of Weil divisors with real coefficients $\sum_i a_i D_i$.

infinite Weil divisor on X such that the sum of divisor classes $\sum_{i=1}^{\infty} a_i[D_i]$ converges in the Néron–Severi space $\mathrm{NS}(X)$ to a finite real big cohomology class $[D]$, then the algebra $\oplus_m H^0(mD)$ is approximable.

The main result of this note proves that the converse is true.

Theorem 1.3. *Let* $\mathbf{B}$ *be an approximable algebra over* $\mathbb{C}$*, and let* $X(\mathbf{B})$ *and* $D(\mathbf{B}) = \sum_{i=1}^{\infty} a_i D_i$ *be the smooth complex variety and infinite Weil divisor constructed in* [Mac17b] *such that* $\mathbf{B}$ *is a full-dimensional subalgebra of the section ring of* $D(\mathbf{B})$*. The sum of cohomology classes* $\sum_{i=1}^{\infty} a_i[D_i]$ *in* $\mathrm{NS}(X)$*, the Néron–Severi space of* X*, is then a convergent series.*

In Section 2, we set notation and recall such results from [Mac17b] as will be necessary. Section 3 contains the proof of Theorem 1.3.

2. Notation and two preliminary results

In this section, we fix some notation and recall some essential preliminary results.

In this paper, we consider a graded approximable algebra $\mathbf{B} = \oplus_m B_m$ such that $B_0 = \mathbb{C}$. For any natural numbers k and n we denote by $\mathrm{Sym}^n(B_k)$ the n^{th} symmetric power of the vector space B_k and by $S^n(B_k)$ the image of $\mathrm{Sym}^n(B_k)$ in B_{nk}.

We say that a graded, not *a priori* approximable, algebra $\oplus_m B_m$ *is of dimension* d *and has volume* v if the sequence

$$\lim_{n\to\infty}\left(\frac{\mathrm{rk}(B_n)}{n^d/d!}\right)$$

converges to the strictly positive real number v. It is proved by Chen in [Che10] that such numbers exist for any approximable algebra $\mathbf{B}$.

We now recall the construction from [Mac17b] of the smooth variety $X(\mathbf{B})$ and the infinite Weil divisor $D(\mathbf{B})$ such that $\mathbf{B}$ is a subalgebra of the section ring of $D(\mathbf{B})$.

2.1. Construction of $X(\mathbf{B})$ and $D(\mathbf{B})$

The variety $X(\mathbf{B})$ is defined up to birational equivalence using the homogeneous field of fractions of $\mathbf{B} = \oplus_m B_m$, which we now define.

Definition 2.1. Let $\mathbf{B} = \oplus_m B_m$ be a graded algebra over $\mathbb{C}$. Then we define its homogeneous fraction field by

$$K^{\mathrm{hom}}(\mathbf{B}) = \left\{\frac{b_1}{b_2} \mid \exists m \text{ such that } b_1, b_2 \in B_m, b_2 \neq 0\right\}/\sim$$

where $\sim$ is the equivalence relation

$$\frac{b_1}{b_2} \sim \frac{c_1}{c_2} \iff b_1 c_2 = c_1 b_2.$$

Note that $K^{\mathrm{hom}}(\mathbf{B})$ is a field extension of $\mathbb{C}$.

Choose n large enough that B_n and B_{n+1} are both non-trivial. Choose $f_1 \in B_n$ and $f_2 \in B_{n+1}$. For any m we can then identify B_m with a subspace of $K^{\mathrm{hom}}(\mathbf{B})$ via the inclusion

$$i_m \colon B_m \hookrightarrow K^{\mathrm{hom}}(\mathbf{B}), \quad b_m \longmapsto \frac{b_m f_1^m}{f_2^m}.$$

(Note that this inclusion depends on the choice of f_1 and f_2). Throughout what follows, we assume that we have fixed $f_1 \in B_n$ and $f_2 \in B_{n+1}$ for some choice of n and that there is therefore for every m a fixed inclusion $i_m \colon B_m \hookrightarrow K^{\mathrm{hom}}(\mathbf{B})$. These inclusions satisfy $i_m(b_m) i_r(b_r) = i_{m+r}(b_m b_r)$ for any $b_m \in B_m$ and $b_r \in B_r$.

This field is proved in [Mac17b] to be finitely generated, enabling the following definition of $X(\mathbf{B})$.

Definition 2.2. The variety $X(\mathbf{B})$ is a smooth projective complex variety such that $K(X(\mathbf{B})) = K^{\mathrm{hom}}(\mathbf{B})$.

Remark 2.3. The variety $X(\mathbf{B})$ is defined only up to birational equivalence. It can be chosen smooth by Hironaka resolution.

We now recall the definition of the infinite Weil divisor $D(\mathbf{B})$, which is constructed as the limit of the sequence of divisors D_m/m, where the divisors D_m are poles of the rational functions $b_m \in B_m$. More precisely, for any $b_m \in B_m$ let $(b_m)_X = \sum_{i=1}^{\infty} a_i D_i$ be the principal divisor(2) on $X(\mathbf{B})$ cut out by the rational function b_m. We let $(b_m)_X^- = \sum_{i|c_i<0} -c_i D_i$ be the poles divisor of b_m and let $(b_m)_X^+ = \sum_{i|c_i>0} c_i D_i$ be the zeros divisor of b_m, so that

$$(b_m)_X = (b_m)_X^+ - (b_m)_X^-.$$

We can now define D_m.

Definition 2.4. For any m such that B_m is non-empty, we define the effective divisor D_m on $X(\mathbf{B})$ by

$$D_m = \sup_{b_m \in B_m} ((b_m)_X^-),$$

where the supremum is taken with respect to the natural partial order on $\mathrm{Weil}(X(\mathbf{B}))$.

It is proved in [Mac17b] that this supremum is a maximum and moreover that for every m such that B_m is non-empty, there exists a b_m such that D_m is the poles divisor of $i_m(b_m)$. We can now define $D(\mathbf{B})$.

Definition 2.5. We set $D(\mathbf{B}) = \lim_{m\to\infty} \left(\frac{D_m}{m}\right)$.(3)

In the next section, we prove that the infinite divisor $D(\mathbf{B})$ has a finite cohomology class.

3. Finiteness of the cohomology class $[D(\mathbf{B})]$

We now give a proof of Theorem 1.3. Note that if $\mathbf{B}_q = \oplus_n B_{nq}$, then $D(\mathbf{B}_q) = qD(\mathbf{B})$, so without loss of generality, we may assume that B_1 is non-zero. Our proof depends on the following observation.

Lemma 3.1. *Let X be a smooth complex variety, and let H be an ample divisor on X. Equip $\mathrm{NS}(X)$, the Néron–Severi group of X, with a norm denoted by $|\cdot|$. Then there is a constant $C > 0$ such that for any pseudo-effective divisor E on X, we have that*

$$E \cdot H^{d-1} > C|[E]|.$$

Proof. Suppose not. Then there exists a sequence of pseudo-effective divisor classes $[E_n]$ such that $|[E_n]| = 1$ for all n and $E_n \cdot H^{d-1} \to 0$. Passing to a convergent subsequence, we may assume that in the Néron–Severi space, $[E_n]$ converges to a non-zero pseudo-effective cohomology class $[E]$ such that $H^{d-1} \cdot E = 0$, but this is impossible. This completes the proof of Lemma 3.1. □

This lemma enables us to give a numerical criterion for convergence of the sequence $[D_m/m]$.

Lemma 3.2. *Let $\mathbf{B}$ be an approximable algebra over $\mathbb{C}$, let $X(\mathbf{B})$ be the associated variety, and consider the divisors D_m/m defined above, converging to an infinite Weil divisor $D(\mathbf{B})$. Let $\mathrm{NS}(X)$ be the Néron–Severi space of X. Let H be an ample divisor on X.*

The sequence $[D_m/m]$ converges in $\mathrm{NS}(X)$ if and only if the numerical sequence $(D_m \cdot H^{d-1})/m$ converges.

Proof. The fact that convergence of $[D_m/m]$ implies convergence of $(D_m \cdot H^{d-1})/m$ is immediate. We assume now that the sequence $(D_m \cdot H^{d-1})/m$ converges, and we will show that the sequence of cohomology classes $[D_m/m]$ also converges. Throughout what follows, we write $[D_1] \geq [D_2]$ if and only if $[D_1 - D_2]$ is the cohomology class of a *pseudo-effective* divisor.

(2)Here, the D_i are prime divisors, and the sum is finite.

(3)It is proved in [Mac17b] that this limit exists and that for any k the divisor $\lfloor kD \rfloor$ is finite.

We first show that if the sequence $[D_{m!}/(m!)]$ converges to a limit $[D]$, then so does the sequence $[D_m/m]$. For any integer m_1 and any $\epsilon > 0$, there exists an integer M_1 such that if $m > M_1$, then

$$[D_m/m] \geq (1-\epsilon)[D_{m_1!}/(m_1!)]. \tag{3.1}$$

We also have that

$$[D_m/m] \leq [D_{m!}/(m!)] \leq [D] \tag{3.2}$$

since the sequence $[D_{m!}/(m!)]$ is increasing in m. (The last equality is immediate because we have defined the partial order $\leq$ in terms of pseudo-effective divisors rather than effective divisors).

It follows from (3.1) that for all $m > M_1$ we have that

$$[D - D_m/m] \leq [(D-(1-\epsilon)D_{m_1!}/(m_1!)]$$

and hence

$$H^{d-1} \cdot (D - D_m/m) \leq [H]^{d-1} \cdot [D-(1-\epsilon)D_{m_1!}/(m_1!)]).$$

Since the right-hand side can be made arbitrarily small by an appropriate choice of ϵ and m_1, we deduce that

$$[H]^{d-1} \cdot ([D - D_m/m]) \longrightarrow_{m\to 0} 0.$$

Since $[D - D_m/m]$ is pseudo-effective by Equation (3.2), it follows from Lemma 3.1 that

$$[D_m/m] \longrightarrow_{m\to\infty} [D].$$

It remains only to show that the sequence $D_{m!}/m!$ is convergent. We note that this sequence of divisors is monotone increasing. In particular, if we set

$$R_m = D_{m!}/(m!) - D_{(m-1)!}/(m-1)!,$$

then R_m is effective and hence pseudo-effective for all m. We have assumed that the sequence $\sum_{m=1}^{\infty} H^{d-1} \cdot R_m$ is convergent. By Lemma 3.1, the series $\sum_{m=1}^{\infty}[R_m]$ is also convergent. This completes the proof of Lemma 3.2. □

A final lemma will be necessary before completing the proof of Theorem 1.3.

Lemma 3.3. *There are constants p, k and N such that for any $m > N$ divisible by p and any n such that $n \geq km$, there are polynomials T_1 and T_2 in $S^n(B_p)$ such that $i_m(b_m) = T_1/T_2$. In particular, such polynomials T_1 and T_2 exist in $S^{km}(B_p)$.*

Proof. The proof of Lemma 3.3 is similar to that of [Mac17b, Proposition 1]. In [Che10], Chen shows that if B is approximable, then there exist a constant d and another constant M such that

$$\dim(B_n) \sim Mn^d.$$

In particular, there exist a constant N and another constant k such that for any $m_1, m_2 > N'$ such that $m_2 > km_1$, we have that

$$\dim\left(B_{m_1+m_2}\right) \leq \tfrac{4}{3}\left(\dim\left(B_{m_2}\right)\right).$$

Pick now a p and n_0 such that we have both of the following:

(1) $p > N$,
(2) $\dim(S^n(B_p)) \geq \frac{2}{3}\dim(B_{np})$ for all $n > n_0$.

Now consider an element b_m in B_m for some $m > \max\{N, n_0 p\}$. We assume furthermore that m is divisible by p; *i.e.* $m = k'p$. Our aim is to give a bound on the poles of $i_m(b_m)$ which depends linearly on m.

Choose n such that $np \geq km$. We note that in particular $np \geq N$. We may assume that $i_m(b_m) = \frac{b_m}{b'_m}$ for some $b_m, b'_m \in B_m$ since we have assumed that $B_1 \neq \{0\}$. Note that

$$\dim(b_m \cdot S^n(B_p)) = \dim(S^n(B_p)) > \frac{2(\dim(B_{np}))}{3} > \frac{\dim(B_{np+m})}{2}.$$

Similarly,

$$\dim(b'_m \cdot S^n(B_p)) = \dim(S^n(B_p)) > \frac{2(\dim(B_{np}))}{3} > \frac{\dim(B_{np+m})}{2},$$

from which it follows that

$$b_m \cdot S^n(B_p)) \cap b'_m \cdot S^n(B_p)) \neq \{0\}$$

and hence

$$\frac{b_m}{b'_m} = \frac{T_1}{T_2}$$

for some $T_1, T_2 \in S^n(B_p)$. This completes the proof of Lemma 3.3. □

We can now complete the proof of Theorem 1.3. Fix n such that $np = km$. For $j = 1, 2$ we set

$$(D(i_{np}(T_j))) = Z_j - P_j,$$

where Z_j and P_j are effective divisors that do not have any common component; since $i_{np}(T_j)$ is a rational function, we have in particular that Z_j and P_j are numerically equivalent. Note that by the definition of D_p, we have that

$$P_j \leq (km)D_p,$$

and it follows that, numerically,

$$Z_j \cdot H^{d-1} \leq (km)D_p \cdot H^{d-1}.$$

But now if we consider the poles divisor of $i_m(b_p)$, we have that

$$P(i_m(b_m)) \leq P_1 + Z_2$$

and hence

$$P(i_m(b_m)) \cdot H^{d-1} < 2(km)D_p \cdot H^{d-1}.$$

We know that there exists a b_m such that $P(i_m(b_m)) = D_m$ and hence

$$\frac{D_m}{m} \cdot H^{d-1} \leq \frac{2km}{m} D_p \cdot H^{d-1} \leq 2kD_p \cdot H^{d-1},$$

and hence this sequence is bounded since k and p were fixed.

It follows that the sequence $\frac{D_m}{m} \cdot H^{d-1}$ is bounded and hence convergent, so that by Lemma 3.2, the sequence $[D_m/m]$ is also convergent in $\mathrm{NS}(X)$. This completes the proof of Theorem 1.3. □

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