Épijournal de Géométrie Algébrique
epiga.episciences.org
Volume 8 (2024), Article No. 4

The universal vector extension of an abeloid variety

Marco Maculan

Abstract. Let A be an abelian variety over a complete non-Archimedean field K. The universal cover of the Berkovich space attached to A reflects the reduction behaviour of A. In this paper the universal cover of the universal vector extension $E(A)$ of A is described. In a forthcoming paper this will be one of the crucial tools to show that rigid analytic functions on $E(A)$ are all constant.

Keywords. Rigid analytic uniformization, universal vector extension, abeloid variety

2020 Mathematics Subject Classification. 14G22, 14K05

Received by the Editors on December 13, 2022, and in final form on October 20, 2023.
Accepted on November 21, 2023.

Marco Maculan
Institut de Mathématiques de Jussieu, Sorbonne Université, 4 place Jussieu, 75252 Paris, France
e-mail: marco.maculan@imj-prg.fr

This research was supported by ANR-18-CE40-0017.

Contents

1. Introduction

1.1. Background

Let K be an algebraically closed non-trivially valued complete non-Archimedean field. The driving force behind Tate's foundation [Tat71] of rigid analysis was the uniformization of elliptic curves over a p-adic field: given an elliptic curve E over K, there is an isomorphism $E(K) \cong K^\times/q^{\mathbb{Z}}$ of rigid analytic spaces for some $|q| < 1$ if and only if the j-invariant of E is not integral. Within the framework of Berkovich spaces, analytic spaces are locally path connected and locally contractible (in contrast with the total disconnectedness of K), so that the usual theory of universal covers and fundamental groups can be applied. Tate's result then can be restated as follows. The topological space underlying the Berkovich space E^{an} attached to E is contractible if and only if E has good reduction; if this is not the case, the universal cover of E^{an} is $\mathbb{G}_m^{\mathrm{an}}$, its fundamental group is identified with $q^{\mathbb{Z}} \subset K^\times$ for some $|q| < 1$, and $E^{\mathrm{an}} = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$. Mumford generalized Tate's theorem both for higher-genus curves, see [Mum72b], and higher-dimensional abelian varieties, see [Mum72a]. Given an abelian variety A over K, the universal cover of A^{an} is of the form E^{an} for a semi-abelian variety

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0, \tag{1.1}$$

where T is a K-torus and B an abelian variety with good reduction. The topological space E^{an} is again seen to be contractible, and the fundamental group of A^{an} is seen to be a free abelian group $\Lambda \subset E(K)$ of rank $\dim T$. Later on Lütkebohmert [Lüt16, Corollary 7.6.5] obtained such a uniformization result for all *abeloid varieties*, that is, proper smooth connected analytic groups over K – the rigid analytic analogue of a complex torus.

1.2. Motivation

In this paper the universal cover of the universal vector extension of an abelian variety is made explicit. Recall that a vector extension of an abelian variety A over a field k is a short exact sequence of algebraic groups

$$0 \longrightarrow \mathbb{V}(F) \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where F is a finite-dimensional k-vector space and $\mathbb{V}(F)$ the vector group attached to it. Such an extension corresponds (up to isomorphism) to elements of the cohomology group $\mathrm{H}^1(A, \mathcal{O}_A) \otimes_K F$. Taking $F = \mathrm{H}^1(A, \mathcal{O}_A)^\vee$, the vector extension with isomorphism class $\mathrm{id} \in \mathrm{End}\,\mathrm{H}^1(A, \mathcal{O}_A)$,

$$0 \longrightarrow \mathbb{V}\left(\mathrm{H}^1(A, \mathcal{O}_A)^\vee\right) \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

is said to be the *universal* one; it has the property that any other vector extension G is obtained as the push-out of $E(A)$ along the linear map $\mathrm{H}^1(A, \mathcal{O}_A)^\vee \to F$ given by the isomorphism class of G.

When $k = \mathbb{C}$, the uniformization of $E(A)$ admits a quite explicit description, which should hopefully clarify the statements in the rigid context. The exponential map $V := \operatorname{Lie} A \to A(\mathbb{C})$ is a universal cover of $A(\mathbb{C})$. We identify the fundamental group $\pi_1(A(\mathbb{C}), 0)$ with the kernel of the exponential map $\Lambda \subseteq \operatorname{Lie} A$. Hodge theory permits one to see $\mathrm{H}^1(A, \mathcal{O}_A)^\vee$ as the complex vector space $\bar{V}$ conjugate to V; consequently, we have an inclusion $\theta_\Lambda \colon \Lambda \to \bar{V}$. Then,

$$E(A)(\mathbb{C}) = (V \times \bar{V})/\Lambda$$

with Λ embedded diagonally. The choice of a basis of Λ induces a biholomorphism $E(A)(\mathbb{C}) \cong (\mathbb{C}^\times)^{2g}$. In particular, the complex manifold $E(A)(\mathbb{C})$ is Stein, meaning that it can be holomorphically embedded in $\mathbb{C}^n$ as a closed subspace for some $n \geqslant 0$. Such an embedding cannot be algebraic, as all algebraic functions on $E(A)$ are constant. In other words, $E(A)$ is an example of a non-affine algebraic variety whose associated complex space is Stein.

The motivation of the present paper comes from the study of the analogous question over K; see [Mac22].

1.3. Results

From now on let us work over K. To ease notation, the superscript ‘an’ is dropped, and all algebraic varieties are treated as analytic spaces. Let

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0$$

be the semi-abelian variety which is the universal cover of A and $\Lambda \subseteq E(K)$ the fundamental group. In order to state the main result, it is necessary to also have at hand the uniformization of the dual abelian variety $\check{A}$. See B as the dual of its dual $\check{B}$; then the group morphism $\Lambda \to B(K)$ defines a semi-abelian variety

$$0 \longrightarrow \check{T} \longrightarrow \check{E} \xrightarrow{\check{p}} \check{B} \longrightarrow 0,$$

where $\check{T}$ is the K-torus with group of characters Λ. Let $\check{\Lambda}$ be the group of characters of the K-torus T. The datum of $\Lambda \subseteq E(K)$ induces an embedding of $\check{\Lambda} \subseteq \check{E}(K)$, and

$$\check{A}^{\mathrm{an}} = \check{E}^{\mathrm{an}}/\check{\Lambda}.$$

For an algebraic group G, let $\omega_G = (\operatorname{Lie} G)^\vee$ denote the dual of its Lie algebra. Since the quotient map $\check{E}^{\mathrm{an}} \to \check{A}^{\mathrm{an}}$ is étale, the spaces of invariant 1-forms $\omega_{\check{E}}$ and $\omega_{\check{A}}$ are identified. The above short exact sequence of algebraic groups gives the following one:

$$0 \longrightarrow \omega_{\check{B}} \xrightarrow{\mathrm{d}\check{p}} \omega_{\check{A}} \longrightarrow \omega_{\check{T}} \longrightarrow 0,$$

where $\mathrm{d}\check{p}$ is the pull-back of 1-forms along $\check{p}$. The first result concerns structure.

Theorem A. *The universal cover $\tilde{E}(A)$ of $E(A)$ is contractible, is the pull-back to $\tilde{A}$ of $E(A)$ and is the push-out of $E(B) \times_B E$ along $\mathrm{d}\check{p}$; i.e. there is a commutative and exact diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{V}(\omega_{\check{B}}) \arrow[r] \arrow[d, "\mathrm{d}\check{p}"] & E(B) \times_B E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{V}(\omega_{\check{A}}) \arrow[r] & \tilde{E}(A) \arrow[r] & E \arrow[r] & 0.
\end{tikzcd}$$

In the statement the isomorphisms $\omega_{\check{A}} \cong \mathrm{H}^1(A, \mathcal{O}_A)^\vee$ and $\omega_{\check{B}} \cong \mathrm{H}^1(B, \mathcal{O}_B)^\vee$ are understood. The fundamental group $\pi_1(E(A), 0)$ of $E(A)$ with base-point 0 can be identified with a subgroup of (the K-rational points of) its universal cover $\tilde{E}(A)$. The projection $\tilde{E}(A) \to E$ given by Theorem A then induces a map

$$\varphi \colon \pi_1(E(A), 0) \longrightarrow \Lambda.$$

In order to understand how $\pi_1(E(A),0)$ sits inside $\tilde{E}(A)$, note that Theorem A gives an isomorphism $\operatorname{Coker}(E(B)\times_B E \to \tilde{E}(A)) \cong \mathbb{V}(\omega_{\check{T}})$. Let

$$\mathrm{pr}_u \colon \tilde{E}(A) \longrightarrow \mathbb{V}(\omega_{\check{T}})$$

be the induced projection. The image of $\pi_1(E(A),0)$ is described by means of the *universal vector hull* of the group Λ, that is, the K-linear map $\theta_\Lambda \colon \Lambda \to \omega_{\check{T}}$ defined as follows. By definition, $\check{T}$ is the K-torus with group of characters Λ. Seeing $\chi \in \Lambda$ as a character $\chi \colon \check{T} \to \mathbb{G}_m$, set

$$\theta_\Lambda(\chi) := \chi^* \tfrac{\mathrm{d}z}{z} \in \omega_{\check{T}},$$

where z is the coordinate function on $\mathbb{G}_m$.

Theorem B. *The map $\varphi \colon \pi_1(E(A),0) \to \Lambda$ is an isomorphism, and*

$$\mathrm{pr}_u \circ \varphi^{-1} = \theta_\Lambda.$$

Concretely, when the abelian variety A has good reduction, the above results simply say that $E(A)$ is contractible, so that it coincides with its universal cover. In the extremely opposite situation, when $A = T/\Lambda$ has totally degenerate reduction, the result is more interesting and gives the following description of $E(A)$:

$$E(A) = (T \times \mathbb{V}(\omega_{\check{T}}))/\{(\chi, \theta_\Lambda(\chi)) : \chi \in \Lambda\}.$$

In the framework of 1-motives, Theorems A and B say that $\tilde{E}(A)$ is the universal vector extension of the 1-motive $M = [\Lambda \to E]$; see [BVS01, Section 1.4] and [Ber09, Section 2.2].

1.4. Content of the paper

In order to show Theorems A and B, the natural approach would be to compare $\tilde{E}(A)$ with the universal vector extension $M^\natural$ of the 1-motive $M = [\Lambda \to E]$. However, $\tilde{E}(A)$ is an *analytic* vector extension of E, and since E is not proper, it is not clear *a priori* why it should be an *algebraic* one. Therefore, we cannot apply the universal property of $M^\natural$ directly. Instead, we will proceed by making an explicit construction of $E(A)$ giving the main results above as a byproduct. Unfortunately, the universal property of the universal vector extension does not say much about how $E(A)$ is constructed. It is instead more insightful to look at the moduli space $A^\natural$ of translation-invariant line bundles on A endowed with a (necessarily integrable) connection – such a moduli space is canonically isomorphic to the universal vector extension. The construction of $A^\natural$ presented here (see Section 2), although quite natural, seemingly does not appear in the literature; it is inspired from [BHR11], even though the Hodge-theoretical reasoning therein had to be circumvented. Such a definition for $A^\natural$ has several advantages. First, it translates right away to the rigid analytic framework for abeloid varieties. Second, its explicit nature permits one to perform the necessary computations (see notably Proposition 4.2, Theorem 4.6 and Theorem 4.11). Third, it allows one to determine the canonical linear isomorphism through which $A^\natural$ is obtained by push-out from $E(A)$ (see Theorem 2.20).(1) The 'universal cover' of $A^\natural$ is then defined by hand (see Section 4.3) and only ultimately shown to be contractible (see Proposition 4.15), so that it is literally the universal cover of $A^\natural$.

1.5. Conventions

Let X be a locally ringed space and

$$(F) \qquad \cdots \longrightarrow F_{i-1} \longrightarrow F_i \longrightarrow F_{i+1} \longrightarrow \cdots$$

(1)This formula, perhaps because of its what-else-could-it-be nature, is missing in the literature. Mazur and Messing prove the canonical isomorphism is functorial in A (see [MM74, Proposition 2.6.7]), but its explicit form is missing. This gap was already pointed out by Crew (see [Cre90, Introduction]), but in the description he proposes (*op.cit.*, Theorem 2.7), he does not determine such a linear map.

a sequence of $\mathcal{O}_X$-modules indexed by integers. For morphisms of locally ringed spaces $f\colon Y \to X$ and $g\colon X \to Z$, and an $\mathcal{O}_X$-module M, let $f^*(F)$, $g_*(F)$, $(F)\otimes M$ denote the sequences obtained from (F) by, respectively, pulling back along f, pushing forward along g and taking the tensor product with M.

Let K be a complete non-trivially valued non-Archimedean field. In this paper K-analytic spaces are considered in the sense of Berkovich (see [Ber93]). By abuse of notation, given a K-analytic space X, an $\mathcal{O}_X$-module here is what is called an $\mathcal{O}_{X_G}$-module in *op.cit.* As soon as the K-analytic space X is good (that is, every point admits an affinoid neighbourhood), the two notions coincide (*op.cit.*, Proposition 1.3.4). For a K-analytic space S, an S-analytic space in groups will be called simply an S-analytic group. An *abeloid variety* over S is a proper, smooth S-analytic group with connected fibers.

2. Moduli of rank 1 connections and universal extension of an abelian scheme

Let S be a scheme, $\alpha\colon A \to S$ an abelian scheme, $e\colon S \to A$ the zero section and $\omega_A = e^*\Omega^1_{A/S}$. Let $\mu, \mathrm{pr}_1, \mathrm{pr}_2\colon A\times_S A \to A$ be, respectively, the group law, the first and the second projections.

2.1. Connections on homogeneous line bundles

A *homogeneous line bundle* on A is the datum of a line bundle on A together with an isomorphism of $\mathcal{O}_A$-modules $\varphi\colon\ \mathrm{pr}_1^* L\otimes \mathrm{pr}_2^* L \to \mu^* L$. The isomorphism φ is called *rigidification.*

Remark 2.1. The isomorphism $(e,e)^*\varphi\colon e^*L\otimes e^*L \to e^*L$ induces a trivialization $u\colon \mathcal{O}_S \to e^*L$ of the line bundle e^*L on S.

Let Δ_1 be the first-order thickening of $A\times_S A$ along the diagonal, $p_i\colon \Delta_1 \to A$ the i^{th} projection for $i = 1, 2$, A_1 the first-order thickening of A along e, $\iota\colon A_1 \to A$ the closed immersion, $\pi\colon A_1 \to S$ the structural morphism and $\tau\colon A_1 \to \Delta_1$ the morphism defined by $p_1\circ\tau = x\circ\pi$ and $p_2\circ\tau = \iota$. Following Grothendieck, a connection on L is an isomorphism of $\mathcal{O}_{\Delta_1}$-modules $\nabla\colon p_1^*L \to p_2^*L$ whose restriction to A is the identity. Similarly, an infinitesimal rigidification[(2)] at e is an isomorphism of $\mathcal{O}_{A_1}$-modules $\rho\colon \pi^*e^* \to \iota^*L$ whose restriction to the zero section is the identity. (Refer to Sections A.4 and A.5 for basics on connections and infinitesimal rigidifications.) In particular, for a connection ∇ on L, $\tau^*\nabla$ is an infinitesimal rigidification of L at e.

Proposition 2.2. *For a homogeneous line bundle L on A, we have a bijection*

$$\{\text{connections on } L\} \longrightarrow \left\{\begin{array}{c}\text{infinitesimal}\\ \text{rigidifications of } L \text{ at } e\end{array}\right\}, \quad \nabla \longmapsto \tau^*\nabla.$$

Proof. The argument is borrowed from (2) and (3) in the proof of [MM74, Proposition 3.2.3, pp. 39–40].

Injectivity. Let $\nabla, \nabla'\colon p_1^*L \to p_2^*L$ be connections on L. Since both isomorphisms ∇ and ∇' are the identity when restricted to the diagonal, they differ by an homomorphism of $\mathcal{O}_X$-modules $L \to \Omega^1_\alpha\otimes L$. The latter is equivalent to the datum of a global section ω of Ω^1_α as L is a line bundle. On the other hand, the evaluation at e homomorphism $\varepsilon\colon \alpha_*\Omega^1_\alpha \to \omega_A$ is also an isomorphism. Therefore, the isomorphisms $\tau^*\nabla$ and $\tau^*\nabla'$ differ by $\varepsilon(\omega)$, which is 0 if and only if ω is.

Surjectivity. Let $\rho\colon \pi^*e^*L \to \iota^*L$ be an infinitesimal rigidification of L. The morphism $p_2 - p_1\colon \Delta_1 \to A$ factors through the closed immersion $\iota\colon A_1 \to A$. Indeed, when restricted to the diagonal, the map $p_2 - p_1$ has constant value e. As A_1 (resp. Δ_1) is the first-order thickening of A along e (resp. of $X\times_S X$ along the diagonal Δ), the morphism $p_2 - p_1$ induces a morphism $\eta\colon \Delta_1 \to A_1$ between first-order thickenings

(2) Mazur–Messing call an ‘infinitesimal rigidification’ simply a ‘rigidification’. Here, the adjective ‘infinitesimal’ is added in order to distinguish the concept from that of a rigidification of a homogeneous line bundle.

such that $\iota \circ \eta = p_2 - p_1$. The pull-back of the rigidification φ along the morphisms of S-schemes $(e \circ \pi \circ \eta, p_1), (p_2 - p_1, p_1) \colon \Delta_1 \to A \times A$ furnishes the following isomorphisms of line bundles over Δ_1:

$$(e \circ \pi \circ \eta, p_1)^* \varphi \colon \eta^* \pi^* e^* L \otimes p_1^* L \longrightarrow p_1^* L,$$
$$(p_2 - p_1, p_1)^* \varphi \colon (p_2 - p_1)^* L \otimes p_1^* L \longrightarrow p_2^* L.$$

On the other hand, taking the tensor product of $\eta^* \rho$ with $p_1^* L$ gives rise the isomorphism of $\mathcal{O}_{\Delta_1}$-modules

$$\eta^* \rho \otimes \mathrm{id} \colon \eta^* \pi^* e^* L \otimes p_1^* L \longrightarrow (p_2 - p_1)^* L \otimes p_1^* L.$$

Consider the unique homomorphism of $\mathcal{O}_{\Delta_1}$-modules $\nabla \colon p_1^* L \to p_2^* L$ making the following diagram commutative:

$$\begin{tikzcd}
\eta^* \pi^* e^* L \otimes p_1^* L \arrow[r, "\eta^* \rho \otimes \mathrm{id}"] \arrow[d, "(e \circ \pi \circ \eta, p_1)^* \varphi"'] & (p_2 - p_1)^* L \otimes p_1^* L \arrow[d, "(p_2 - p_1, p_1)^* \varphi"] \\
p_1^* L \arrow[r, "\nabla"] & p_2^* L.
\end{tikzcd} \tag{2.1}$$

To conclude, one has to show that the infinitesimal rigidification $\tau^* \nabla$ is ρ. Notice that the endomorphism $\eta \circ \tau$ of the S-scheme A_1 is the identity. Indeed, since ι is a closed immersion (thus a monomorphism of schemes), it suffices to show the equality $\iota \circ \eta \circ \tau = \iota$. But $\iota \circ \eta \circ \tau = (p_2 - p_1) \circ \tau = \iota - e \circ \pi = \iota$ as A_1-valued points of A because $e \circ \pi$ is the neutral element of the group $A(A_1)$. Now, pulling back the diagram (2.1) along τ gives the following commutative diagram of $\mathcal{O}_{A_1}$-modules:

$$\begin{tikzcd}
\pi^* e^* L \arrow[r, "\tau^* \nabla"] \arrow[d, "\mathrm{id} \otimes \pi^* u"'] & \iota^* L \arrow[d, "\mathrm{id} \otimes \pi^* u"] \\
\pi^* e^* L \otimes \pi^* e^* L \arrow[r, "\rho \otimes \mathrm{id}"] & \iota^* L \otimes \pi^* e^* L,
\end{tikzcd}$$

where $u \colon \mathcal{O}_S \to e^* L$ is the trivialization introduced in Remark 2.1. (Here we used the equalities $p_1 \circ \tau = x \circ \pi$ and $p_2 \circ \tau = \iota$ holding by the definition of τ.) The equality $\tau^* \nabla = \rho$ follows. This concludes the proof. $\square$

For a homogeneous line bundle L on A, let

$$0 \longrightarrow \Omega^1_{A/S} \longrightarrow \mathrm{At}_{A/S}(L) \longrightarrow \mathcal{O}_A \longrightarrow 0 \tag{$\mathrm{At}_{A/S}(L)$}$$

be its Atiyah extension (see Section A.4). Recall that connections on L correspond to splittings of the Atiyah extension of L.

Proposition 2.3. *Let (L, φ) be a homogeneous line bundle on A.*

(1) *If the cohomology group $\mathrm{H}^1(S, \omega_A)$ vanishes, then the line bundle L admits a connection.*

(2) *The following sequence of $\mathcal{O}_S$-modules is short exact:*

$$0 \longrightarrow \alpha_* \Omega^1_{A/S} \longrightarrow \alpha_* \mathrm{At}_{A/S}(L) \longrightarrow \alpha_* \mathcal{O}_A \longrightarrow 0. \tag{$\alpha_*(\mathrm{At}_{A/S}(L))$}$$

(3) *The homomorphism $\alpha_* \alpha^* (\mathrm{At}_{A/S}(L)) \to (\mathrm{At}_{A/S}(L))$ of short exact sequences of $\mathcal{O}_A$-modules obtained by adjunction is an isomorphism.*

Proof. (1) According to Proposition 2.2, it suffices to show that the line bundle L admits an infinitesimal rigidification at e. With the notation of Proposition A.6, an infinitesimal rigidification corresponds to a splitting of the short exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \omega_A \otimes e^* L \longrightarrow \pi_* \iota^* L \longrightarrow e^* L \longrightarrow 0. \tag{$\pi_*(\iota^* L)$}$$

The latter is an extension of the line bundle $e^* L$ on S by the vector bundle $\omega_A \otimes e^* L$; therefore, its isomorphism class lies in $\mathrm{H}^1(S, \mathcal{H}om(e^* L, \omega_A \otimes e^* L)) = \mathrm{H}^1(S, \omega_A)$. By hypothesis, the cohomology group $\mathrm{H}^1(S, \omega_A)$ vanishes; hence the short exact sequence $\pi_*(\iota^* L)$ splits, and the line bundle L admits a connection.

(2) The push-forward along α is a left-exact functor; therefore, it suffices to show that the natural map $p\colon \alpha_* \operatorname{At}_{A/S}(L) \to \alpha_* \mathcal{O}_A$ is surjective. The statement is local on S; thus the scheme S may be supposed to be affine. Under this assumption, the cohomology group $\mathrm{H}^1(S, \omega_A)$ vanishes, and because of (1), the line bundle L admits a connection. In other words, by Proposition A.4, the Atiyah extension $(\operatorname{At}_{A/S}(L))$ of L admits a splitting $s\colon \mathcal{O}_A \to \operatorname{At}_{A/S}(L)$. The homomorphism of $\mathcal{O}_S$-modules $\alpha_* s\colon \alpha_* \mathcal{O}_A \to \alpha_* \operatorname{At}_{A/S}(L)$ is a section of p; that is, it satisfies $p \circ \alpha_* s = \mathrm{id}$. In particular, the homomorphism p is surjective.

(3) The homomorphism of short exact sequences of $\mathcal{O}_A$-modules in question is the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \alpha^*\alpha_*\Omega^1_{A/S} & \longrightarrow & \alpha^*\alpha_* \operatorname{At}_{A/S}(L) & \longrightarrow & \alpha^*\alpha_*\mathcal{O}_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_{A/S} & \longrightarrow & \operatorname{At}_{A/S}(L) & \longrightarrow & \mathcal{O}_A & \longrightarrow & 0, \end{array}$$

where the three vertical arrows are given by adjunction. The leftmost and the rightmost vertical arrows are isomorphisms; thus the central vertical arrow must be so by the five lemma. □

Remark 2.4. The Atiyah extension $(\operatorname{At}_{A/S}(L))$ is obtained as the tensor product with $L^\vee$ of the short exact sequence

$$(\mathcal{J}^1_{A/S}(L)) \qquad 0 \longrightarrow \Omega^1_{A/S} \otimes L \longrightarrow \mathcal{J}^1_{A/S}(L) \longrightarrow L \longrightarrow 0,$$

where $\mathcal{J}^1_{A/S}(L)$ is the $\mathcal{O}_X$-module of first-order jets of L. Proposition A.7 furnishes an isomorphism $\varphi\colon e^*(\mathcal{J}^1_{A/S}(L)) \to \pi_*(\iota^* L)$ of short exact sequences of $\mathcal{O}_S$-modules. Taking the tensor product with $e^*L^\vee$ induces an isomorphism

$$e^*(\operatorname{At}_{A/S}(L)) \longrightarrow \pi_*(\iota^* L) \otimes e^* L^\vee$$

of short exact sequences of $\mathcal{O}_S$-modules. Also note that, by Remark 2.1, the line bundle e^*L on S is trivial, whence we have an isomorphism

$$e^*(\operatorname{At}_{A/S}(L)) \cong \pi_*(\iota^* L).$$

2.2. The canonical extension of the trivial line bundle

The functor associating with an S-scheme S' the set of isomorphism classes of homogeneous line bundles on $A \times_S S'$ is representable by an abelian scheme $\check{\alpha}\colon \check{A} \to S$, called the ***dual abelian scheme*** (this is [FC90, Theorem 1.9]; to see that the definition in *loc. cit.* is equivalent to the one here, see [Oor66, Proposition 18.4]). Let $\mathcal{L}$ be the Poincaré bundle (that is, the universal homogeneous line bundle on $A \times_S \check{A}$), $q\colon A \times_S \check{A} \to \check{A}$ the projection onto the second factor and $(\operatorname{At}_q(\mathcal{L}))$ the Atiyah extension relative to q of the Poincaré bundle $\mathcal{L}$. Set $\mathcal{U}_{\check{A}} := q_* \operatorname{At}_q(\mathcal{L})$. The sequence of $\mathcal{O}_{\check{A}}$-modules

$$(\mathcal{U}_{\check{A}}) \qquad 0 \longrightarrow \check{\alpha}^*\omega_A \longrightarrow \mathcal{U}_{\check{A}} \longrightarrow \mathcal{O}_{\check{A}} \longrightarrow 0$$

obtained by pushing forward the short exact sequence $(\operatorname{At}_q(\mathcal{L}))$ along q, is short exact by Proposition 2.3(2) applied to the abelian scheme $A \times_S \check{A}$ over $\check{A}$.

Definition 2.5. The extension $(\mathcal{U}_{\check{A}})$ is called the ***canonical extension*** of $\mathcal{O}_{\check{A}}$.

Remark 2.6. The pull-back of the Atiyah extension of $\mathcal{L}$ along the unramified morphism $(\mathrm{id}_A, \check{e})$ is the Atiyah extension of the trivial line bundle $\mathcal{O}_A$. Therefore, the canonical derivation $\mathrm{d}_{A/S}\colon \mathcal{O}_A \to \Omega^1_{A/S}$, being a connection on the trivial bundle, defines a splitting of $\check{e}^*(\mathcal{U}_{\check{A}})$, called the ***canonical splitting***.

Remark 2.7. Let A_1 be the first-order thickening of A along the zero section e, $\iota\colon A_1 \times_S \check{A} \to A \times_S \check{A}$ the morphism obtained from the closed immersion $A_1 \to A$ by base change along $\check{\alpha}$ and $\pi\colon A_1 \times_S \check{A} \to \check{A}$ the second projection. Remark 2.4 (applied to the abelian scheme $A \times_S \check{A}$ over $\check{A}$) furnishes an isomorphism of sequences of $\mathcal{O}_{\check{A}}$-modules

$$(e, \mathrm{id}_{\check{A}})^*(\mathrm{At}_q(\mathcal{L})) \xrightarrow{\sim} \pi_*(\iota^*\mathcal{L}).$$

On the other hand, by Proposition 2.3(3), the evaluation at the zero section

$$(\mathcal{U}_{\check{A}}) = q_*(\mathrm{At}_q(\mathcal{L})) \longrightarrow (e, \mathrm{id}_{\check{A}})^*(\mathrm{At}_q(\mathcal{L}))$$

is an isomorphism of sequences of $\mathcal{O}_{\check{A}}$-modules. Composing these isomorphisms yields an isomorphism of sequences of $\mathcal{O}_{\check{A}}$-modules

$$(\mathcal{U}_{\check{A}}) \cong \pi_*(\iota^*\mathcal{L}).$$

2.3. Moduli space of connections

Definition 2.8. The S-scheme $\check{A}^\natural = \mathbb{P}(\mathcal{U}_{\check{A}}) \smallsetminus \mathbb{P}(\check{\alpha}^*\omega_A)$ is called the *moduli space of rank* 1 *connections* on A.

Theorem 2.9. *The S-scheme $\check{A}^\natural$ represents the functor associating with an S-scheme S' the set of isomorphism classes of triples (L, φ, ∇) made up of a homogeneous line bundle (L, φ) on the abelian scheme $A_{S'}$ and a connection $\nabla \colon L \to \Omega^1_{A_{S'}/S'} \otimes L$.*

Proof. For an S-scheme S', an S'-valued point of $\check{A}^\natural$ consists of the datum of a morphism of S-schemes $f \colon S' \to \check{A}$ and a splitting $s \colon \mathcal{O}_{S'} \to f^*\mathcal{U}_{\check{A}}$ of the short exact sequence of $\mathcal{O}_{S'}$-modules $f^*(\mathcal{U}_{\check{A}})$. Let (L, φ) be the homogeneous line bundle on the abelian scheme $A' := A_{S'}$ obtained as the pull-back of the Poincaré bundle $\mathcal{L}$ along the morphism $\mathrm{id}_A \times f \colon A' := A \times_S S' \to A \times_S \check{A}$. Let $\alpha' \colon A' \to S'$ be the morphism obtained from α by base change. Then, by Proposition 2.3(3), the short exact sequence of $\mathcal{O}_{A'}$-modules $\alpha'^* f^*(\mathcal{U}_{\check{A}})$ is the Atiyah extension $(\mathrm{At}_{\alpha'}(L))$ of the homogeneous line bundle L. By Proposition A.4, the splitting $\alpha'^* s \colon \mathcal{O}_{A'} \to \mathrm{At}_{\alpha'}(L)$ corresponds to a connection on L (notice the equality $\mathcal{E}nd L = \mathcal{O}_S$, due to L being a line bundle). □

The tensor product of line bundles together with a connection endows $\check{A}^\natural$ with the structure of a group S-scheme. The natural projection $\pi \colon \check{A}^\natural \to \check{A}$, $(L, \varphi, \nabla) \mapsto (L, \varphi)$ is a faithfully flat morphism of group S-schemes. The kernel of π is by definition made of connections on the trivial bundle $\mathcal{O}_A$. Now, a connection on the trivial line bundle is nothing but $\mathrm{d}_{A/S} + \omega$ for a global differential form on A, where $\mathrm{d}_{A/S}$ is the canonical $\alpha^{-1}\mathcal{O}_S$-linear derivation. The isomorphism $\alpha_*\Omega^1_{A/S} \cong e^*\Omega^1_{A/S} =: \omega_A$ thus yields a short exact sequence of group S-schemes

$$0 \longrightarrow \mathbb{V}(\omega_A) \longrightarrow \check{A}^\natural \longrightarrow \check{A} \longrightarrow 0.$$

The following remark will only be needed once, quite further in text (see the proof of Theorem 4.6). The reader may harmlessly skip it at first.

Remark 2.10. As any affine bundle on an abelian scheme, the group structure on $\check{A}^\natural$ is defined by a unique isomorphism of $\mathcal{O}_{\check{A}}$-modules

$$\check{\psi} \colon \mathrm{pr}_1^*\mathcal{U}_{\check{A}} +_{\mathrm{B}} \mathrm{pr}_2^*\mathcal{U}_{\check{A}} \longrightarrow \check{\mu}^*\mathcal{U}_{\check{A}},$$

where $\check{\mu}, \mathrm{pr}_1, \mathrm{pr}_2 \colon \check{A} \times_S \check{A} \to \check{A}$ are, respectively, the group law, the first and the second projection, and $+_{\mathrm{B}}$ is the Baer sum of extensions (see Examples 2.15 and 2.17 below). According to Example 2.18, the isomorphism $\check{\psi}$ is the push-forward along the morphism $(A_1 \times_S \check{A}) \times_{A_1} (A_1 \times_S \check{A}) \to \check{A} \times_S \check{A}$ of the rigidification

$$\check{\varphi} \colon (\mathrm{id}, \mathrm{pr}_1)^*\mathcal{L} \otimes (\mathrm{id}, \mathrm{pr}_2)^*\mathcal{L} \longrightarrow (\mathrm{id}, \check{\mu})^*\mathcal{L}$$

of the Poincaré bundle $\mathcal{L}$ on the abelian scheme $A \times_S \check{A}$ over A.

2.4. The Lie algebra of the dual abelian scheme (redux)

Let $\check{A}_1$ be the first-order thickening of $\check{A}$ along its zero section $\check{e}$, $\pi\colon A \times \check{A}_1 \to A$ the projection onto the first factor, $\iota\colon A \times_S \check{A}_1 \to A \times_S \check{A}$ the morphism obtained from the closed embedding $\check{A}_1 \to \check{A}$ by base change with respect to $\alpha\colon A \to S$ and $\alpha_1 : A \times_S \check{A}_1 \to S$ the structural morphism. Consider the isomorphism class

$$[\iota^*\mathcal{L}] \in \mathrm{H}^1\left(A \times_S \check{A}_1, \mathcal{O}^{\times}_{A\times\check{A}_1}\right)$$

of the line bundle $\iota^*\mathcal{L}$ on $A \times_S \check{A}_1$, where $\mathcal{L}$ is the Poincaré bundle on $A \times_S \check{A}$. Let $c_1(\iota^*\mathcal{L})$ be the global section of the sheaf of abelian groups $\mathrm{R}^1\alpha_{1*}\mathcal{O}^{\times}_{A\times\check{A}_1}$ on S defined as the image of $[\iota^*\mathcal{L}]$ via the homomorphism of abelian groups

$$\mathrm{H}^1\left(A \times_S \check{A}_1, \mathcal{O}^{\times}_{A\times\check{A}_1}\right) \longrightarrow \mathrm{H}^0\left(S, \mathrm{R}^1\alpha_{1*}\mathcal{O}^{\times}_{A\times\check{A}_1}\right)$$

given by the Grothendieck–Leray spectral sequence

$$\mathrm{H}^p\left(S, \mathrm{R}^q\alpha_{1*}\mathcal{O}^{\times}_{A\times\check{A}_1}\right) \implies \mathrm{H}^{p+q}\left(A \times_S \check{A}_1, \mathcal{O}^{\times}_{A\times\check{A}_1}\right).$$

The first projection $\pi\colon A \times_S \check{A}_1 \to A$ induces a homeomorphism on the underlying topological spaces, whence an isomorphism $\mathrm{R}^1\alpha_*(\pi_*\mathcal{O}^{\times}_{A\times\check{A}_1}) \cong \mathrm{R}^1\alpha_{1*}\mathcal{O}^{\times}_{A\times\check{A}_1}$ that will be treated as understood in what follows. The homomorphism $\mathcal{O}_A \to \pi_*\mathcal{O}_{A\times\check{A}_1}$ defines a splitting of $0 \to \alpha^*\omega_{\check{A}} \to \pi_*\mathcal{O}_{A\times\check{A}_1} \to \mathcal{O}_A \to 0$. It follows that the short exact sequence of sheaves abelian groups on A

$$0 \longrightarrow \alpha^*\omega_{\check{A}} \xrightarrow{\exp} \pi_*\mathcal{O}^{\times}_{A\times\check{A}_1} \longrightarrow \mathcal{O}^{\times}_A \longrightarrow 0$$

splits, where $\exp\colon v \mapsto 1+v$ is the truncated exponential. We therefore have an exact sequence of sheaves of abelian groups on S

$$0 \longrightarrow \mathrm{R}^1\alpha_*\alpha^*\omega_{\check{A}} \longrightarrow \mathrm{R}^1\alpha_*\left(\pi_*\mathcal{O}^{\times}_{A\times\check{A}_1}\right) \longrightarrow \mathrm{R}^1\alpha_*\mathcal{O}_A.$$

The pull-back of the line bundle $\iota^*\mathcal{L}$ along the morphism $(\mathrm{id},\check{e})\colon A \to A \times_S \check{A}_1$ is trivial; thus $c_1(\iota^*\mathcal{L})$ is (the image via the truncated exponential of) a section of $\mathcal{O}_S$-module $\mathrm{R}^1\alpha_*\alpha^*\omega_{\check{A}}$ still written $c_1(\iota^*\mathcal{L})$. The projection formula yields an isomorphisms $\mathrm{R}^1\alpha_*\alpha^*\omega_{\check{A}} \cong \mathrm{R}^1\alpha_*\mathcal{O}_A \otimes \omega_{\check{A}} = \mathcal{H}om(\mathrm{Lie}\,\check{A}, \mathrm{R}^1\alpha_*\mathcal{O}_A)$ through which the class $c_1(\iota^*\mathcal{L})$ corresponds to an isomorphism (see [BLR90, Section 8.4, Theorem 1])

$$\Phi_A\colon \mathrm{Lie}\,\check{A} \longrightarrow \mathrm{R}^1\alpha_*\mathcal{O}_A.$$

2.5. Universal property of the canonical extension

We borrow notation from Section 2.4. For a vector bundle F on S, the Grothendieck–Leray spectral sequence $\mathrm{H}^p(S, \mathrm{R}^q\alpha_*\alpha^*F) \Rightarrow \mathrm{H}^{p+q}(A, \alpha^*F)$ yields an exact sequence of $\Gamma(S,\mathcal{O}_S)$-modules

$$0 \longrightarrow \mathrm{H}^1(S,F) \xrightarrow{\alpha^*} \mathrm{H}^1(A,\alpha^*F) \xrightarrow{p_F} \mathrm{H}^0(S, \mathrm{R}^1\alpha_*\alpha^*F). \tag{2.2}$$

For an extension of $\mathcal{O}_A$ by α^*F,

$$0 \longrightarrow \alpha^*F \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_A \longrightarrow 0, \tag{$\mathcal{F}$}$$

let $[\mathcal{F}] \in \mathrm{H}^1(A,\alpha^*F)$ be its isomorphism class. The global section $c(\mathcal{F}) := p_F([\mathcal{F}])$ of the $\mathcal{O}_S$-module $\mathrm{R}^1\alpha_*\alpha^*F$ can also be seen as the connecting homomorphism in the long exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow F \longrightarrow \alpha_*\mathcal{F} \longrightarrow \mathcal{O}_S \xrightarrow{c(\mathcal{F})} \mathrm{R}^1\alpha_*\alpha^*F \longrightarrow \cdots.$$

The abelian scheme A coincides with the dual abelian scheme of $\check{A}$. By means of this identification, consider the canonical extension $(\mathcal{U}_A)$ on A and its isomorphism

$$[\mathcal{U}_A] \in \mathrm{H}^1\left(A, \mathcal{H}om\left(\mathcal{O}_A, \alpha^*\omega_{\check{A}}\right)\right) = \mathrm{H}^1\left(A, \alpha^*\omega_{\check{A}}\right).$$

The truncated exponential map $\exp\colon \alpha^*\omega_{\check{A}} \to \pi_*\mathcal{O}^\times_{A\times\check{A}_1}$, $v \mapsto 1+v$ induces a homomorphism of abelian groups

$$\exp\colon \mathrm{H}^1\left(A, \alpha^*\omega_{\check{A}}\right) \longrightarrow \mathrm{H}^1\left(A, \pi_*\mathcal{O}^\times_{A\times\check{A}_1}\right).$$

On the other hand, the morphism π is a homeomorphism on the underlying topological spaces, thus an isomorphism of abelian groups

$$\mathrm{H}^1\left(A\times_S \check{A}_1, \mathcal{O}^\times_{A\times\check{A}_1}\right) \xrightarrow{\sim} \mathrm{H}^1\left(A, \pi_*\mathcal{O}^\times_{A\times\check{A}_1}\right).$$

The image of $[\iota^*\mathcal{L}]$ via the preceding isomorphism is the isomorphism class $[\pi_*\iota^*\mathcal{L}]$ of the invertible $\pi_*\mathcal{O}_{A\times\check{A}_1}$-module $\pi_*\iota^*\mathcal{L}$.

Proposition 2.11. *With the notation above,*

$$[\pi_*\iota^*\mathcal{L}] = \exp([\mathcal{U}_A]), \tag{1}$$

$$c(\mathcal{U}_A) = c_1(\iota^*\mathcal{L}). \tag{2}$$

The second statement can be reformulated by saying that, via $\mathrm{R}^1\alpha_*\alpha^*\omega_{\check{A}} \cong \mathcal{H}om(\operatorname{Lie}\check{A}, \mathrm{R}^1\alpha_*\mathcal{O}_A)$, the following equality holds:

$$c(\mathcal{U}_A) = \Phi_A.$$

Proof. (1) Remark 2.7 furnishes an isomorphism of short exact sequence of $\mathcal{O}_A$-modules $(\mathcal{U}_A) \cong \pi_*(\iota^*\mathcal{L})$. For an affine open cover $\{A_i\}_{i\in I}$ of A, the isomorphism class $[\mathcal{U}_A]$ is represented by a 1-cocycle, say $f_{ij} \in \Gamma(A_i\cap A_j, \alpha^*\omega_{\check{A}})$ for $i,j\in I$. The invertible $\pi_*\mathcal{O}_{A\times\check{A}_1}$-module $\pi_*\iota^*\mathcal{L}$ is the glueing of the $\mathcal{O}_{A_i}$-modules $(\pi_*\mathcal{O}_{A\times\check{A}_1})_{|A_i}$ along the transition maps $\exp(f_{ij}) = 1 + f_{ij}$. Relation (2) follows immediately from (1). ☐

For a homomorphism $\varphi\colon \omega_{\check{A}} \to F$ with F a vector bundle on S, let $(\mathcal{F}_\varphi)$ be the short exact sequence obtained by push-out of $(\mathcal{U}_A)$ along the homomorphism $\alpha^*\varphi$. Let $\gamma_F(\varphi)$ denote its isomorphism class $[\mathcal{F}_\varphi] \in \mathrm{H}^1(A, \alpha^*F)$; then this construction defines a map

$$\gamma_F\colon \operatorname{Hom}(\omega_{\check{A}}, F) \longrightarrow \mathrm{H}^1(A, \alpha^*F).$$

Theorem 2.12. *The map γ_F is injective and $\Gamma(S,\mathcal{O}_S)$-linear, and its image is the set of isomorphism classes of extensions*

$$0 \longrightarrow \alpha^*F \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_A \longrightarrow 0 \tag{$\mathcal{F}$}$$

such that the short exact sequence of $\mathcal{O}_S$-modules $e^*(\mathcal{F})$ splits.

This statement is an immediate consequence of the following more precise fact. To state it, consider the homomorphism $e^*\colon \mathrm{H}^1(A,\alpha^*F) \to \mathrm{H}^1(S,F)$ given by the pull-back of an extension along the morphism e. Since e is a section of α, the composite map $e^*\circ\alpha^*$ is the identity of $\mathrm{H}^1(S,F)$. Also, note that an extension $(\mathcal{F})$ as above splits if and only if $e^*[\mathcal{F}] = 0$.

Lemma 2.13. *For a vector bundle F on S,*

1. *$p_F\circ\gamma_F$ is the isomorphism induced by $\Phi_A\otimes\mathrm{id}_F\colon \mathcal{H}om(\omega_{\check{A}}, F) \to \mathrm{R}^1\alpha_*\alpha^*F$ on global sections,*
2. *the following sequence of $\Gamma(S,\mathcal{O}_S)$-modules is short exact:*

$$0 \longrightarrow \operatorname{Hom}(\omega_{\check{A}}, F) \xrightarrow{\gamma_F} \mathrm{H}^1(A,\alpha^*F) \xrightarrow{e^*} \mathrm{H}^1(S,F) \longrightarrow 0.$$

Proof. (1) The diagram of $\mathcal{O}_S$-modules

$$\begin{tikzcd}
0 \arrow[r] & \omega_{\check{A}} \arrow[r] \arrow[d, "\varphi"] & \alpha_*\mathcal{U}_A \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r, "c(\mathcal{U}_A)"] \arrow[d, equal] & \mathrm{R}^1\alpha_*\mathcal{O}_A\otimes\omega_{\check{A}} \arrow[r] \arrow[d, "\mathrm{id}\otimes\varphi"] & \cdots \\
0 \arrow[r] & F \arrow[r] & \alpha_*\mathcal{F}_\varphi \arrow[r] & \mathcal{O}_S \arrow[r, "c(\mathcal{F}_\varphi)"] & \mathrm{R}^1\alpha_*\mathcal{O}_A\otimes F \arrow[r] & \cdots
\end{tikzcd}$$

is commutative. It follows from Proposition 2.11(2) that the composite map $p_F \circ \gamma_F$ is the one induced on global sections by $\Phi_A \otimes \mathrm{id}_F \colon \operatorname{Lie} \check{A} \otimes F \to \mathrm{R}^1 \alpha_* \mathcal{O}_A \otimes F$.

(2) First of all, notice that the map γ_F is injective and $\Gamma(S, \mathcal{O}_S)$-linear. Second, the composite map $e^* \circ \gamma_F$ vanishes: for a homomorphism $\varphi \colon \omega_{\check{A}} \to F$, the splitting of the short exact sequence of $\mathcal{O}_S$-module $e^*(\mathcal{U}_A)$ induced by the derivation $\mathrm{d}_{\check{A}/S}$ (see Remark 2.6) induces a splitting of $e^*(\mathcal{F}_\varphi)$. Therefore, it remains to show that the image of γ_F is the whole $\operatorname{Ker}(e^*)$. Set $r_F := (p_F \circ \gamma_F)^{-1} \circ p_F$. Then $r_F \circ \gamma_F$ is the identity of $\operatorname{Hom}(\omega_{\check{A}}, F)$, and the sequence of $\Gamma(S, \mathcal{O}_S)$-modules

$$0 \longrightarrow \mathrm{H}^1(S, F) \xrightarrow{\alpha^*} \mathrm{H}^1(A, \alpha^* F) \xrightarrow{r_F} \operatorname{Hom}(\omega_{\check{A}}, F) \longrightarrow 0$$

is short exact by (2.2). Since γ_F and e^* are sections of, respectively, r_F and α^*, the result follows. □

2.6. Preliminaries on extensions of an abelian scheme

Let G be an affine, commutative, faithfully flat and finitely presented group S-scheme.

2.6.1. Principal bundles.— A ***principal*** G***-bundle*** on A is a faithfully flat A-scheme P endowed with an action of G such that the morphism $(\sigma, \mathrm{pr}_P) \colon G \times_S P \to P \times_A P$ is an isomorphism, where $\sigma, \mathrm{pr}_P \colon G \times_S P \to P$ are, respectively, the morphism defining the action and the projection onto A. Let $\mathrm{H}^1_{\mathrm{fppf}}(A, G)$ denote the set of isomorphism classes of principal G-bundles on A.

Let $\rho \colon G \to G'$ be a morphism of group S-schemes, where G' has the same properties of G, and let P be a principal G-bundle on A. The quotient $\rho_* P$ of S-scheme $G' \times_S P$ via the action $g(g', x) = (\rho(g)g', gx)$ of G exists and is a principal G'-bundle on A, called the ***push-out*** (see [Ols16, Propositions 4.5.6 and 12.1.2]). For principal G-bundles P and P' on A, the push-out

$$P \wedge P' := \mu_{G*}(P \times_A P')$$

of the principal $G \times_S G$-bundle $P \times_A P'$ along the sum map $\mu_G \colon G \times_S G \to G$ is called the ***sum***. This operation endows the set $\mathrm{H}^1_{\mathrm{fppf}}(A, G)$ with the structure of an abelian group with neutral element $G \times_S A$ and inverse $P \mapsto [-1]_* P$, where $[-1]$ is the inverse map on G. Furthermore, the push-out of an endomorphism of G as a group S-scheme induces an endomorphism of the abelian group $\mathrm{H}^1_{\mathrm{fppf}}(A, G)$. This equips $\mathrm{H}^1_{\mathrm{fppf}}(A, G)$ with the structure of a module on the ring $\operatorname{End}(G)$ of endomorphisms of G. For instance, if $G = \mathbb{V}(F)$ for some vector bundle F on S, the set $\mathrm{H}^1_{\mathrm{fppf}}(A, G)$ is naturally a $\Gamma(S, \mathcal{O}_S)$-module.

Example 2.14. Suppose $G = \mathbb{G}_m$. The principal $\mathbb{G}_m$-bundle associated with a line bundle L on A is the total space of L deprived of its zero section $\mathbb{V}(L)^\times$. The so-defined map $\operatorname{Pic}(A) \to \mathrm{H}^1_{\mathrm{fppf}}(A, \mathbb{G}_m)$, $L \mapsto \mathbb{V}(L)^\times$ is an isomorphism of abelian groups (see [SGA1, Exposé XI, Proposition 5.1]), that is, a sum of principal $\mathbb{G}_m$-bundles corresponds to a tensor product of line bundles.

Example 2.15. Suppose $G = \mathbb{V}(F)$ for some vector bundle F on S, and let

$$0 \longrightarrow \alpha^* F \longrightarrow \mathcal{F} \xrightarrow{p} \mathcal{O}_A \longrightarrow 0 \tag{$\mathcal{F}$}$$

be a short exact sequence of $\mathcal{O}_A$-modules. The A-scheme $\mathbb{A}(F) := \mathbb{P}(\mathcal{F}) \smallsetminus \mathbb{P}(\alpha^* F)$ is a principal $\mathbb{V}(F)$-bundle. The map $\mathrm{H}^1(A, \alpha^* F) \to \mathrm{H}^1_{\mathrm{fppf}}(A, \mathbb{V}(F))$, $(\mathcal{F}) \mapsto \mathbb{A}(\mathcal{F})$ is an isomorphism of $\Gamma(S, \mathcal{O}_S)$-modules (see [SGA1, Exposé XI, Proposition 5.1]). In particular, the sum of such principal $\mathbb{V}(F)$-bundles is the $\mathbb{V}(F)$-bundle associated with the Baer sum of the corresponding extensions.

2.6.2. Law groups on principal bundles.— An *extension* of A by G is the datum of a short exact sequence of commutative group S-schemes

$$0 \longrightarrow G \xrightarrow{i_E} E \xrightarrow{p_E} A \longrightarrow 0, \tag{E}$$

where the morphism p_E is faithfully flat. Note that an extension E of A by G is naturally a principal G-bundle over A. Since the natural homomorphism $\mathcal{O}_S \to f_* \mathcal{O}_A$ is an isomorphism, the commutativity of E

is automatic. Morphisms of extensions are defined in the evident way. An isomorphism of extensions is a morphism inducing the identity on A and G. Let $\mathrm{Ext}(A,G)$ be set of isomorphism classes of extensions of A by G. For a morphism of S-group schemes $\rho\colon G\to G'$, with G' having the same properties as G, and an extension (E), the cokernel $\varphi_*(E)$ of the morphism $(i_E,\rho)\colon G\to E\times_S G'$ is called the ***push-out*** of (E). The ***Baer sum*** of extensions (E) and (E') of A by G is the push-out of $E\times_A E$ along the sum morphism $G\times_S G\to G$. The Baer sum endows $\mathrm{Ext}(A,G)$ with the structure of an abelian group. Similarly to the case of principal G-bundles, the abelian group $\mathrm{Ext}(A,G)$ is endowed with the structure of an $\mathrm{End}(G)$-module. Seeing an extension as a principal bundle gives rise to a homomorphism of $\mathrm{End}(G)$-modules

$$\lambda_G\colon \mathrm{Ext}(A,G)\longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(A,G).$$

A ***rigidification*** of a principal G-bundle P over A is an isomorphism

$$\varphi\colon \mathrm{pr}_1^*P\wedge \mathrm{pr}_2^*P\longrightarrow \mu_A^*P$$

of principal G-bundles over $A\times_S A$.[(3)] The map λ_G is injective, and its image is the set of isomorphism classes of principal G-bundles admitting a rigidification (see [Ser59, Théorème 15.5]).

Example 2.16. Suppose $G=\mathbb{G}_m$ and identify principal $\mathbb{G}_m$-bundles with line bundles as in Example 2.14. A rigidification of a line bundle L on A is an isomorphism of $\mathcal{O}_{A\times_S A}$-modules $\varphi\colon \mathrm{pr}_1^*L\otimes\mathrm{pr}_2^*L\to\mu_A^*L$.

Example 2.17. Suppose $G=\mathbb{V}(F)$ for some vector bundle F on S. Identify principal $\mathbb{V}(F)$-bundles with extensions of $\mathcal{O}_A$ by f^*F as in Example 2.15. For a short exact sequence of $\mathcal{O}_A$-modules

$$(\mathcal{F})\qquad 0\longrightarrow f^*F\longrightarrow\mathcal{F}\longrightarrow\mathcal{O}_A\longrightarrow 0,$$

a rigidification of $(\mathcal{F})$ is an isomorphism of short exact sequences of $\mathcal{O}_{A\times_S A}$-modules

$$\varphi\colon \mathrm{pr}_1^*(\mathcal{F})+_{\mathrm{B}}\mathrm{pr}_2^*(\mathcal{F})\xrightarrow{\sim}\mu_A^*(\mathcal{F}),$$

where $\mathrm{pr}_1^*(\mathcal{F})+_{\mathrm{B}}\mathrm{pr}_2^*(\mathcal{F})$ is the Baer sum of the extensions $\mathrm{pr}_1^*(\mathcal{F})$ and $\mathrm{pr}_2^*(\mathcal{F})$.

Example 2.18. For a vector bundle F on S, let S' be the first-order thickening of $\mathbb{V}(F^\vee)$ along its zero section. Let L be a line bundle on $A':=A\times_S S'$ endowed with a rigidification $\varphi\colon \mathrm{pr}_1^*L\otimes\mathrm{pr}_2^*L\to\mu_{A'}^*L$, with the obvious notation. Let $s\colon A\to A'$ be the closed immersion induced by the zero section of $F^\vee$ and $\pi\colon A'\to A$ the projection onto A. The $\mathcal{O}_A$-module $\mathcal{F}:=\pi_*L$ sits into the following short exact sequence of $\mathcal{O}_A$-modules:

$$(\mathcal{F})\qquad 0\longrightarrow f^*F\otimes s^*L\longrightarrow\mathcal{F}\longrightarrow s^*L\longrightarrow 0.$$

According to Proposition A.2, the push-forward of the $\mathcal{O}_{A'\times A'}$-module $\mathrm{pr}_1^*L\otimes\mathrm{pr}_2^*L$ along the morphism $\pi\times\pi\colon A'\times_{S'}A'\to A\times_S A$ is the Baer sum of the extensions $\mathrm{pr}_1^*\mathcal{F}\otimes\mathrm{pr}_2^*s^*L$ and $\mathrm{pr}_1^*s^*L\otimes\mathrm{pr}_2^*\mathcal{F}$. Assume further that the line bundle s^*L is isomorphic to $\mathcal{O}_A$. Then, the vector bundle $\mathcal{F}$ is an extension of $\mathcal{O}_A$ by f^*F, and the isomorphism $(\pi\times\pi)_*\varphi\colon \mathrm{pr}_1^*\mathcal{F}+_{\mathrm{B}}\mathrm{pr}_2^*\mathcal{F}\to\mu_A^*\mathcal{F}$ is a rigidification like the one considered in Example 2.17.

2.7. The universal vector extension

A *vector extension* of the abelian scheme A is an extension of A by $\mathbb{V}(F)$ for some vector bundle F on S called its *vector part*. Recall the injective homomorphism of $\Gamma(S,\mathcal{O}_S)$-modules

$$\lambda_F:=\lambda_{\mathbb{V}(F)}\colon \mathrm{Ext}(A,\mathbb{V}(F))\longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(A,\mathbb{V}(F))=\mathrm{H}^1(A,\alpha^*F)$$

defined in Section 2.6.2 and the map $p_F\colon \mathrm{H}^1(A,\alpha^*F)\to\mathrm{H}^0(S,\mathrm{R}^1\alpha_*\alpha^*F)$ introduced in (2.2). According to [MM74, Proposition 1.10], the composite homomorphism

$$\lambda_{F/S}:=p_F\circ\lambda_F\colon \mathrm{Ext}(A,F)\longrightarrow\mathrm{H}^0(S,\mathrm{R}^1\alpha_*\alpha^*F)$$

(3) The definition of a rigidification in [SGA7-I, Exposé VII, Section 1] involves the commutativity of two diagrams which is automatic over abelian schemes.

is bijective. For the vector bundle $F = (\mathrm{R}^1\alpha_*\mathcal{O}_A)^\vee$ on S, this gives an isomorphism

$$\mathrm{Ext}(A, \mathbb{V}((\mathrm{R}^1\alpha_*\mathcal{O}_A)^\vee)) \cong \mathrm{End}\,\mathrm{R}^1\alpha_*\mathcal{O}_A.$$

Definition 2.19. The universal vector extension is the one corresponding to the identity via the above isomorphism:

$$0 \longrightarrow \mathbb{V}((\mathrm{R}^1\alpha_*\mathcal{O}_A)^\vee) \longrightarrow \mathrm{E}(A) \longrightarrow A \longrightarrow 0. \tag{E(A)}$$

The extension $\mathrm{E}(A)$ deserves the 'universal' title because of the following property. First notice that, for vector bundles F and F' on S and a homomorphism of $\mathcal{O}_S$-modules $\varphi\colon F \to F'$, the diagram of $\Gamma(S, \mathcal{O}_S)$-modules

$$\begin{tikzcd}
\mathrm{Ext}(A,F) \arrow[r, "\lambda_{F/S}"] \arrow[d] & \mathrm{H}^0(S, \mathrm{R}^1\alpha_*\alpha^*F) \arrow[d, "\mathrm{id}\otimes\varphi"] \\
\mathrm{Ext}(A,F') \arrow[r, "\lambda_{F'/S}"] & \mathrm{H}^0(S, \mathrm{R}^1\alpha_*\alpha^*F')
\end{tikzcd}$$

is commutative, where the leftmost vertical arrow is given by push-out along φ. Moreover, for a vector extension G, the global section $\lambda_{F/S}(G)$ of $\mathrm{R}^1\alpha_*\alpha^*F$ defines a homomorphism of $\mathcal{O}_S$-modules $\lambda_{F/S}(G)\colon (\mathrm{R}^1\alpha_*\mathcal{O}_A)^\vee \to F$. The extension G is isomorphic (in a unique way!) to the push-out of (E(A)) along $\lambda_{F/S}(G)$. The moduli space $A^\natural$ of rank 1 connections on $\check{A}$ is a vector extension of the abelian scheme A with vector part $\omega_{\check{A}}$. Recall the isomorphism of $\mathcal{O}_S$-modules $\Phi_A\colon \mathrm{Lie}\,\check{A} \to \mathrm{R}^1\alpha_*\mathcal{O}_A$ introduced in Section 2.4. Proposition 2.11(2) implies the following.

Theorem 2.20. *The vector extension $A^\natural$ is the push-out of $\mathrm{E}(A)$ along*

$$\Phi_{\check{A}}^\vee \colon (\mathrm{R}^1\alpha_*\mathcal{O}_A)^\vee \xrightarrow{\sim} (\mathrm{Lie}\,\check{A})^\vee = \omega_{\check{A}}.$$

As a consequence, one obtains a down-to-earth description of how vector extensions are constructed. For a homomorphism of $\mathcal{O}_S$-modules $\varphi\colon \omega_{\check{A}} \to F$, let E_φ be the push-out of $A^\natural$ along φ.

Corollary 2.21. *The map* $\mathrm{Hom}(\omega_{\check{A}}, F) \to \mathrm{Ext}(A, F)$, $\varphi \mapsto [E_\varphi]$ *is the inverse of* $(\Phi_{\check{A}}^\vee \otimes \mathrm{id}_F) \circ \lambda_{F/S}$.

Observe that the affine bundle underlying E_φ is $\mathbb{P}(\mathcal{F}_\varphi) \smallsetminus \mathbb{P}(\alpha^*F)$, where $(\mathcal{F}_\varphi)$ is the push-out of $(\mathcal{U}_A)$ along φ.

3. Preliminaries on Tate–Raynaud uniformization

Let K be a complete non-trivially valued non-Archimedean field and R its ring of integers.

3.1. Raynaud's generic fiber of a formal abelian scheme

Let $\mathcal{S}$ be an admissible formal R-scheme, $\mathcal{B}$ a formal abelian scheme over the formal R-scheme $\mathcal{S}$, $\check{\mathcal{B}}$ its dual and $\mathcal{L}_{\mathcal{B}}$ the Poincaré bundle on $\mathcal{B} \times_{\mathcal{S}} \check{\mathcal{B}}$. As customary, Raynaud's generic fibers of formal schemes are referred to by straight letters (as opposed to curly ones for formal schemes). More explicitly, let S, B and $\check{B}$ be Raynaud's generic fibers of $\mathcal{S}$, $\mathcal{B}$ and $\check{\mathcal{B}}$. Let $\mathcal{L}_B$ be the line bundle on $B \times_S \check{B}$ deduced from $\mathcal{L}_{\mathcal{B}}$. The K-analytic space $\check{B}$ represents the functor associating with a S-analytic space the group of isomorphism classes of homogeneous line bundles on $B \times_S S'$. Moreover, the universal object is the line bundle $\mathcal{L}_B$ on $B \times_S \check{B}$ (see [BL91, Proposition 6.2]). Let $\beta\colon B \to S$ and $\check{\beta}\colon \check{B} \to S$ be the structural morphisms. For an S-analytic space $S' \to S$ and morphisms $b\colon S' \to B$, $\check{b}\colon S' \to \check{B}$ of S-analytic spaces, let

$$\mathcal{L}_{B,(b,\check{b})} := (b, \check{b})^*\mathcal{L}_B$$

be the line bundle on S' obtained by pulling back the Poincaré bundle $\mathcal{L}_B$ on $B \times_S \check{B}$ along the morphism $(b, \check{b})\colon S' \to B \times_S \check{B}$.

3.2. Datum of a toric extension

Let $\check{\Lambda}$ be a free abelian group of rank equal to that of Λ, $\check{\Lambda}_S$ the constant S-analytic group with value $\check{\Lambda}$ and T the split S-torus with group of characters $\check{\Lambda}$. From now on suppose that S is connected, so that sections of the morphism $\check{\Lambda}_S \to S$ are naturally in one-to-one correspondence with elements $\check{\Lambda}$ and will be henceforth identified with those. Let $\check{c}\colon \check{\Lambda}_S \to \check{B}$ be a morphism of S-analytic groups, and consider the extension $\varepsilon\colon E \to S$ of the proper K-analytic group B by the torus T determined by $\check{c}$:

$$0 \longrightarrow T \longrightarrow E \xrightarrow{p} B \longrightarrow 0.$$

The extension E is described as follows. For an S-analytic space S', an S'-valued point g of the S-analytic space E is the datum of

- an S'-valued point $b = p(g)$ of B and,
- for $\check{\chi} \in \check{\Lambda}$, a trivialization $\langle g, \check{\chi}\rangle_E$ of the line bundle $\mathscr{L}_{B,(b,\check{c}(\check{\chi}))}$ on S'.

Moreover, the trivializations are required to satisfy the following compatibility: for $\check{\chi}, \check{\chi}' \in \check{\Lambda}$,

$$\langle g, \check{\chi} + \check{\chi}'\rangle_E = \langle g, \check{\chi}\rangle_E \otimes \langle g, \check{\chi}'\rangle_E,$$

where the equality is meant to be understood via the isomorphism

$$\mathscr{L}_{B,(b,\check{c}(\check{\chi}))} \otimes \mathscr{L}_{B,(b,\check{c}(\check{\chi}'))} \cong \mathscr{L}_{B,(b,\check{c}(\check{\chi}+\check{\chi}'))} \tag{3.1}$$

induced by the implied rigidification of the homogeneous line bundle $\mathscr{L}_B$.

3.3. Dual datum

Let Λ be a free abelian group of finite rank, Λ_S the constant S-analytic group with value Λ, $\check{T}$ the split S-torus with group of characters Λ and $i\colon \Lambda_S \to E$ a morphism of S-analytic groups which is a closed immersion. See the abeloid variety B as the dual of $\check{B}$; then the group morphism $c = p \circ i\colon M_S \to B$ determines an extension $\check{E}$ of the proper S-analytic group $\check{B}$ by the torus $\check{T}$:

$$0 \longrightarrow \check{T} \longrightarrow \check{E} \xrightarrow{\check{p}} \check{B} \longrightarrow 0.$$

Just to fix notation, for an S-analytic space S', an S'-valued point $\check{g}$ of $\check{E}$ corresponds to the datum of

- an S'-valued point $\check{b} = \check{p}(\check{g})$ of $\check{B}$ and,
- for $\chi \in \Lambda$, a trivialization $\langle \chi, \check{g}\rangle_{\check{E}}$ of the line bundle $\mathscr{L}_{B,(c(\chi),\check{b})}$ on S'.

As before, the trivializations are subsumed to the relation, for $\chi, \chi' \in \Lambda$,

$$\langle \chi + \chi', \check{g}\rangle_{\check{E}} = \langle \chi, \check{g}\rangle_{\check{E}} \otimes \langle \chi', \check{g}\rangle_{\check{E}}.$$

Now, for $\check{\chi} \in \check{\Lambda}$, by the symmetry of the Poincaré bundle, there is a unique S-valued point $\check{\imath}(\check{\chi})$ of $\check{E}$ such that, for $\chi \in \Lambda$,

$$\langle \chi, \check{\imath}(\check{\chi})\rangle_{\check{E}} = \langle i(\chi), \check{\chi}\rangle_E.$$

This defines an injective morphism of S-analytic groups $\check{\imath}\colon \check{\Lambda}_S \to \check{E}$.

In an attempt of unburdening the (already overwhelming) notation, in what follows Λ_S is identified with the image of i, and the subscript E is dropped from the pairing $\langle -, -\rangle_E$, and similarly for $\check{\Lambda}_S$, $\check{\imath}$ and $\langle -, -\rangle_{\check{E}}$.

3.4. Quotient

Since the subgroup Λ_S is closed (hence fiberwise discrete) in E, the (topological) quotient $A := E/\Lambda_S$ exists. Moreover, assume that the structural morphism $\alpha\colon A \to S$ is proper. Under these working hypotheses, the bilinear pairing $\langle -, -\rangle$ on $\Lambda \times \check{\Lambda}$ is non-degenerate, the subgroup $\check{\Lambda}_S$ is closed in $\check{E}$, the quotient $\check{A} := \check{E}/\check{\Lambda}_S$ exists, and the structural morphism $\check{\alpha}\colon \check{A} \to S$ is proper (see [BL91, Proposition 3.4]). Let $u\colon E \to A$ and $\check{u}\colon \check{E} \to \check{A}$ be the quotient maps. The situation is summarized in the following diagrams:

$$\begin{tikzcd} & \Lambda_S \arrow[d, "i"] \arrow[dr, "c"] & \\ T \arrow[r] & E \arrow[r, "p"] \arrow[d, "u"] & B \\ & A, & \end{tikzcd} \qquad \begin{tikzcd} & \check{\Lambda}_S \arrow[d, "\check{\imath}"] \arrow[dr, "\check{c}"] & \\ \check{T} \arrow[r] & \check{E} \arrow[r, "\check{p}"] \arrow[d, "\check{u}"] & \check{B} \\ & \check{A}. & \end{tikzcd}$$

3.5. Coherent sheaves on the quotient

Descent of modules along the morphism u can be restated in terms of Λ_S-linearizations. More precisely, the datum of a coherent $\mathcal{O}_A$-module is equivalent to that of a coherent $\mathcal{O}_E$-module V endowed with a Λ_S-linearization

$$\lambda\colon \operatorname{pr}_E^* V \longrightarrow \sigma^* V,$$

where $\sigma, \operatorname{pr}_E\colon \Lambda_S \times_S E \to E$ are, respectively, the restriction to $\Lambda_S \times_S E$ of the group law of E and the projection onto the second factor. Quite concretely, the group Λ_S being constant, the datum of a Λ_S-linearization of a coherent $\mathcal{O}_E$-module V boils down to that, for $\chi \in \Lambda$, of an isomorphism of $\mathcal{O}_E$-modules

$$\lambda_\chi\colon V \longrightarrow \operatorname{tr}_\chi^* V,$$

where tr_χ is the translation by χ on E. Additionally, the collection of isomorphisms λ_χ is required to fulfil the following compatibility, for $\chi, \chi' \in \Lambda$:

$$\lambda_{\chi+\chi'} = \operatorname{tr}_\chi^* \lambda_{\chi'} \circ \lambda_\chi.$$

3.6. Homogeneous line bundles

Owing to [BL91, Theorem 6.7], given a homogeneous line bundle L on A, its pull-back u^*L on E is isomorphic to p^*M for some line bundle M on B. Moreover, the natural rigidification

$$\operatorname{pr}_1^* p^*M \otimes \operatorname{pr}_2^* p^*M \longrightarrow \mu_E^* p^*M,$$

where μ_E is the group law on E, is equivariant with respect to the Λ_S-linearization λ on p^*M induced by the isomorphism $u^*L \cong p^*M$. It follows that the Λ_S-linearization λ can be expressed as the datum of isomorphisms, for $\chi \in \Lambda$,

$$\lambda_\chi\colon p^*M \longrightarrow \operatorname{tr}_\chi^* p^*M, \quad s \longmapsto s \otimes r(\chi),$$

where the isomorphism $\operatorname{tr}_\chi^* p^*M \cong p^*M \otimes \varepsilon^* c(\chi)^*M$ coming from the homogeneity of M is taken into account (recall that ε is the structural morphism of E) and r is a trivialization of the line bundle c^*M on Λ_S. What is more, the trivialization r must satisfy, for $\chi, \chi' \in \Lambda$, the relation

$$r(\chi) \otimes r(\chi') = r(\chi + \chi'),$$

where, as is customary at this stage, the above equality is meant to be understood via the isomorphism $c(\chi)^*M \otimes c(\chi')^*M \cong c(\chi+\chi')^*M$ given by the rigidification of M.

3.7. Duality

By [BL91, Theorem 6.8] the S-analytic group $\check{A}$ represents the functor associating with an S-analytic space S' the set of isomorphism classes of homogeneous line bundles on $A \times_S S'$. Furthermore, let $\mathcal{L}_A$ be the Poincaré bundle on $A \times_S \check{A}$. According to *loc. cit.*, there is a (necessarily unique) isomorphism of line bundles

$$\xi\colon (u, \check{u})^* \mathcal{L}_A \xrightarrow{\sim} \mathcal{L}_E := (p, \check{p})^* \mathcal{L}_B$$

on $E \times_S \check{E}$ compatible with the implied rigidifications of the homomogeneous line bundles $\mathcal{L}_A$ and $\mathcal{L}_B$. The isomorphism ξ endows the line bundle $\mathcal{L}_E$ with a $(\Lambda \times \check{\Lambda})_S$-linearization λ, which can be described as follows. For characters $\chi \in \Lambda$ and $\check{\chi} \in \check{\Lambda}$, an S-analytic space S' and S'-valued points x of E and $\check{x}$ of $\check{E}$, the isomorphism of $\mathcal{O}_{S'}$-modules

$$\lambda_{(\chi,\chi'),(x,x')} \colon \mathcal{L}_{E,(x,\check{x})} \xrightarrow{\sim} \mathcal{L}_{E,(x+\chi,\check{x}+\check{\chi})}$$

induced by the linearization λ is

$$v \longmapsto (\langle x, \check{\chi}\rangle_E \otimes \langle \chi, \check{\chi}\rangle_E) \otimes (\langle \chi, \check{x}\rangle_{\check{E}} \otimes v). \tag{3.2}$$

In order to make sense of (3.2), observe that $\langle x, \check{\chi}\rangle_E$ is by definition a section of the line bundle $\mathcal{L}_{E,(x,\check{\chi})}$, while $\langle \chi, \check{\chi}\rangle_E$ is a section of the line bundle $\mathcal{L}_{E,(\chi,\check{\chi})}$ and $\langle x, \check{\chi}\rangle_E \otimes \langle \chi, \check{\chi}\rangle_E$ is seen as a section of the line bundle $\mathcal{L}_{E,(x+\chi,\check{\chi})}$ via the isomorphism of $\mathcal{O}_{S'}$-modules

$$\mathcal{L}_{E,(x,\check{\chi})} \otimes \mathcal{L}_{E,(\chi,\check{\chi})} \cong \mathcal{L}_{E,(x+\chi,\check{\chi})}$$

induced by the rigidification of the homogeneous line bundle $\mathcal{L}_B$. Arguing similarly, $\langle \chi, \check{x}\rangle_{\check{E}} \otimes v$ is a section of line bundle $\mathcal{L}_{E,(x+\chi,x')}$, so that the right-hand side of (3.2) is the section of the line bundle $\mathcal{L}_{E,(x+\chi,\check{x}+\check{\chi})}$ via the isomorphism of $\mathcal{O}_{S'}$-modules

$$\mathcal{L}_{E,(x+\chi,\check{\chi})} \otimes \mathcal{L}_{E,(x+\chi,\check{x})} \cong \mathcal{L}_{E,(x+\chi,\check{x}+\check{\chi})}.$$

4. The universal vector extension of an abeloid variety

Let K be a non-trivially valued complete non-Archimedean field and S a K-analytic space.

4.1. The canonical extension

4.1.1. Definition.— Let A be an abeloid variety over S, that is, a proper and smooth S-analytic group with (geometrically) connected fibers. Suppose that the functor associating with an S-analytic space S' the set of isomorphism classes of (fiberwise) homogeneous line bundles on $A_{S'}$ is represented by an abeloid variety $\check{A}$ over S. For instance, this is the case if S is a K-rational point (see [Lüt16, Corollary 7.6.5]) or if $\check{A}$ admits a uniformization as the one described in Section 3 (see [BL91, Theorem 6.8]). Under this assumption, translating the arguments of Section 2 into the rigid analytic framework permits one to define the canonical extension $(\mathcal{U}_A)$ on the abeloid variety A and the moduli space $A^\natural$ of rank 1 connections on A.

4.1.2. Canonical extension on the universal cover.— From now on, and up until Section 4.6, the abeloid variety A is supposed to admit a uniformization as the one described in Section 3. We borrow notation introduced therein and consider the canonical extension $(\mathcal{U}_B)$ on (Raynaud's generic fiber of) the formal abelian scheme B.

Definition 4.1. The pull-back of differential forms along the morphism $\check{p} \colon \check{E} \to \check{B}$ induces a homomorphism of $\mathcal{O}_S$-modules $\mathrm{d}\check{p} \colon \omega_{\check{B}} \to \omega_{\check{E}}$. The short exact sequence of $\mathcal{O}_E$-modules

$$0 \longrightarrow \varepsilon^*\omega_{\check{E}} \longrightarrow \mathcal{U}_E \longrightarrow \mathcal{O}_E \longrightarrow 0 \tag{$\mathcal{U}_E$}$$

obtained as the push-out of short exact sequence of $\mathcal{O}_E$-modules $p^*(\mathcal{U}_B)$ along $\mathrm{d}\check{p}$ is called the *canonical extension* of $\mathcal{O}_E$.

4.1.3. Alternative description.— To perform 'explicit' computations, it is often useful to have at hand a down-to-earth expression for $\mathcal{U}_E$, similar to that in Remark 2.7. Let $X = A$, B, E and, respectively, $\check{X} = \check{A}$, $\check{B}$, $\check{E}$. Let $\check{X}_1 = \check{A}_1$, $\check{B}_1$, $\check{E}_1$ be the first-order thickenings of $\check{X}$, and

$$\iota_X \colon X \times_S \check{X}_1 \longrightarrow X \times_S \check{X}, \quad \pi_X \colon X \times_S \check{X}_1 \longrightarrow X,$$

respectively, the closed immersion and the projection onto the first factor. With this notation, the considerations in Remark 2.7 furnish isomorphisms

$$\mathcal{U}_A \cong \pi_{A*}\iota_A^*\mathcal{L}_A, \quad \mathcal{U}_B \cong \pi_{B*}\iota_B^*\mathcal{L}_B,$$

where $\mathcal{L}_A$ and $\mathcal{L}_B$ are, respectively, the Poincaré bundles on $A \times_S \check{A}$ and $B \times_S \check{B}$. Recall the isomorphism

$$\xi \colon (u, \check{u})^*\mathcal{L}_A \xrightarrow{\sim} \mathcal{L}_E := (p, \check{p})^*\mathcal{L}_B$$

of line bundles on $E \times_S \check{E}$ considered in Section 3.7. By definition, the canonical extension $\mathcal{U}_E$ is the push-forward along the morphism π_E of the line bundle $\iota_E^*\mathcal{L}_E$ on $E \times_S \check{E}_1$:

$$\mathcal{U}_E = \pi_{E*}\iota_E^*\mathcal{L}_E.$$

Pushing forward the isomorphism $\iota_E^*\xi$ along the map π_E yields an isomorphism of $\mathcal{O}_E$-modules

$$\pi_{E*}\iota_E^*\xi \colon u^*\mathcal{U}_A \xrightarrow{\sim} \mathcal{U}_E.$$

4.2. Linearization of the canonical extension

By means of the isomorphism $u^*\mathcal{U}_A \cong \mathcal{U}_E$ described above, the canonical extension $\mathcal{U}_E$ acquires a natural Λ_S-linearization. The task undertaken here is to give an explicit expression for it. Unfortunately, this point is as crucial as dreadfully technical.

4.2.1. — Describing the Λ_S-linearization of the line bundle $\iota_E^*\mathcal{L}_E$ is easily achieved. Indeed, such a linearization is the pull-back along the morphism ι_E of the $(\Lambda \times \check{\Lambda})_S$-linearization of the line bundle $\mathcal{L}_E$. Let $\check{\jmath} \colon \check{E}_1 \to \check{E}$ denote the closed immersion and, for $\chi \in \Lambda$, consider the trivialization $\langle \chi, \check{\jmath} \rangle$ of the line bundle

$$\mathcal{L}_{E,(\chi,\check{\jmath})} = \left(\chi, \mathrm{id}_{\check{E}_1}\right)^* \iota_E^*\mathcal{L}_E$$

on $\check{E}_1$. Evaluating (3.2) at $\check{\chi} = 0$, $x = \mathrm{id}_E$ and $\check{x} = \check{\jmath}$ shows that the Λ_S-linearization of the line bundle $\iota_E^*\mathcal{L}_E$ is given, for $\chi \in \Lambda$, by the isomorphism

$$\iota_E^*\mathcal{L}_E \longrightarrow \left(\mathrm{tr}_\chi, \mathrm{id}_{\check{E}_1}\right)^* \iota_E^*\mathcal{L}_E, \quad v \longmapsto v \otimes \langle \chi, \check{\jmath} \rangle, \tag{4.1}$$

where tr_χ is the translation by χ on E. To make sense of the formula, notice that the homogeneity of the line bundle $\iota_E^*\mathcal{L}_E$ furnishes an isomorphism

$$\left(\mathrm{tr}_\chi, \mathrm{id}\right)^* \iota_E^*\mathcal{L}_E \cong \iota_E^*\mathcal{L}_E \otimes \varepsilon_1^*\mathcal{L}_{E,(\chi,\check{\jmath})}, \tag{4.2}$$

where $\varepsilon_1 \colon E \times \check{E}_1 \to \check{E}_1$ is the projection onto the second factor.

4.2.2. — The Λ_S-linearization of the unipotent bundle $\mathcal{U}_E$ is somewhat trickier to come by. Rather formally, for $\chi \in \Lambda$, the isomorphism $\mathcal{U}_E \to \mathrm{tr}_\chi^* \mathcal{U}_E$ is just the push-forward of the isomorphism (4.1) along the Λ_S-equivariant morphism π_E. The key observation (see Proposition A.2) is that the isomorphism (4.2) becomes, after pushing forward along π_E, an isomorphism of $\mathcal{O}_E$-modules

$$\mathrm{tr}_\chi^* \mathcal{U}_E \cong \mathcal{U}_E +_{\mathrm{B}} \varepsilon^*\chi^*\mathcal{U}_E.$$

(Notice that both the vector bundles $\mathcal{U}_E$ and $\varepsilon^*\chi^*\mathcal{U}_E$ are extensions of $\mathcal{O}_E$ by $\varepsilon^*\omega_{\check{E}}$; thus their Baer sum is well defined.)

Accordingly, the tensor product in (4.1) is now replaced by a 'sum'. To be more precise about what this possibly means, recall how the Baer sum of the extensions $(\mathcal{U}_E)$ and $(\varepsilon^*\chi^*\mathcal{U}_E)$ is constructed. First, consider the $\mathcal{O}_E$-submodule

$$V \subseteq \mathcal{U}_E \oplus \varepsilon^*\chi^*\mathcal{U}_E$$

made of pairs (v, w) whose components have same projection in $\mathcal{O}_E$. The $\mathcal{O}_E$-module V is an extension of $\mathcal{O}_E$ by $\varepsilon^*(\omega_{\check{E}} \oplus \omega_{\check{E}})$, and the Baer sum in question is its push-out along the sum map $\omega_{\check{E}} \oplus \omega_{\check{E}} \to \omega_{\check{E}}$. For a pair (v,w), the aforementioned 'sum' $v+w$ is the image of (v,w) in $\mathcal{U}_E +_{\mathrm{B}} \varepsilon^*\chi^*\mathcal{U}_E$. Summing up, we proved the following.

Proposition 4.2. *With the notation above, the Λ_S-linearization of the unipotent bundle $\mathcal{U}_E$ is given, for $\chi \in \Lambda$, by the isomorphism*

$$\mathcal{U}_E \longrightarrow \mathrm{tr}_\chi^* \mathcal{U}_E, \quad v \longmapsto v + q(v).\langle \chi, \check{\jmath} \rangle, \tag{4.3}$$

where $q\colon \mathcal{U}_E \to \mathcal{O}_E$ is the projection in the datum of the extension $(\mathcal{U}_E)$.

The meaning of the preceding formula is unveiled by the following.

Remark 4.3. The equality $\mathcal{U}_E = \pi_{E*}\iota_E^*\mathcal{L}_E$ implies

$$\chi^*\mathcal{U}_E = \check{\varepsilon}_{1*}\mathcal{L}_{E,(\chi,\check{\jmath})},$$

where $\check{\varepsilon}_1\colon \check{E}_1 \to S$ is the structural morphism. By means of the preceding, the trivialization $\langle \chi, \check{\jmath}\rangle$ of the line bundle $\mathcal{L}_{E,(\chi,\check{\jmath})}$ can be seen as a section of the vector bundle $\chi^*\mathcal{U}_E$ on S. Moreover, the sum in expression (4.3) does make sense for the following reason: the projection in $\mathcal{O}_S = \mathcal{L}_{E,(\chi,\check{e})}$ of the section $\langle \chi, \check{\jmath}\rangle$ of the extension $\chi^*\mathcal{U}_E$ is $\langle \chi, \check{e}\rangle = 1$ (where $\check{e}$ is the neutral section of $\check{E}$); thus the sections v and $q(v).\langle \chi, \check{\jmath}\rangle$ have equal projection in $\mathcal{O}_S$.

4.3. Universal cover of the universal vector extension

Definition 4.4. The *universal cover* of the universal vector extension $A^\natural$ is

$$E^\natural := \mathbb{P}(\mathcal{U}_E) \smallsetminus \mathbb{P}(\varepsilon^*\omega_{\check{E}}).$$

Remark 4.5. When S is a point, the name 'universal cover' is well deserved as the topological space $E^\natural$ is contractible (see Proposition 4.15) and comes with a covering map $E^\natural \to A^\natural$ (see Theorem 4.6).

Let $\check{E}_1$ be the first-order thickening of $\check{E}$ along the neutral section and $\check{\jmath}\colon \check{E}_1 \to \check{E}$ the closed immersion. As explained in Remark 4.3, for $\chi \in \Lambda$, the trivialization $\langle \chi, \check{\jmath}\rangle$ of the line bundle $\mathcal{L}_{E,(\chi,\check{\jmath})}$ on $\check{E}_1$ defines a section of the vector bundle $\chi^*\mathcal{U}_E$ on S. Moreover, the projection of $\langle \chi, \check{\jmath}\rangle$ in $\mathcal{O}_S$ is 1; thus the section $\langle \chi, \check{\jmath}\rangle$ defines a splitting of the short exact sequence $\chi^*(\mathcal{U}_E)$. Therefore, $\langle \chi, \check{\jmath}\rangle$ defines an S-valued point $\chi^\natural$ of the universal cover $E^\natural$. Let

$$i^\natural\colon \Lambda_S \longrightarrow E^\natural, \quad \chi \longmapsto \chi^\natural := \langle \chi, \check{\jmath}\rangle$$

be the so-defined morphism of S-analytic spaces. For $\chi \in \Lambda$, the projection of $\chi^\natural$ in E is by definition χ; thus the morphism $i^\natural$ is a closed immersion. The isomorphism of $\mathcal{O}_E$-modules $u^*\mathcal{U}_A \cong \mathcal{U}_E$ introduced in Section 4.1.3 induces an isomorphism of E-analytic spaces $A^\natural \times_A E \cong E^\natural$. It follows that $E^\natural$ is naturally endowed with a structure of S-analytic space in groups and the natural morphism of S-analytic spaces

$$u^\natural\colon E^\natural \longrightarrow A^\natural$$

is a group morphism.

Theorem 4.6. *The map $i^\natural$ is a group morphism with image the kernel of $u^\natural$; that is, the following sequence of S-analytic groups is short exact:*

$$0 \longrightarrow \Lambda_S \xrightarrow{i^\natural} E^\natural \xrightarrow{u^\natural} A^\natural \longrightarrow 0.$$

Proof. The group law on the affine bundle $A^\natural = \mathbb{P}(\mathcal{U}_A) \smallsetminus \mathbb{P}(\alpha^*\omega_{\check{A}})$ is given by an isomorphism of $\mathcal{O}_A$-modules

$$\psi_A\colon \mathrm{pr}_1^*\mathcal{U}_A +_{\mathrm{B}} \mathrm{pr}_2^*\mathcal{U}_A \xrightarrow{\sim} \mu_A^*\mathcal{U}_A,$$

where $\mu_A, \mathrm{pr}_1, \mathrm{pr}_2\colon A \times_S A \to A$ are, respectively, the group law, the first and the second projection on A. According to Remark 2.10, or better its rigid analytic analogue, such an isomorphism ψ_A is the push-forward along the projection

$$(A \times_S \check{A}_1) \times_{\check{A}_1} (A \times_S \check{A}_1) = A \times_S A \times \check{A}_1 \longrightarrow A \times_S A$$

of the isomorphism $(\mathrm{id}_A \times \check{\jmath}, \mathrm{id}_A \times \check{\jmath})^* \varphi_A$, where

$$\varphi_A \colon \mathrm{pr}_1^* \mathscr{L}_A \otimes \mathrm{pr}_2^* \mathscr{L}_A \xrightarrow{\sim} \mu_A^* \mathscr{L}_A$$

is the rigidification of the Poincaré bundle $\mathscr{L}_A$ and

$$(\mathrm{id}_A \times \check{\jmath}, \mathrm{id}_A \times \check{\jmath}) \colon \left(A \times_S \check{A}_1\right) \times_{\check{A}_1} \left(A \times_S \check{A}_1\right) \longrightarrow \left(A \times_S \check{A}\right) \times_{\check{A}} \left(A \times_S \check{A}\right)$$

the closed immersion. By design, the group law of $E^\natural$ is constructed from that of $A^\natural$. That is, the group law on the affine bundle $E^\natural = \mathbb{P}(\mathscr{U}_E) \smallsetminus \mathbb{P}(\varepsilon^* \omega_{\check{E}})$ is defined by the isomorphism of $\mathscr{O}_E$-modules

$$\psi_E := (u \times u)^* \psi_A \colon \mathrm{pr}_1^* \mathscr{U}_E +_{\mathrm{B}} \mathrm{pr}_2^* \mathscr{U}_E \xrightarrow{\sim} \mu_E^* \mathscr{U}_E,$$

the isomorphism $u^* \mathscr{U}_A \cong \mathscr{U}_E$ (and with the obvious notation) being allowed for. Thanks to (4.3) and with the notation therein, the Λ_S-linearization of the unipotent bundle $\mathscr{U}_E$ is the datum of the isomorphisms, for $\chi \in \Lambda$,

$$\lambda_\chi \colon v \longmapsto v + q(v)\chi^\natural,$$

where $q \colon \mathscr{U}_E \to \mathscr{O}_E$ is the projection in the datum of the extension $(\mathscr{U}_E)$.

Given $\chi, \chi' \in \Lambda$, via the isomorphism ψ_E, the wanted additivity $\chi^\natural + \chi'^\natural = (\chi + \chi')^\natural$ is seen to be nothing but the compatibility $\lambda_{\chi+\chi'} = \mathrm{tr}_\chi^* \lambda_{\chi'} \circ \lambda_\chi$ to which is subsumed the Λ_S-linearization of $\mathscr{U}_E$.

This substantially also proves the remaining assertions. Indeed, the affine bundle $A^\natural$ is the quotient of $E^\natural$ by the natural action of Λ_S on it induced by the Λ_S-linearization of $\mathscr{U}_E$. Moreover, from this point of view, the map $u^\natural$ is just the quotient morphism, and the above considerations show that the action of Λ_S on $E^\natural$ is described, for $\chi \in \Lambda$, as the translation by $\chi^\natural$. This amounts to saying that the sequence of S-analytic groups

$$0 \longrightarrow \Lambda_S \xrightarrow{i^\natural} E^\natural \xrightarrow{u^\natural} A^\natural \longrightarrow 0$$

is short exact, thus concluding the proof. □

4.4. A computation on toric bundles

The next task is acquiring a better understanding of the map $i^\natural$. This is done via a computation on toric bundles whose proof is clearer when stated in a broader generality. So let us momentarily reset our notation, and let X be a separated S-analytic space, Λ a free abelian group of finite rank n, $\Lambda \to \mathrm{Pic}(X)$, $\chi \mapsto L_\chi$ a group homomorphism and P the X-analytic space whose points s with values on a X-analytic space $f \colon X' \to X$ form the set of data, for $\chi \in \Lambda$, of a trivialization $\langle \chi, s \rangle$ of the line bundle $f^* L_\chi$. Moreover, the trivializations above satisfy, for $\chi, \chi' \in \Lambda$, the relation[4]

$$\langle \chi, s \rangle \otimes \langle \chi, s \rangle = \langle \chi + \chi', s \rangle. \tag{4.4}$$

Let $p \colon P \to X$ be the projection. The split torus T over S with group of characters Λ acts naturally on P by the rule defined, for an S-analytic space S', S'-valued points s of P and t of T and a character $\chi \in \Lambda$, by

$$\langle \chi, ts \rangle = \chi(t) \langle \chi, s \rangle.$$

Let $u \colon S \to P$ be a morphism of S-analytic spaces, P_1 the first-order thickening of P along the section u and $j \colon P_1 \to P$ the closed immersion. To make the notation more flexible, given a finite morphism of K-analytic spaces $Z \to S$ and a coherent $\mathscr{O}_Z$-module F, let us denote again by F its push-forward onto

[4]Several abuses of notation have been perpetrated here. Rather than isomorphism classes of line bundles, one should fix, for $\chi \in \Lambda$, a line bundle L_χ and, for $\chi, \chi' \in \Lambda$, isomorphisms $L_\chi \otimes L_{\chi'} \cong L_{\chi+\chi'}$ through which the formula $\langle \chi, s \rangle \otimes \langle \chi, s \rangle = \langle \chi + \chi', s \rangle$ ought to be understood.

S. Consider the S-valued point $x = p(u)$ of X, the first-order thickening X_1 of X along the section x, the closed immersion $i\colon X_1 \to X$ and the following commutative and exact diagram of $\mathcal{O}_S$-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & x^*\Omega^1_{X/S} \otimes x^*L_\chi \arrow[r] \arrow[d, "\mathrm{d}p\otimes\mathrm{id}"] & i^*L_\chi \arrow[r] \arrow[d] & x^*L_\chi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & u^*\Omega^1_{P/S} \otimes x^*L_\chi \arrow[r] \arrow[d] & j^*p^*L_\chi \arrow[r] \arrow[d] & x^*L_\chi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & u^*\Omega^1_{P/X} \otimes x^*L_\chi \arrow[r, "\sim"] \arrow[d] & j^*p^*L_\chi/i^*L_\chi \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd} \tag{4.5}$$

The map $i^*L_\chi \to j^*p^*L_\chi$ is given by adjunction with respect to the morphism $p_1\colon P_1 \to X_1$ induced by p. The isomorphism $T \times_S P \to P \times_X P$ given by the action of T yields an isomorphism of $\mathcal{O}_S$-modules $u^*\Omega^1_{P/X} \cong e^*\Omega^1_{T/S} =: \omega_T$, where e is the neutral section of T. Consider the homomorphism

$$q\colon j^*p^*L_\chi \longrightarrow \omega_T \otimes x^*L_\chi$$

defined via the isomorphisms $u^*\Omega^1_{P/X} \otimes x^*L_\chi \cong j^*p^*L_\chi/i^*L_\chi$ and $u^*\Omega^1_{P/X} \cong \omega_T$.

Proposition 4.7. *With the notation above,*

$$q(\langle \chi, j\rangle) = \chi^* \tfrac{\mathrm{d}z}{z} \otimes u,$$

where z is the coordinate function on $\mathbb{G}_{m,S}$ and χ is seen as a character $T \to \mathbb{G}_{m,S}$.

Proof. The argument is just a tedious *dévissage* until reaching the case $\Lambda = \mathbb{Z}$, $\chi = 1$, $X = S$, $L_1 = \mathcal{O}_S$, $u = 1$, for which the result is substantially trivial.

First step. To begin with, reduce to the case $\Lambda = \mathbb{Z}$ and $\chi = 1$. Consider the X-analytic space P_χ parametrizing trivializations of the line bundle L_χ and the morphism $\mathrm{pr}_\chi\colon P \to P_\chi$ sending a point s of P with values in an X-analytic space X' to the trivialization $\langle \chi, s\rangle$. Then, by design, the trivialization $\langle \chi, j\rangle$ can be seen as the composite morphism $\mathrm{pr}_\chi \circ j\colon P_1 \to P_\chi$. The latter factors through the first-order thickening $P_{\chi,1}$ of P_χ along the section $\langle \chi, u\rangle = \mathrm{pr}_\chi(u)$, giving rise to the commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "j"] \arrow[d, "\mathrm{pr}_{\chi,1}"'] & P \arrow[d, "\mathrm{pr}_\chi"] \\
P_{\chi,1} \arrow[r, "j_\chi"] & P_\chi,
\end{tikzcd}$$

where j_χ is the closed immersion and $\mathrm{pr}_{\chi,1}$ the factorization of pr_χ. Upon letting $p_\chi\colon P_\chi \to X$ be the projection, the homomorphism $q_\chi\colon j_\chi^* p_\chi^* L_\chi \to \omega_{\mathbb{G}_m} \otimes x^*L_\chi$, defined analogously to q, fits in the commutative diagram of $\mathcal{O}_S$-modules

$$\begin{tikzcd}
j_\chi^* p_\chi^* L_\chi \arrow[r, "q_\chi"] \arrow[d] & \mathrm{pr}_\chi(u)^*\omega_{\mathbb{G}_m} \otimes x^*L_\chi \arrow[d, "\mathrm{d}\chi\otimes\mathrm{id}"] \\
j^*p^*L_\chi \arrow[r, "q"] & \omega_T \otimes x^*L_\chi,
\end{tikzcd}$$

where the leftmost vertical arrow is given by adjunction with respect to the map $\mathrm{pr}_{\chi,1}$ and the rightmost one by pull-back of differential forms along the character χ. In particular, it suffices to show that the trivialization of the line bundle L_χ given by the $P_{\chi,1}$-valued point j_χ of P_χ is mapped to $\frac{\mathrm{d}z}{z} \otimes u$ by q_χ.

Second step. Suppose $\Lambda = \mathbb{Z}$ and $\chi = 1$. To simplify notation, simply write L instead of L_χ. The aim of this second step is to reduce to the case $X = S$, $L = \mathcal{O}_S$ and $u = 1$. In order to do this, consider the fiber P_x of P at x, which can also be seen as the principal $\mathbb{G}_m$-bundle over S associated with the line bundle x^*L on S. In this case, the identity map of S plays the role of the section x, and the line bundle j^*p^*L on P_1 is replaced by the line bundle π^*x^*L on $P_{x,1}$, where $P_{x,1}$ is the first-order thickening of P_x at u and $\pi\colon P_{x,1} \to S$ is the structural morphism. Diagram (4.5) for the principal bundle P_x reads more simply as the following:

$$\begin{tikzcd}
 & 0 \arrow[d, equal] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, equal] & 0 \arrow[r] \arrow[d] & x^*L \arrow[r, equal] \arrow[d] & x^*L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & x^*L \otimes u^*\Omega^1_{P_x/S} \arrow[r] \arrow[d, equal] & \pi^*x^*L \arrow[r] \arrow[d] & x^*L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x^*L \otimes u^*\Omega^1_{P_x/S} \arrow[r, "\sim"] \arrow[d] & \pi^*x^*L/x^*L \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0, & &
\end{tikzcd}$$

where $x^*L \to \pi_*x^*L$ is natural map. The restriction map from P_1 to $P_{x,1}$ furnishes a homomorphism of commutative diagrams of $\mathcal{O}_S$-modules from (4.5) to the above, matching the entries in the obvious manner. In particular, the projection $q_x\colon \pi^*x^*L \to x^*L \otimes \omega_{\mathbb{G}_m}$ defined similarly to q sits in the following diagram:

$$\begin{tikzcd}
j^*p^*L \arrow[r, "q"] \arrow[d] & x^*L \otimes \omega_{\mathbb{G}_m} \arrow[d, equal] \\
\pi^*x^*L \arrow[r, "q_x"] & x^*L \otimes \omega_{\mathbb{G}_m}.
\end{tikzcd}$$

Strictly speaking, the above argument permits one to reduce to the case $X = S$ and to a line bundle L on S. However, by means of the isomorphism $\mathcal{O}_S \cong L$ induced by the given trivialization u, one is finally led back to the case of the trivial line bundle on S and $u = 1$.

Third step. Suppose $X = S$, $L = \mathcal{O}_S$ and $u = 1$. Let z be the coordinate function on $P = \mathbb{G}_m$. The $\mathcal{O}_S$-algebra $\mathcal{O}_{P_1}$ is then identified with $\mathcal{O}_S[z]/(z-1)^2$ and its ideal generated by $z-1$ with the $\mathcal{O}_S$-module $\omega_{\mathbb{G}_m}$, so that

$$\mathcal{O}_S[z]/(z-1)^2 = \mathcal{O}_S \oplus \omega_{\mathbb{G}_m}.$$

In these terms, the map q is just the projection onto $\omega_{\mathbb{G}_m}$. Moreover, and tautologically enough, the closed immersion $j\colon P_1 \to P$ corresponds to the invertible function z on P_1. Writing $z = 1 + (z-1)$ shows that the image of z in $\omega_{\mathbb{G}_m}$ coincides with that of $z-1$, which in turn corresponds to the invariant differential $\frac{\mathrm{d}z}{z}$. □

4.5. Relation with the universal vector hull of the fundamental group

In this section the map $i^\natural$ is related to the universal vector hull of the group Λ. To define the latter, recall that the group Λ is by definition the lattice of characters of the torus $\check{T}$. Given $\chi \in \Lambda$, let $\chi\colon \check{T} \to \mathbb{G}^{\mathrm{an}}_{m,S}$ again denote the corresponding character.

Definition 4.8. The *universal vector hull* of Λ_S is the S-analytic group morphism

$$\theta_\Lambda\colon \check{T} \longrightarrow \mathbb{V}(\omega_{\check{T}}), \quad \chi \longmapsto \chi^*\tfrac{\mathrm{d}z}{z},$$

where z is the coordinate function on $\mathbb{G}_m$ and $\frac{\mathrm{d}z}{z}$ the corresponding invariant differential form.

Lemma 4.9. *For a basis $\chi_1, \dots, \chi_n$ of Λ, where $n = \operatorname{rk} \Lambda$, the sections $\theta_\Lambda(\chi_i)$ form a basis of the vector bundle $\omega_{\check{T}}$ on S.*

The map θ_Λ deserves the name 'universal vector hull' because of the following property: given a vector bundle E over S and a morphism of S-analytic groups $f : \Lambda_S \to \mathbb{V}(E)$, there exists a unique homomorphism of $\mathcal{O}_S$-modules $\varphi \colon \omega_{\check{T}} \to E$ such that $f = \varphi \circ \theta_\Lambda$; see [MM74, Example 1.3 a) and Proposition 1.4]. Now, the canonical extension $(\mathcal{U}_E)$ is by definition the push-out along the injective map $\mathrm{d}\check{p} \colon \omega_{\check{B}} \to \omega_{\check{E}}$ of the extension $p^*(\mathcal{U}_B)$. The quotient $\mathcal{U}_E / p^* \mathcal{U}_B$ therefore sits in the following commutative and exact diagram of $\mathcal{O}_E$-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \varepsilon^* \omega_{\check{B}} \arrow[r] \arrow[d] & p^* \mathcal{U}_B \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \varepsilon^* \omega_{\check{E}} \arrow[r] \arrow[d] & \mathcal{U}_E \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon^* \omega_{\check{T}} \arrow[r, "\sim"] \arrow[d] & \mathcal{U}_E / p^* \mathcal{U}_B \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd} \tag{4.6}$$

The isomorphism $\varepsilon^* \omega_{\check{T}} \cong \mathcal{U}_E / p^* \mathcal{U}_B$ allows one to define a projection $q \colon \mathcal{U}_E \to \varepsilon^* \omega_{\check{T}}$. In particular, for $\chi \in \Lambda$, this defines a homomorphism of $\mathcal{O}_S$-modules $q \colon \chi^* \mathcal{U}_E \to \omega_{\check{T}}$. Recall that $\chi^\natural = \langle \chi, \check{\jmath} \rangle$ can be seen as a section of $\chi^* \mathcal{U}_E$ (see Remark 4.3).

Proposition 4.10. *For $\chi \in \Lambda$ we have $q(\langle \chi, \check{\jmath} \rangle) = \theta_\Lambda(\chi)$.*

Proof. Up to transliteration, the statement is a special case of Proposition 4.7. Needless to say,

- the S-analytic space therein X plays the role of $\check{B}$,
- the principal bundle P that of $\check{E}$,
- the line bundle L_χ that of $\mathcal{L}_{B,(c(\chi),\mathrm{id})}$ and
- the section u that of $\langle \chi, \check{e} \rangle$, where $\check{e}$ is the neutral section of $\check{E}$.

For $\chi \in \Lambda$ the section $\langle \chi, j \rangle$ corresponds via this dictionary to the trivialization $\langle \chi, \check{\jmath} \rangle$. Nonetheless, the reader might still be lost in translation while trying to see why diagram (4.5) reads as (4.6). To remedy that, first, notice that in the current framework, the line bundle $\check{e}^* \mathcal{L}_{B,(c(\chi),\mathrm{id})}$ is always understood to be trivialized via $\langle \chi, \check{e} \rangle$. This should elucidate the omnipresence of the line bundle $x^* L_\chi$ as opposed to the absence of the corresponding line bundle $\check{e}^* \mathcal{L}_{B,(c(\chi),\mathrm{id})}$. Second, the statement of Proposition 4.10 revolves around the vector bundles $\mathcal{U}_E$ and $p^* \mathcal{U}_B$ on E (rather, their fibers at χ), whereas in Proposition 4.7 the line bundles $j^* p^* L_\chi$ and $i^* L_\chi$ are considered (better, their push-forward onto S). However, the line bundle $j^* p^* L_\chi$ translates to $\mathcal{L}_{E,(\chi,\check{\jmath})}$ and, as already observed in Remark 4.3, the push-forward of $\mathcal{L}_{E,(\chi,\check{\jmath})}$ onto S coincides with $\chi^* \mathcal{U}_E$. Along a similar line, the vector bundle $i^* L_\chi$ on S plays the role of $\chi^* p^* \mathcal{U}_B$. Third, in diagram (4.6), there is nothing whatsoever like $u^* \Omega^1_{P/X}$. This is because the projection q in the statement of Proposition 4.7 is constructed by further taking into account the isomorphism $\omega_T \cong u^* \Omega^1_{P/X}$ already implied here. $\square$

Diagram (4.6) permits one to define a morphism of S-analytic groups

$$\mathrm{pr}_u \colon E^\natural \longrightarrow \mathbb{V}(\omega_{\check{T}}).$$

Unwinding the definitions, Proposition 4.10 is rephrased as follows.

Theorem 4.11. *For $\chi \in \Lambda$ we have $\mathrm{pr}_u(\chi^\natural) = \theta_\Lambda(\chi)$.*

Example 4.12. Theorem 4.11 is more eloquent when A is an abeloid variety over K with totally degenerate reduction, that is, $A = T/\Lambda$. If $\check{T}$ is the torus with group of characters Λ and $\theta_\Lambda \colon \Lambda \to \mathbb{V}(\omega_{\check{T}})$ the universal vector hull of Λ, then

$$A^\natural = (T \times \mathbb{V}(\omega_{\check{T}}))/\{(\chi, \theta_\Lambda(\chi)) : \chi \in \Lambda\}.$$

4.6. Universal cover of affine bundles

4.6.1. Extensions.— Let F be a vector bundle over S, $\varphi_A \colon \omega_{\check{A}} \to F$ a homomorphism of $\mathcal{O}_S$-modules and $(\mathcal{F}_A)$ the short exact sequence of $\mathcal{O}_S$-modules obtained by push-out of the canonical extension $(\mathcal{U}_A)$ along φ_A. Via the isomorphism $\mathcal{U}_E \cong u^*\mathcal{U}_A$ obtained in Section 4.1.3, the short exact sequence of $\mathcal{O}_E$-modules $(\mathcal{F}_E) := u^*(\mathcal{F}_A)$ is seen to be the push-out of $(\mathcal{U}_E)$ along the homomorphism of $\mathcal{O}_S$-modules

$$\varphi_E \colon \omega_{\check{E}} \xrightarrow{(\mathrm{d}\check{u})^{-1}} \omega_{\check{A}} \xrightarrow{\varphi_A} F,$$

where $\mathrm{d}\check{u} \colon \omega_{\check{A}} \to \omega_{\check{E}}$ is the isomorphism given by pull-back of differential forms along the étale morphism $\check{u}$. On the other hand, by its very definition, the short exact sequence $(\mathcal{U}_E)$ is itself the push-out of $p^*(\mathcal{U}_B)$ along the K-linear homomorphism $\mathrm{d}\check{p} \colon \omega_{\check{B}} \to \omega_{\check{E}}$. Thus, the preceding considerations furnish an isomorphism $(\mathcal{F}_E) \cong p^*(\mathcal{F}_B)$ of short exact sequences of $\mathcal{O}_E$-modules, where $(\mathcal{F}_B)$ is the push-out of the canonical extension $(\mathcal{U}_B)$ along the homomorphism of $\mathcal{O}_S$-modules

$$\varphi_B \colon \omega_{\check{B}} \xrightarrow{\mathrm{d}\check{p}} \omega_{\check{E}} \xrightarrow{\varphi_E} F.$$

4.6.2. Affine bundles.— Consider the affine bundle $\pi_A \colon \mathbb{V}(\mathcal{F}_A) \to A$ and the morphism of S-analytic spaces

$$\Phi_A \colon A^\natural \longrightarrow \mathbb{A}(\mathcal{F}_A)$$

induced by the homomorphism $(\mathcal{U}_A) \to (\mathcal{F}_A)$ of short exact sequences of $\mathcal{O}_A$-modules given by the definition of $(\mathcal{F}_A)$ as a push-out. The affine bundle $\mathbb{A}(\mathcal{F}_A)$ carries a unique S-analytic group structure for which the morphism Φ_A is a group morphism. The pull-back $\mathbb{V}(\mathcal{F}_A) \times_A E$ to E is by definition the affine bundle $\pi_E \colon \mathbb{A}(\mathcal{F}_E) \to E$ associated with the extension $(\mathcal{F}_E)$. By transport of structure, the E-analytic space $\mathbb{A}(\mathcal{F}_E)$ is an S-analytic group such that the natural morphism

$$\Phi_E \colon E^\natural \longrightarrow \mathbb{A}(\mathcal{F}_E)$$

deduced from Φ_A is a group morphism. The natural action of Λ_S on $\mathbb{V}(\mathcal{F}_E)$ given by the natural Λ_S-linearization of $\mathcal{F}_E$ is described, for $\chi \in \Lambda$, as the translation by the point $\Phi_E(\chi^\natural)$, where $\chi^\natural$ is the S-point of the universal vector extension $E^\natural$ considered in Section 4.3.

Now, the isomorphism of short exact sequences of $\mathcal{O}_E$-modules $(\mathcal{F}_E) \cong p^*(\mathcal{F}_B)$ permits one to identify the affine bundle $\mathbb{A}(\mathcal{F}_E)$ with the fibered product $\mathbb{A}(\mathcal{F}_B) \times_B E$, where $\pi_B \colon \mathbb{A}(\mathcal{F}_B) \to B$ is the affine bundle associated with $(\mathcal{F}_B)$. This identification respects the natural S-analytic group structures involved and will be implied in what follows. Let $q \colon \mathbb{A}(\mathcal{F}_E) \to \mathbb{A}(\mathcal{F}_B)$ be the projection, so that the following square of K-analytic space is Cartesian:

$$\begin{tikzcd} \mathbb{A}(\mathcal{F}_E) \arrow[r, "q"] \arrow[d, "\pi_E"] & \mathbb{A}(\mathcal{F}_B) \arrow[d, "\pi_B"] \\ E \arrow[r, "p"] & B. \end{tikzcd}$$

Let C_B be the cokernel of φ_B. Arguing as for the map $\mathrm{pr}_u \colon E^\natural \to \mathbb{V}(\omega_{\check{T}})$ permits one to define a morphism of S-analytic groups $\mathrm{pr}_{u,B} \colon \mathbb{A}(\mathcal{F}_B) \to \mathbb{V}(C_B)$. Let $\varphi_T \colon \omega_{\check{T}} \to C_B$ the unique map fitting in the following commutative and exact diagram of homomorphism of $\mathcal{O}_S$-modules:

$$
\begin{tikzcd}
0 \arrow[r] & \omega_{\check{B}} \arrow[r, "\mathrm{d}\check{p}"] \arrow[d, "\varphi_B"] & \omega_{\check{E}} \arrow[r] \arrow[d, "\varphi_E"] & \omega_{\check{T}} \arrow[r] \arrow[d, "\varphi_T"] & 0 \\
0 \arrow[r] & \operatorname{Im}\varphi_B \arrow[r] & F \arrow[r] & C_B \arrow[r] & 0.
\end{tikzcd}
\tag{4.7}
$$

The construction of $(\mathscr{F}_E)$ as a push-out of $(\mathscr{U}_E)$ along φ_E implies that the diagram of S-analytic spaces

$$
\begin{tikzcd}
E^\natural \arrow[r] \arrow[d, "\Phi_E"] & \mathbb{V}(\omega_{\check{T}}) \arrow[d, "\mathrm{pr}_{u,E}"] \\
\mathbb{A}(\mathscr{F}_E) \arrow[r, "\varphi_T"] & \mathbb{V}(C_B)
\end{tikzcd}
$$

is commutative, where the upper horizontal arrow is the projection considered in Theorem 4.11 and $\mathrm{pr}_{u,E} := \mathrm{pr}_{u,B} \circ q \colon \mathbb{A}(\mathscr{F}_E) \to \mathbb{V}(C_B)$. These considerations together with Theorem 4.11 prove the following.

Corollary 4.13. *With the notation above, for $\chi \in \Lambda$ we have*

$$\mathrm{pr}_{u,E}(\Phi_E(\chi^\natural)) = \varphi_T(\theta_\Lambda(\chi)).$$

4.6.3. Contractibility of the universal cover.— In this final section the contractibility of the space $\mathbb{A}(\mathscr{F}_E)$ above is addressed when S is a K-rational point.

Lemma 4.14. *Let $\mathfrak{X}$ be a smooth connected admissible formal R-scheme with Raynaud's generic fiber $X := \mathfrak{X}_\eta$. Then, for any closed analytic subspace of $Z \subsetneq X$, the open subset $X \smallsetminus Z$ is contractible.*

Proof. Let $h \colon X \times [0,1] \to X$ be the deformation retraction onto the skeleton $\mathrm{Sk}(\mathfrak{X})$ of $\mathfrak{X}$ given by [Ber99, Theorem 5.2]. Since the formal scheme $\mathfrak{X}$ is smooth connected, the skeleton $\mathrm{Sk}(\mathfrak{X})$ is a singleton, namely the unique preimage of the generic point of $\mathfrak{X}$ under the reduction map $X \to \mathfrak{X}$. According to item (v) of *loc. cit.*, for $0 < t \leqslant 1$ and $x \in X$, the local ring at the point $h(x,t)$ is a field. Thus the only closed analytic subspace of X containing $h(x,t)$ is X itself. In particular, the point $h(x,t)$ belongs to $X \smallsetminus Z$, and the statement follows. □

Proposition 4.15. *The topological space $\mathbb{A}(\mathscr{F}_E)$ is contractible and is a universal cover of $\mathbb{A}(\mathscr{F}_A)$.*

Proof. Let F_0 be the image of the R-module $\omega_{\check{\mathscr{B}}}$ via the map $\varphi_B \colon \omega_{\check{B}} \to F$ and $\mathscr{F}_0$ the push-out of the canonical extension $(\mathscr{U}_{\mathscr{B}})$ on $\mathscr{B}$ along $\omega_{\check{\mathscr{B}}} \to F_0$. For $\check{\chi} \in \check{\Lambda}$, the line bundle $\mathscr{L}_{B,\check{\chi}} = \mathscr{L}_{B|B\times\{\check{c}(\check{\chi})\}}$ extends to a line bundle $\mathscr{L}_{\mathscr{B},\check{\chi}}$ on $\mathscr{B}$. Consider a basis $\check{\chi}_1, \dots, \check{\chi}_n$ of Λ and the smooth connected formal R-scheme

$$\mathfrak{X} := \mathbb{P}(\mathscr{F}_0) \times_{\mathscr{B}} \mathbb{P}\left(\mathcal{O}_B \oplus \mathscr{L}_{\check{\chi}_1}\right) \times_{\mathscr{B}} \cdots \times_{\mathscr{B}} \mathbb{P}\left(\mathcal{O}_B \oplus \mathscr{L}_{\check{\chi}_n}\right).$$

Now $\mathbb{A}(\mathscr{F}_E)$ is the complement of a Cartier divisor in $X := \mathfrak{X}_\eta$; thus by Lemma 4.14 it is contractible. Since $\mathbb{A}(\mathscr{F}_A)$ is the quotient of $\mathbb{A}(\mathscr{F}_E)$ by the (free) action of Λ, the statement follows. □

Applying this with $F = \omega_{\check{A}}$ and $\varphi_A = \mathrm{id}$ gives that $E^\natural$ is contractible and $E^\natural \to A^\natural$ is a universal cover, which justifies the name universal cover for $E^\natural$.

Appendix. Connections

A.1. Vector bundles on first-order thickenings

Let X_0 and X_1 be schemes endowed with a closed immersion $s \colon X_0 \to X_1$ and a morphism $f \colon X_1 \to X_0$ such that $f \circ s = \mathrm{id}_{X_0}$. Suppose that the sheaf of ideals $I := \mathrm{Ker}(\mathcal{O}_{X_1} \to \mathcal{O}_{X_0})$ is of square zero. For a quasi-coherent $\mathcal{O}_{X_1}$-module F, consider the short exact sequence

$$0 \longrightarrow IF \longrightarrow F \longrightarrow F/IF \longrightarrow 0. \tag{F}$$

The sequence of $\mathcal{O}_{X_0}$-modules

$$f_*(F) \qquad\qquad 0 \longrightarrow f_*(IF) \longrightarrow f_*F \longrightarrow s^*F \longrightarrow 0$$

obtained by pushing forward (F) along f is short exact because the morphism f is affine (affineness only depends on the underlying reduced structure). Pushing forward a homomorphism of $\mathcal{O}_{X_1}$-modules $\varphi\colon F \to F'$ yields a homomorphism of short exact sequences of $\mathcal{O}_{X_0}$-modules $f_*\varphi\colon f_*(F) \to f_*(F')$. The so-defined functor

$$\left\{\mathcal{O}_{X_1}\text{-modules}\right\} \longrightarrow \left\{\begin{array}{c}\text{short exact sequences}\\ \text{of } \mathcal{O}_{X_0}\text{-modules}\end{array}\right\}, \quad F \longmapsto f_*(F)$$

is faithful. Moreover, an isomorphism of $\mathcal{O}_{X_1}$-modules $f^*s^*F \to F$ induces a splitting of the short exact $f_*(F)$.

Proposition A.1. *The bijection* $\operatorname{Hom}(f^*s^*F, F) \to \operatorname{Hom}(s^*F, f_*F)$ *given by adjunction for a vector bundle* F *on* X_1 *induces a bijection*

$$\left\{\begin{array}{c}\text{isomorphisms } \rho\colon f^*s^*F \to F\\ \text{such that } s^*\rho = \mathrm{id}_{s^*F}\end{array}\right\} \cong \left\{\begin{array}{c}\text{splittings of the short}\\ \text{exact sequence } f_*(F)\end{array}\right\}.$$

Proof. The only thing to show is that, for a splitting $\varphi\colon s^*F \to f_*F$ of the short exact sequence $f_*(F)$, the homomorphism $\Phi\colon f^*s^*F \to F$ obtained by extending φ $\mathcal{O}_{X_1}$-linearly is an isomorphism. To check this, one may reason locally on X_0 and choose a splitting $f_*F \cong s^*F \oplus s^*F \otimes f_*I$ of the short exact sequence $f_*(F)$. This allows for the identities $IF = I \otimes F$ and $f_*(IF) = f_*I \otimes s^*F$, which hold, respectively, because F is flat and because the ideal I is of square zero. Via these identifications, the splitting φ is of the form $v \mapsto (v, \varepsilon(v))$ for a homomorphism of $\mathcal{O}_X$-modules $\varepsilon\colon s^*F \to s^*F \otimes f_*I$. Write a section of f^*s^*F as (v, v') for sections v of s^*F and v' of $s^*F \otimes f_*I$; then the map Φ is defined as $(v, v') \mapsto (v, \varepsilon(v) + v')$, where the term $\varepsilon(v')$ vanished because the ideal I is of square zero. Such an expression clearly defines an isomorphism, which concludes the proof. $\square$

A.2. Tensor product and Baer sums

For simplicity, assume X_1 to be the first-order thickening of $\mathbb{V}(E^\vee)$ along its zero section s, where E is a vector bundle on X_0 and $\mathbb{V}(E^\vee)$ the total space of its dual. Concretely, the scheme X_1 is the spectrum of the $\mathcal{O}_{X_0}$-module $\mathcal{O}_{X_0} \oplus E$ endowed with an $\mathcal{O}_{X_0}$-algebra structure defined by the formula $(a, v) \cdot (b, w) = (ab, aw + bv)$. In what follows, it will be important to distinguish whether tensor products are taken with respect to $\mathcal{O}_{X_0}$ or $\mathcal{O}_{X_1}$. To mark the difference but at the same time make the notation lighter, for $i = 0, 1$ and $\mathcal{O}_{X_i}$-modules V and V', write $V \otimes_i V'$ instead of $V \otimes_{\mathcal{O}_{X_i}} V'$. For a vector bundle V on X_1, set $V_0 := s^*V$, so that the vector bundle f_*V on X_0 is an extension of V_0 by $V_0 \otimes_0 E$. In order to make sense of the following statement, observe that, for vector bundles V and V' on X_1, the $\mathcal{O}_{X_0}$-modules $f_*V \otimes_0 V_0'$, $f_*V' \otimes_0 V_0$ and $f_*(V \otimes_1 V')$ are all extensions of $V_0 \otimes_0 V_0'$ by $E \otimes_0 V_0 \otimes_0 V_0'$.

Proposition A.2. *The extension* $f_*(V \otimes_1 V')$ *is the Baer sum of* $f_*V \otimes_0 V_0'$ *and* $f_*V' \otimes_0 V_0$.

Proof. Unfortunately, the argument is quite clumsy and goes through the explicit construction of the Baer sum in question. To recall it, let $p\colon f_*V \to V_0$ and $p'\colon f_*V' \to V_0'$ denote the homomorphisms given by restriction to X_0, and consider the $\mathcal{O}_{X_0}$-submodule $W \subseteq (V_0 \otimes_0 f_*V') \oplus (f_*V \otimes_0 V_0')$ made of pairs whose components have the same image in $V_0 \otimes_0 V_0'$ via, respectively, $p \otimes \mathrm{id}_{V_0'}$ and $\mathrm{id}_{V_0} \otimes p'$. The vector bundle W on X_0 fits into the following short exact sequence of $\mathcal{O}_{X_0}$-modules:

$$(W) \qquad\qquad 0 \longrightarrow (E \otimes_0 V_0 \otimes_0 V_0')^{\oplus 2} \longrightarrow W \longrightarrow V_0 \otimes_0 V_0' \longrightarrow 0.$$

The Baer sum mentioned in the statement is the push-out of the short exact sequence (W) along the sum map $(E \otimes_0 V_0 \otimes_0 V_0')^{\oplus 2} \to E \otimes_0 V_0 \otimes_0 V_0'$. This being said, consider the natural epimorphism of $\mathcal{O}_{X_0}$-modules $\varphi\colon f_*V \otimes_0 f_*V' \to f_*(V \otimes_1 V')$ given by universal property of tensor product. The vector bundle E, seen as a sheaf of ideals of $f_*\mathcal{O}_{X_1}$, is of square zero; hence $\operatorname{Ker}\varphi = (E \otimes_0 V_0) \otimes_0 (E \otimes_0 V_0')$. The right-hand side of the previous equality is seen to also be the intersection of the kernels of the homomorphisms $p \otimes \mathrm{id}_{f_*V'}$ and $\mathrm{id}_{f_*V'} \otimes p'$. That is, upon setting

$$\psi = (p \otimes \mathrm{id}_{f_*V'}, \mathrm{id}_{f_*V'} \otimes p') \colon f_*V \otimes_0 f_*V' \longrightarrow (V_0 \otimes_0 f_*V') \oplus (f_*V \otimes_0 V_0'),$$

the homomorphisms φ and ψ share the same kernel. Moreover, the identity

$$(p \otimes \mathrm{id}_{V_0'}) \circ (\mathrm{id}_{f_*V} \otimes p') = p \otimes p' = (\mathrm{id}_{V_0} \otimes p') \circ (p \otimes \mathrm{id}_{f_*V'})$$

of maps $f_*V \otimes_0 f_*V' \to V_0 \otimes_0 V_0'$ implies that the image of ψ is W. Consequently, the homomorphism φ factors uniquely through an epimorphism $\tilde{\varphi} \colon W \to f_*(V \otimes_1 V')$ of $\mathcal{O}_{X_0}$-modules. The kernel of φ is contained in that of $p \otimes p'$, and the homomorphism $\tilde{\varphi}$ acts on the quotient

$$\frac{\operatorname{Ker}(p \otimes p')}{\operatorname{Ker}\varphi} = \frac{K + K'}{K \cap K'} \cong \frac{K}{K \cap K'} \oplus \frac{K'}{K \cap K'} = (E \otimes_0 V_0 \otimes_0 V_0')^{\oplus 2}$$

as the sum map, where K and K' are the kernels of the homomorphisms $p \otimes \mathrm{id}_{f_*V'}$ and $\mathrm{id}_{f_*V} \otimes p'$, respectively. In other words, the homomorphism $\tilde{\varphi}$ fits into the following commutative and exact diagram of $\mathcal{O}_{X_0}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & (E \otimes_0 V_0 \otimes_0 V_0')^{\oplus 2} \arrow[r] \arrow[d, "\mathrm{sum}"] & W \arrow[r] \arrow[d, "\tilde{\varphi}"] & V_0 \otimes_0 V_0' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \otimes_0 V_0 \otimes_0 V_0' \arrow[r] & f_*(V \otimes_1 V') \arrow[r] & V_0 \otimes_0 V_0' \arrow[r] & 0,
\end{tikzcd}$$

where the upper row is the short exact sequence (W). To put it differently, the lower row is the push-out of the upper one along the sum map; that is, the extension $f_*(V \otimes_1 V')$ is the desired Baer sum. □

A.3. Connections

Let S be a scheme, $f : X \to S$ a separated morphism of schemes, $\Delta_X \colon X \to X \times_S X$ the diagonal morphism and $I := \operatorname{Ker}(\mathcal{O}_{X \times_S X} \to \Delta_{X*}\mathcal{O}_X)$ its augmentation ideal. With this notation, the sheaf of differentials (denoted by Ω^1_f or $\Omega^1_{X/S}$) relative to f is the $\mathcal{O}_X$-module $\Delta_X^* I$. Let $\Delta_{X,1}$ be the first infinitesimal neighbourhood of the diagonal, that is, the closed subscheme of $X \times_S X$ defined by the sheaf of ideals I^2. For $i = 1, 2$, let $p_i \colon \Delta_{X,1} \to X$ be the morphism induced by the i^{th} projection. The $\mathcal{O}_X$-module $\mathcal{J}^1_f := p_{1*}\mathcal{O}_{\Delta_{X,1}}$ is called the *sheaf of first-order jets.* Each p_i induces a homomorphism of $f^{-1}\mathcal{O}_S$-algebras $j_i \colon \mathcal{O}_X \to \mathcal{J}^1_f$. The homomorphism of $f^{-1}\mathcal{O}_S$-modules $\mathrm{d}_f := j_2 - j_1 : \mathcal{O}_X \to \Omega^1_f$ is called the *canonical derivation.*

Definition A.3. A *connection* on a vector bundle F on X is an isomorphism $\nabla \colon p_1^* F \to p_2^* F$ of vector bundles over $\Delta_{X,1}$ whose restriction to the diagonal $\Delta_X^* \nabla$ is the identity of F.

A.4. Atiyah extension

The kernel of the restriction map $\mathcal{O}_{\Delta_{X,1}} \to i_*\mathcal{O}_X$, where $i \colon X \to \Delta_{X,1}$ is the closed immersion induced by the diagonal, is by definition of square zero. This allows one to adopt the notation introduced in Section A.1 with $X_0 = X$, $X_1 = \Delta_{X,1}$, $s = i$ and $f = p_1$. Pushing forward the $\mathcal{O}_{\Delta_{X,1}}$-module $p_2^* F$ along p_1 yields the short exact sequence

$$(\mathcal{J}^1_f(F)) \qquad 0 \longrightarrow \Omega^1_f \otimes F \longrightarrow \mathcal{J}^1_f(F) \longrightarrow F \longrightarrow 0$$

of $\mathcal{O}_S$-modules, where $\mathcal{J}^1_f(F) := p_{1*}p_2^* F$ is the $\mathcal{O}_X$*-module of first-order jets of* F. Applied to the $\mathcal{O}_{\Delta_{X,1}}$-module $p_2^* F$, Proposition A.1 implies the following.

Proposition A.4. *The bijection* $\mathrm{Hom}(p_1^*F, p_2^*F) \to \mathrm{Hom}(F, \mathcal{J}^1_{X/S}(F))$ *given by adjunction induces a bijection*

$$\left\{ \text{ connections on } F \text{ } \right\} \cong \left\{ \begin{array}{c} \text{splittings of the short} \\ \text{exact sequence } (\mathcal{J}^1_{X/S}(F)) \end{array} \right\}.$$

Set $\mathrm{At}_f(F) := \mathcal{H}om(E, \mathcal{J}^1_f(F))$. The short exact sequence of $\mathcal{O}_X$-modules

$$(\mathrm{At}_f(F)) \qquad 0 \longrightarrow \Omega^1_f \otimes \mathcal{E}nd F \longrightarrow \mathrm{At}_f(F) \longrightarrow \mathcal{E}nd F \longrightarrow 0$$

obtained as the tensor product of $(\mathcal{J}^1_f(F))$ with $F^\vee$ is called the *relative Atiyah extension of* F. When F is a line bundle, $\mathcal{E}nd F \cong \mathcal{O}_X$, and splittings of the Atiyah extension $(\mathrm{At}_f(F))$ are in bijection with those of $(\mathcal{J}^1_f(F))$, thus with connections on F.

A.5. Infinitesimal rigidifications

Let $\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times_S X \to X$ be, respectively, the first and the second projection and $x \colon S \to X$ a section of the structural morphism $f : X \to S$. Let X_1 be the first-order thickening of X along x, $\iota \colon X_1 \to X$ the closed immersion and $\pi = f \circ \iota \colon X_1 \to S$ the structural morphism.

Definition A.5. An *infinitesimal rigidification of* F *at* x is an isomorphism of $\mathcal{O}_{X_1}$-modules $\rho \colon \pi^* x^* F \to \iota^* F$ such that $x^*\rho$ is the identity.

Note that if $\nabla \colon p_1^* F \to p_2^* F$ is a connection on F, then the homomorphism of $\mathcal{O}_{X_1}$-modules $\tau^*\nabla$ is an infinitesimal rigidification of F at x. Now, it is possible to give a characterization of infinitesimal rigidifications similar to that of connections. For, the augmentation ideal of the closed immersion $x_1 : S \to X_1$ induced by x is of square zero. This permits one to employ the conventions introduced in Section A.1 with $X_0 = S$, $s = x_1$ and $f = \pi$. For instance, if F is a vector bundle, then pushing forward the $\mathcal{O}_{X_1}$-module $\iota^* F$ along π yields the short exact sequence

$$\pi_*(\iota^*F) \qquad 0 \longrightarrow x^*\Omega^1_{X/S} \otimes x^*F \longrightarrow \pi_*\iota^*F \longrightarrow x^*F \longrightarrow 0$$

of $\mathcal{O}_S$-modules. Applied to $\mathcal{O}_{X_1}$-module ι^*F, Proposition A.1 reads as follows.

Proposition A.6. *The bijection* $\mathrm{Hom}(\pi^*x^*F, \iota^*F) \to \mathrm{Hom}(x^*F, \pi_*\iota^*F)$ *given by adjunction induces a bijection*

$$\left\{ \begin{array}{c} \text{infinitesimal} \\ \text{rigidifications of } F \text{ at } x \end{array} \right\} \cong \left\{ \begin{array}{c} \text{splittings of the short} \\ \text{exact sequence } \pi_*(\iota^*F) \end{array} \right\}.$$

Let $\tau \colon X_1 \to \Delta_{X,1}$ be the morphism determined by $p_1 \circ \tau = x \circ \pi$ and $p_2 \circ \tau = \iota$. Consider the homomorphism $\psi \colon p_2^*F \to \tau_*\iota^*F$ of $\mathcal{O}_{\Delta_{X,1}}$-modules given by adjunction (note the equality $\iota^*F = \tau^*p_2^*F$). Pushing it forward along the morphism p_1 gives a homomorphism

$$\pi_*\psi \colon (\mathcal{J}^1_{X/S}(F)) \longrightarrow p_{1*}\tau_*(\iota^*F)$$

of short exact sequences of $\mathcal{O}_X$-modules. Now, by the definition of τ, the following square is commutative:

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi} & S \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle x} \\ \Delta_{X,1} & \xrightarrow{p_1} & X. \end{array}$$

Therefore, the short exact sequence $p_{1*}\tau_*(\iota^*F)$ of $\mathcal{O}_X$-modules is nothing but the push-forward along the closed immersion x of the short exact sequence $\pi_*(\iota^*F)$ of $\mathcal{O}_S$-modules.

Proposition A.7. *The homomorphism* $\varphi \colon x^*(\mathcal{J}^1_{X/S}(F)) \to \pi_*(\iota^*F)$ *of short exact sequence of* $\mathcal{O}_S$*-modules adjoint to* $\pi_*\psi$ *is an isomorphism.*

Proof. The homomorphism $\psi\colon p_2^*F \to \tau_*\iota^*F$ restricted to the diagonal is the evaluation at x. Therefore, the homomorphism $\pi_*\psi$ of short exact sequence of $\mathcal{O}_X$-modules is the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_{X/S}\otimes F \arrow[r] \arrow[d, "\mathrm{ev}_x"] & \mathcal{J}^1_{X/S}(F) \arrow[r] \arrow[d, "\pi_*\psi"] & F \arrow[r] \arrow[d, "\mathrm{ev}_x"] & 0 \\ 0 \arrow[r] & x_*x^*\Omega^1_{X/S}\otimes x_*x^*F \arrow[r] & x_*\pi_*\iota^*F \arrow[r] & x_*x^*F \arrow[r] & 0, \end{tikzcd}$$

where ev_x is the evaluation at x. The homomorphism φ, which is the adjoint to $\pi_*\psi$, reads as the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & x^*\Omega^1_{X/S}\otimes x^*F \arrow[r] \arrow[d, equal] & x^*\mathcal{J}^1_{X/S}(F) \arrow[r] \arrow[d, "\varphi"] & x^*F \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & x^*\Omega^1_{X/S}\otimes x^*F \arrow[r] & \pi_*\iota^*F \arrow[r] & x^*F \arrow[r] & 0 \end{tikzcd}$$

of $\mathcal{O}_S$-modules. The five lemma implies that φ is an isomorphism. □

References

[BVS01] L. Barbieri-Viale and V. Srinivas, *Albanese and Picard 1-motives*, Mém. Soc. Math. Fr. (N.S.) (2001), no. 87.

[Ber93] V. G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161.

[Ber99] ———, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84.

[Ber09] A. Bertapelle, *Deligne's duality for de Rham realizations of 1-motives*, Math. Nachr. **282** (2009), no. 12, 1637–1655.

[BHR11] I. Biswas, J. Hurtubise, and A. K. Raina, *Rank one connections on abelian varieties*, Internat. J. Math. **22** (2011), no. 11, 1529–1543.

[BL91] S. Bosch and W. Lütkebohmert, *Degenerating abelian varieties*, Topology **30** (1991), no. 4, 653–698.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergeb. Math. Grenzgeb. (3), vol. 21, Springer-Verlag, Berlin, 1990.

[Cre90] R. Crew, *Universal extensions and p-adic periods of elliptic curves*, Compos. Math. **73** (1990), no. 1, 107–119.

[FC90] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties* (with an appendix by D. Mumford), Ergeb. Math. Grenzgeb. (3), vol. 22, Springer-Verlag, Berlin, 1990.

[Lüt16] W. Lütkebohmert, *Rigid geometry of curves and their Jacobians*, Ergeb. Math. Grenzgeb. (3), vol. 61, Springer, Cham, 2016.

[Mac22] M. Maculan, *Rigid analytic Stein algebraic groups are affine*, preprint arXiv:2007.04659v3 (2022).

[MM74] B. Mazur and W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Lecture Notes in Math., vol. 370, Springer-Verlag, Berlin-New York, 1974.

[Mum72a] D. Mumford, *An analytic construction of degenerating abelian varieties over complete rings*, Compos. Math. **24** (1972), 239–272.

[Mum72b] ———, *An analytic construction of degenerating curves over complete local rings*, Compos. Math. **24** (1972), 129–174.

[Ols16] M. Olsson, *Algebraic spaces and stacks*, Amer. Math. Soc. Colloq. Publ., vol. 62, Amer. Math. Soc., Providence, RI, 2016.

[Oor66] F. Oort, *Commutative group schemes*, Lecture Notes in Math., vol. 15, Springer-Verlag, Berlin-New York, 1966.

[Ser59] J.-P. Serre, *Groupes algébriques et corps de classes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VII, Hermann, Paris, 1959.

[SGA7-I] *Groupes de monodromie en géométrie algébrique. I* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969 (SGA 7 I). Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim), Lecture Notes in Math., vol. 288, Springer-Verlag, Berlin-New York, 1972.

[SGA1] *Revêtements étales et groupe fondamental (SGA 1)* (Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1). Directed by A. Grothendieck. With two papers by M. Raynaud. Updated and annotated reprint of the 1971 original), Doc. Math. (Paris), vol. 3, Soc. Math. France, Paris, 2003.

[Tat71] J. Tate, *Rigid analytic spaces*, Invent. Math. **12** (1971), 257–289.