

On the finite generation of valuation semigroups on toric surfaces

Klaus Altmann, Christian Haase, Alex Küronya, Karin Schaller, and Lena Walter

Abstract. We provide a combinatorial criterion for the finite generation of a valuation semigroup associated with an ample divisor on a smooth toric surface and a non-toric valuation of maximal rank. As an application, we construct a lattice polytope such that none of the valuation semigroups of the associated polarized toric variety coming from one-parameter subgroups and centered at a non-toric point are finitely generated.

Keywords. Newton–Okounkov bodies, toric surfaces, finite generation of valuation semigroups

2020 Mathematics Subject Classification. 14C20, 14M25, 52B20

Received by the Editors on May 31, 2023, and in final form on November 23, 2023.

Accepted on December 15, 2023.

Klaus Altmann

Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

e-mail: altmann@math.fu-berlin.de

Christian Haase

Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

e-mail: haase@math.fu-berlin.de

Alex Küronya

Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6, 60325 Frankfurt am Main, Germany

e-mail: kuronya@math.uni-frankfurt.de

Karin Schaller

Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

e-mail: kschaller@math.fu-berlin.de

Lena Walter

Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

e-mail: lenawalter@math.fu-berlin.de

The first, second, fourth, and fifth authors have been partially supported by the Polish-German grant “ATAG – Algebraic Torus Actions: Geometry and Combinatorics” [project number 380241778] of the Deutsche Forschungsgemeinschaft (DFG). The third author acknowledges support by the Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 “Geometry and Arithmetic of Uniformized Structures” [project number 444845124].

Contents

| | |
|---|----|
| 1. Introduction | 2 |
| 2. Notation and preliminaries | 4 |
| 3. Valuation semigroups associated with non-toric flags | 7 |
| 4. Shape of the Newton–Okounkov body | 12 |
| 5. Criterion for the finite generation of certain valuation semigroups | 14 |
| 6. Finite generation criterion | 15 |
| References | 21 |

1. Introduction

Finite generation of semigroups or rings arising from geometric situations has been a question of interest for a long time. As a salient example, we can recall the finite generation of canonical or adjoint rings from birational geometry, which motivated the field through the minimal model program for several decades; *cf.* [BCHM10]. The question of finite generation of valuation semigroups arising from Newton–Okounkov theory appears to be equally difficult in general, with little progress beyond the completely toric situation, but potentially great benefits such as the existence of toric degenerations, *cf.* [And13], and completely integrable systems, *cf.* [HK15], to name but a few. In this article, we take a few steps away from the situation where every participant is toric: we consider valuation semigroups associated with torus-invariant divisors on toric surfaces with respect to a *non-toric* valuation.

The main idea behind Newton–Okounkov theory is to attach combinatorial/convex-geometric objects to geometric situations to facilitate their analysis, in other words, to partially replicate the setup of toric geometry in settings without any useful group action. The basis for the theory was developed by Kaveh–Khovanskii [KK12] and Lazarsfeld–Mustață [LM09], building on earlier work of Okounkov [Oko96], but the subject has seen substantial growth in the last decade. By now applications of Newton–Okounkov theory range from combinatorics and representation theory through birational geometry, *cf.* [KL17a, KL17b, KL19, KL18a], to mirror symmetry, *cf.* [RW19], and geometric quantization in mathematical physics.

Given a projective variety X and a divisor D on X , Newton–Okounkov theory associates to (X, D) a semigroup $S_{Y_\bullet}(D)$, the valuation semigroup, and a convex body $\Delta_{Y_\bullet}(D)$, the Newton–Okounkov body of D . Both the valuation semigroup and the Newton–Okounkov body depend, however, on a maximal rank valuation of the function field of X coming from an admissible flag Y_\bullet of subvarieties.

The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ is an asymptotic version of $S_{Y_\bullet}(D)$ and is, accordingly, a lot easier to determine. Newton–Okounkov bodies on surfaces end up being almost rational polygons; *cf.* [KLM12]. If the section ring of D is finitely generated, then a suitably general flag valuation will yield a rational simplex as its Newton–Okounkov body; *cf.* [AKL14]. In the case of a toric variety X with torus-invariant D and Y_\bullet , the associated Newton–Okounkov body recovers the moment polytope of the polarized toric variety.

In this paper, we will focus on the valuation semigroup $S_{Y_\bullet}(D)$, more concretely, on the question of whether or not it is finitely generated. It is a classical fact that $S_{Y_\bullet}(D)$ is often not finitely generated even if X is a smooth projective curve.

It is known that $S_{Y_\bullet}(D)$ is a finitely generated semigroup if X is a toric variety, D a torus-invariant divisor, and Y_\bullet an admissible flag of torus-invariant subvarieties. We consider the next open question, namely, the case of toric surfaces and non-torus-invariant flag valuations. Although the divisorial geometry of toric surfaces themselves is not particularly complicated, things get out of control once we start blowing up non-toric points. Blowing up just one point on a toric surface can lead to surfaces with infinitely many negative curves on them, as recent research of Castravet–Laface–Tevelev–Ugaglia [CLTU23] illustrates. Blowing up many general points quickly leads to notoriously difficult situations like Nagata’s conjecture.

1.1. Newton–Okounkov bodies

Assume that $X = \mathbb{T}\mathbb{V}(\Sigma)$ is a smooth projective toric variety of dimension n . Then ample torus-invariant divisors D or their associated line bundles $\mathcal{O}_X(D)$ can be understood as lattice polytopes $\Delta(D)$ in the character lattice $M \cong \mathbb{Z}^n$ of the torus \mathbb{T} acting on X . Using this description, the starting fan Σ can be recovered as the normal fan of these polytopes.

Moreover, it is a well-known feature of the toric theory that the vector space $\Gamma(X, \mathcal{O}_X(D))$ has the set of lattice points $\Delta(D) \cap M$ as its distinguished basis. That is, the polytope $\Delta(D)$ gives $\Gamma(X, \mathcal{O}_X(D))$ not just a dimension but also a shape.

In [LM09] and [KK12], this concept was generalized to arbitrary projective varieties X (still of dimension n). If we are given a so-called *admissible flag*

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$$

of nested (irreducible) varieties with $\dim Y_i = n - i$ that are smooth in the special point Y_n , then for every ample divisor D , there is an associated convex body $\Delta_{Y_\bullet}(D)$ (“Newton–Okounkov body”) in \mathbb{R}^n reflecting many properties of D ; see Section 2.1 for details.

Note that \mathbb{Z}^n has ceased to be a character lattice because there is no longer a torus around. While the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ depends not on D but only on its numerical class, the dependence on the chosen flag is striking.

If, for instance, X is toric as at the very beginning, then the construction of $\Delta_{Y_\bullet}(D)$ recovers the correspondence between divisors and polytopes we mentioned above. But to make this true requires a *toric flag*; *i.e.*, all subvarieties Y_i are supposed to be orbit closures.

The fact that toric varieties with toric flags lead to well-known polytopes is not a one-way street. In fact, if, for a general variety X , the semigroups $S_{Y_\bullet}(D)$ are finitely generated, then they provide a toric degeneration of X . This was shown in [And13]. Observe that in general the semigroup $S_{Y_\bullet}(D)$ being finitely generated is much stronger than $\Delta_{Y_\bullet}(D)$ being polyhedral. This finite generation of the valuation semigroup $S_{Y_\bullet}(D)$ is the main point of this paper.

1.2. Results

In [IM19] the finite generation was shown for complexity one \mathbb{T} -varieties with toric flags. Note that this implies the toric case. Whenever one is only interested in the Newton–Okounkov body (instead of the semigroup), there are more results: The most general one solves the question for surfaces using Zariski decomposition; *cf.* [LM09, Theorem 6.4]. In particular, the Newton–Okounkov bodies are polyhedral in this case. Specializing this situation, [HKW20] has provided an explicit combinatorial description of $\Delta_{Y_\bullet}(D)$ for toric surfaces with certain *non-toric* flags.

In the present paper, partially inspired by [AP20], we consider the very same setup of toric surfaces, namely with the following admissible flag:

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2,$$

where Y_1 is the closure of a one-parameter subgroup of the torus, which is non-torus-invariant and Y_2 is a general smooth point. Then we prove that, in dependence on Y_1 , the semigroups $S_{Y_\bullet}(D)$ can be both finitely generated and not finitely generated.

The main result of the paper (Theorem 6.8) comes from understanding the relationship between the Newton–Okounkov body of D and the Newton polygon associated with the *non-toric* flag curve Y_1 . The significance of our contribution lies in the fact that we infer the finite generation of valuation semigroups from asymptotic/convex-geometric data and provide a combinatorial criterion.

In order to state our Theorem, we introduce a bit of terminology. For our flag Y_\bullet , we consider a non-torus-invariant curve Y_1 given as the closure of the one-parameter subgroup determined by a primitive vector $v_N \in N$; our flag point is going to be $0 \otimes_{\mathbb{Z}} 1$ on the torus $\mathbb{T} \cong \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{C}^\times$. For a strongly convex cone $\sigma \subseteq N_{\mathbb{R}}$ and a lattice point $u \in \text{int}(\sigma) \cap N$, we say that u is *strongly decomposable in σ* if $u = u' + u''$ for suitable $u', u'' \in \text{int}(\sigma) \cap N$. Given a divisor D on X , we construct strongly convex cones σ^+ and σ^- associated with v_N that are spanned by certain rays of the fan of X (see Definition 6.7). With this said, our main result goes as follows.

Theorem (Theorem 6.8). *Let X be a smooth toric surface associated with a fan Σ and D an ample divisor on X . The valuation semigroup $S_{Y_\bullet}(D)$ is finitely generated if and only if v_N is not strongly decomposable in σ^+ and $-v_N$ is not strongly decomposable in σ^- .*

To illustrate the combinatorial content, Figure 1 pictures the situation in the case of the 7-gon of [CLTU23], where the blow-up surface $X = \mathbb{T}\mathbb{V}(\Sigma)$ accomodates infinitely many negative curves (cf. Figure 10 for a complete picture).

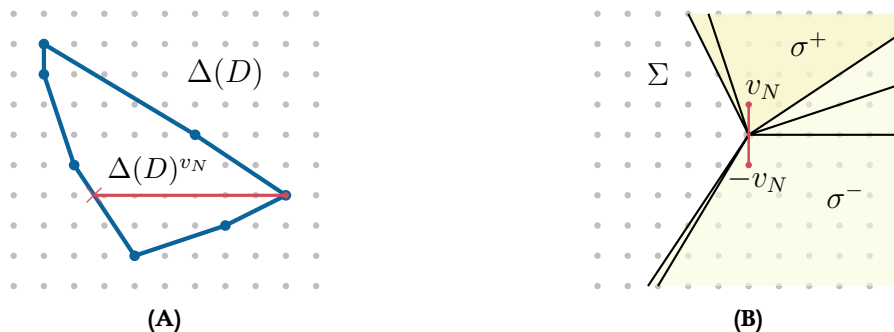


Figure 1. The good 7-gon $\Delta(D)$. (A) The polytope $\Delta(D)$ and the rational line segment $\Delta(D)^{v_N}$. (B) The normal fan Σ of $\Delta(D)$ together with the two cones $\sigma^+ \ni v_N$ and $\sigma^- \ni -v_N$.

As an application of the above theory, we construct in Example 6.11 a lattice polygon with a strong non-finite-generation property. To be more concrete, we look at the ample divisor D associated with the polytope $\Delta(D)$ given in Figure 11A on the toric variety $X = \mathbb{T}\mathbb{V}(\Sigma)$ corresponding to the fan Σ in Figure 11B. Example 6.11 shows that the semigroup $S_{Y_\bullet}(D)$ will not be finitely generated for any $v_N \in N$ we pick.

2. Notation and preliminaries

Let X be a two-dimensional smooth projective variety, and assume that we are given an admissible flag

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2,$$

as in Section 1.1, and an ample divisor D on X .

2.1. Newton–Okounkov bodies and valuation semigroups

Following [LM09], we obtain a rank two valuation-like function (or, equivalently, a rank two valuation of the function field of X , see [KMR21])

$$\text{val}_{Y_\bullet} : \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} \longrightarrow \mathbb{Z}^2$$

as follows: Let f be an equation for Y_1 near Y_2 . For a non-trivial section $s \in \Gamma(X, \mathcal{L})$ of a line bundle \mathcal{L} , e.g., $\mathcal{L} = \mathcal{O}_X(D)$, we define

$$\text{val}_{Y_\bullet}(s) = (\text{val}_1(s), \text{val}_2(s)) := (\text{ord}_{Y_1}(s), \text{ord}_{Y_2}(\tilde{s}|_{Y_1})),$$

where $\tilde{s} := s/f^{\text{val}_1(s)}$ is a section of $\mathcal{L} \otimes \mathcal{O}_X(-\text{val}_1(s) \cdot Y_1)$.

The valuation semigroup $S_{Y_\bullet}(D)$ of D (with respect to the flag Y_\bullet) is defined as

$$S_{Y_\bullet}(D) := \{(\ell, \text{val}_{Y_\bullet}(s)) \mid s \in \Gamma(X, \mathcal{O}_X(\ell D)) \setminus \{0\}, \ell \in \mathbb{N}\} \subseteq \mathbb{N}^3.$$

The Newton–Okounkov body of D (with respect to the flag Y_\bullet) is defined as the set

$$\Delta_{Y_\bullet}(D) := \overline{\bigcup_{\ell \geq 1} \frac{1}{\ell} \{ \text{val}_{Y_\bullet}(s) \mid s \in \Gamma(X, \mathcal{O}_X(\ell D)) \setminus \{0\}, \ell \in \mathbb{N} \}} \subseteq \mathbb{R}^2.$$

For Newton–Okounkov theory on surfaces, see [KL18b].

2.2. Toric setup

Let $N \cong \mathbb{Z}^2$ be a two-dimensional lattice with dual lattice M and Σ a smooth complete fan associated with the toric surface $X = \mathbb{T}\mathbb{V}(\Sigma)$. We may assume that our ample divisor D is toric, hence is represented by a polytope $\Delta(D)$. Recall that $\mathbb{T} = \text{Spec}(\mathbb{C}[M])$ is our torus acting on X ; hence M becomes its character lattice and N the associated lattice of one-parameter subgroups.

Next, fix an admissible flag $Y_\bullet : X \supseteq Y_1 \supseteq Y_2$ as follows: Choose a primitive element $v_N \in N$. The embedding

$$\iota : \overline{N} := \mathbb{Z}v_N \hookrightarrow N$$

induces a map of fans, hence a toric map $\iota : \mathbb{P}^1 \rightarrow X = \mathbb{T}\mathbb{V}(\Sigma)$. Set $Y_1 := \iota(\mathbb{P}^1)$ and $Y_2 := \iota(1 \in \mathbb{P}^1)$.

Within the torus \mathbb{T} , the curve $C := Y_1$ is given by the binomial equation $f := x^{v_M} - 1$ with $v_M \in M$ being one of the two primitive elements of $v_N^\perp \subset M_{\mathbb{R}}$. The associated Newton polytope $\Delta^{\text{newt}} := \text{newt}(f)$ is the line segment $[0, v_M]$ connecting 0 and v_M in $M_{\mathbb{R}}$. This way $\iota : \mathbb{P}^1 \rightarrow C$ is the normalization map.

For a toric line bundle \mathcal{L} on X , the pullback $\iota^* : \Gamma(X, \mathcal{L}) \rightarrow \Gamma(\mathbb{P}^1, \iota^*\mathcal{L})$ corresponds to the projection

$$\pi : M \longrightarrow M/\mathbb{Z}v_M =: \overline{M},$$

which we will identify with $v_N : M \rightarrow \mathbb{Z} \cong \overline{M}$.

Observe that this almost fits the setup of [IM19] as Y_1 is invariant under the codimension one torus given by v_N . We only deviate from [IM19] by choosing a non-invariant flag point Y_2 .

2.3. Torifying the curve

Note that C is also a prime (Cartier) divisor on X , which properly intersects all torus-invariant curves, i.e., all boundary curves of X . Therefore, C is nef and $C' := C - \text{div}(f)$ is its \mathbb{T} -invariant representative.

Lemma 2.1. *The polygon $\Delta^{\text{nef}} := \Delta(C')$ is given by*

$$\Delta^{\text{nef}} = \{m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq \min\{0, \langle v_M, \rho \rangle\} \text{ for all } \rho \in \Sigma(1)\},$$

where the rays $\rho \in \Sigma(1)$ are identified with their first (hence primitive) lattice points.

Proof. The prime divisor $D_\rho = \overline{\text{orb}(\rho)}$ associated with a ray $\rho \in \Sigma(1)$ appears in the torus-invariant Weil divisor C' as often as it does in the (non-equivariant) principal divisor $-\text{div}(f)$. Thus,

$$\Delta^{\text{nef}} = \left\{ m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq \text{ord}_\rho(f) \text{ for all } \rho \in \Sigma(1) \right\},$$

and it remains to discuss $\text{ord}_\rho(f)$. If $\rho \neq \pm v_N$, then the ρ -orders of the two summands of $f = x^{v_M} - 1$ are different, hence

$$\text{ord}_\rho(f) = \min\{\text{ord}_\rho(1), \text{ord}_\rho(x^{v_M})\} = \min\{0, \langle v_M, \rho \rangle\}.$$

If $\rho = \pm v_N$, then the situation looks locally like $\text{ord}_y(x-1) = 0$ in \mathbb{A}^2 . \square

See [HKW20, Proposition 3.1] for a different proof.

Example 2.2. Let $X = \mathbb{T}\mathbb{V}(\Sigma)$ be the toric surface associated with the fan Σ , where $\Sigma(1) = \{\rho_i \mid 0 \leq i \leq 3\}$ with

$$\rho_0 = (-1, 0), \rho_1 = (0, -1), \rho_2 = (1, 2), \text{ and } \rho_3 = (0, 1).$$

As in Lemma 2.1, we identify the rays ρ_i with their generating lattice points (cf. Figure 2A). We denote by $D_i := \overline{\text{orb}(\rho_i)}$ ($0 \leq i \leq 3$) the toric prime divisors on X . Then, $D := 8D_2 + 3D_3$ is an ample divisor on X , which corresponds to the polytope

$$\Delta(D) = \text{conv}([0, 0], [-8, 0], [-2, -3], [0, -3]).$$

We take $C = \{(t^{-2}, t^3) \mid t \in \mathbb{C}\}$ as our curve for the non-toric flag. This means that $v_N = (-2, 3) \in N$ and $v_M = [-3, -2] \in M$, hence

$$\Delta^{\text{newt}} = \text{conv}([0, 0], [-3, -2]).$$

Since the boundary part of $\text{div}(f)$ equals $-7D_2 - 2D_3$, we obtain $C' = 7D_2 + 2D_3$ with nef polytope (cf. Figure 2B)

$$\Delta^{\text{nef}} = \text{conv}([0, 0], [-7, 0], [-3, -2], [0, -2]).$$

2.4. An alternative view on Δ^{nef}

Beside the explicit description of Lemma 2.1, it is possible to describe the shape of Δ^{nef} in the following more combinatorial way. The relation $v_M \in v_N^\perp$ among our curve parameters means

$$\langle 0, v_N \rangle = \langle v_M, v_N \rangle = 0;$$

i.e., $\Delta^{\text{newt}} = \text{conv}(0, v_M)$ is contained in the level set $[v_N = 0]$.

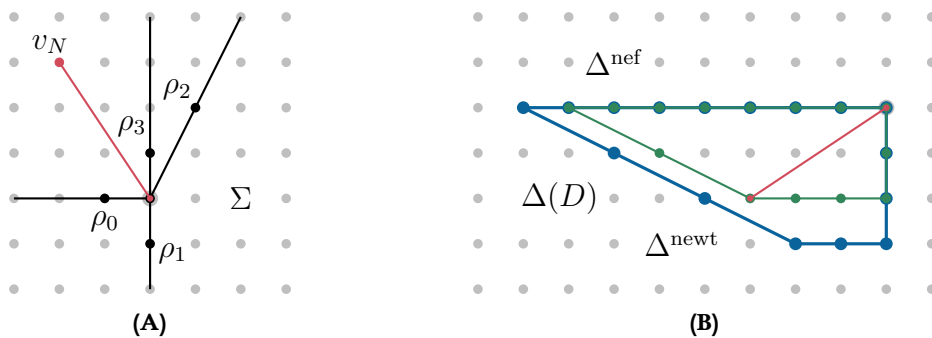


Figure 2. Input data. (A) The fan Σ together with its set of rays $\{\rho_i \mid 0 \leq i \leq 3\}$ and a primitive element $v_N = (-2, 3) \in N$. (B) The polytope $\Delta(D)$ corresponding to the ample divisor $D = 8D_2 + 3D_3$ on $X = \mathbb{T}\mathbb{V}(\Sigma)$, the Newton polytope Δ^{newt} given by $v_M = [-3, -2] \in M$, and the polytope Δ^{nef} corresponding to C' .

We denote by $r_{\max}, r_{\min} \in M$ the vertices of $\Delta(D)$, where $\langle \Delta(D), v_N \rangle$ admits its extremal values (cf. Figure 3A). Moreover, we define $\sigma_{\max}, \sigma_{\min}$ to be the two-dimensional cones generated by the two edges of $\Delta(D)$ that contain the vertices r_{\max} and r_{\min} , respectively.

We take the line segment Δ^{newt} and fit it inside the cone σ_{\max} until it hits both rays of this cone. In this way, we construct a lattice triangle Δ_{\max} with base Δ^{newt} and top vertex r_{\max} . We construct Δ_{\min} (cf. Figure 3A) in a similar way. In other words, the cones σ_{\max} and σ_{\min} are cut off along v_N -constant lines producing edges of Δ_{\max} and Δ_{\min} , respectively, of lattice length one. Note that both cut lines are parallel translates. Gluing Δ_{\max} and Δ_{\min} along Δ^{newt} (cf. Figure 3B) yields

$$\Delta^{\text{nef}} = \Delta_{\max} \cup_{\Delta^{\text{newt}}} \Delta_{\min}.$$

Note that $\Delta^{\text{nef}} \supseteq \Delta^{\text{newt}}$ is the smallest polytope containing Δ^{newt} and having Σ as a refinement of its normal fan. Actually, either Δ^{nef} is a quadrangle with Δ^{newt} serving as one of its diagonals, or it is a triangle with Δ^{newt} as a side.

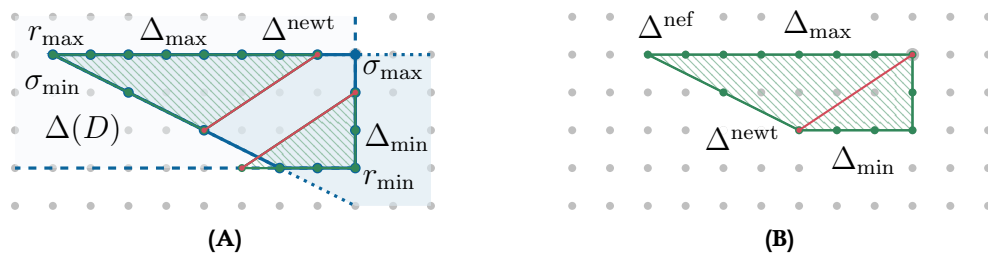


Figure 3. Alternative view on Δ^{nef} . (A) The two cones $\sigma_{\max}, \sigma_{\min}$ and polytopes $\Delta_{\max}, \Delta_{\min}$ constructed as in Section 2.4 with $v_M = [-3, -2] \in M$. (B) Gluing Δ_{\max} and Δ_{\min} along Δ^{newt} yields Δ^{nef} .

3. Valuation semigroups associated with non-toric flags

In this section, we determine the valuation semigroup $S_{Y_\bullet}(D)$ associated with an ample (Cartier) divisor D and a *non-toric* flag Y_\bullet as a subset of \mathbb{N}^3 . The main result is Theorem 3.11, where the abstract semigroup $S_{Y_\bullet}(D)$ is described in terms of lattice points coming from a polyhedral construction in M .

Let us fix $\ell \geq 1$ and $k \geq 0$. The space of sections $s \in \Gamma(X, \mathcal{O}_X(\ell D))$ which have vanishing order at least k along C is the image of

$$\Gamma(X, \mathcal{O}_X(\ell D - k C)) \xrightarrow{f^k} \Gamma(X, \mathcal{O}_X(\ell D)), \quad \tilde{s} \mapsto s.$$

We set $\mathcal{L}(\ell, k) := \mathcal{O}_X(\ell D - k C)$ and $L(\ell, k) := \Gamma(X, \mathcal{L}(\ell, k))$. If we have $\text{ord}_C(s) = k$, then $\text{ord}_{Y_2}(\tilde{s}|_C) = \text{ord}_{1 \in \mathbb{P}^1}(\iota^*(\tilde{s}))$ using the pullback via $\iota: \mathbb{P}^1 \rightarrow X$.

3.1. Return to toric geometry

Our goal is to understand the restriction of global sections via toric geometry. Therefore, we implement two changes. First, we will shift the linear series of the flag curve, which enables us to replace some of the line bundles we study with torus-invariant ones. Second, we will normalize the restriction.

We are going to use $C' = C - \text{div}(f)$ from Section 2.3. In terms of the associated sheaves, this means

$$\mathcal{O}_X(C') = f \cdot \mathcal{O}_X(C),$$

where f is the equation $x^{v_M} - 1$ of C mentioned earlier. This leads to the possibility of replacing $\mathcal{L}(\ell, k)$ by the isomorphic, but torus-invariant, line bundle

$$\mathcal{L}'(\ell, k) := \mathcal{O}_X(\ell D - k C') = f^{-k} \cdot \mathcal{L}(\ell, k) \subseteq \mathbb{C}(X).$$

Accordingly, we set

$$L'(\ell, k) := \Gamma(X, \mathcal{L}'(\ell, k))$$

and replace $\tilde{s} \in L(\ell, k)$ with $s' := f^{-k} \cdot \tilde{s} \in L'(\ell, k)$.

Recall that the nef invertible sheaves $\mathcal{O}_X(\ell D)$ and $\mathcal{O}_X(k C')$ correspond to the polytopes $\ell\Delta(D)$ and $k\Delta^{\text{nef}}$, respectively. This implies that $L'(\ell, k)$ has a monomial base provided by

$$\begin{aligned} \Theta(\ell, k) &:= (\ell\Delta(D) : k\Delta^{\text{nef}}) := \{m \in M_{\mathbb{R}} \mid m + k\Delta^{\text{nef}} \subseteq \ell\Delta(D)\} \\ &= (\ell\Delta(D) : k\Delta^{\text{newt}}) := \{m \in M_{\mathbb{R}} \mid m + k\Delta^{\text{newt}} \subseteq \ell\Delta(D)\}; \end{aligned}$$

see, e.g., the ($i = 0$)-case of [AP20, Theorem 2].

Remark 3.1. In toric geometry, there is a well-known way to associate to every divisor D a polytope P_D reflecting the global sections of $\mathcal{O}_X(D)$. If (and only if) D is nef, this mapping $D \mapsto P_D$ allows one to recover D , i.e., creates a one-to-one correspondence. Therefore, we introduce another polyhedral gadget $\Delta(D)$ providing a complete characterization for more general divisors. If D is nef, then $\Delta(D) = P_D$ coincides with the former construction. However, for general divisors $D = A - B$ (A, B nef), $\Delta(D)$ becomes a virtual polytope, i.e., a formal difference of two true polyhedra that is Minkowski-additive in its arguments. More concretely, we have $\Delta(A - B) = P_A - P_B$. The global section polytope P_D for arbitrary divisors $D = A - B$ can be recovered as $P_D = (\Delta(A) : \Delta(B))$. If D was nef, then we could choose B as the zero divisor, i.e., $D = A$, and obtain again $P_A = \Delta(A)$.

Example 3.2. Continuing Example 2.2, we obtain

$$\begin{aligned} \Theta(\ell, k) &= (\ell\Delta(D) : k\Delta^{\text{nef}}) \\ &= \{m \in M_{\mathbb{R}} \mid \langle m, (-1, 0) \rangle \geq 0, \langle m, (0, -1) \rangle \geq 0, \\ &\quad \langle m, (1, 2) \rangle \geq 7k - 8\ell, \langle m, (0, 1) \rangle \geq 2k - 3\ell\}. \end{aligned}$$

In particular, $\Theta(1, 1) = \text{conv}([0, 0], [-1, 0], [0, -1/2])$ (cf. the yellow polytope in Figure 4). The projection looks like $\pi: M \rightarrow \bar{M} = M/\mathbb{Z}v_M$. This map (cf. Figure 4) can be identified with $v_N: M \rightarrow \mathbb{Z}$, that is,

$$\pi: [x, y] \mapsto -2x + 3y.$$

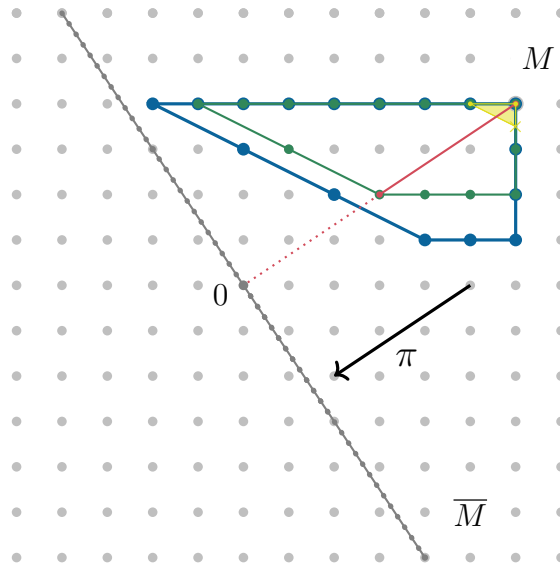


Figure 4. Projection map. $\pi: M \rightarrow \bar{M} = M/\mathbb{Z}[-3, -2]$, $[x, y] \mapsto -2x + 3y$.

3.2. An alternative view on $\Theta(\ell, k)$

In general, Σ is not the normal fan of $\Theta(\ell, k)$ as it was of $\Delta(D)$. Geometrically, this means that $\Theta(\ell, k)$ does, in general, *not* encode a nef Cartier divisor on X . While $\Theta(\ell, k)$ is defined as some kind of a difference of polytopes, it is in general not true that the inclusions

$$\Theta(\ell, k) + k\Delta^{\text{nef}} \subseteq \ell\Delta(D) \quad \text{and} \quad \Theta(\ell, k) + k\Delta^{\text{newt}} \subseteq \ell\Delta(D)$$

become equalities (*cf.* Example 3.4). We present a suggestion on how to overcome this.

Recall from Section 2.4 that we had denoted by $r_{\max}, r_{\min} \in M$ the vertices of $\Delta(D)$ where $\langle \Delta(D), v_N \rangle$ attains its extremal values. Similarly, we denote by $r_{\max}'(\ell, k), r_{\min}'(\ell, k) \in M_{\mathbb{R}}$ the v_N -extremal vertices of $\Theta(\ell, k)$. The latter lead to the line segments $r_{\max}'(\ell, k) + k\Delta^{\text{newt}}$ and $r_{\min}'(\ell, k) + k\Delta^{\text{newt}}$, which cut the polytope $\ell\Delta(D)$ into three subpolytopes which we call $\square_{\max}(\ell, k)$, $\Delta(D)^{C(\ell, k)}$, and $\square_{\min}(\ell, k)$.

More concretely, here is how we obtain $\square_{\max}(\ell, k)$ and $\square_{\min}(\ell, k)$: We take the line segment $k\Delta^{\text{newt}}$ and fit it inside the polytope $\ell\Delta(D)$ until it hits the boundary twice. This way, we construct the lattice polygon (not necessarily a triangle) $\square_{\max}(\ell, k)$ such that $k\Delta^{\text{newt}}$ is one of its edges and $r_{\max}(\ell, k)$ is one of its vertices. In a similar way, we construct $\square_{\min}(\ell, k)$ using $r_{\min}(\ell, k)$.

As $\ell\Delta(D)$ before, the polytope $\Delta(D)^{C(\ell, k)}$ just defined still fulfills the equality

$$\Theta(\ell, k) = (\ell\Delta(D)^{C(\ell, k)} : k\Delta^{\text{newt}});$$

however, now we also have the equality

$$\Theta(\ell, k) + k\Delta^{\text{newt}} = \ell\Delta(D)^{C(\ell, k)}.$$

Remark 3.3. After this point, we will use the shorter notation $r_{\max}' = r_{\max}'(\ell, k)$, $r_{\min}' = r_{\min}'(\ell, k)$, $\square_{\max} = \square_{\max}(\ell, k)$, $\Delta(D)^C = \Delta(D)^{C(\ell, k)}$, $\square_{\min} = \square_{\min}(\ell, k)$. Nevertheless, one should keep in mind that all of these quantities depend on ℓ, k .

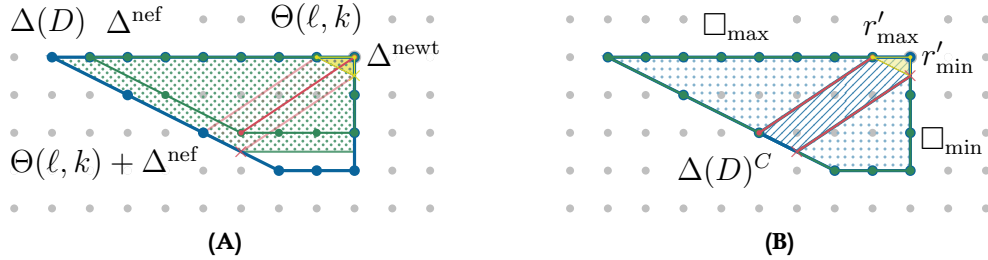


Figure 5. Alternative view on $\Theta(\ell, k)$. (A) The Minkowski sum $\Theta(\ell, k) + k\Delta^{\text{nef}}$ and $\Theta(\ell, k) + k\Delta^{\text{newt}}$. (B) The cut of $\Delta(D)$ along $r'_{\max} + k\Delta^{\text{newt}}$ and $r'_{\min} + k\Delta^{\text{newt}}$ into \square_{\max} , \square_{\min} , and $\Delta(D)^C$ with $\Delta(D)^C = \Theta(\ell, k) + k\Delta^{\text{newt}}$.

Example 3.4. Continuing Example 3.2, Figure 5A shows that the inclusions

$$\Theta(\ell, k) + k\Delta^{\text{nef}} = \text{conv}([0, 0], [-8, 0], [-3, -5/2], [0, -5/2]) \subsetneq \Delta(D)$$

and

$$\Theta(\ell, k) + k\Delta^{\text{newt}} = \text{conv}([0, 0], [-1, 0], [-4, -2], [-3, -5/2], [0, -1/2]) \subsetneq \Delta(D)$$

are strict in general for $(\ell, k) = (1, 1)$. We cut the polytope $\Delta(D)$ along the line segments $r'_{\max} + k\Delta^{\text{newt}}$ and $r'_{\min} + k\Delta^{\text{newt}}$ into the subpolytopes $\square_{\max} = \text{conv}([-1, 0], [-8, 0], [-4, -2])$, $\Delta(D)^C = \Theta(\ell, k) + k\Delta^{\text{newt}}$, and $\square_{\min} = \text{conv}([0, -1/2], [0, -3], [-2, -3], [-3, -5/2])$, where $r'_{\max} = [-1, 0]$ and $r'_{\min} = [0, -1/2]$ (*cf.* Figure 5B).

3.3. Projections of polytopes

We start by pulling back the sheaf $\mathcal{L}'(\ell, k)$. To this end, we define

$$\overline{d(\ell, k)} := \ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta^{\text{nef}}),$$

where $\text{width}_{v_N}(\cdot)$ denotes the lattice width of a polytope with respect to the linear functional $v_N \in N$; i.e., if $\Delta \subseteq M_{\mathbb{R}}$ is a polytope, then $\text{width}_{v_N}(\Delta) := \max_{m, m' \in \Delta} |\langle m, v_N \rangle - \langle m', v_N \rangle|$. Note that this equals the length of the line segment $\overline{\Delta} := \pi(\Delta)$; i.e.,

$$\overline{d(\ell, k)} = \ell \cdot \text{length}(\overline{\Delta(D)}) - k \cdot \text{length}(\overline{\Delta^{\text{nef}}}).$$

Proposition 3.5. *The pullback $i^*\mathcal{L}'(\ell, k)$ is a line bundle on \mathbb{P}^1 of degree $\overline{d(\ell, k)}$.*

Proof. We obtain

$$\begin{aligned} i^*\mathcal{L}'(\ell, k) &= i^*\mathcal{O}_X(\ell D - kC') = i^*\mathcal{O}_X(\ell\Delta(D) - k\Delta^{\text{nef}}) \\ &= \mathcal{O}_{\mathbb{P}^1}(\ell \cdot (D \cdot C') - k \cdot (C' \cdot C')) \\ &= \mathcal{O}_{\mathbb{P}^1}(\ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta^{\text{nef}})). \quad \square \end{aligned}$$

Remark 3.6. Altogether, this yields the sequence of inclusions

$$\pi(\Theta(\ell, k) \cap M) \subseteq \overline{\Theta(\ell, k)} = \overline{(\ell\Delta(D) : k\Delta^{\text{nef}})} \subseteq \overline{(\ell\overline{\Delta(D)} : k\overline{\Delta^{\text{nef}}})} =: \Xi(\ell, k),$$

which might be strict, where $\overline{\Delta} = \pi(\Delta)$ is the projection of any polytope $\Delta \subseteq M$ along v_M .

Example 3.7. Continuing Example 3.4, let us fix $(\ell, k) = (1, 1)$. Then the projected polytopes along π are

$$\begin{aligned} \overline{\Delta(D)} &= \text{conv}([-9], [16]), & \overline{\Delta^{\text{nef}}} &= \text{conv}([-6], [14]), \\ \overline{\Theta(\ell, k)} &= \text{conv}([-3/2], [2]), & \text{and } \Xi(\ell, k) &= \text{conv}([-3], [2]), \end{aligned}$$

where $\overline{d(\ell, k)} = 25 - 20 = 5$ (cf. Figure 6).

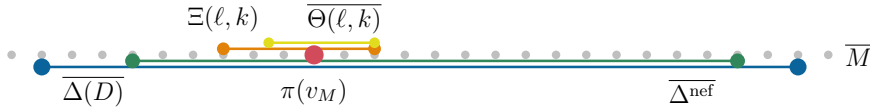


Figure 6. Projected polytopes.

Recall that the (torus-equivariant) global sections of $\mathcal{L}'(\ell, k)$ are encoded by the elements of $\Theta(\ell, k) \cap M$. Under this identification, their pullbacks via i^* are given by their images under π . Denote their number by

$$\overline{e(\ell, k)} := \#\pi(\Theta(\ell, k) \cap M).$$

Summarizing what we have done so far, we obtain the following.

Proposition 3.8. *The pullback $i^*\mathcal{L}'(\ell, k) = \mathcal{O}_{\mathbb{P}^1}(\Xi(\ell, k))$ is a line bundle on \mathbb{P}^1 of degree $\overline{d(\ell, k)}$. Its global sections correspond to the elements of $\Xi(\ell, k) \cap \overline{M}$. Under this identification, the subspace $i^*L'(\ell, k)$ coincides with $\pi(\Theta(\ell, k) \cap M)$. In particular,*

$$\dim(i^*L'(\ell, k)) = \overline{e(\ell, k)}.$$

Proof. Proposition 3.5 yields the degree $\overline{d(\ell, k)}$ of the pullback of $\mathcal{L}'(\ell, k)$. It remains to show that this sheaf is precisely given via $\Xi(\ell, k)$ as $\mathcal{O}_{\mathbb{P}^1}(\Xi(\ell, k))$, which is slightly finer information. The statement holds since if $\Delta \subseteq M_{\mathbb{R}}$ is a nef polytope (e.g., $\ell\Delta(D)$ or $k\Delta^{\text{nef}}$), then

$$i^*\mathcal{O}_X(\Delta) = \mathcal{O}_{\mathbb{P}^1}(\pi(\Delta)) = \mathcal{O}_{\mathbb{P}^1}(\overline{\Delta}).$$

This claim is valid for any toric map and does not depend on having \mathbb{P}^1 as a target. \square

Example 3.9. Continuing Example 3.7, we obtain $\Theta(\ell, k) \cap M = \{[0, 0], [-1, 0]\}$ and therefore $\pi(\Theta(\ell, k) \cap M) = \{0, 2\}$; i.e., $\overline{e(\ell, k)} = 2$ for $(\ell, k) = (1, 1)$.

3.4. Shape of the semigroup

As we did before, let us fix a pair (ℓ, k) . We know from Section 2.1 that we are supposed to collect the values $\text{ord}_{Y_2}(\overline{s}|_C)$ for all possible sections \overline{s} , where $Y_2 = \{1\}$ is a smooth point on C . In Section 3.1, we have transferred this setup to $\text{ord}_{1 \in \mathbb{P}^1}(i^*s')$, where s' runs through all global sections represented by the polytope $\Theta(\ell, k) \subseteq M_{\mathbb{R}}$.

Proposition 3.8 implies that the pullbacks i^*s' run through all $\overline{e(\ell, k)}$ elements of $\pi(\Theta(\ell, k) \cap M) \subseteq \overline{M} \cong \mathbb{Z}$. Each element of \mathbb{Z} represents a rational monomial function on \mathbb{P}^1 . We are supposed to find the orders of vanishing at $1 \in \mathbb{P}^1$ of all of their linear combinations.

Lemma 3.10. *Let $Z \subset \mathbb{Z}$ be a finite subset with e elements leading to the e -dimensional vector space*

$$\mathbb{C}[Z] := \{f \in \mathbb{C}[t, t^{-1}] \mid \text{supp}(f) \subseteq Z\}.$$

Then $\text{ord}_1 \mathbb{C}[Z] = \{0, 1, \dots, e-1\} = \text{ord}_c \mathbb{C}[Z]$ for all $c \in \mathbb{C}^ \subseteq \mathbb{P}^1$.*

Proof. Set $Z = \{p_1, \dots, p_e\}$. For an element $f \in \mathbb{C}[Z]$ with $f = \lambda_1 \cdot t^{p_1} + \dots + \lambda_e \cdot t^{p_e}$ and $d \in \mathbb{N}$, the rows of the matrix P given as

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_e \\ p_1(p_1-1) & p_2(p_2-1) & \dots & p_e(p_e-1) \\ p_1(p_1-1)(p_1-2) & p_2(p_2-1)(p_2-2) & \dots & p_e(p_e-1)(p_e-2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1 \cdot \dots \cdot (p_1 - (d-1)) & p_2 \cdot \dots \cdot (p_2 - (d-1)) & \dots & p_e \cdot \dots \cdot (p_e - (d-1)) \end{pmatrix}$$

encode $f(1) = \lambda_1 \cdot 1 + \dots + \lambda_e \cdot 1$, $f'(1) = \lambda_1 \cdot p_1 \cdot 1 + \dots + \lambda_e \cdot p_e \cdot 1$, $f''(1), \dots, f^{(d)}(1)$. Let p be an arbitrary variable. Then the linear spaces

$$L_1(p) := \text{span}_{\mathbb{Q}}\{1, p, p^2, \dots, p^d\} \subseteq \mathbb{Q}[p]$$

and

$$L_2(p) := \text{span}_{\mathbb{Q}}\{0! \binom{p}{0}, 1! \binom{p}{1}, 2! \binom{p}{2}, \dots, d! \binom{p}{d}\} \subseteq \mathbb{Q}[p]$$

coincide because $L_1(p) \supseteq L_2(p)$ and $\dim(L_1(p)) = \dim(L_2(p))$. In particular, there exists an invertible lower triangular matrix F such that $P = F \cdot V$, where $V := (p_j^i)_{0 \leq i \leq d, 1 \leq j \leq e}$ is the transposed Vandermonde matrix. If we choose $d = e-1$, the matrices V and thus P are invertible. Hence the system of linear equations

$$P \cdot (\lambda_1, \dots, \lambda_e)^T = 0,$$

which is equivalent to the system $f(1) = f'(1) = f''(1) = \dots = f^{(e-1)}(1) = 0$, has the unique solution $f = 0$, so $\text{ord}_1(f) \geq e$ is impossible.

On the other hand, replacing the equation $f^{(k)}(1) = 0$ by $f^{(k)}(1) = 1$ ($0 \leq k \leq e-1$) yields an f with $\text{ord}_1(f) = k$. \square

As a direct consequence, we obtain the following statement.

Theorem 3.11. *The valuation semigroup is given as*

$$S_{Y_\bullet}(D) = \{(\ell, k, \delta) \in \mathbb{N}^3 \mid 0 \leq \delta \leq \overline{e(\ell, k)} - 1\},$$

and $\overline{e(\ell, k)} = 0$ for large $k \gg \ell$.

Proof. The definition of the valuation semigroup can be reformulated as

$$S_{Y_\bullet}(D) = \{(\ell, k, \delta) \in \mathbb{N}^3 \mid s' \in \Gamma(X, \mathcal{L}'(\ell, k)) \setminus \{0\}, \text{ord}_{1 \in \mathbb{P}^1}(t^* s') = \delta\}.$$

Then everything follows from Proposition 3.8. \square

4. Shape of the Newton–Okounkov body

Building on Section 3, we determine the Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ in Theorem 4.3. Consider the assignment

$$d(\ell, k) := \text{width}_{v_N}(\Theta(\ell, k)) = \text{length}(\pi(\Theta(\ell, k))) = \text{length}(\overline{\Theta(\ell, k)}) \in \mathbb{Q} \sqcup \{-\infty\}.$$

We extend this definition to all $\ell, k \in \mathbb{R}_{\geq 0}$ using the convention $\text{width}_{v_N}(\emptyset) = -\infty$. This becomes necessary when $k \cdot \Delta^{\text{nef}}$ does not fit inside $\ell \cdot \Delta(D)$, which will happen for $k \gg \ell$.

This should not be confused with $\overline{d(\ell, k)}$, which was defined on page 10 as

$$\overline{d(\ell, k)} = \ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta^{\text{nef}}).$$

The chain of inclusions at the end of Section 3.3 gives rise to the inequalities

$$\overline{e(\ell, k)} - 1 \leq d(\ell, k) \leq \overline{d(\ell, k)}.$$

Example 4.1. Continuing Example 3.9, i.e., $(\ell, k) = (1, 1)$, $\overline{d(\ell, k)} = 5$, and $\overline{e(\ell, k)} = 2$, we obtain $d(\ell, k) = 7/2$ satisfying the inequalities $1 \leq 7/2 \leq 5$.

Moreover, we observe the following.

Lemma 4.2. *For $q \in \mathbb{R}_{\geq 0}$, the assignment $d(q) := d(\ell, \ell q)/\ell$ does not depend on ℓ . In particular, $d(q) = d(1, q)$.*

Proof. The width function is linear in its polyhedral argument. \square

Note that the same statement holds true for $\overline{d(\ell, k)}$ but not for $\overline{e(\ell, k)}$.

From now on, we return to $(\ell, k) \in \mathbb{N}^2$.

Theorem 4.3. *The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ coincides with the convex hull of the set*

$$\{[q, t] \in (\mathbb{R}_{\geq 0})^2 \mid 0 \leq t \leq d(q)\} = \bigcup_{q \in \mathbb{R}_{\geq 0}, d(q) \geq 0} \text{conv}([q, 0], [q, d(q)]).$$

Moreover, $d(q)$ is a decreasing piecewise linear function with $d(q) = -\infty$ for $q \gg 0$.

Proof. Let $\epsilon > 0$ and denote by A, B the vertices of $\pi(\Theta(\ell, k)) = \overline{\Theta(\ell, k)}$. First assume that the dimension of $\Theta(\ell, k)$ equals two. Then the two fibers

$$\pi^{-1}(T) \cap \Theta(\ell, k)$$

with $T = A + \epsilon$ or $B - \epsilon$ have positive lengths greater than (or equal to) some $\mu > 0$. In particular, all fibers in between do so as well. Hence, setting $\lambda = 1/\mu$, the fibers

$$\pi^{-1}(T') \cap \lambda\Theta(\ell, k)$$

have at least length one for $\lambda(A + \epsilon) \leq T' \leq \lambda(B - \epsilon)$. If in addition $T' \in \overline{M}$, then all of these fibers have to contain lattice points in M . Thus, we obtain

$$\text{conv}(\lambda(A + \epsilon), \lambda(B - \epsilon)) \subseteq \pi(\lambda\Theta(\ell, k) \cap M).$$

Keeping $q = k/\ell$ constant and using Lemma 4.2, we see that $\overline{e(\ell, k)}$ behaves like $d(\ell, k)$ asymptotically with respect to dilations. The result then follows by recalling the fact that Newton–Okounkov bodies are closed.

It remains to consider the pathological case $\dim(\Theta(1, q)) = 1$. Here, we can approximate $[q, t]$ by $[q - \epsilon, t]$ so that the resulting $\Theta(1, q - \epsilon)$ is full-dimensional and $t \leq d(q) \leq d(q - \epsilon)$. Then the previous argument shows that $[q - \epsilon, t] \in \Delta_{Y_\bullet}(D)$. As Newton–Okounkov bodies are closed by definition, $[q, t] \in \Delta_{Y_\bullet}(D)$. \square

We remark that the case $\dim(\Theta(1, q)) = 1$ from the previous proof requires special v_N and a unique $q_0 = k/\ell$. This configuration is characterized by the fact that (a shift of) $q_0 \Delta^{\text{newt}}$ connects two parallel edges of $\Delta(D)$. Note that $\Theta(\ell, k)$ is also parallel to these edges. In contrast to the general case, for $k/\ell = q_0$, the number $e(\ell, k)$ behaves asymptotically like $1/g \cdot d(\ell, k)$, where

$$g := d(\ell, k) / \text{length}_M(\Theta(\ell, k)).$$

Despite that $\overline{e(\ell, k)}$ for $k/\ell = q_0$ does not approach $d(\ell, k)$ at all, this does not cause a problem: as we have seen in the proof, for $q \leq q_0$, the general case applies, and for $q > q_0$, we have $d(q) = -\infty$ anyway.

Example 4.4. We continue Example 4.1 and apply Theorem 4.3. The Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ (cf. Figure 7 and [HKW20]) is given as

$$\Delta_{Y_\bullet}(D) = \text{conv}([0, 0], [0, 25], [8/7, 0], [2/3, 35/3]).$$

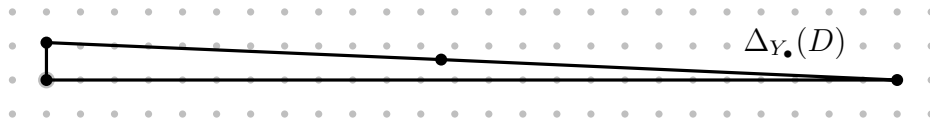


Figure 7. Newton–Okounkov body $\Delta_{Y_\bullet}(D)$ with flipped coordinates.

This example already gives an instance of a vertex that does not lift to the semigroup (cf. Definition 5.3) when building the Newton–Okounkov body in question. Let us consider the vertex $[\frac{2}{3}, \frac{35}{3}]$ and fix $\ell = 3$, $k = 2$. The respective polyhedra $3\Delta(D)$, $2\Delta^{\text{newt}}$, $2\Delta^{\text{nef}}$, and $\Theta(\ell, k) = (3\Delta(D) : 2\Delta^{\text{nef}})$ are pictured in Figure 8. To hit the vertex $[\frac{2}{3}, \frac{35}{3}]$, the value of $e(3, 2)$ would have to coincide with $d(3, 2) = 35$. However, we only obtain $e(3, 2) = \#(\pi(\Theta(\ell, k) \cap M)) = 30$. The red lines in Figure 8 indicate the gaps, i.e., the fibers of π with no lattice points in $\Theta(\ell, k)$. No matter how big of a multiple of (ℓ, k) we consider, the gaps will not be closed in any scaled version of the situation. Hence, the vertex $[\frac{2}{3}, \frac{35}{3}]$ is never hit, and the associated valuation semigroup $S_{Y_\bullet}(D)$ is therefore not finitely generated.

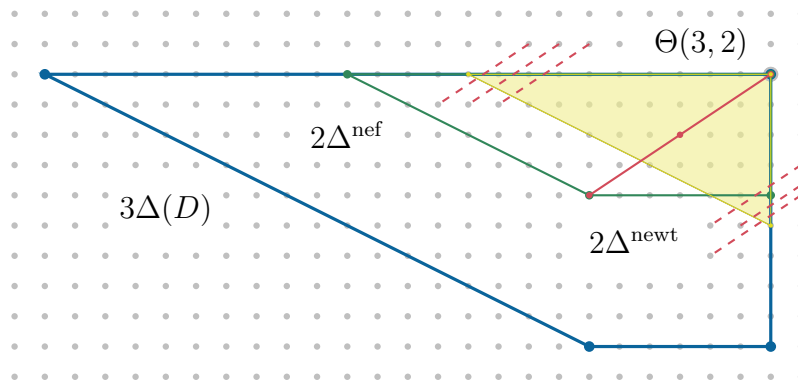


Figure 8. Combinatorial view on a non-finitely generated semigroup $S_{Y_\bullet}(D)$. The polytopes $3\Delta(D)$, $2\Delta^{\text{newt}}$, $2\Delta^{\text{nef}}$, and $\Theta(3, 2) = (3\Delta(D) : 2\Delta^{\text{nef}})$. The dashed red lines indicate the difference between $e(3, 2) - 1 = 29$ and $d(3, 2) = 35$.

5. Criterion for the finite generation of certain valuation semigroups

We provide a criterion for the finite generation of strictly positive (with respect to their height functions) semigroups in terms of their limit polyhedra.

5.1. Semigroups with polyhedral limit

We start with a free abelian group M of rank n , i.e., $M \cong \mathbb{Z}^n$, and a linear form $h: M \rightarrow \mathbb{Z}$ which we call a *height function*. This induces $h_{\mathbb{R}}: M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}$, which we will often denote by h as well.

Let $S \subseteq h^{-1}(\mathbb{N})$ be a semigroup that is *strictly positive*; i.e., $S \cap \ker(h) = \{0\}$. In order to refer to the individual layers of a given height, we will write

$$S_k := S \cap h^{-1}(k); \text{ i.e., we have } S = \bigcup_{k \in \mathbb{N}} S_k \text{ with } S_0 = \{0\}.$$

This setup allows us to define the enveloping cone

$$C_S := \overline{\text{cone } S} \subseteq M_{\mathbb{R}}$$

as well as the convex limit figure

$$\Delta_S := C_S \cap h_{\mathbb{R}}^{-1}(1) \supseteq S_1.$$

In the case of a valuation semigroup $S_Y(D)$, the height $h: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ is the projection on ℓ , which then leads to the Newton–Okounkov body $\Delta_{Y_\bullet}(D) = C_{S_{Y_\bullet}(D)} \cap h_{\mathbb{R}}^{-1}(1) \subseteq h_{\mathbb{R}}^{-1}(1) \cong \mathbb{R}^2$.

Definition 5.1. We say that S has a *polyhedral limit* if Δ_S is a polytope, i.e., if Δ_S equals the convex hull of its (finitely many) vertices.

This property is fulfilled whenever the semigroup S is finitely generated. In this case, Δ_S even has rational vertices; it is a rational polytope. However, the following standard example shows that the converse implication does not hold in general.

Example 5.2. Let $h: \mathbb{Z}^2 \rightarrow \mathbb{Z}, [x, y] \mapsto x + y$ be the summation map. Then $S := \{0\} \cup (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1})$ is not finitely generated, but $C_S = \mathbb{R}_{\geq 0}^2$ and Δ_S equals the line segment connecting the points $[1, 0]$ and $[0, 1]$.

5.2. Equivalent conditions for finite generation

We assume that $S \subseteq M$ is a strictly positive (with respect to h) semigroup that has polyhedral limit Δ_S .

Definition 5.3. We say that a point $p \in \Delta_S$ *lifts to the semigroup* S (i.e., is a valuation point) if there exists some scalar $c \in \mathbb{R}_{>0}$ with $c \cdot p \in S$.

Note that in this case, both p and c have to be rational; i.e., $p \in \Delta_S \cap M_{\mathbb{Q}}$ and $c \in \mathbb{Q}_{>0}$. Hence, it is not a surprise that the assumption of the next lemma is automatically fulfilled if the semigroup S is finitely generated.

Lemma 5.4. *If all vertices of Δ_S lift to S , then they are rational (i.e., Δ_S is a rational polytope) and every rational point $p \in \Delta_S \cap M_{\mathbb{Q}}$ lifts to S .*

Proof. Let $v^1, \dots, v^d \in \Delta_S$ be linearly independent (rational) vertices such that p is contained in their convex hull. Then the unique coefficients λ_i in the representation $p = \sum_{i=1}^d \lambda_i v^i$ have to be rational, too. Thus, we may choose an integer μ such that $\mu \cdot \lambda_i \in \mathbb{N}$. On the other hand, there is a joint factor $c \in \mathbb{Z}_{\geq 1}$ such that all multiples $c \cdot v^i$ belong to S . This implies

$$\mu c \cdot p = \sum_{i=1}^d \mu \lambda_i \cdot (c \cdot v^i) \in S. \quad \square$$

Next, we formulate the main point of this section.

Proposition 5.5. *A semigroup S with a polyhedral limit Δ_S is finitely generated if and only if all vertices of Δ_S lift to S .*

We have already seen that this condition is necessary for the finite generation. Now we will show that it is sufficient, too. Note that, in Example 5.2, the two vertices of the line segment Δ_S indeed do not lift to the semigroup.

Let $S \subseteq M$ be a subsemigroup with rational polyhedral limit $\Delta(S)$ (with respect to some height function $h: M \rightarrow \mathbb{Z}$). Assume that the vertices and thus, by Lemma 5.4, all rational points of Δ_S lift to S . Moreover, we may assume that C_S is a full-dimensional cone.

Proof of Proposition 5.5. Assume that S is not finitely generated. Then S has infinitely many indecomposable elements; *i.e.*,

$$\mathcal{H} := S \setminus \left((S \setminus \{0\}) + (S \setminus \{0\}) \right)$$

is infinite; *cf.* [CLS11, Proposition 1.2.23].

By taking a simplicial subdivision we may, w.l.o.g., assume that the cone C_S is simplicial and given as $C_S = \text{cone}(s_1, \dots, s_n)$. Consider the lattice Λ generated by s_1, \dots, s_n . As M/Λ is finite, there must be a coset $m + \Lambda$ which contains infinitely many elements of \mathcal{H} . Here we may choose m to be a minimal representative in C_S : $m \in C_S \cap M$ so that $m - s_i \notin C_S$ for $i = 1, \dots, n$.

As the elements in $C_S \cap \mathcal{H}$ were indecomposable in S , they certainly are indecomposable in $C_S \cap S$. In particular, if we identify $(m + \Lambda) \cap C_S$ with \mathbb{N}^n , we obtain an infinite set of pairwise incomparable elements, in contradiction to Dickson's lemma [CLO15, Chapter 4, Theorem 5]. \square

6. Finite generation criterion

6.1. Characterising the lifting property

The following theorem gives a purely combinatorial criterion to check if our valuation semigroup $S_Y(D)$ (in the language of Section 2.2) is finitely generated. Recall the definition

$$d(k/\ell) = \text{length}(\pi(\Theta(\ell, k)))$$

from Section 4 on page 12.

Theorem 6.1. *The point $(1, k/\ell, d(k/\ell)) = (1, q, d(q))$ is a valuation point (*i.e.*, a multiple of it lies in $S_Y(D)$) if and only if there exists a $\lambda \in \mathbb{N}$ such that*

$$\pi: \lambda\Theta(1, q) \cap M \longrightarrow \pi(\lambda\Theta(1, q)) \cap \overline{M}$$

*is surjective (*i.e.*, $\lambda \cdot d(q) = \overline{e(\lambda, \lambda q)} - 1$) and $\pi(\lambda\Theta(1, q))$ has endpoints in \overline{M} .*

Proof. By definition, $(1, q, d(q))$ is a valuation point if and only if there exists a $\lambda \in \mathbb{N}$ such that we have $\lambda \cdot (1, q, d(q)) \in S_Y(D) \subseteq \mathbb{N}^3$. By Theorem 3.11, the latter happens exactly if

$$0 \leq \lambda \cdot d(q) \leq \overline{e(\lambda, \lambda q)} - 1,$$

where $\overline{e(\lambda, \lambda q)} = \#\pi(\lambda\Theta(1, q) \cap M)$. In addition, we see that

$$\begin{aligned} \lambda \cdot d(q) &= \lambda \cdot d(1, q) = \lambda \cdot \text{length}(\pi(\Theta(1, q))) \\ &= \text{length}(\lambda\pi(\Theta(1, q))) = \text{length}(\pi(\lambda\Theta(1, q))) \\ &\geq \#\pi(\lambda\Theta(1, q)) \cap \overline{M} - 1 \\ &\geq \#\pi(\lambda\Theta(1, q) \cap M) - 1 = \overline{e(\lambda, \lambda q)} - 1. \end{aligned}$$

Combining all these inequalities, we obtain the equations

$$(6.1) \quad \lambda \cdot d(q) = \#\pi(\lambda\Theta(1, q)) \cap \overline{M} - 1$$

and

$$(6.2) \quad \#\pi(\lambda\Theta(1, q)) \cap \overline{M} = \#\pi(\lambda\Theta(1, q) \cap M),$$

where Equation (6.1) is equivalent to $\pi(\lambda\Theta(1, q))$ having end points in \overline{M} , and Equation (6.2) to

$$\pi: \lambda\Theta(1, q) \cap M \longrightarrow \pi(\lambda\Theta(1, q)) \cap \overline{M}$$

being surjective (*i.e.*, π meets all possible lattice points in $\pi(\lambda\Theta(1, q)) \cap \overline{M}$). \square

In the following, the *tangent cone* of a polygon Θ at a point $r \in \Theta$ is the cone generated by $\Theta - r$. It is pointed if and only if r is a vertex.

Lemma 6.2. *Suppose that the functional $v_N: M \rightarrow \mathbb{Z}$ takes its minimum and maximum values over $\Theta(1, q)$ at the two vertices r_{\min}' and r_{\max}' , respectively. Denote their tangent cones by σ_{\min}' and σ_{\max}' .*

Then $(1, q, d(q))$ is a valuation point if and only if $1 \in \pi(\sigma_{\min}' \cap M)$ and $-1 \in \pi(\sigma_{\max}' \cap M)$.

Proof. If $(1, q, d(q))$ is a valuation point, then according to Theorem 6.1, there exists a $\lambda \in \mathbb{N}$ such that $\pi(\lambda\Theta(1, q))$ has vertices in \overline{M} and such that $\pi: \lambda\Theta(1, q) \cap M \rightarrow \pi(\lambda\Theta(1, q)) \cap \overline{M}$ is surjective. Hence, $1 \in \pi(\lambda(\Theta(1, q) - r_{\min}') \cap M) \subset \pi(\sigma_{\min}' \cap M)$ and similarly for σ_{\max}' .

For the converse, assume $1 \in \pi(\sigma_{\min}' \cap M)$ and $-1 \in \pi(\sigma_{\max}' \cap M)$. We will construct a suitable scaling factor $\lambda \in \mathbb{N}$ for which

$$\pi: \lambda\Theta(1, q) \cap M \longrightarrow \pi(\lambda\Theta(1, q)) \cap \overline{M}$$

is surjective and $\pi(\lambda\Theta(1, q))$ is a lattice polytope, in order to again apply Theorem 6.1. To this end, choose levels $\delta_{\min} \in \mathbb{Q}_{>0}$ and $\delta_{\max} \in \mathbb{Q}_{>0}$ such that

$$\sigma_{\min}' \cap [v_N \leq \delta_{\min}] \subset \Theta(1, q) - r_{\min}' \quad \text{and} \quad \sigma_{\max}' \cap [v_N \geq \delta_{\max}] \subset \Theta(1, q) - r_{\max}'.$$

We set

$$\varepsilon_{\min} = \text{length}(\sigma_{\min}' \cap [v_N = \delta_{\min}]) \quad \text{and} \quad \varepsilon_{\max} = \text{length}(\sigma_{\max}' \cap [v_N = \delta_{\max}]).$$

Choose a $\lambda \in \mathbb{N}$ with $\lambda > \frac{1}{\varepsilon_{\min}}$ and $\lambda > \frac{1}{\varepsilon_{\max}}$ such that $\lambda r_{\min}'$ and $\lambda r_{\max}'$ are lattice points in M . Then the corresponding projection π is surjective. To show that, we divide the image $\pi(\lambda\Theta(1, q))$ into three parts. The first $\lambda\delta_{\min}$ lattice points in $\pi(\lambda\Theta(1, q)) \cap \overline{M}$ are in the image of π because π restricted to σ_{\min}' is surjective by assumption. The same argument holds for the last $\lambda\delta_{\max}$ points since π is surjective on σ_{\max}' . The points in between are hit by projecting the lattice points in $\lambda\Theta(1, q)$ because all the respective fibers have length greater than $\text{length}(v_M) = 1$, by construction. Thus π is surjective and $(1, q, d(q))$ is a valuation point, according to Theorem 6.1. \square

Remark 6.3. The case where v_N takes its minimum (or maximum) not at a vertex but at an edge is actually easier to handle. As soon as $\lambda\Theta(1, q)$ is a lattice polytope, the edge is an integral multiple of v_M . So we can omit the first (or last) of the three parts in the above proof.

6.2. Strong decomposability

We have seen that it is important to decide the surjectivity of the projection of lattice points in a cone in $M_{\mathbb{R}}$. Next, we will translate this surjectivity into a statement in N . To this end, we introduce the following notion.

Definition 6.4. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. A lattice point $u \in \text{int}(\sigma) \cap N$ is *strongly decomposable* in σ if $u = u' + u''$ for suitable $u', u'' \in \text{int}(\sigma) \cap N$.

Lemma 6.5. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone and $u \in \text{int}(\sigma) \cap N$ a direction. Then the following statements are equivalent:*

- (i) We have $1 \notin \langle \mathcal{H}_{\sigma^\vee}, \mathbf{u} \rangle$, where $\mathcal{H}_{\sigma^\vee}$ denotes the Hilbert basis of σ^\vee .
- (ii) The direction \mathbf{u} is strongly decomposable in σ .
- (iii) The closure of the one-parameter subgroup $\lambda^{\mathbf{u}}(\mathbb{C}^*) \subseteq \mathbb{T}$ in $\mathbb{T}\mathbb{V}(\sigma)$ is singular.

Proof. (i) and (iii) are equivalent: The 1-parameter subgroup represented by \mathbf{u} can always be extended to $\lambda^{\mathbf{u}}: \mathbb{C} \rightarrow \mathbb{T}\mathbb{V}(\sigma)$. On the dual level of regular functions, however, this corresponds to

$$\langle \cdot, \mathbf{u} \rangle: \mathbb{C}[\sigma^\vee \cap M] \longrightarrow \mathbb{C}[\mathbb{N}].$$

The latter map is surjective if and only if $1 \in \langle \sigma^\vee \cap M, \mathbf{u} \rangle$.

(i) \Rightarrow (ii): By assumption, there exist primitive lattice points $s^0, s^1 \in M \setminus \sigma^\vee$ such that the line segment between them lying on $[\mathbf{u} = 1]$ contains no interior lattice point but intersects σ^\vee . Note that $\{s^0, s^1\}$ is a \mathbb{Z} -basis because the lattice triangle $\text{conv}(0, s^0, s^1)$ is unimodular. Then

$$\text{int}(\text{cone}(s^0, s^1)) \supset \sigma^\vee \quad \text{and thus} \quad \text{cone}(t^0, t^1) \subset \text{int}(\sigma),$$

where $\{t^0, t^1\}$ denotes the basis dual to $\{s^0, s^1\}$. By the definition of the dual basis, this yields $\langle s^i, \mathbf{u} \rangle = 1 = \langle s^i, t^0 + t^1 \rangle$ for all i ; *i.e.*, the two linear functionals coincide on the basis. Therefore, $\mathbf{u} = t^0 + t^1$.

(ii) \Rightarrow (i): Let \mathbf{u} be strongly decomposable in σ . Then $\mathbf{u} = \mathbf{u}' + \mathbf{u}''$ for some $\mathbf{u}', \mathbf{u}'' \in \text{int}(\sigma) \cap N$. Therefore, we have $\langle m, \mathbf{u} \rangle = \langle m, \mathbf{u}' \rangle + \langle m, \mathbf{u}'' \rangle \in \mathbb{Z}$ for $m \in \mathcal{H}_{\sigma^\vee}$. Since \mathbf{u}' and \mathbf{u}'' lie in the interior of σ , both summands are positive. Thus, $1 \notin \langle \mathcal{H}_{\sigma^\vee}, \mathbf{u} \rangle$. \square



Figure 9. Strongly decomposable primitive element. (A) The polytope $\Theta(3, 2)$ with tangent cone σ_{\min}' at the v_N -extremal vertex r_{\min}' . (B) The dual cone $(\sigma_{\min}')^\vee$ together with the strong decomposition $(-2, 3) = (-2, 1) + (0, 2)$ of v_N inside it.

Example 6.6. We continue Example 4.4 and fix $(\ell, k) = (3, 2)$. Then

$$\Theta(\ell, k) = \text{conv}([0, 0], [-10, 0], [0, -5])$$

(*cf.* Figure 8). Its two v_N -extremal vertices are $r_{\max}' = [-10, 0]$ and $r_{\min}' = [0, -5]$. Thus, the corresponding tangent cones σ_{\min}' and σ_{\max}' are given as $\text{cone}([0, 1], [-2, 1])$ and $\text{cone}([1, 0], [2, -1])$, respectively (*cf.* Section 3.2). Hence the direction $v_N = (-2, 3)$ is contained in the interior of $(\sigma_{\min}')^\vee = \text{cone}((-1, 0), (1, 2))$. It is strongly decomposable in $(\sigma_{\min}')^\vee$ because $v_N = (-2, 1) + (0, 2)$ (*cf.* Figure 9). An application of Lemma 6.5 yields that we do not obtain 1 via $\langle \mathcal{H}_{\sigma_{\min}'}, v_N \rangle$.

Now we are ready to formulate our finite generation criterion. The essence is that it suffices to check strong decomposability of v_N in two specific cones which we now define.

Definition 6.7. Given $\Delta(D)$ and v_N , we define the rational line segment $\Delta(D)^{v_N} \subseteq \Delta(D)$ as a line segment $\Delta(D) \cap [v_N = c]$ of maximal length orthogonal to v_N . We call its vertices v_1 and $v_2 \in M_{\mathbb{R}}$ and its length \hat{q} so that $v_2 - v_1 = \hat{q}v_M$. Moreover, we denote by e_1^+ the part of the edge of $\Delta(D)$ with vertex v_1 lying in the half-plane $[v_N \geq c]$, and by e_2^+ the part of the edge of $\Delta(D)$ with vertex v_2 also lying in the half-plane $[v_N \geq c]$. The cone $\sigma^- \subseteq N_{\mathbb{R}}$ is the cone generated by the inner normal vectors of e_1^+ and e_2^+ .

In the same manner, we define the line segments e_1^- and e_2^- contained in $[v_N \leq c]$, which yield the cone σ^+ .

Observe that $\Theta(1, q) \neq \emptyset$ (and thus $d(q) \geq 0$) if and only if $q \in [0, \hat{q}]$.

Theorem 6.8. *The valuation semigroup $S_{Y_\bullet}(D)$ is finitely generated if and only if v_N is not strongly decomposable in σ^+ and $-v_N$ is not strongly decomposable in σ^- .*

Proof. Combining Lemmas 6.2 and 6.5, we see that $S_{Y_\bullet}(D)$ is finitely generated if and only if for every rational $q \in [0, \hat{q}]$, the vector v_N is not strongly decomposable in $(\sigma_{\min}')^\vee$ and $-v_N$ is not strongly decomposable in $(\sigma_{\max}')^\vee$.

Now we use that for all $q \in [0, \hat{q}]$,

$$(\sigma^+)^\vee \subseteq \sigma_{\min}' \quad \text{and} \quad (\sigma^-)^\vee \subseteq \sigma_{\max}',$$

so v_N is strongly decomposable in $(\sigma_{\min}')^\vee$ for some q if and only if it is strongly decomposable in σ^+ , and correspondingly for $-v_N$. \square

Corollary 6.9. *The valuation semigroup $S_{Y_\bullet}(D)$ is finitely generated if and only if the morphism $\mathbb{P}^1 \rightarrow X'$ given by v_N is a smooth embedding, where X' is the toric variety associated with the fan generated by σ^+ and σ^- .*

Example 6.10. We apply Theorem 6.8 to the 7-gon $\Delta(D)$ (cf. Figure 10A) with vertices $[4, 1]$, $[7, 2]$, $[9, 3]$, $[6, 5]$, $[1, 8]$, $[1, 7]$, and $[2, 4]$ from [CLTU23, Example 4.8], which is a *good polytope* in the language of *op. cit.*

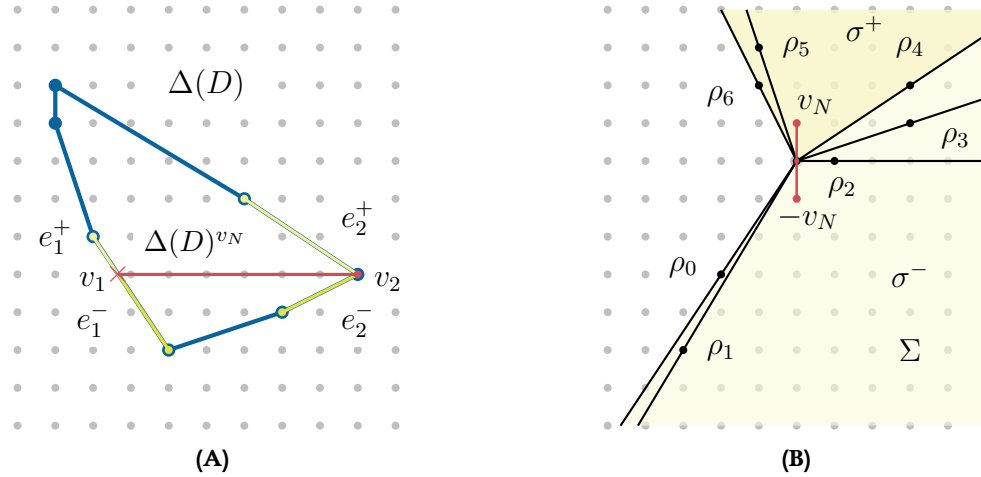


Figure 10. The good 7-gon $\Delta(D)$. (A) The polytope $\Delta(D)$ having seven vertices together with the rational line segment $\Delta(D)^{v_N} = \Delta(D) \cap [v_N = 3]$ and its two vertices $v_1 = [8/3, 3]$, $v_2 = [9, 3]$. (B) The normal fan Σ of $\Delta(D)$ having seven rays ρ_i ($0 \leq i \leq 6$) together with the two cones $\sigma^+ = \text{cone}(\rho_4, \rho_6) \ni v_N$ and $\sigma^- = \text{cone}(\rho_0, \rho_4) \ni -v_N$.

In *op. cit.*, the authors construct examples of projective toric surfaces whose blow-ups at the general point have a non-polyhedral pseudo-effective cone. In particular, this is the case for projective toric surfaces associated with good polytopes; cf. [CLTU23, Definition 4.3, Theorem 4.4].

Consider the projective toric surface $X = \mathbb{T}\mathbb{V}(\Sigma)$ associated with the normal fan Σ of $\Delta(D)$ with rays

$$\begin{aligned} \rho_0 &= (-2, -3), \quad \rho_1 = (-3, -5), \quad \rho_2 = (1, 0), \quad \rho_3 = (3, 1), \\ \rho_4 &= (3, 2), \quad \rho_5 = (-1, 3), \quad \text{and} \quad \rho_6 = (-1, 2) \end{aligned}$$

(cf. Figure 10B) and an admissible flag $Y_\bullet: X \supseteq C \supseteq \{1\}$ on X with $v_N = (0, 1)$.

To apply Theorem 6.8, we compute the following data:

$$\Delta(D)^{v_N} = \Delta(D) \cap [v_N = 3] = \text{conv}([8/3, 3], [9, 3]),$$

i.e., $v_1 = [8/3, 3]$ and $v_2 = [9, 3]$. The inner normal vectors of e_1^+ , e_2^+ and e_1^- , e_2^- are ρ_4 , ρ_0 and ρ_4 , ρ_6 , respectively. Therefore,

$$\sigma^+ = \text{cone}(\rho_4, \rho_6) \ni (0, 1) = v_N \quad \text{and} \quad \sigma^- = \text{cone}(\rho_4, \rho_0) \ni (0, -1) = -v_N.$$

Thus the associated semigroup $S_{Y_\bullet}(D)$ is finitely generated because $v_N = (0, 1)$ is not strongly decomposable in σ^+ and $-v_N = (0, -1)$ is not strongly decomposable in σ^- .

6.3. Varying v_N and D

The strategy to obtain a finitely generated semigroup by choosing an appropriate direction v_N does not always work. For the following example, there is no direction v_N that works.

Example 6.11. Consider the ample divisor D associated with the polytope $\Delta(D)$ depicted in Figure 11A on the toric variety $X = \mathbb{T}\mathbb{V}(\Sigma)$ corresponding to the fan Σ depicted in Figure 11B. We claim that no matter what $v_N \in N$ we pick, the resulting semigroup $S_{Y_\bullet}(D)$ will not be finitely generated.



Figure 11. $S_{Y_\bullet}(D)$ non-finitely generated for all v_N . (A) The polytope $\Delta(D)$ associated with an ample divisor D on $X = \mathbb{T}\mathbb{V}(\Sigma)$. (B) The fan Σ of $\mathbb{T}\mathbb{V}(\Sigma)$ with 16 rays.

We will use our characterization in Theorem 6.8. As $\Delta(D)$ is centrally symmetric, the longest line segment $\Delta(D)^{v_N}$ in Definition 6.7 will pass through the origin, whatever v_N .

We distinguish two cases: either the endpoints of the segment $\Delta(D)^{v_N}$ are vertices of $\Delta(D)$, or they belong to the interior of an edge. Up to symmetry, there are four vertices and four edges to consider. We will carry out the argument for one vertex and for one edge. The others are left to the reader.

If $\Delta(D)^{v_N}$ hits the interior of the edges e_1^\pm, e_2^\pm indicated in Figure 12A, then v_N must belong to the interior of the red region in Figure 12B. In this case, the cones σ^\pm from Definition 6.7 will be the two half-planes bounded by the dotted line which is perpendicular to the direction of e_1^\pm, e_2^\pm . But all lattice vectors in the red region are strongly decomposable in their half-plane as they all have lattice distance greater than one from the dotted line. (The vectors which are not strongly decomposable are the vectors at distance one; *i.e.*, they lie on the dashed lines.)

If, on the other hand, $\Delta(D)^{v_N}$ contains the vertices v_1, v_2 indicated in Figure 12C, then v_N is determined up to sign, as are σ^\pm (*cf.* Figure 12D). Again, we see that v_N is strongly decomposable. (And again, the vectors which are not strongly decomposable are the vectors which lie on the dashed lines.)

Proposition 6.12. *Given a fan Σ and a direction v_N , the valuation semigroup $S_{Y_\bullet}(D)$ is finitely generated for all ample divisors D on X if and only if v_N and $-v_N$ are not strongly decomposable in σ for all cones $\sigma \subseteq N_{\mathbb{R}}$ generated by rays of the given fan Σ .*

Proof. “ \Leftarrow ”: Theorem 6.8. “ \Rightarrow ”: Assume there exists a cone σ built from rays of Σ such that v_N is strongly decomposable in σ . We will construct an ample divisor D such that $S_{Y_\bullet}(D)$ is not finitely generated.

In a first step, we build a torus-invariant divisor $D_{\Theta(\ell, k)} = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho$ whose associated polytope $\Theta(\ell, k)$ has a vertex r_{\min}' with tangent cone $\sigma_{\min}' = \sigma^\vee$. For that we choose coefficients $b_\rho \in \mathbb{Z}$ such that $D_{\Theta(\ell, k)}$ has positive intersection numbers with all curves corresponding to rays $\rho \not\subseteq \text{int}(\sigma)$. Then we relax the inequalities at r_{\min}' by choosing the coefficients $b_\rho \gg 0$ for all $\rho \subseteq \text{int}(\sigma)$. This guarantees that σ^\vee is the tangent cone of $\Theta(\ell, k)$ at its vertex r_{\min}' .

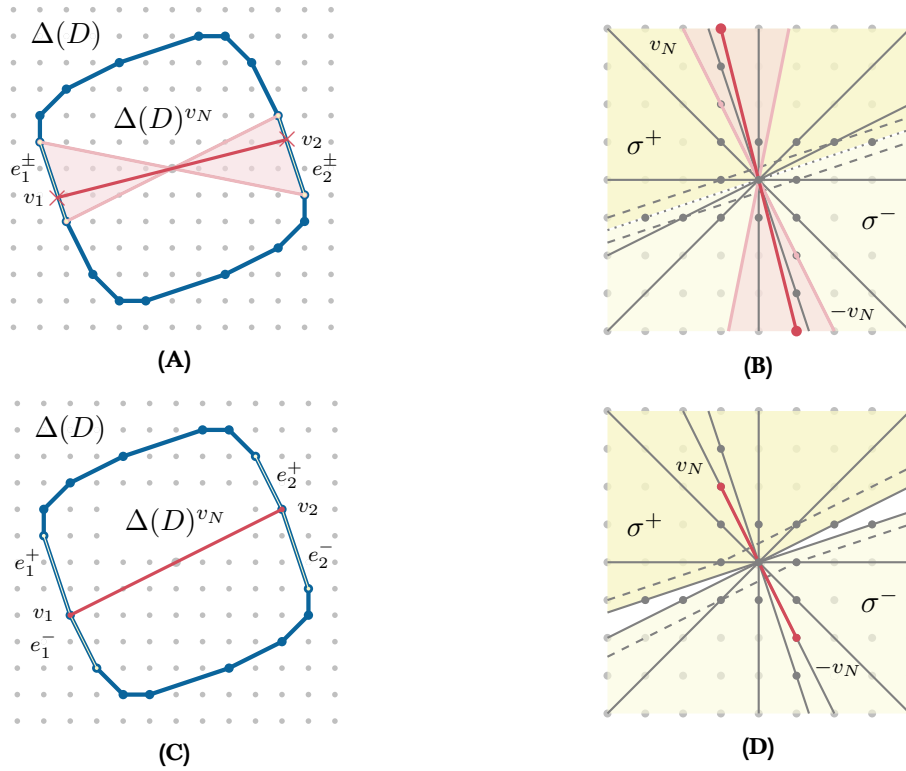


Figure 12. Illustration of Example 6.11. (A) The line segment $\Delta(D)^{v_N}$ containing the interior of the edges e_1^\pm, e_2^\pm . (B) The possible region for $\pm v_N$ in red together with the two cones σ^\pm . (C) The line segment $\Delta(D)^{v_N}$ hitting two vertices v_1, v_2 of $\Delta(D)$. (D) The cones σ^\pm containing $\pm v_N$.

Now, we want to define an ample torus-invariant divisor D such that $\Theta(\ell, k) = (\Delta(D) : \Delta^{\text{nef}})$. As an intermediate step, set $D' = \sum_{\rho \in \Sigma(1)} a'_\rho D_\rho := C' + D_{\Theta(\ell, k)}$. Then by construction, the associated polytope $\Delta(D')$ contains the Minkowski sum $\Theta(\ell, k) + \Delta^{\text{nef}}$. Moreover, the defining inequalities of $\Theta(\ell, k) + \Delta^{\text{nef}}$ coincide with the ones for the polytope $\Delta(D')$ for all rays $\rho \not\subseteq \text{int}(\sigma)$.

In general, D' is not nef and in particular not ample because it might have negative intersection with curves associated with remaining rays $\rho \subseteq \text{int}(\sigma)$. Hence, as a last step, we define a torus-invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ with $a_\rho = a'_\rho$ for all $\rho \in \Sigma(1) \setminus \text{int}(\sigma)$. For the remaining rays $\rho \subseteq \text{int}(\sigma)$, choose coefficients a_ρ small enough such that D is ample and big enough such that $\Theta(\ell, k) + \Delta^{\text{nef}} \subseteq \Delta(D)$. Then we have $\Theta(\ell, k) = (\Delta(D) : \Delta^{\text{nef}})$, and $\Theta(\ell, k)$ has a vertex r_{\min}' with tangent cone $\sigma_{\min}' = \sigma^\vee$. Since v_N is strongly decomposable in σ , it follows from Theorem 6.8 that the semigroup $S_{Y_\bullet}(D)$ is not finitely generated. \square

Example 6.13. We illustrate the construction from the proof of Proposition 6.12 for a modification of our running Example 6.6 with $v_N = (-2, 3) \in N$. Let $X' = \mathbb{T}\mathbb{W}(\Sigma')$ be the toric surface associated with the fan Σ' , where $\Sigma'(1) = \Sigma(1) \cup \{\rho_4, \rho_5\}$ and $\rho_4 = (1, 0)$, $\rho_5 = (-1, 1)$. Then (as in Example 6.6) $v_N = (-2, 1) + (0, 2)$ is strongly decomposable in $\sigma = \text{cone}(\rho_0, \rho_2)$ (cf. Figure 13A). Moreover,

$$C' = 7D_2 + 2D_3 + 3D_4$$

in $X' = \mathbb{T}\mathbb{W}(\Sigma')$ with $\Delta^{\text{nef}} = \text{conv}([0, 0], [-3, 0], [-3, -2], [-2, -2])$ (cf. Figure 13B). Choose the divisor $D_{\Theta(\ell, k)}$ as

$$D_{\Theta(\ell, k)} = 6D_2 + 4D_3 + 2D_4 + 6D_5,$$

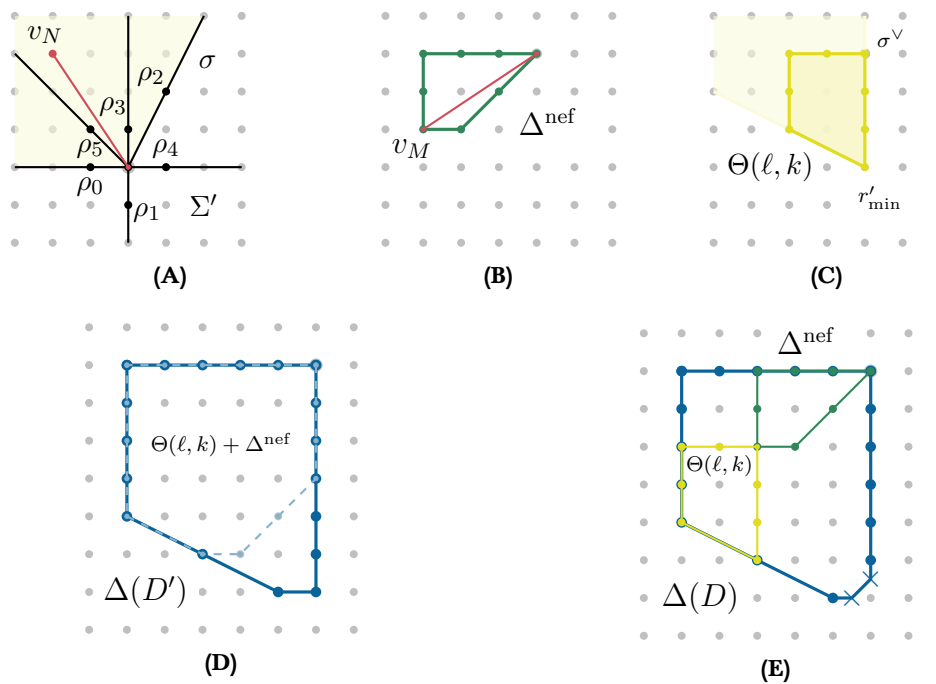


Figure 13. Illustration of Proposition 6.12. (A) The set of rays $\Sigma'(1) = \Sigma(1) \cup \{\rho_4, \rho_5\}$, a primitive element $v_N = (-2, 3) \in N$, and the cone σ in which v_N is strongly decomposable. (B) The Newton polytope Δ^{neft} given by $v_M = [-3, -2]$ and the polytope Δ^{nef} corresponding to C' in $X' = \mathbb{T}\mathbb{V}(\Sigma')$. (C) The polytope $\Theta(\ell, k)$ having a vertex r'_{\min} with tangent cone σ^\vee . (D) The polytope $\Delta(D')$ corresponding to the non-ample divisor D' containing $\Theta(\ell, k) + \Delta^{\text{nef}}$. (E) The polytope $\Delta(D)$ corresponding to the ample divisor D with $\Theta(\ell, k) = (\Delta(D) : \Delta^{\text{nef}})$.

i.e., $\Theta(\ell, k) = \text{conv}([0, 0], [-2, 0], [-2, -2], [0, -3])$ (*cf.* Figure 13C). We obtain the divisor

$$D' = D_{\Theta(\ell, k)} + C' = 13D_2 + 6D_3 + 5D_4 + 6D_5$$

with $\Delta(D') = \text{conv}([0, 0], [-5, 0], [-5, -4], [-1, -6], [0, -6])$ and $\Theta(\ell, k) + \Delta^{\text{nef}} \subseteq \Delta(D')$ (*cf.* Figure 13D). Note that D' is not ample since we have $D' \cdot D_5 = 0$.

To make it ample, adjust the coefficient of D_5 to 5.5. Then D is ample with $\Delta(D) = \text{conv}([0, 0], [-5, 0], [-5, -4], [-1, -6], [-0.5, -6], [0, -5.5])$ (*cf.* Figure 13E). Moreover, we have $\Theta(\ell, k) = (\Delta(D) : \Delta^{\text{nef}})$.

References

- [AP20] K. Altmann and D. Ploog, *Displaying the cohomology of toric line bundles*, *Izv. Math.* **84** (2020), no. 4, 683–693.
- [And13] D. Anderson, *Okounkov bodies and toric degenerations*, *Math. Ann.* **356** (2013), no. 3, 1183–1202.
- [AKL14] D. Anderson, A. Küronya, and V. Lozovanu, *Okounkov bodies of finitely generated divisors*, *Int. Math. Res. Not. IMRN* (2014), no. 9, 2343–2355.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468.
- [CLTU23] A.-M. Castravet, A. Laface, J. Tevelev, and L. Ugaglia, *Blown-up toric surfaces with non-polyhedral effective cone*, *J. reine angew. Math.* **800** (2023), 1–44.
- [CLO15] D. A. Cox, J. Little, and D. O’Shea, *Ideals, varieties, and algorithms*, 4th ed., Undergrad. Texts Math., Springer, Cham, 2015.

- [CLS11] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Grad. Stud. Math., vol. 124, Amer. Math. Soc., Providence, RI, 2011.
- [HKW20] C. Haase, A. Küronya, and L. Walter, *Toric Newton-Okounkov functions with an application to the rationality of certain Seshadri constants on surfaces*, preprint [arXiv:2008.04018](https://arxiv.org/abs/2008.04018) (2020).
- [HK15] M. Harada and K. Kaveh, *Integrable systems, toric degenerations and Okounkov bodies*, Invent. Math. **202** (2015), no. 3, 927–985.
- [IM19] N. Ilten and C. Manon, *Rational complexity-one T -varieties are well-poised*, Int. Math. Res. Not. IMRN (2019), no. 13, 4198–4232.
- [KK12] K. Kaveh and A.G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. Math. (2) **176** (2012), no. 2, 925–978.
- [KL17a] A. Küronya and V. Lozovanu, *Infinitesimal Newton-Okounkov bodies and jet separation*, Duke Math. J. **166** (2017), no. 7, 1349–1376.
- [KL17b] ———, *Positivity of line bundles and Newton-Okounkov bodies*, Doc. Math. **22** (2017), 1285–1302.
- [KL18a] ———, *Geometric aspects of Newton-Okounkov bodies*, in: *Phenomenological approach to algebraic geometry*, pp. 137–212, Banach Center Publ., vol. 116, Polish Acad. Sci. Inst. Math., Warsaw, 2018.
- [KL18b] ———, *Local positivity of linear series on surfaces*, Algebra Number Theory **12** (2018), no. 1, 1–34.
- [KL19] ———, *A Reider-type theorem for higher syzygies on abelian surfaces*, Algebr. Geom. **6** (2019), no. 5, 548–570.
- [KLM12] A. Küronya, V. Lozovanu, and C. Maclean, *Convex bodies appearing as Okounkov bodies of divisors*, Adv. Math. **229** (2012), no. 5, 2622–2639.
- [KMR21] A. Küronya, C. Maclean, and J. Roé, *Newton-Okounkov theory in an abstract setting*, Beitr. Algebra Geom. **62** (2021), no. 2, 375–395.
- [LM09] R.K. Lazarsfeld and M. Mustață, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835.
- [Oko96] A. Okounkov. *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), no. 3, 405–411.
- [RW19] K. Rietsch and L. Williams. *Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians*, Duke Math. J. **168** (2019), no. 18, 3437–3527.