*Épijournal de Géométrie Algébrique*
epiga.episciences.org
Volume 8 (2024), Article No. 11



# Ngô's support theorem and polarizability of quasi-projective commutative group schemes

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**Abstract.** In this short note, we prove that any commutative group scheme $G \to B$ over an arbitrary base scheme $B$ of finite type over a field with connected fibers and admitting a relatively ample line bundle is polarizable in the sense of Ngô. More precisely, we associate a polarization to any relatively ample line bundle on $G \to B$. This extends the applicability of Ngô's support theorem to new cases, such as Lagrangian fibrations with integral fibers, and has consequences for the construction of algebraic classes.

**Keywords.** Group schemes, abelian varieties, Hodge theory, hyper-kähler varieties, algebraic cycles

**2020 Mathematics Subject Classification.** 14K05, 14K20, 14L15, 14C25, 14C20

Received by the Editors on September 28, 2023, and in final form on January 26, 2024.
Accepted on February 25, 2024.

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This research was partly supported by grant ANR-18-CE40-0017 of the Agence Nationale de la Recherche.

## Contents



## 1. Introduction

Let $\pi\colon G \to B$ be a commutative group scheme of relative dimension $d$ with connected fibers over a scheme $B$ of finite type over $\mathbb{C}$. The relative Tate module is

$$T(G) := R^{2d-1}\pi_! \underline{\mathbb{Q}}_G(d),$$

where we denote by $(d)$ the Tate twist and $\underline{\mathbb{Q}}_G$ is the constant sheaf on $G$.

By Chevalley's structure theorem, for all $b \in B$, the fiber $G_b$ of $\pi$ over $b$ sits in an exact sequence

$$1 \longrightarrow L_b \longrightarrow G_b \longrightarrow A_b \longrightarrow 1, \tag{1.1}$$

where $L_b$ is the unique connected affine subgroup of $G_b$ such that the quotient group $A_b$ is an abelian variety.

The following is the definition of polarization after [Ngô10, Section 7.1.4].

**Definition 1.1.** A polarization on $G$ is an alternating bilinear map

$$\eta\colon T(G) \otimes T(G) \longrightarrow \underline{\mathbb{Q}}_B(1)$$

such that for every geometric point $b \in B$, the induced bilinear form

$$\eta_b\colon T(G_b) \otimes T(G_b) \longrightarrow \mathbb{Q}(1) \tag{1.2}$$

has kernel exactly $T(L_b)$, where $L_b$ is as in (1.1).

If such a polarization exists, we say that $T(G)$, or simply $G$, is polarizable.

Our main result is the following.

**Theorem 1.2.** *Any quasi-projective group scheme is polarizable. More precisely, let $\mathcal{L}$ be a relatively ample line bundle on $G \to B$. Then its first Chern class induces a polarization*

$$\eta_{\mathcal{L}}\colon T(G) \otimes T(G) \longrightarrow \underline{\mathbb{Q}}_B(1).$$

The original definition of Ngô uses $\ell$-adic coefficients instead of rational coefficients and makes sense over very general bases. Our result also holds in this setting (see Section 6).

Recently, Ngô's support theorem has been extensively used to study Lagrangian fibrations. One of its crucial hypotheses is the polarizability of the underlying group scheme. Hence, our Theorem 1.2 allows one to apply Ngô's support theorem to more general Lagrangian fibrations, and in particular we can deduce the following result.

**Corollary 1.3**. *Let $X$ be a projective hyper-kähler variety and $f: X \to B$ be a Lagrangian fibration with integral fibers. Then all perverse sheaves appearing in the decomposition theorem for $f$ have dense support.*

*Remark* 1.4. During the final stages of this work, we learned that similar results with different proofs have been obtained by Mark de Cataldo, Roberto Fringuelli, Andrés Fernandez-Herrero and Mirko Mauri. Their article is in preparation. We warmly thank them for the friendly exchanges during our parallel work.

### Idea of the proof of Theorem 1.2

The construction of $\eta_{\mathcal{L}}$ is purely formal from the adjunction $(R\pi_!, \pi^!)$. In particular, base change ensures functoriality by pullback on the base, and the problem is therefore reduced to the absolute case $B = \mathrm{pt} = \{b\}$ and $\mathcal{L}_b$ an ample line bundle on the commutative algebraic group $G_b$. Such a reduction step is the main reason why we construct a polarization using Chern classes instead of constructing it using purely Hodge-theoretical techniques. Notice that it is easy to show that every $G_b$ is polarizable using Hodge theory; however, it is unclear in general how to put those polarizations in a family. The use of Chern classes allows us to give a global definition of a pairing which restricts well on each fiber. The point then becomes to check that the restriction on each fiber is a polarization.

Let us now write $G = G_b$ and consider Chevalley's structure theorem applied to $G$:

$$1 \longrightarrow L \longrightarrow G \xrightarrow{p} A \longrightarrow 1.$$

One shows that $p$ induces a surjective map

$$p^*\colon \operatorname{Pic}(A) \longrightarrow \operatorname{Pic}(G).$$

We moreover show, and it is essential, that if $\mathcal{L} := p^*\mathcal{M}$ is ample on $G$, then $\mathcal{M}$ is ample as well (see Proposition 6.4, or Proposition 4.5 for a weaker version). This reduces the problem to the case of an abelian variety, where one has to show that the pairing is non-degenerate.

When $G = A$, we check that the pairing constructed in the paper is related to more classical polarizations from Hodge theory, which are known to be non-degenerate. This is possible through the explicit analytic description of Chern classes on abelian varieties given by the Appell–Humbert theorem (see [Mum08, Section I.2, p. 20]). An alternative algebraic argument is provided in Section 6.

### Organization of the paper

In Section 2, we explain the construction of the polarization associated with a line bundle. The special absolute case of complex abelian varieties is studied in Section 3. The general absolute case is explained in Section 4. The reduction to the absolute case is explained in Section 5, where one will also find the proofs of Theorem 1.2 and Corollary 1.3. In Section 6, we explain how to modify the arguments in order to extend the results from complex bases to general schemes.

### Acknowledgments

We thank Marian Aprodu, Olivier Benoist, Michel Brion, Mattia Cavicchi, Mark de Cataldo and Giulia Saccà for their interest in our work and for fruitful discussions. We warmly thank the referee for a careful reading.

## 2. Construction of the polarization

Let $\pi\colon G \to B$ be a commutative group scheme of relative dimension $d$. Recall that we defined the Tate module to be $T(G) := R^{2d-1}\pi_!\underline{\mathbb{Q}}_G(d)$. In this section, we construct, for any class in cohomology $\omega \in \mathrm{H}^2(G, \mathbb{Q})(1)$, a map

$$\eta_\omega\colon \Lambda^2 T(G) \longrightarrow \underline{\mathbb{Q}}_B(1).$$

We will apply it in later sections to $\omega = c_1(\mathcal{L})$ for $\mathcal{L}$ a relatively ample line bundle on $G \to B$.

We can view $\omega$ as a map in the derived category of constructible sheaves on $G$

$$\omega\colon \underline{\mathbb{Q}}_G \longrightarrow \underline{\mathbb{Q}}_G[2](1).$$

For a smooth map of finite type schemes $f: X \to Y$ of relative dimension $d$, we have $f^* = f^![-2d](-d)$. This is a relative version of Poincaré duality (see [AGV73, Theorem XVIII.3.2.5] and its proof). Applying this to the smooth map $\pi$, we get

$$\underline{\mathbb{Q}}_G = \pi^! \underline{\mathbb{Q}}_B[-2d](-d),$$

and $\omega$ can now be written as

$$\omega\colon \underline{\mathbb{Q}}_G \longrightarrow \pi^! \underline{\mathbb{Q}}_B[-2d+2](-d+1).$$

By the adjunction $(R\pi_!, \pi^!)$, we can view $\omega$ equivalently as

$$\omega\colon R\pi_! \underline{\mathbb{Q}}_G \longrightarrow \underline{\mathbb{Q}}_B[-2d+2](-d+1), \tag{2.1}$$

and taking the $2d-2$ cohomology sheaf in Equation (2.1) gives the map

$$\eta_\omega\colon R^{2d-2}\pi_! \underline{\mathbb{Q}}_G \longrightarrow \underline{\mathbb{Q}}_B(-2d+1). \tag{2.2}$$

Through the canonical identification $R^{2d-2}\pi_! \underline{\mathbb{Q}}_G = \Lambda^2 R^{2d-1}\pi_! \underline{\mathbb{Q}}_G$ (see for example [AHPL16]), we have indeed constructed

$$\eta_\omega\colon \Lambda^2 T(G) \longrightarrow \mathbb{Q}_B(1). \tag{2.3}$$

By its very definition, this construction is compatible with changing the base $B$.

**Lemma 2.1.** *Let $f: B' \to B$ be a map from a finite type scheme $B'$, and denote by $\pi'\colon G' \to B'$ the base change to $B'$. Let $\omega \in H^2(G, \mathbb{C})(1)$, and let $\omega' \in H^2(G', \mathbb{C})(1)$ be its pullback. Then we have $f^*(\eta_\omega) = \eta_{\omega'}$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} f^*T(G) \otimes f^*T(G) & \xrightarrow{f^*(\eta_\omega)} & f^*\underline{\mathbb{Q}}_B(1) \\ \downarrow\wr & & \downarrow \\ T(G') \otimes T(G') & \xrightarrow{\eta_{\omega'}} & \underline{\mathbb{Q}}_{B'}(1), \end{array}$$

*where the left vertical isomorphism comes from base change.*

## 3. The case of abelian varieties: The Appell–Humbert theorem

The goal of this section is to prove that $\eta_\omega$ as in (2.3) is non-degenerate when $\omega$ is the Chern class of an ample line bundle $\mathcal{L}$ on an abelian variety $A$. After some preliminaries, we will recall (following [Mum08, Section I.2]) the main ingredient: the Appell–Humbert theorem.

Let $A$ be a complex abelian variety, and write it as $V/\Gamma$, where $V = T_eA$ and $\Gamma = \mathrm{H}_1(A, \mathbb{Z})$. The quotient is taken in the realm of complex analytic varieties.

Any line bundle $\mathcal{L}$ on $A$ becomes trivial when pulled back to $V$. To recover $\mathcal{L}$, some descent datum must be specified; in this situation, it can be made very explicit in terms of linear algebra on $V$ and $\Gamma$. Consider pairs $(H, \rho)$, where

$$H\colon V \otimes_{\mathbb{R}} V \longrightarrow \mathbb{C}$$

is a hermitian form and

$$\rho\colon \Gamma \longrightarrow \mathbb{S}^1$$

is an $H$-pseudo-character; *i.e.*, $\rho$ satisfies

$$\rho(u+v) = e^{i\pi E(u,v)}\rho(u)\rho(v) \quad \text{for all } u, v \in \Gamma, \tag{3.1}$$

where $E := \operatorname{Im} H\colon V \otimes_{\mathbb{R}} V \to \mathbb{R}$ is the imaginary part of $H$.

With such a datum $(H,\rho)$, assuming that $E(\Gamma \times \Gamma) \subset \mathbb{Z}$, we can associate the line bundle $\mathcal{L}(H,\rho)$ on $A$ given by

$$\mathcal{L}(H,\rho) := (V \times \mathbb{C})/\Gamma, \tag{3.2}$$

where the action of $\Gamma$ is given by

$$u \cdot (v,z) := (u+v, \rho(u)e^{\pi H(v,u)+\frac{1}{2}\pi H(u,u)}z).$$

We list some useful properties whose proofs are immediate.

**Lemma 3.1.** *With the above notation, the following hold:*

(1) *If* $(H_1,\rho_1)$ *and* $(H_2,\rho_2)$ *are two pairs as above, then*
$$\mathcal{L}(H_1,\rho_1) \otimes \mathcal{L}(H_2,\rho_2) = \mathcal{L}(H_1+H_2, \rho_1\rho_2).$$
(2) *If* $f\colon A' \to A$ *is a morphism of abelian varieties with* $A' = V'/\Gamma'$*, then*
$$f^*(\mathcal{L}(H,\rho)) = \mathcal{L}(f^*H, f^*\rho),$$
*where* $f^*H$ *is the hermitian form on* $V'$ *induced from* $H$ *through the linear map* $V' \to V$ *and similarly for* $f^*\rho$*.*

For the comfort of the reader, let us also recall the following relationship between $H$ and $E$.

**Lemma 3.2** (*cf.* [Mum08, Section I.2, p. 20]). *There is a bijection between hermitian forms* $H$ *on* $V$ *and real skew-symmetric forms* $E$ *on* $V$ *satisfying the identity* $E(ix,iy) = E(x,y)$*, which is given by*

$$E(x,y) = \operatorname{Im} H(x,y), \tag{3.3}$$

$$H(x,y) = E(ix,y) + iE(x,y). \tag{3.4}$$

*Moreover, the kernel of* $H$ *is equal to the kernel of* $E$*. In particular,* $H$ *is non-degenerate if and only if* $E$ *is non-degenerate.*

The following result describes the Néron–Severi group

$$NS(A) := \operatorname{Image}(\mathrm{H}^1(A, \mathcal{O}_A^\times) \longrightarrow \mathrm{H}^2(A,\mathbb{Z}))$$

under the identification

$$\mathrm{H}^2(A,\mathbb{Z}) = (\Lambda^2\Gamma)^\vee.$$

**Theorem 3.3** (Appell–Humbert, *cf.* [Mum08, Section I.2, p. 20]). *Any line bundle on* $A$ *is of the form* $\mathcal{L}(H,\rho)$ *for a unique pair* $(H,\rho)$ *as above. Moreover, we have isomorphic short exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(\Gamma, \mathbb{S}^1) & \longrightarrow & \left\{\begin{array}{c}(H,\rho)\\ \textit{satisfying}\\ (3.1)\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}H \textit{ hermitian}\\ \textit{on } V \textit{ s.t.}\\ \operatorname{Im}(H)(\Gamma\times\Gamma) \subseteq \mathbb{Z}\end{array}\right\} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \longrightarrow & \operatorname{Pic}^0(A) & \longrightarrow & \operatorname{Pic}(A) & \xrightarrow{c_1(-)} & NS(A) & \longrightarrow & 0,
\end{array}$$

*where* $\gamma(H) = E := \operatorname{Im}(H)$ *is the imaginary part of* $H$*.*

**Corollary 3.4.** *With the above notation, we have*

$$c_1(\mathcal{L}(H,\rho)) = E \in (\Lambda^2\Gamma)^\vee = \mathrm{H}^2(A,\mathbb{Z}), \quad \textit{where } E = \operatorname{Im}(H).$$

**Theorem 3.5** (Lefschetz's theorem, *cf.* [Mum08, Section I.3, p. 28]). *Keeping the above notation, we have that* $\mathcal{L}(H,\rho)$ *is ample if and only if* $H$ *is positive definite.*

We now have the ingredients to prove the main result of the section, which is a special case of Theorem 1.2.

**Theorem 3.6.** *Let $\mathcal{L}(H,\rho)$ be an ample line bundle on the abelian variety $A = V/\Gamma$. Then the map $\eta_\omega$ constructed in (2.3) associated with $\omega = c_1(\mathcal{L}(\rho, H))$,*

$$\eta_\omega\colon \Lambda^2 T(A) \longrightarrow \mathbb{Q}(1),$$

*is non-degenerate.*

*Proof.* Recall that, starting with a class $\omega \in H^2(A)$, the construction of the pairing $\eta_\omega$ is based on the identification $H^2(A) = \Lambda^2 H^1(A)$ and on Poincaré duality. On the other hand, notice that the description of $c_1(\mathcal{L}(H,\rho))$ in Corollary 3.4 is also based on the equality $H^2(A) = \Lambda^2 H^1(A)$ and on duality.

Now we have the identification $T(A) = \Gamma$, as they are both the dual of $H^1(A)$. Under this identification, the pairing $\eta_{c_1(\mathcal{L}(\rho,H))}$ coincides with $E = \mathrm{Im}(H)$, by Corollary 3.4. As $\mathcal{L}(H,\rho)$ is ample, the hermitian form $H$ is non-degenerate by Theorem 3.5, which implies that $E$ is non-degenerate as well by Lemma 3.2. □

## 4. Picard groups of commutative algebraic groups

Let $G$ be a connected, commutative group scheme of finite type over $\mathbb{C}$, and use Chevalley's structure theorem to get

$$1 \longrightarrow L \longrightarrow G \xrightarrow{p} A \longrightarrow 1$$

with $L$ an affine algebraic group and $A$ an abelian variety. The purpose of this section is to relate the Picard group of $G$ with that of $A$. This is achieved in Proposition 4.5, which is the only result of this section using analytic methods. In Proposition 6.4, we propose an algebraic alternative for it.

**Proposition 4.1.** *With the above notation, the map $p\colon G \to A$ induces a surjection*

$$p^*\colon \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(G).$$

*Proof.* If $L$ has a non-trivial unipotent radical, then by $\mathbb{A}^1$-homotopy, we clearly have $\mathrm{Pic}(G) \simeq \mathrm{Pic}(G/L_u)$, where $L_u$ is the unipotent radical of $L$. We can therefore assume that $L$ has no unipotent subgroups, in other words, that $L \simeq \mathbb{G}_m^r$ for some $r \geq 1$.

Consider the natural action of $\mathbb{G}_m^r$ on $\mathbb{A}^r$, and take the associated $\mathbb{A}^r$-bundle

$$\overline{G} := G \times^L \mathbb{A}^r := (G \times \mathbb{A}^r)/L$$

giving the following commutative diagram, with $j$ an open immersion:

$$\begin{array}{ccc} G & \xhookrightarrow{\ j\ } & \overline{G} \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle \overline{q}} & \\ A. & & \end{array}$$

Consider the restriction map

$$j^*\colon \mathrm{CH}^1(\overline{G}) \longrightarrow \mathrm{CH}^1(G).$$

It is surjective since any cycle on $G$ is the restriction of its closure in $\overline{G}$. Moreover, since $\overline{q}^*\colon \mathrm{CH}^1(A) \to \mathrm{CH}^1(\overline{G})$ is an isomorphism by $\mathbb{A}^1$-homotopy, we deduce that $q^*\colon \mathrm{CH}^1(A) \to \mathrm{CH}^1(G)$ is also surjective. □

For the next result, we need to use some sort of numerical criterion for ample divisors on open varieties.[(1)]

**Proposition 4.2.** *Let $X$ be a smooth projective variety, and let $U \subset X$ be an open subvariety. Let $\mathcal{L}_1, \mathcal{L}_2$ be two line bundles on $X$ such that their first Chern classes coincide $c_1(\mathcal{L}_1) = c_1(\mathcal{L}_2)$. If $\mathcal{L}_1|_U$ is ample, then $\mathcal{L}_2|_U$ is also ample.*

[(1)] We warmly thank Olivier Benoist for telling us about these results.

*Proof.* It is basically [Ben13, Proposition 7]. In *loc. cit.*, the statement is for algebraic equivalence. However, for divisors on smooth projective varieties, algebraic equivalence coincides with numerical equivalence by a theorem of Matsusaka (1956); see [Kle66, Corollary 1]. □

In our context, we will apply it as follows

**Lemma 4.3.** *Let $X$ be a smooth quasi-projective variety and $\mathcal{L}$ an ample line bundle on $X$. Suppose the first Chern class of $\mathcal{L}$ vanishes. Then $X$ is quasi-affine.*

*Proof.* Let $X \subset \overline{X}$ be a smooth compactification with complement a normal crossing divisor (this is possible in characteristic zero, by Nagata's theorem and the resolution of singularities). Denote by $Z$ the divisor at infinity.

Extend the line bundle $\mathcal{L}$ to a line bundle $\overline{\mathcal{L}}$ over $\overline{X}$ by taking the closure of the corresponding Weil divisor. The Chern class of $\overline{\mathcal{L}}$ is going to be supported on $Z$ because $c_1(\overline{\mathcal{L}}|_U) = c_1(\mathcal{L}) = 0$. Hence, there is a divisor supported on $Z$, call it $D$, such that $c_1(\overline{\mathcal{L}}) = c_1(\mathcal{O}_{\overline{X}}(D))$.

We can now apply Proposition 4.2 and deduce that $\mathcal{O}_X = \mathcal{O}_{\overline{X}}(D)|_X$ is ample on $X$; this means precisely that $X$ is quasi-affine (see [Sta23, Tag 01QE]). □

The next lemma is contained in [ABD+65, Proposition VIB.11.11], but we include a short proof for completeness.

**Lemma 4.4.** *Let $G$ be a commutative group scheme. If $G$ is quasi-affine, then $G$ is affine.*

*Proof.* Let us put $G' := \mathrm{Spec}(\mathcal{O}(G))$. We have a morphism of group schemes $G \to G'$ that is moreover an open dense embedding because of the assumption that $G$ is quasi-affine. Since images of group morphisms are closed (see for example [Sta23, Lemma 047T]), we deduce that $G = G'$. □

Recall the notation $\mathcal{L}(H,\rho)$ from Theorem 3.3 and the preceding paragraphs.

**Proposition 4.5.** *Assume that the line bundle $p^*\mathcal{L}(H,\rho)$ on $G$ is ample. Then $H$ is a non-degenerate hermitian form.*

*Proof.* Let us use the complex uniformization of $A$: put $V = T_eA$ and $\Gamma := \mathrm{H}_1(A,\mathbb{Z})$. We have $V/\Gamma \simeq A$ (quotient in the realm of complex analytic varieties).

Put $N := \ker(H)$. It is a complex vector subspace of $V$ which moreover has the property that $N \cap \Gamma$ is cocompact in $N$, as the imaginary part $E := \mathrm{Im}(H)$ of $H$ takes integral values on $\Gamma$ (see the condition on $\mathrm{Im}(H)$ in Theorem 3.3).

Therefore, we can consider $A' := N/(N \cap \Gamma)$ as a complex subtorus of $A$. By Chow's lemma, we have that $A'$ is projective, hence an abelian variety. Moreover, the line bundle $\mathcal{L}(H,\rho)$ restricted to $A'$ is precisely $\mathcal{L}(0,\rho|_{\Lambda\cap N})$, again by Theorem 3.3. In particular, using Corollary 3.4, we have that its Chern class is zero.

The restriction of $p\colon G \to A$ to $A'$ gives a commutative group scheme with Chevalley decomposition

$$1 \longrightarrow L \longrightarrow G' \longrightarrow A' \longrightarrow 1$$

with $G' := p^{-1}(A')$ and such that $(p^*\mathcal{L}(H,\rho))|_{G'}$ is ample. The Chern class of this line bundle is zero by functoriality; hence $G'$ is quasi-affine by Lemma 4.3 and therefore affine by Lemma 4.4. It follows that $A' = 1$ and therefore that $N = \{0\}$ or, in other words, that $H$ is non-degenerate. □

*Remark* 4.6. As already pointed out, Proposition 4.5 is the only result in this section using analytic methods. In Proposition 6.4, we propose an algebraic alternative for it.

## 5. Proof of the main theorem

We are now ready to give the proof of our main result Theorem 1.2 and of its application Corollary 1.3. For the convenience of the reader, we recall the statements.

**Theorem 5.1**. *Let $B$ be a scheme of finite type over the complex numbers. Let $\pi\colon G \to B$ be a commutative group schemes with connected fibers. Assume that there is a relatively ample line bundle $\mathcal{L}$ on $\pi\colon G \to B$. Let $\omega$ be the first Chern class of $\mathcal{L}$. Then*

$$\eta_\omega\colon T(G) \otimes T(G) \longrightarrow \underline{\mathbb{Q}}_B(1)$$

(*as constructed in Section 2*) *is a polarization in the sense of Definition 1.1.*

*Proof.* By the functoriality of the construction, see Lemma 2.1, we are reduced to the case $B = \mathrm{Spec}(\mathbb{C})$.

Let $1 \to L \to G \to A \to 1$ be the Chevalley decomposition of $G$ with $L$ an affine algebraic group and $A$ an abelian variety. We want to check that $\eta_\omega$ has kernel precisely $T(L)$. Notice that for weight reasons, the kernel contains $T(L)$; hence we have to show that the induced pairing on $T(A)$ is non-degenerate.

By Proposition 4.1, $\mathcal{L}$ is the pullback of a line bundle on $A$. Hence, by Theorem 3.3, we can write $\mathcal{L} = p^*\mathcal{L}(H,\rho)$, where $p\colon G \to A$ is the projection. By Lemma 2.1, the pairing we want to study on $T(A)$ is the one induced by $\mathcal{L}(H,\rho)$. This is nothing but $\mathrm{Im}\, H$, by Corollary 3.4. On the other hand, Proposition 4.5 shows that $H$ is non-degenerate, hence so is $\mathrm{Im}\, H$ by virtue of Lemma 3.2. □

**Corollary 5.2**. *Let $X$ be a projective hyper-kähler variety and $f: X \to B$ be a Lagrangian fibration with integral fibers. Then all perverse sheaves appearing in the decomposition theorem for $f$ have dense support.*

*Proof.* In [AF16], under the same hypotheses, the authors construct a group scheme $\pi\colon G \to B$ as a subgroup of $\mathrm{Aut}(f)$ and verify that all hypotheses needed to apply Ngô's support theorem are satisfied, except possibly the polarization (see also [ACLS23, Proposition 9.5]). On the other hand, under the hypothesis of irreducible fibers, Ngô's support theorem implies that the supports of the perverse sheaves are dense. We are then reduced to showing that $\pi$ is polarizable; hence, by Theorem 5.1, we have to show that $\pi$ admits a relatively ample line bundle.

By the construction of $G$, it is enough to show that $\mathrm{Aut}(f)$ admits a relatively ample line bundle. If we associate with each automorphism its graph, we obtain an embedding of $\mathrm{Aut}(f)$ in the Hilbert scheme of $X \times_B X$. The latter is a disjoint union of projective schemes over $B$, so we can conclude. □

*Remark* 5.3. This corollary has applications to the construction of algebraic classes on Lagrangian fibrations; see the main results of [ACLS23].

## 6. The $\ell$-adic setting

In this section, we extend our main theorem to group schemes over more general bases, in particular in positive characteristic and in mixed characteristic.

Let $\ell$ be a prime number and $B$ be a base scheme such that $\ell$ is invertible on $B$. This assumption is needed in order to have $\ell$-adic cohomology. Let $\pi\colon G \to B$ be a commutative group scheme with connected fibers of relative dimension $d$. The Tate module is defined as before as

$$T(G) := R^{2d-1}\pi_!\underline{\mathbb{Q}}_\ell(d).$$

**Definition 6.1**. (*cf.* [Ngô10, Section 7.1.4]) A polarization on $\pi\colon G \to B$ is a pairing on the Tate module

$$\eta\colon \Lambda^2 T(G) \longrightarrow \underline{\mathbb{Q}}_\ell(1)$$

such that for every geometric point $b$ of $B$, if

$$1 \longrightarrow L_b \longrightarrow G_b \longrightarrow A_b \longrightarrow 1$$

is the Chevalley decomposition, then the pairing $\eta_b$ induces a non-degenerate pairing

$$\eta_b\colon T(A_b)\otimes T(A_b)\longrightarrow \mathbb{Q}_\ell(1).$$

If such a polarization exists, we say that $G$ is polarizable.

**Theorem 6.2.** *Assume that $B$ is quasi-compact and quasi-separated and that $\pi$ is quasi-projective. Then $G$ is polarizable. More precisely, a relative ample line bundle on $\pi$ induces a polarization through its first Chern class.*

*Remark* 6.3. The assumptions of quasi-compact and quasi-separated are those in [AGV73, Chapter XVIII] that are needed for duality. Ngô's definition is, anyway, in the context of $B$ being of finite type over a field.

*Proof.* The proof of Theorem 1.2 applies to the $\ell$-adic context *mutatis mutandis*, up to three exceptions.

First, the construction of the pairing $\eta_\omega$ associated with a cohomology class $\omega$ used the duality $\pi^* = \pi^![-2d](-d)$. In characteristic zero, such a duality holds as $\pi$ is always smooth, but in our generality, we need an extra argument.

Notice that $\pi$ is compactifiable in the sense of [AGV73, Définition XVII.3.2.1], as it is quasi-projective. This gives a trace map relating $\pi^*$ and $\pi^![-2d](-d)$ by [AGV73, Equation XVIII.(3.2.1.2)]. As the trace map is compatible with base change (by *loc. cit.*), it is enough to check that it is an isomorphism on geometric fibers. The fibers are not smooth, but they become smooth with the reduced scheme structure (as they are algebraic groups). Since $\ell$-adic cohomology is insensitive to the scheme structure, we can conclude that the trace map $\pi^* \to \pi^![-2d](-d)$ is an isomorphism.

Second, the reduction to the case of an abelian variety used analytic methods (Proposition 4.5). It can be made purely algebraic by using the functoriality of the pairing Lemma 2.1 and Proposition 6.4 below.

Third, the proof in the case of an abelian variety (see Theorem 3.6) was also of analytic nature, relying on the Appell–Humbert theorem and analytic uniformization. We give here an alternative argument of algebraic nature, which is less precise but sufficient for our purposes. It is in the same style as the proof of [Anc21, Lemma 5.3].

Let $A$ be an abelian variety and $\mathcal{L}$ be an ample line bundle. There are two ways to associate with $\mathcal{L}$ an alternating pairing on the Tate module $T(A)$. The first is by considering the Weil pairing between $T(A)$ and the Tate module of the dual abelian variety $T(A^\vee)$ and then using $\mathcal{L}$ to have an isogeny between $A$ and $A^\vee$ and hence identify $T(A)$ with $T(A^\vee)$. Notice that this pairing has the advantage of being non-degenerate (but it is unclear how to extend it to general commutative group schemes).

The second pairing is that in the present paper. Look at the Chern class of $\mathcal{L}$ in the $\ell$-adic cohomology $H^2(A) = \Lambda^2 H^1(A)$. By dualizing, such a class becomes an alternating pairing on $T(A)$. These two pairings are probably equal, but we do not know how to show it. We instead show that they are the same up to an invertible scalar, which implies in particular that the second pairing is non-degenerate as well.

To do so notice that, after fixing a suitable level structure, $A$ becomes a point of the moduli space $M$ parameterizing abelian varieties of the same dimension as $A$ together with line bundles of the same degree as $\mathcal{L}$. Now both pairings extend to the whole family of abelian varieties parameterized by $M$, hence define pairings at the level of the local systems. This means that both pairings on $T(A)$ are equivariant with respect to the monodromy action. Such an action on $T(A)$ is geometrically irreducible (it is the standard representation of the group GSp); hence by Schur's lemma, the two pairings are the same up to a scalar. Since both pairings are non-zero (the Chern class of an ample line bundle on a projective variety is non-zero), we conclude that such a scalar is invertible. □

The following is [Ray70, Lemme XI.1.11]. We warmly thank Michel Brion for pointing out the original reference to us. Nevertheless, we include a proof for completeness.

**Proposition 6.4.** *Let $\mathcal{L}$ be an ample line bundle on $G$ which by Proposition 4.1 is isomorphic to $p^*\mathcal{M}$ with $\mathcal{M}$ a line bundle on $A$. Then $\mathcal{M}$ is ample. In other words, any ample line bundle on $G$ is the pullback of an ample line bundle on $A$.*

*Proof.* We know from [Mum08, Section II.6, Application 1, p. 60] that $\mathcal{M}$ is ample on $A$ if and only if the closed subgroup

$$K(\mathcal{M}) := \{x \in A \mid t_x^*\mathcal{M} \simeq \mathcal{M}\}$$

is finite. Suppose toward a contradiction that $K(\mathcal{M})$ is not finite, and let $Y$ be its identity component. Notice that $Y$ is itself an abelian variety. Denote by $\mathcal{M}' = \mathcal{M}|_Y$ the restriction of $\mathcal{M}$ to $Y$. The proof from *loc.cit.* shows that on $Y$, we have

$$\mathcal{M}' \otimes (-1_Y)^*\mathcal{M}' \simeq \mathcal{O}_Y. \tag{6.1}$$

On the subgroup $H := p^{-1}Y$, we have that the ample line bundle $\mathcal{L}|_H$ is isomorphic to $p^*\mathcal{M}'$. However, if $p^*\mathcal{M}'$ is ample, then $(-1_H)^*p^*\mathcal{M}'$ is also ample, and hence their tensor product is also ample. By (6.1), we get that $\mathcal{O}_H$ is ample on $H$, *i.e.*, that $H$ is quasi-affine, hence affine by Lemma 4.4. This gives a contradiction; therefore, $\mathcal{M}$ is ample on $A$. □

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