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Local Euler characteristics of A_n-singularities and their application to hyperbolicity

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Abstract. Wahl's local Euler characteristic measures the local contributions of a singularity to the usual Euler characteristic of a sheaf. Using tools from toric geometry, we study the local Euler characteristic of sheaves of symmetric differentials for isolated surface singularities of type A_n. We prove an explicit formula for the local Euler characteristic of the m^{th} symmetric power of the cotangent bundle; this is a quasi-polynomial in m of period $n+1$. We also express the components of the local Euler characteristic as a count of lattice points in a non-convex polyhedron, again showing it is a quasi-polynomial. We apply our computations to obtain new examples of algebraic quasi-hyperbolic surfaces in $\mathbb{P}^3$ of low degree. We show that an explicit family of surfaces with many singularities constructed by Labs has no genus 0 curves for the members of degree at least 8 and no curves of genus 0 or 1 for degree at least 10.

Keywords. Local Euler characteristic, singularities, toric geometry, toric vector bundles, algebraic quasi-hyperbolicity

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1. Introduction

1.1. Motivation and local Euler characteristic

We present a full analysis of the local Euler characteristic of the cotangent sheaf at a surface singularity of type A_n. Our main motivation is its application to a method for proving that certain surfaces are algebraically quasi-hyperbolic by showing that sufficiently high symmetric powers of the cotangent sheaf have global sections.

Let Y be a non-singular projective surface over a field $\mathbf{k}$ of characteristic 0. For simplicity we assume that $\mathbf{k}$ is algebraically closed. We say that Y is *algebraically quasi-hyperbolic* if it contains only finitely many curves of genus 0 and 1. Coskun and Riedl [CR23] proved that very general surfaces in $\mathbb{P}^3$ of degree $d \geq 5$ are algebraically hyperbolic, which is a property that implies they are algebraically quasi-hyperbolic as well. However, no surface defined over a number field is very general, so for many specific surfaces, the question about their quasi-hyperbolicity remains open.

For surfaces of general type, Bogomolov [Bog77] shows that if the cotangent bundle on Y is big, then Y is algebraically quasi-hyperbolic.

The cotangent bundle of a non-singular surface $X \subset \mathbb{P}^3$ is never big. However, Bogomolov and de Oliveira [BDO06] observed that the resolution $\phi\colon Y \to X$ of a normal surface X may have a big cotangent bundle if X has sufficiently many singularities for its degree. One way to see this is by considering the m^{th} symmetric power $\mathcal{F} = S^m\Omega_Y$ of the cotangent sheaf on Y. This is a vector bundle, and in particular reflexive. We take the direct image of $\mathcal{F}$ on X and take its reflexive hull $\mathcal{F}'$. We have $\mathcal{F}' = \hat{S}^m\Omega_X$, where $\hat{S}^m\Omega_X$ denotes the reflexive hull of $S^m\Omega_X$.

As Blache [Bla96, Section 3.9] shows, if the singular locus S of X consists of ADE-singularities, then *local Euler characteristics* as defined by Wahl in [Wah76] can be used to express the difference in Euler characteristics as a sum of local contributions at the singularities $s \in S$ as

$$\chi(X, \mathcal{F}') = \chi(Y, \mathcal{F}) + \sum_{s \in S} \chi_{\mathrm{loc}}(s, \mathcal{F}), \tag{1.1}$$

where $\chi_{\mathrm{loc}}(s,\mathcal{F})$ is defined as follows. For a sufficiently small open affine neighbourhood X° of s, together with $Y^\circ = \phi^{-1}X^\circ$ and $E_s = \phi^{-1}(s)$, we set

$$\begin{aligned}\chi_{\mathrm{loc}}(s,\mathcal{F}) &= \chi^0(s,\mathcal{F}) + \chi^1(s,\mathcal{F}), \text{ where}\\ \chi^0(s,\mathcal{F}) &= \dim \mathrm{H}^0(Y^\circ - E_s,\mathcal{F})/\mathrm{H}^0(Y^\circ,\mathcal{F}) \text{ and}\\ \chi^1(s,\mathcal{F}) &= \dim \mathrm{H}^1(Y^\circ,\mathcal{F}).\end{aligned}$$

We note that $\mathrm{H}^0(Y^\circ - E_s,\mathcal{F}) \simeq \mathrm{H}^0(X^\circ - s,\mathcal{F}')$ and that, thanks to reflexivity, $\mathrm{H}^0(X^\circ - s,\mathcal{F}') \simeq \mathrm{H}^0(X^\circ,\mathcal{F}')$. Hence, $\chi^0(s,S^m\Omega_Y)$ gives a bound on the number of conditions that sections of $\mathcal{F}'$ need to satisfy to extend to regular sections on E_s upon pull-back. Globally, this yields

$$h^0(Y,S^m\Omega_Y) \geq h^0(X,\hat{S}^m\Omega_X) - \sum_s \chi^0(s,S^m\Omega_Y).$$

For Y of general type, we have $h^2(X,\hat{S}^m\Omega_X) = 0$ for $m > 2$ by [BDO06, Proposition 2.3], which implies that $h^0(X,\hat{S}^m\Omega_X) \geq \chi(X,\hat{S}^m\Omega_X)$ for $m \geq 3$. We obtain

$$h^0(Y,S^m\Omega_Y) \geq \chi(Y,S^m\Omega_Y) + \sum_s \chi^1(s,S^m\Omega_Y) \quad \text{for } m \geq 3. \tag{1.2}$$

By [BTVA22, Proposition 3.7] we have $\chi^1(s,S^m\Omega_Y) = \frac{4}{27}m^3 + O(m)$ for an A_1-singularity. It follows (see [BTVA22, Example 4.2 and Remark 4.3]) that a surface $X \subset \mathbb{P}^3$ of degree $d \geq 5$ with $r > \frac{9}{4}(2d^2 - 5d)$ singularities of type A_1 has a big cotangent bundle.

1.2. Local Euler characteristics at A_n-singularities

We compute the local Euler characteristic and its components for an A_n-singularity s_n. Specifically, we prove the following in Section 3.5.

Theorem 1.3. *For an A_n-singularity s_n on a surface X with minimal resolution $Y \to X$, we have*

$$\chi_{\mathrm{loc}}(s_n,S^m\Omega^1_Y) = \frac{(n+1)^2-1}{(n+1)}\left(\frac{1}{6}m^3 + \frac{1}{2}m^2 + \frac{1}{4}m\right) + \frac{b_n(m)}{4(n+1)}\cdot m + \frac{c_n(m)}{12(n+1)},$$

where

$$b_n(m) = \begin{cases} 0 & n \text{ even},\\ 1 & n \text{ odd}, q \text{ even},\\ -1 & n \text{ odd}, q \text{ odd},\end{cases}$$

$$c_n(m) = \begin{cases} 2q^3 - 3(n-1)q^2 + (n^2-4n-2)q & n \text{ even}, q \text{ even},\\ 2q^3 - 3(n-1)q^2 + (n^2-4n-2)q - 3(n+1) & n \text{ even}, q \text{ odd},\\ 2q^3 - 3(n-1)q^2 + (n^2-4n-5)q & n \text{ odd}, q \text{ even},\\ 2q^3 - 3(n-1)q^2 + (n^2-4n+1)q - 3(n+1) & n \text{ odd}, q \text{ odd},\end{cases}$$

and q is the remainder of m divided by $n+1$.

We also compute $\chi^0(s_n,S^m\Omega^1_Y)$ as a sum of lattice point counts in rational polytopes. In order to formulate the result, we need to define some vertices. Consider

$$\begin{aligned} P_i &= \left(-\frac{1}{i+1},0,0\right), \quad Q_i = \left(-\frac{2}{(i+1)(i+2)},-\frac{i}{i+2},\frac{i}{i+2}\right) \quad \text{for } i = 0,1,\dots,\\ Z &= (0,-1,0),\\ P'_n &= \left(\frac{1}{n+1},-1,0\right), \quad Q'_n = \left(\frac{2}{(n+1)(n+2)},-1,\frac{n}{n+2}\right).\end{aligned}$$

Writing Conv V for the convex hull of a set V, we consider the half-open convex polytopes

$$\begin{aligned} \mathcal{P}_i &= \operatorname{Conv}\{P_{i-1}, Q_{i-1}, P_i, Q_i, Z\} \setminus \operatorname{Conv}\{P_i, Q_i, Z\} \setminus \operatorname{Conv}\{P_{i-1}, P_i, Z\}, \\ \mathcal{C}_n &= \operatorname{Conv}\{P_n, P_n', Q_n, Q_n', Z\} \setminus \operatorname{Conv}\{P_n, P_n', Z\}. \end{aligned} \tag{1.4}$$

For a polytope $\mathcal{P}$ we consider the Ehrhart function counting integral point in dilations of the polytope $\mathcal{P}$,

$$L(\mathcal{P}, t) = \#\left(t\mathcal{P} \cap \mathbb{Z}^3\right) \quad \text{for } t = 0, 1, 2, \ldots.$$

For a convex polytope spanned by vertices with rational coordinates, this function is a *quasi-polynomial.* The same holds for a non-convex polytope. In Section 4.3 we prove the following.

Theorem 1.5. *Let $Y \to X$ be the minimal resolution of a surface singularity $s_n \in X$ of type A_n. Then*

$$\chi^0\left(s_n, S^m\Omega^1_Y\right) = L(\mathcal{C}_n, m+1) + 2\sum_{i=1}^{n} L(\mathcal{P}_i, m+1),$$

which is the lattice point count of the dilations of a half-open non-convex polytope with volume

$$\operatorname{vol}\mathcal{C}_n + 2\sum_{i=1}^{n}\operatorname{vol}\mathcal{P}_i.$$

See the appendix for the generating functions of $L(\mathcal{P}_n, m+1)$ and $L(\mathcal{C}_n, m+1)$ for small values of n.

By considering the non-convex polytope as $n \to \infty$, we obtain some extra information; see Section 4.4 for the proof.

Proposition 1.6.

(1) *The function $\chi^0(s_n, S^m\Omega^1_Y)$ is non-decreasing both in n and in m.*
(2) *The function $\chi^0(s_n, S^m\Omega^1_Y)$ is constant in n for $n > m$.*
(3) *For any fixed n we have $\chi^0(s_n, S^m\Omega^1_Y) \le (\frac{2}{9}\pi^2 - 2)(m+1)^3 + O(m^2)$, where π denotes Archimedes' constant, so $\frac{2}{9}\pi^2 - 2 \approx 0.1932$.*

For the proofs of the above results, we employ tools from toric geometry. Consider the affine variety $X\colon x_1x_2 = x_3^{n+1} \subset \mathbb{A}^3$ with its A_n-singularity $s = (0,0,0)$, as well as its minimal resolution $Y \to X$. Both X and Y are toric varieties; see Examples 2.1 and 2.3.

The reflexive hull of the symmetric powers of the cotangent sheaf on X, and the symmetric powers of the cotangent bundle on Y, are torus-equivariant reflexive sheaves. In general, for any equivariant reflexive sheaf $\mathcal{F}$ on a toric variety Z, the equivariant structure provides a grading of the cohomology parametrized by the character lattice M of the maximal torus:

$$\mathrm{H}^i(Z, \mathcal{F}) = \bigoplus_{u \in M} \mathrm{H}^i(Z, \mathcal{F})_u.$$

Klyachko [Kly89] gives a very detailed description of these graded pieces in terms of combinatorial data associated to Z and $\mathcal{F}$; this applies in particular to the sheaves $\hat{S}^m\Omega_X$ and $S^m\Omega_Y$. We can express the quantities in Theorems 1.3 and 1.5 as sums of graded parts as well. Using Klyachko's machinery we find that only finitely many of these graded parts are non-trivial and that we can express them as lattice point counts in a non-convex polytope dilated by a factor of $m+1$. For Theorem 1.3 this expression significantly simplifies through the use of lattice-preserving scissor operations and manipulations of generating functions.

1.3. Applications to algebraic quasi-hyperbolicity

Comparing the results from Theorem 1.3 and Proposition 1.6, we see that the coefficient of m^3 in $\chi^0(s_n, S^m\Omega^1_Y)$ is bounded in n, whereas in $\chi_{\mathrm{loc}}(s_n, S^m\Omega^1_Y)$ it grows linearly with n. As a result, we see that the inequality (1.2) improves as n grows.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$d=5$	57	27	18	13	11	–
$d=6$	95	46	30	22	18	15
$d=7$	142	68	45	33	27	22
$d=8$	199	95	62	46	37	31
$d=9$	264	126	83	61	49	41
$d=10$	338	162	106	78	62	52

Table 1.1. Values for $r(d,n)$ for small d,n.

A hypersurface $X \subset \mathbb{P}^3$ of degree $d \geq 5$ with r singularities of type A_n is of general type. For a minimal desingularization Y of X, one can compute $\chi(Y, S^m\Omega^1_Y)$ by Atiyah's observation that this equals $\chi(Z, S^m\Omega^1_Z)$ for a smooth degree d surface $Z \subset \mathbb{P}^3$, combined with a standard application of Hirzebruch–Riemann–Roch and a Chern class computation. This is outlined in [BDO06], and the explicit full formulae are given in [BTVA22, Appendix]. The leading term is

$$\chi\left(Y, S^m\Omega^1_Y\right) = -\tfrac{1}{3}(2d^2-5d)m^3 + O\left(m^2\right). \tag{1.7}$$

By combining Theorems 1.3 and 1.5, we can compute for any particular n the formula for $\chi^1(s_n, S^m\Omega^1_Y)$ as a quasi-polynomial in m and therefore compute the bound (1.2) explicitly as a function of m and r. It is then a matter of simple algebra to determine a bound $r(d,n)$ such that for $r \geq r(d,n)$ we have that $\chi(Y, S^m\Omega^1_Y)$ is a quasi-polynomial of degree 3 in m with a positive coefficient for m^3. This ensures that for large enough m the sheaf $S^m\Omega^1_Y$ has global sections, guaranteeing the algebraic quasi-hyperbolicity; in fact, it guarantees that the cotangent sheaf is big. We tabulate some values for $r(d,n)$ for small d,n in Table 1.1. Code to compute the requisite quasi-polynomials is available; see [BIX23].

Miyaoka [Miy84] shows that a degree d surface has at most $\frac{2}{3}(d-1)^2 d(n+1)/(2n+1)$ singularities of type A_n. Hence, we see that for $n=1$ the smallest realizable degree would be $d=10$, and indeed Barth's decic surface has $r=345$ singularities of type A_1 and therefore has big cotangent bundle. For $n \geq 2$ we see that Miyaoka's bound does not exclude any d.

Labs [Lab06, Corollary A], using a construction attributed to Segre [Seg52] and generalized by Galliarti [Gal52], describes surfaces of degree $d=2k$ with $4k^2$ singularities of type A_{k-1}. An explicit equation (see Section 5.2) for such surfaces is

$$X_k:\ \xi_0^{2k}+\xi_1^{2k}+\xi_2^{2k}+\xi_3^{2k}-\xi_0^k\xi_1^k-\xi_0^k\xi_2^k-\xi_0^k\xi_3^k-\xi_1^k\xi_2^k-\xi_1^k\xi_3^k-\xi_2^k\xi_3^k=0.$$

For $k \geq 4$ these surfaces have enough singularities to force the cotangent bundle on their minimal resolutions to be big, and hence these surfaces are algebraically quasi-hyperbolic, as was also found by Weiss; see [Wei20, Corollary 1.1.17].

While very general surfaces of degree at least 5 are algebraically hyperbolic by [CR23], no surface defined over a number field is very general. The surface X_4 is an explicit degree 8 surface in $\mathbb{P}^3$ that is algebraically quasi-hyperbolic. To our knowledge this is the lowest-degree explicit example.

We can in fact prove a little more by computing a regular symmetric differential on X_k for $k \geq 4$; see Section 5.3 for the proof.

Theorem 1.8. *For $k \geq 4$ the surface X_k contains no genus 0 curves. For $k \geq 5$ the surface X_k contains no curves of genus 0 or 1.*

1.4. Literature

Bogomolov and de Oliveira [BDO06] first considered algebraic quasi-hyperbolicity of hypersurfaces with A_1-singularities. Due to an error in their computations, they are led to consider an alternative inequality to

(1.2) that is established through Serre duality,

$$h^0\left(Y,S^m\Omega_Y^1\right)\geq\chi\left(Y,S^m\Omega_Y^1\right)+\sum_s\chi^0\left(s,S^m\Omega_Y^1\right). \tag{1.9}$$

Bruin–Thomas–Várilly-Alvarado [BTVA22] correct the error and compute $\chi^0(s_1,S^m\Omega_Y^1)$ and $\chi^1(s_1,S^m\Omega_Y^1)$ exactly. They also generalize the results to complete intersection surfaces and give several examples of algebraically quasi-hyperbolic ones.

Using an orbifold approach, Roulleau–Rousseau [RR14] approximate the local Euler characteristic of an A_n-singularity s_n by $\chi_{\mathrm{loc}}(s_n,S^m\Omega_Y^1)=\frac{n(n+2)}{6(n+1)}m^3+O(m^2)$, consistent with Theorem 1.3. They combine equations (1.2) and (1.9) to a weaker inequality

$$h^0\left(Y,S^m\Omega_Y^1\right)\geq\chi\left(Y,S^m\Omega_Y^1\right)+\tfrac{1}{2}\sum_s\chi_{\mathrm{loc}}\left(s,S^m\Omega_Y^1\right),$$

which allows them to identify examples of degree $d\geq 13$ with sufficient A_1-singularities for their bound to imply algebraic quasi-hyperbolicity. From Proposition 1.6 it follows that (1.2) gives the stronger result for $n\geq 2$.

De Oliveira–Weiss [DOW19] consider A_2-singularities and reference an approximation to $\chi^0(s_2,S^m\Omega_Y)$ that is consistent with Theorem 1.5. They also reference [Lab06] for an example of a degree 9 surface with sufficiently many A_2-singularities to conclude it has big cotangent bundle. Theorem 1.5 and Proposition 1.6 largely follow the exposition in the third author's master's thesis [Xu23]. The leading coefficient in m for $\chi^0(s_n,S^m\Omega_Y)$ for A_n-singularities is derived independently by Weiss [Wei20], and the top two coefficients are determined independently by Asega–de Oliveira–Weiss [ADOW23].

Explicit computations with symmetric differentials as in Section 5.3 go back to Vojta [Voj00]. See also [BTVA22] for more elaborate examples.

2. Preliminaries

2.1. Toric varieties

We recall here the necessary basics of toric geometry. See [CLS11] for more details. Let N be a finitely generated free abelian group with dual $M=\mathrm{Hom}(N,\mathbb{Z})$. Given a pointed polyhedral cone $\sigma\subseteq N\otimes\mathbb{R}$, its dual is

$$\sigma^\vee=\{u\in M\otimes\mathbb{R}\mid\langle v,u\rangle\geq 0\ \forall\ v\in\sigma\}.$$

Here $\langle v,u\rangle$ is the natural pairing induced by the duality of N and M. The semigroup $\sigma^\vee\cap M$ is finitely generated, and

$$X_\sigma=\mathrm{Spec}\,\mathbf{k}[\sigma^\vee\cap M]$$

is the ***affine toric variety*** associated to the cone σ. The dimension of X_σ is simply the rank of N. The M-grading of $\mathbf{k}[\sigma^\vee\cap M]$ induces an inclusion of the torus $T=\mathrm{Spec}\,\mathbf{k}[M]=N\otimes\mathbf{k}$ in X_σ, with the action of T on itself extending to X_σ.

Example 2.1 (An A_n-singularity). We take $M=N=\mathbb{Z}^2$, with $\langle\cdot,\cdot\rangle$ the standard inner product. Let σ_{A_n} be the cone generated by $(0,1)$ and $(n+1,1)$. Its dual $\sigma_{A_n}^\vee$ is generated by $(1,0)$ and $(-1,n+1)$. The semigroup $\sigma_{A_n}^\vee\cap M$ is generated by $(1,0)$, $(-1,n+1)$, and $(0,1)$. See Figure 2.1. These generators satisfy the relation

$$(1,0)+(-1,n+1)=(n+1)\cdot(0,1),$$

so the toric variety $X_{\sigma_{A_n}}$ is isomorphic to the vanishing locus of $x_1x_2-x_3^{n+1}$ in $\mathbb{A}^3$. This is an isolated surface singularity of type A_n.

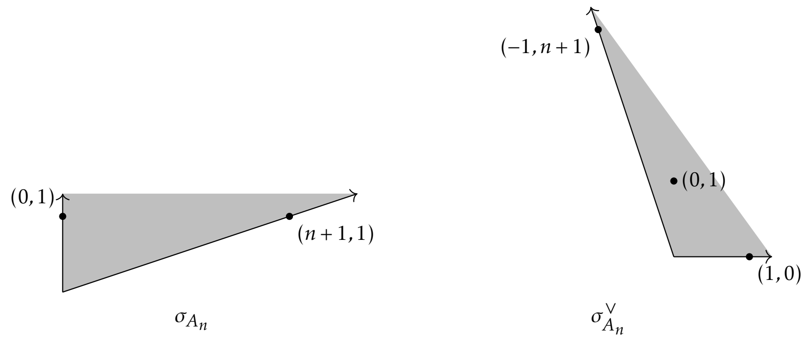


Figure 2.1. The cone and dual cone for an A_n-singularity

The above construction globalizes. Let Σ be a ***fan*** in $N \otimes \mathbb{R}$, that is, a collection of pointed polyhedral cones that is closed under taking faces, and such that any two elements intersect in a common face. Any face relation $\tau \prec \sigma$ for $\sigma \in \Sigma$ induces an open inclusion $X_\tau \hookrightarrow X_\sigma$. The toric variety X_Σ is constructed by gluing together the affine toric varieties

$$\{X_\sigma\}_{\sigma \in \Sigma}$$

along the open immersions induced by face relations; see [CLS11, Section 3.1] for precise details. Moreover, any normal variety X equipped with an effective action of the torus T can be constructed in this fashion; see [CLS11, Corollary 3.1.8].

Many aspects of the geometry of X_Σ can be read directly from Σ. For instance, X_Σ is non-singular if and only if the fan Σ is ***smooth***, that is, the primitive lattice generators for each cone in Σ can be completed to a basis of N; see [CLS11, Theorem 3.1.19]. For any natural number i, let $\Sigma^{(i)}$ be the set of i-dimensional cones in Σ. Torus-invariant prime divisors on X_Σ are in bijection with elements of $\Sigma^{(1)}$; see [CLS11, Section 4.1]. Given a ray $\rho \in \Sigma^{(1)}$, we denote the corresponding prime divisor by D_ρ. We will denote the primitive lattice generator of the ray ρ by ν_ρ. The valuation determined by a divisor D_ρ is easily described: for any ray $\rho \in \Sigma^{(1)}$ and $u \in M$, we have

$$\operatorname{ord}_{D_\rho}(x^u) = \langle \nu_\rho, u \rangle, \tag{2.2}$$

where x^u is the rational function on the torus corresponding to u and $\operatorname{ord}_{D_\rho}(x^u)$ denotes its order of vanishing along D_ρ.

Example 2.3 (The minimal resolution of an A_n-singularity). Continuing with $M = N = \mathbb{Z}^2$, for $i = 0, 1, \ldots, n+1$ we let ρ_i be the ray in $N \otimes \mathbb{R}$ generated by $(i, 1)$. Consider the fan Σ whose $n+1$ top-dimensional cones are generated by ρ_i, ρ_{i+1} for $i = 0, \ldots, n$. See Figure 2.2.

The fan Σ is smooth, so the resulting surface X_Σ is non-singular. In fact, the toric variety X_Σ is the minimal resolution of the A_n-surface singularity from Example 2.1. Indeed, the inclusion of each cone of Σ in the cone σ_{A_n} generated by ρ_0, ρ_{n+1} induces a birational morphism $\phi : X_\Sigma \to X_\sigma$. The morphism ϕ is proper since the union of the cones in Σ is just σ_{A_n}; see [CLS11, Theorem 3.4.11].

Since the subfan of Σ consisting of ρ_0, ρ_{n+1}, and the origin is the non-singular locus of $X_{\sigma_{A_n}}$, the exceptional locus E of ϕ is the union of the prime divisors $E_1 = D_{\rho_1}, \ldots, E_n = D_{\rho_n}$. Using *e.g.* [CLS11, Theorem 10.4.4] one computes that each E_i is a (-2)-curve, so the resolution ϕ is indeed minimal.

2.2. Torus-equivariant reflexive sheaves

Let $\mathcal{F}$ be a T-equivariant reflexive sheaf on the toric variety X_Σ. In [Kly89, Kly91] Klyachko associates a collection of filtrations to $\mathcal{F}$ as follows. We first set

$$V_{\mathcal{F}} = \mathrm{H}^0\left(T, \mathcal{F}_{\restriction T}\right)^T;$$

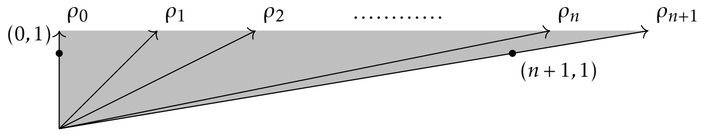


Figure 2.2. The minimal resolution of an A_n-singularity

that is, $V_{\mathcal{F}}$ is the $\mathbf{k}$-vector space obtained as the T-invariant sections of the restriction of $\mathcal{F}$ to the torus T. The restriction of $\mathcal{F}$ to T is a vector bundle, and $V_{\mathcal{F}}$ may be identified with the fibre of this bundle over the identity element of T. In particular, it is a vector space of dimension equal to the rank of $\mathcal{F}$.

For each ray $\rho \in \Sigma^{(1)}$, we may consider the decreasing $\mathbb{Z}$-filtration $V_{\mathcal{F}}^{\rho}$ defined as

$$V_{\mathcal{F}}^{\rho}(i) = \{z \in V_{\mathcal{F}} \mid \operatorname{ord}_{D_\rho}(z) \geq i\}.$$

As before $\operatorname{ord}_{D_\rho}(z)$ denotes the order of vanishing of a section z along the prime divisor D_ρ. When the sheaf $\mathcal{F}$ is clear from the context, we will omit the subscript and use the notation V and $V^\rho(i)$.

Example 2.4 (The reflexive hull of the cotangent sheaf). Let X_Σ be a toric variety with cotangent sheaf $\Omega = \Omega_{X_\Sigma}$. This bundle has a natural T-equivariant structure. The corresponding filtrations for its reflexive hull $\hat{\Omega}$ are as follows:

$$V = M \otimes \mathbf{k},$$
$$V^\rho(i) = \begin{cases} V & i < 0, \\ \ker\left(\nu_\rho\right) \subset V & i = 0, \\ 0 & i > 0. \end{cases}$$

If X_Σ is smooth, then $\hat{\Omega} = \Omega$ and this is just [Kly89, Section 2.3, Example 5]. For the singular case we note that $\hat{\Omega}$ agrees with Ω on the non-singular locus of X_Σ. Since any toric variety is smooth in codimension 1, the filtrations for $\hat{\Omega}$ agree with the filtrations for the restriction of Ω to the non-singular locus of X_Σ, which are exactly the filtrations above.

Let $\mathcal{F}$ be an equivariant reflexive sheaf on X_Σ. It is straightforward to describe the filtration data of the reflexive hull of its symmetric powers $\hat{S}^m\mathcal{F}$ in terms of the filtration data of $\mathcal{F}$:

$$V_{\hat{S}^m\mathcal{F}} = S^m V_{\mathcal{F}},$$
$$V^\rho_{\hat{S}^m\mathcal{F}}(i) = \sum_{j_1+j_2+\ldots+j_m=i} V^\rho_{\mathcal{F}}(j_1)\cdot\ldots\cdot V^\rho_{\mathcal{F}}(j_m) \subseteq S^m V_{\mathcal{F}}.$$

See [Gon11, Corollary 3.5] for the locally free case; the reflexive case follows immediately.

Example 2.5 (Symmetric powers of the cotangent sheaf). Combining the above with Example 2.4, we obtain that for the reflexive sheaf $\hat{S}^m\Omega$, we have

$$V_{\hat{S}^m\Omega} = S^m(M \otimes \mathbf{k}),$$
$$V^\rho_{\hat{S}^m\Omega}(i) = \begin{cases} S^m(M \otimes \mathbf{k}) & i \leq -m, \\ S^{i+m}(\rho^\perp)\cdot S^{-i}(M\otimes\mathbf{k}) & -m \leq i \leq 0, \\ 0 & i \geq 1. \end{cases}$$

For any T-equivariant reflexive sheaf $\mathcal{F}$, T acts on the cohomology groups $\mathrm{H}^p(X_\Sigma,\mathcal{F})$, and so these decompose as a direct sum of eigenspaces

$$\mathrm{H}^p(X_\Sigma,\mathcal{F}) = \bigoplus_{u\in M} \mathrm{H}^p(X_\Sigma,\mathcal{F})_u.$$

Global sections are especially easy to describe. For any ray $\rho \in \Sigma$ and $u \in M$, let $\rho(u) = \langle \nu_\rho, u\rangle$. We have that $\mathrm{H}^0(T, \mathcal{F}|_T)_u \simeq V_{\mathcal{F}}$ via $z \mapsto x^u z$. From $\mathrm{ord}_{D_\rho}(x^u z) = \mathrm{ord}_{D_\rho}(z) + \rho(u)$ we obtain

$$\left\{z \in \mathrm{H}^0(T, \mathcal{F}|_T)_u \mid \mathrm{ord}_{D_\rho}(z) \geq i\right\} = x^{-u} V_{\mathcal{F}}^{\rho}(i + \rho(u)). \tag{2.6}$$

In particular,

$$\mathrm{H}^0(X_\Sigma, \mathcal{F})_u \cong \bigcap_{\rho \in \Sigma^{(1)}} V_{\mathcal{F}}^{\rho}(\rho(u)). \tag{2.7}$$

Higher cohomology groups of $\mathcal{F}$ may also be recovered from the filtration data. For $\sigma \in \Sigma$ and $u \in M$, set

$$W_{\mathcal{F}}^{\sigma}(u) = V_{\mathcal{F}} / \sum_{\rho \in \Sigma^{(1)} \cap \sigma} V_{\mathcal{F}}^{\rho}(\rho(u)).$$

Klyachko uses these vector spaces to construct a complex

$$0 \longrightarrow \bigoplus_{\sigma \in \Sigma^{(0)}} W^{\sigma}(u) \longrightarrow \bigoplus_{\sigma \in \Sigma^{(1)}} W^{\sigma}(u) \longrightarrow \bigoplus_{\sigma \in \Sigma^{(2)}} W^{\sigma}(u) \longrightarrow \cdots \tag{2.8}$$

whose p^{th} cohomology may be identified with $\mathrm{H}^p(X_\Sigma, \mathcal{F})_u$; see [Kly89, Theorem 4.1.1]. In particular, we have the following.

Proposition 2.9. *Let $\mathcal{F}$ be a T-equivariant reflexive sheaf on the toric variety X_Σ. For any $u \in M$ the quantity*

$$\chi_u(\mathcal{F}) := \sum_{p \geq 0} (-1)^p \dim \mathrm{H}^p(X_\Sigma, \mathcal{F})_u$$

may be computed as

$$\chi_u(\mathcal{F}) = \sum_{p \geq 0} (-1)^p \sum_{\sigma \in \Sigma^{(p)}} \dim W_{\mathcal{F}}^{\sigma}(u).$$

Proof. Since the cohomology of the complex (2.8) computes $\mathrm{H}^p(X_\Sigma, \mathcal{F})_u$, the alternating sum of the dimensions of the terms of the complex computes $\chi_u(\mathcal{F})$. ☐

Remark 2.10. Klyachko initially constructs the complex (2.8) when $\mathcal{F}$ is locally free. However, it is straightforward to check that the result [Kly89, Theorem 4.1.1] is also true in the reflexive case; the proof in *loc. cit.* goes through verbatim.

2.3. Ehrhart theory

We briefly recall some basics of Ehrhart theory. See *e.g.* [BR15] for details. For the purposes of this article, a *convex polytope* is the convex hull of a finite set in $\mathbb{R}^d$. A *non-convex polytope* is a connected finite union of convex polytopes. A *half-open polytope* is a polytope with some of its faces removed. A *quasi-polynomial* $f(t)$ is a function from $\mathbb{N}$ to $\mathbb{N}$ that may be written in the form

$$f(t) = a_d(t)t^d + a_{d-1}(t)t^{d-1} + \cdots + a_0(t),$$

where the coefficient functions $a_i(t)$ are periodic of integral period. The degree of such an $f(t)$ is the largest exponent d such that $a_d(t)$ is not identically zero; the period is the least common multiple of the periods of all coefficient functions.

For a rational polytope $\mathcal{P} \subset \mathbb{R}^d$, we may consider its Ehrhart function

$$L(\mathcal{P}, t) = \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right) \quad \text{for } t = 0, 1, 2, \ldots.$$

This function is a quasi-polynomial in t whose period divides the smallest integer λ such that $\lambda \cdot \mathcal{P}$ is integral. The degree of $L(\mathcal{P}, t)$ is the dimension of $\mathcal{P}$. Assuming that $\mathcal{P}$ has dimension d, the leading coefficient of $L(\mathcal{P}, t)$ is simply the volume of $\mathcal{P}$.

Given a subset $A \subset \mathbb{R}^d$, we define its *lattice point transform* to be

$$\mathfrak{S}_A = \sum_{u \in A \cap \mathbb{Z}^d} z^u.$$

This is a formal power series in $z_1, \dots, z_d$ and is a useful tool for computing the generating series of $L(\mathcal{P}, t)$. We will make use of the following.

Proposition 2.11 (*cf.* [BR15, Theorem 3.5]). *Let $C \subset \mathbb{R}^d$ be a simplicial cone whose rays are generated by primitive vectors $w_1, \dots, w_k \in \mathbb{Z}^d$. Set*

$$\Pi(C) = \left\{ \sum \alpha_i w_i \mid 0 \le \alpha_i < 1 \right\}.$$

Then

$$\mathfrak{S}_C(z) = \frac{\mathfrak{S}_{\Pi(C)}}{(1 - z^{w_1}) \cdots (1 - z^{w_k})}.$$

3. Computation of χ_{loc}

3.1. A recursive formula

Let $Y \to X$ be a minimal resolution of a surface X with an A_n-singularity s_n. We are interested in computing

$$\chi(n, m) := \chi_{\mathrm{loc}}\left(s_n, S^m \Omega_Y^1\right).$$

Our approach is to use the machinery described in Section 2.2. It will be advantageous to first develop a recursive formula for $\chi(n, m)$. For $n = 0$ we set $\chi(n, m) = \chi(0, m) = 0$.

Fix $N = \mathbb{Z}^2$. As in Example 2.3 we let $\rho_i \subset \mathbb{R}^2$ be the ray generated by $(i, 1)$. We additionally consider the rays $\rho_+, \rho_-, \rho_\infty$ generated by $(1, 0)$, $(-1, 0)$, and $(0, -1)$, respectively. Fixing $n \ge 1$, we let $\widetilde{\Sigma}$, $\overline{\Sigma}$, and Σ be the unique complete fans in $\mathbb{R}^2$ whose rays are as follows:

$$\begin{aligned}
\widetilde{\Sigma}^{(1)} &= \{\rho_0, \dots, \rho_{n+1}, \rho_+, \rho_\infty, \rho_-\}, \\
\overline{\Sigma}^{(1)} &= \{\rho_0, \rho_1, \rho_{n+1}, \rho_+, \rho_\infty, \rho_-\}, \\
\Sigma^{(1)} &= \{\rho_0, \rho_{n+1}, \rho_+, \rho_\infty, \rho_-\}.
\end{aligned}$$

See Figure 3.1.

For any $m \ge 0$ and $u \in M = \mathbb{Z}^2$, we define

$$\delta_n(m, u) := \chi_u\left(\hat{S}^m \Omega_{X_\Sigma}\right) - \chi_u\left(\hat{S}^m \Omega_{X_{\overline{\Sigma}}}\right).$$

We will see in Section 3.2 how to calculate $\delta_n(m, u)$ explicitly. We set

$$\delta_n(m) = \sum_{u \in M} \delta_n(m, u).$$

Since both $X_{\overline{\Sigma}}$ and X_Σ are complete, $\delta_n(m, u) = 0$ for all but finitely many $u \in M$, and the above sum is finite.

Lemma 3.1. *For any $m \ge 0$,*

$$\chi(n, m) - \chi(n - 1, m) = \delta_n(m).$$

Proof. The toric varieties $X_{\widetilde{\Sigma}}$, $X_{\overline{\Sigma}}$, and X_Σ are all complete surfaces. Similarly to Example 2.3, there is a sequence of toric morphisms

$$X_{\widetilde{\Sigma}} \longrightarrow X_{\overline{\Sigma}} \longrightarrow X_\Sigma.$$

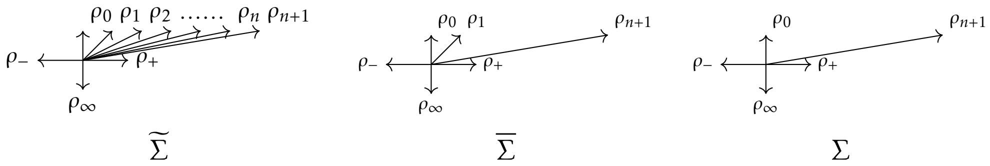


Figure 3.1. The fans $\widetilde{\Sigma}$, $\overline{\Sigma}$, and Σ

The surface X_Σ has a single A_n-singularity (see Example 2.1). The surface $X_{\overline{\Sigma}}$ has a single A_{n-1}-singularity: this may be seen by applying the lattice isomorphism

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

Here, an A_0-singularity is just a smooth point. As in Example 2.3, $X_{\widetilde{\Sigma}}$ is the minimal resolution of both X_Σ and $X_{\overline{\Sigma}}$.

By applying (1.1) for both the A_n- and the A_{n-1}-singularity, we obtain

$$\begin{aligned} \chi(n,m) - \chi(n-1,m) &= \left(\chi\left(\hat{S}^m\Omega_{X_\Sigma}\right) - \chi\left(S^m\Omega_{X_{\widetilde{\Sigma}}}\right)\right) - \left(\chi\left(\hat{S}^m\Omega_{X_{\overline{\Sigma}}}\right) - \chi\left(S^m\Omega_{X_{\widetilde{\Sigma}}}\right)\right) \\ &= \chi\left(\hat{S}^m\Omega_{X_\Sigma}\right) - \chi\left(\hat{S}^m\Omega_{X_{\overline{\Sigma}}}\right), \end{aligned}$$

and the claim follows. $\square$

3.2. Computing $\delta_n(m)$

Define

$$\lambda_m(i) = \begin{cases} 0 & i \le -m, \\ i+m & -m \le i \le 1, \\ m+1 & i \ge 1. \end{cases}$$

Lemma 3.2. *For any $u \in M = \mathbb{Z}^2$,*

$$\begin{aligned} \delta_n(m,u) = (m+1) - \lambda_m(\rho_1(u)) &- \max\{m+1-\lambda_m(\rho_0(u)) - \lambda_m(\rho_1(u)), 0\} \\ &- \max\{m+1-\lambda_m(\rho_1(u)) - \lambda_m(\rho_{n+1}(u)), 0\} \\ &+ \max\{m+1-\lambda_m(\rho_0(u)) - \lambda_m(\rho_{n+1}(u)), 0\}. \end{aligned}$$

Proof. We let V and $\{V^\rho(i)\}$ be the vector space and filtrations associated to the reflexive hull of the m^{th} symmetric power of the cotangent sheaf on any toric surface. Then by Example 2.5 we have

$$\begin{aligned} \dim V &= m+1, \\ \dim V^\rho(i) &= m+1-\lambda_m(i), \\ \dim V^\rho(i) \cap V^{\rho'}(j) &= \max\{m+1-\lambda_m(i)-\lambda_m(j), 0\} \quad \text{if } \rho \ne \rho'. \end{aligned}$$

For $0 \le i,j \le n+1$ let σ_{ij} denote the cone in $\mathbb{R}^2$ spanned by ρ_i and ρ_j. We have that $\Sigma^{(0)} = \overline{\Sigma}^{(0)}$, and the rays of Σ and $\overline{\Sigma}$ differ only by ρ_1 (which belongs to $\overline{\Sigma}$). The sets $\Sigma^{(2)}$ and $\overline{\Sigma}^{(2)}$ differ only by $\sigma_{01}, \sigma_{1(n+1)}$, which belong to $\overline{\Sigma}^{(2)}$, and $\sigma_{0(n+1)}$, which belongs to $\Sigma^{(2)}$. Applying Proposition 2.9 to both $\chi_u(S^m\hat{\Omega}_{X_{\overline{\Sigma}}})$ and

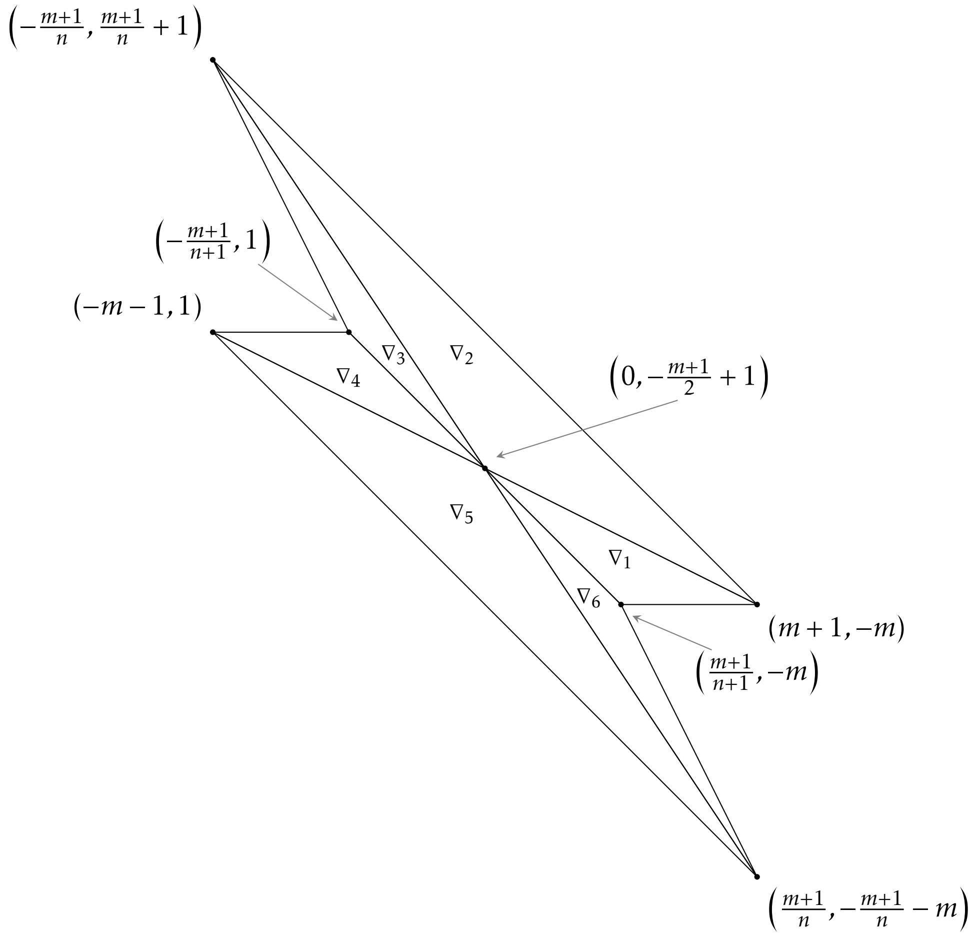


Figure 3.2. Regions of linearity of $\delta_n(m,u)$

$\chi_u(S^m\hat{\Omega}_{X_\Sigma})$ and cancelling terms, we obtain

$$\begin{aligned}
\delta_n(m,u) &= \dim W^{\rho_1}(u) - \dim W^{\sigma_{01}}(u) - \dim W^{\sigma_{1(n+1)}}(u) + \dim W^{\sigma_{0(n+1)}}(u)\\
&= -\dim V^{\rho_1}(\rho_1(u)) + \dim(V^{\rho_0}(\rho_0(u)) + V^{\rho_1}(\rho_1(u)))\\
&\qquad + \dim(V^{\rho_1}(\rho_1(u)) + V^{\rho_{n+1}}(\rho_{n+1}(u)))\\
&\qquad - \dim(V^{\rho_0}(\rho_0(u)) + V^{\rho_{n+1}}(\rho_{n+1}(u)))\\
&= (m+1) - \lambda_m(\rho_1(u)) - \max\{m+1-\lambda_m(\rho_0(u)) - \lambda_m(\rho_1(u)), 0\}\\
&\qquad - \max\{m+1-\lambda_m(\rho_1(u)) - \lambda_m(\rho_{n+1}(u)), 0\}\\
&\qquad + \max\{m+1-\lambda_m(\rho_0(u)) - \lambda_m(\rho_{n+1}(u)), 0\}.
\end{aligned}$$

The second equality follows by writing W^σ in terms of V^ρ. The third follows by using

$$\dim(V^\rho(i) + V^{\rho'}(j)) = \dim V^\rho(i) + \dim V^{\rho'}(j) - \dim(V^\rho(i) \cap V^{\rho'}(j))$$

and the above computation of $\dim(V^\rho(i) \cap V^{\rho'}(j))$. □

Using the formula for $\delta_n(m,u)$ in Lemma 3.2, we may extend $\delta_n(m,u)$ to a function in u on all of $\mathbb{R}^2$; this function is piecewise linear.

Lemma 3.3. *Outside of the six polytopes $\nabla_1,\ldots,\nabla_6$ pictured in Figure 3.2, the function $\delta_n(m,u)$ vanishes. The regions of linearity of $\delta_n(m,u)$ are exactly the six polytopes $\nabla_1,\ldots,\nabla_6$. On each of these six simplices, $\delta_n(m,u)$ takes value $(m+1)/2$ at the vertex $(0,-(m+1)/2+1)$ and 0 at the other two vertices.*

Proof. From the description of $\delta_n(m,u)$ in Lemma 3.2 and the definition of $\lambda_m(i)$, it follows that the non-linear locus of $\delta_n(m,u)$ is contained in the lines $\rho_i(u) = 1$, $\rho_j(u) = -m$ for $i,j = 0,1,n+1$ along with the lines $\rho_0(u) + \rho_1(u) = 1-m$, $\rho_1(u) + \rho_{n+1}(u) = 1-m$, and $\rho_0(u) + \rho_{n+1}(u) = 1-m$.

Since $\delta_n(m,u) = 0$ for all but finitely many $u \in \mathbb{Z}^2$, we know that $\delta_n(m,u) = 0$ on any unbounded region in the above subdivision of $\mathbb{R}^2$. For each of the remaining bounded regions, we may calculate the linear function representing $\delta_n(m,u)$ on that region. In doing so, and combining regions with the same linear function, one obtains the result of the lemma. □

3.3. Counting lattice points

For this subsection we introduce some notation for subsets of $\mathbb{R}^2$. Let $\gamma = (0,1/2) \in \mathbb{R}^2$. For $a \le b \in \mathbb{Q}$ set

$$[a:b] = \mathrm{Conv}\{(a,0),(b,0),(0,1/2)\} \setminus \gamma.$$

We further define

$$[a] = [a:a], \quad (a:b] = [a:b] \setminus [a], \quad [a:b) = [a:b] \setminus [b], \quad (a:b) = [a:b] \setminus ([a] \cup [b]).$$

For sets $A, B \subset \mathbb{R}^2$ we will use the notation $A + B$ to denote a disjoint union of A and B as abstract sets. Likewise, for $\ell \in \mathbb{Z}$ we use $\ell * A$ to denote the disjoint union of A with itself ℓ times (again as an abstract set). In particular, $\#((\ell * A) \cap \mathbb{Z}^2) = \ell \cdot (\#(A \cap \mathbb{Z}^2))$.

We set

$$\begin{aligned}\square_n :=& 2 * \left(-\frac{1}{n} : -\frac{1}{n+1}\right) + 2 * \left(-1 : \frac{1}{n}\right) + 2 * \left(\frac{1}{n+1} : 1\right) \\ &+ 2 * \left[\frac{1}{n+1}\right] + 2 * \left[\frac{1}{n}\right] + 2 * [1] + \gamma.\end{aligned}$$

By $(m+1) \cdot \square_n$ we denote the $(m+1)^{\text{st}}$ dilate of $\square_n$, where the dilate of a disjoint union is the disjoint union of the dilates.

Lemma 3.4. *For any $m \ge 1$,*

$$\delta_n(m) = \sum_{(x,y) \in ((m+1)\cdot \square_n) \cap \mathbb{Z}^2} y.$$

Proof. To each polytope ∇_i from Figure 3.2, we will apply an invertible integral affine linear transformation ϕ_i:

Polytope	Transformation ϕ_i	Image
∇_1	$(x,y) \mapsto (x, y+m)$	$(m+1) \cdot [\frac{1}{n+1} : 1]$
∇_2	$(x,y) \mapsto (-x, -x-y+1)$	$(m+1) \cdot [-1 : \frac{1}{n}]$
∇_3	$(x,y) \mapsto (x, (n+1)x + y + m)$	$(m+1) \cdot [\frac{-1}{n} : \frac{-1}{n+1}]$
∇_4	$(x,y) \mapsto (-x, -y+1)$	$(m+1) \cdot [\frac{1}{n+1} : 1]$
∇_5	$(x,y) \mapsto (x, x+y+m)$	$(m+1) \cdot [-1 : \frac{1}{n}]$
∇_6	$(x,y) \mapsto (-x, -(n+1)x - y + 1)$	$(m+1) \cdot [\frac{-1}{n} : \frac{-1}{n+1}]$

Note that the transformations ϕ_i and ϕ_{i+1} agree along $\nabla_i \cap \nabla_{i+1}$, with indices taken modulo 6. It follows from Lemma 3.3 that for each i and each $(x,y) \in \phi_i(\nabla_i)$, we have

$$\delta_n\left(m, \phi_i^{-1}((x,y))\right) = y.$$

Again using Lemma 3.3, we have

$$\delta_n(m) = \sum_{u \in (\bigcup \nabla_i) \cap M} \delta_n(m,u).$$

Applying ϕ_i to each ∇_i and using inclusion-exclusion, we obtain the claim of the lemma. □

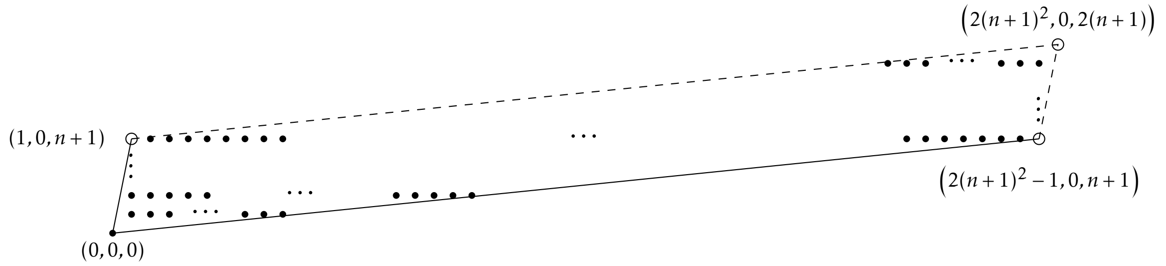


Figure 3.3. Lattice points of the region $\Pi(C_1)$

We are now able to use induction to obtain a formula for $\chi(n,m)$ as a weighted lattice point count. Using notation introduced at the start of this subsection, define

$$\Delta_n = 2 * \left(\frac{1}{n+1} : 2(n+1) - \frac{1}{n+1}\right] + n * \gamma.$$

Theorem 3.5. *For $n,m \geq 1$ we have*

$$\chi(n,m) = \sum_{(x,y)\in((m+1)\cdot\Delta_n)\cap\mathbb{Z}^2} y.$$

Proof. Up to integral translation in the x-direction and reflection around the line $x=0$, we have

$$\square_n \equiv 2 * \left(\left(-\frac{1}{n} : -\frac{1}{n+1}\right] + \left(\frac{1}{n+1} : 2 + \frac{1}{n}\right]\right) + \gamma.$$

Indeed,

$$\left(-\frac{1}{n} : -\frac{1}{n+1}\right) + \left[\frac{1}{n+1}\right] \equiv \left(-\frac{1}{n} : -\frac{1}{n+1}\right) + \left[\frac{-1}{n+1}\right] \equiv \left(-\frac{1}{n} : -\frac{1}{n+1}\right]$$

and

$$\left(-1 : \frac{1}{n}\right) + \left(\frac{1}{n+1} : 1\right) + \left[\frac{1}{n}\right] + [1] \equiv \left(1 : 2 + \frac{1}{n}\right) + \left(\frac{1}{n+1} : 1\right) + \left[2 + \frac{1}{n}\right] + [1]$$
$$\equiv \left(\frac{1}{n+1} : 2 + \frac{1}{n}\right].$$

By translating the set $(\frac{1}{k+1} : 2 + \frac{1}{k}]$ in $\square_k$ by $2(n-k)$ to the right, it is straightforward to see that

$$\square_1 + \cdots + \square_n \equiv 2 * \left(\left(-1 : -\frac{1}{n+1}\right] + \left(\frac{1}{n+1}, 2n+1\right]\right) + n * \gamma \equiv \Delta_n.$$

Since $\chi(n,m) = \delta_1(m) + \cdots + \delta_n(m)$ by Lemma 3.1, the claim of the theorem follows from Lemma 3.4. $\square$

3.4. Generating functions

Lemma 3.6. *The regular generating function for $\chi(n,m)$ as a function of m is*

$$\sum_{m\geq 0} \chi(n,m) z^m = \frac{z \cdot \left((n+1)(1+z+\cdots+z^n)^2 - (1+z^2+\cdots+z^{2n})\right)}{(1-z)^2(1-z^{n+1})^2}.$$

Proof. Consider the cones

$$C_1 = \mathrm{Pos}\left\{(1,0,n+1), (2(n+1)^2-1, 0, n+1), (0,1,2)\right\},$$
$$C_2 = \mathrm{Pos}\{(1,0,n+1), (0,1,2)\},$$
$$C_3 = \mathrm{Pos}\{(0,1,2)\},$$

where Pos denotes the positive hull. These are the cones in $\mathbb{R}^3$ over $[\frac{1}{n+1} : 2(n+1) - \frac{1}{n+1}]$, $[\frac{1}{n+1}]$, and γ.

Using variables x, y, z and following notation from Proposition 2.11, we have

$$\mathfrak{S}_{\Pi(C_1)} = 1 + \left(\sum_{k=1}^{n} \sum_{j=1}^{2(n+1)k-1} (x^j z^k + x^{2(n+1)^2-j} z^{2(n+1)-k}) \right) + \sum_{j=2}^{2(n+1)^2-2} x^j z^{n+1}. \tag{3.7}$$

Indeed, fixing the third coordinate equal to some integer k with $1 \leq k \leq n$, $\Pi(C_1)$ contains lattice points with first coordinate ranging from

$$\left\lceil \frac{k}{n+1} \right\rceil = 1 \qquad \text{to} \qquad \left\lfloor 2(n+1)k - \frac{k}{n+1} \right\rfloor = 2(n+1)k - 1.$$

For third coordinate equal to $n+1$, we have lattice points with first coordinate ranging from 2 up to $2(n+1)^2 - 2$. Points with third coordinate larger than $n+1$ are obtained by reflecting points with $1 \leq k \leq n$ through the point $(n+1, 0, (n+1)^2)$. See Figure 3.3.

Further, we note that $\mathfrak{S}_{\Pi(C_2)} = \mathfrak{S}_{\Pi(C_3)} = 1$. By Proposition 2.11 we conclude that

$$\begin{aligned}
\mathfrak{S}_{C_1} &= \frac{\mathfrak{S}_{\Pi(C_1)}}{(1 - xz^{n+1})\left(1 - x^{2(n+1)^2-1} z^{n+1}\right)} \cdot \frac{1}{(1 - yz^2)}, \\
\mathfrak{S}_{C_2} &= \frac{1}{(1 - xz^{n+1})} \cdot \frac{1}{(1 - yz^2)}, \\
\mathfrak{S}_{C_3} &= \frac{1}{(1 - yz^2)}.
\end{aligned}$$

By the definition of Δ_n, $\#((m+1) \cdot \Delta_n) \cap \mathbb{Z}^2$ is the coefficient of z^{m+1} in

$$2 \cdot \mathfrak{S}_{C_1}(1,1,z) - 2 \cdot \mathfrak{S}_{C_2}(1,1,z) + n \cdot \mathfrak{S}_{C_3}(1,1,z).$$

Similarly,

$$\sum_{(x,y) \in ((m+1)\cdot\Delta_n) \cap \mathbb{Z}^2} y$$

is the coefficient of z^{m+1} in

$$2 \cdot \frac{\partial \mathfrak{S}_{C_1}}{\partial y}(1,1,z) - 2 \cdot \frac{\partial \mathfrak{S}_{C_2}}{\partial y}(1,1,z) + n \cdot \frac{\partial \mathfrak{S}_{C_3}}{\partial y}(1,1,z).$$

Applying Theorem 3.5 and using the above expressions for the lattice point transforms, we obtain

$$\begin{aligned}
\sum_{m \geq 0} \chi(n,m) z^m &= \frac{1}{z} \left(2 \cdot \frac{\partial \mathfrak{S}_{C_1}}{\partial y}(1,1,z) - 2 \cdot \frac{\partial \mathfrak{S}_{C_2}}{\partial y}(1,1,z) + n \cdot \frac{\partial \mathfrak{S}_{C_3}}{\partial y}(1,1,z) \right) \\
&= \frac{z}{(1-z^2)^2} \cdot \frac{2\mathfrak{S}_{\Pi(C_1)}(1,1,z) - 2(1 - z^{n+1}) + n(1 - z^{n+1})^2}{(1 - z^{n+1})^2}.
\end{aligned}$$

The claim of the lemma follows from Lemma 3.8 below. □

Lemma 3.8. *We have that*

$$\begin{aligned}
&2\mathfrak{S}_{\Pi(C_1)}(1,1,z) - 2\left(1 - z^{n+1}\right) + n\left(1 - z^{n+1}\right)^2 \\
&= (1+z)^2 \cdot \left((n+1)(1 + z + \cdots + z^n)^2 - \left(1 + z^2 + \cdots + z^{2n}\right) \right).
\end{aligned}$$

Proof. Using (3.7) we have that

$$\mathfrak{S}_{\Pi(C_1)}(1,1,z) = 1 + \left(\sum_{k=1}^{n} (2(n+1)k - 1) \cdot \left(z^k + z^{2(n+1)-k}\right) \right) + (2(n+1)^2 - 2) z^{n+1}.$$

Thus, the coefficients of z^k on the left-hand side of the claimed equality in the statement of the lemma are symmetric around z^{n+1} and are equal to

$$\begin{cases} n & k=0, \\ 4(n+1)k-2 & 1\leq k\leq n, \\ 4(n+1)^2-2-2n & k=n+1. \end{cases}$$

The coefficients of the expansion of the right-hand side are clearly also symmetric around z^{n+1}. It is straightforward to expand the right-hand side and compare coefficients with the above. □

To determine a formula for $\chi(n,m)$, we will extract coefficients from its generating function. We note that

$$\frac{z\cdot\left((n+1)(1+z+\cdots+z^n)^2-\left(1+z^2+\cdots+z^{2n}\right)\right)}{(1-z)^2(1-z^{n+1})^2}=f(z)-g(z)$$

for

$$f(z)=\frac{(n+1)z}{(1-z)^4},\quad g(z)=\frac{z\cdot\left(1+z^2+\cdots+z^{2n}\right)}{(1-z)^2(1-z^{n+1})^2}.$$

Lemma 3.9. *There is an expansion of $g(z)$ as*

$$g(z)=\frac{a_1z+\cdots+a_{4n+1}z^{4n+1}}{(1-z^{n+1})^4}$$

for some coefficients $a_1,\ldots,a_{4n+1}$. Set additionally $a_0=a_{4n+2}=a_{4n+3}=0$. Then for $q=0,\ldots,n$, we have

$$\begin{aligned} a_q+a_{(n+1)+q}+a_{2(n+1)+q}+a_{3(n+1)+q}&=(n+1)^2,\\ 2a_q+a_{(n+1)+q}-a_{3(n+1)+q}&=(n+1)(q+1),\\ 11a_q+2a_{(n+1)+q}-1a_{2(n+1)+q}+2a_{3(n+1)+q}&=\begin{cases}\frac{1}{2}(n+1)^2+3q(q+2) & n \text{ odd}, q \text{ even},\\ \frac{1}{2}(n+1)^2+3q(q+2)+3 & n \text{ odd}, q \text{ odd},\\ \frac{1}{2}(n+1)^2+3q(q+2)+\frac{3}{2} & n \text{ even},\end{cases}\\ a_q&=\begin{cases}\frac{q(q+2)}{4} & q \text{ even},\\ \frac{(q+1)^2}{4} & q \text{ odd}.\end{cases}\end{aligned}$$

Proof. The expansion is obtained by multiplying numerator and denominator of $g(z)$ by $(1+z+z^2+\cdots+z^n)^2$. Doing so we obtain

$$\left(z+z^3+\cdots+z^{2n+1}\right)(1+z+z^2+\cdots+z^n)^2=a_1z+\cdots+a_{4n+1}z^{4n+1}. \tag{3.10}$$

To compute the coefficients in the expansion of the left-hand side of (3.10), we consider an $n\times(4n+1)$ array. The columns are labelled by $1,2,\ldots,4n+1$. The first row consists of the entries $1,2,3,\ldots,n+1,n,\ldots,2,1$, followed by zeros. More generally, the i^{th} row has non-zero entries obtained by shifting the non-zero entries of the first row $2i-2$ positions to the right. See Figure 3.4 for the examples $n=5$ and $n=6$. Since the coefficients of $(1+z+z^2+\cdots+z^n)^2$ are exactly the non-zero entries of the first row of the array, the coefficient a_i is the sum of the entries of the i^{th} column of the array.

When n is even, we see by inspection that for $q=0,\ldots,n$,

$$a_q+a_{(n+1)+q}+a_{2(n+1)+q}+a_{3(n+1)+q}=1+2+3+\cdots+(n+1)+n+\cdots+1.$$

Similarly, when n is odd, for $q\leq n$ with q even

$$a_q+a_{(n+1)+q}+a_{2(n+1)+q}+a_{3(n+1)+q}=2\cdot(2+4+\cdots+(n+1)+(n-1)+\cdots+2),$$

and for q odd we instead have

$$a_q+a_{(n+1)+q}+a_{2(n+1)+q}+a_{3(n+1)+q}=2\cdot(1+3+\cdots+n+n+(n-1)+\cdots+1).$$

All three of these quantities evaluate to $(n+1)^2$. This shows the first desired identity.

For $i = 0, \dots, n+1$ we have by inspection

$$a_i = \begin{cases} \sum_{j=1}^{i/2} 2j = \frac{i(i+2)}{4} & i \text{ even,} \\ \sum_{j=1}^{(i+1)/2} (2j-1) = \frac{(i+1)^2}{4} & i \text{ odd.} \end{cases}$$

In particular, this implies the fourth identity.

We next consider the quantity $a_{(n+1)+q} - a_{3(n+1)+q}$ for $0 \le q \le n$. This is the sum of the first $q+1$ entries in column $(n+1)+q$ and has the form

$$a_{(n+1)+q} - a_{3(n+1)+q} = \begin{cases} (n+1) + 2\sum_{j=1}^{q/2} (n+1-2j) & q \text{ even,} \\ 2\sum_{j=1}^{(q+1)/2} (n+2-2j) & q \text{ odd.} \end{cases}$$

Considering instead $2a_q + a_{(n+1)+q} - a_{3(n+1)+q}$, we obtain

$$(n+1) + 2\sum_{j=1}^{q/2} ((n+1-2j) + 2j) = (n+1)(q+1)$$

for q even and

$$2\sum_{j=1}^{(q+1)/2} ((n+2-2j) + 2j - 1) = (n+1)(q+1)$$

for q odd, proving the second identity.

For the coefficients $a_{2(n+1)+i}$ for $i \ge 0$, we have

$$a_{2(n+1)+i} = a_{2n-i}.$$

We thus obtain

$$\begin{aligned} &11a_q + 2a_{(n+1)+q} - 1a_{2(n+1)+q} + 2a_{3(n+1)+q} \\ &\quad = 6a_q + 6a_{3(n+1)+q} - (a_q + a_{(n+1)+q} + a_{2(n+1)+q} + a_{3(n+1)+q}) \\ &\qquad + 3(2a_q + a_{(n+1)+q} - a_{3(n+1)+q}) \\ &\quad = 6(a_q + a_{n-q-1}) - (n+1)^2 + 3(n+1)(q+1). \end{aligned}$$

Using the above formula for a_i $(i \le n+1)$ and substituting, one obtains the third identity. □

3.5. Proof of Theorem 1.3

We extract the coefficients in front of z^m in the power series $f(z)$ and $g(z)$. For

$$f(z) = (n+1)z \cdot \left(\sum_{i\ge 0} z^i\right)^4$$

this coefficient extraction $[z^m]f(z)$ is straightforward, and we obtain

$$\begin{aligned} [z^m]f(z) &= (n+1) \cdot \binom{m+2}{3} = \frac{(n+1)(m+2)(m+1)m}{6} \\ &= \frac{(n+1)}{6}m^3 + \frac{(n+1)}{2}m^2 + \frac{(n+1)}{3}m. \end{aligned}$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	2	3	4	5	6	5	4	3	2	1										
		1	2	3	4	5	6	5	4	3	2	1								
				1	2	3	4	5	6	5	4	3	2	1						
						1	2	3	4	5	6	5	4	3	2	1				
								1	2	3	4	5	6	5	4	3	2	1		
										1	2	3	4	5	6	5	4	3	2	1

$$n = 5$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	2	3	4	5	6	7	6	5	4	3	2	1												
		1	2	3	4	5	6	7	6	5	4	3	2	1										
				1	2	3	4	5	6	7	6	5	4	3	2	1								
						1	2	3	4	5	6	7	6	5	4	3	2	1						
								1	2	3	4	5	6	7	6	5	4	3	2	1				
										1	2	3	4	5	6	7	6	5	4	3	2	1		
												1	2	3	4	5	6	7	6	5	4	3	2	1

$$n = 6$$

Figure 3.4. Example arrays from the proof of Lemma 3.9

For $[z^m]g(z)$ we use the form of $g(z)$ from Lemma 3.9 and obtain that $g(z)$ is equal to

$$\sum_{\substack{k\geq 0\\ q=0,\dots,n}} \left(a_q\binom{k+3}{3} + a_{(n+1)+q}\binom{k+2}{3} + a_{2(n+1)+q}\binom{k+1}{3} + a_{3(n+1)+q}\binom{k}{3}\right)\cdot z^{k(n+1)+q}.$$

For $m = k(n+1)+q$ with $q = 0,\dots,n$, and setting $p = n+1$ to simplify notation, it follows that

$$[z^m]g(z) = a_q\binom{\frac{m-q}{p}+3}{3} + a_{p+q}\binom{\frac{m-q}{p}+2}{3} + a_{2p+q}\binom{\frac{m-q}{p}+1}{3} + a_{3p+q}\binom{\frac{m-q}{p}}{3}.$$

We now expand as a polynomial in m to obtain that $[z^m]g(z)$ is

$$\begin{aligned}
&\frac{1}{6p^3}\left(a_q + a_{p+q} + a_{2p+q} + a_{3p+q}\right)m^3\\
&+\frac{1}{2p^3}\left(p\left(2a_q + a_{p+q} - a_{3p+q}\right) - q\left(a_q + a_{p+q} + a_{2p+q} + a_{3p+q}\right)\right)m^2\\
&+\frac{1}{6p^3}\left(p^2\left(11a_q + 2a_{p+q} - 1a_{2p+q} + 2a_{3p+q}\right) - 6qp\left(2a_q + a_{p+q} - a_{3p+q}\right)\right.\\
&\qquad\qquad\qquad\qquad \left.+3q^2\left(a_q + a_{p+q} + a_{2p+q} + a_{3p+q}\right)\right)m\\
&+\frac{1}{6p^3}\left(6p^3a_q - qp^2\left(11a_q + 2a_{p+q} - 1a_{2p+q} + 2a_{3p+q}\right) + 3p^3\left(2a_q + a_{p+q} - a_{3p+q}\right)\right.\\
&\qquad\qquad\qquad\qquad \left.-q^3\left(a_q + a_{p+q} + a_{2p+q} + a_{3p+q}\right)\right).
\end{aligned}$$

Setting

$$\begin{aligned}
\alpha_1 &= a_q + a_{(n+1)+q} + a_{2(n+1)+q} + a_{3(n+1)+q},\\
\alpha_2 &= 2a_q + a_{(n+1)+q} - a_{3(n+1)+q},\\
\alpha_3 &= 11a_q + 2a_{(n+1)+q} - 1a_{2(n+1)+q} + 2a_{3(n+1)+q},
\end{aligned}$$

we thus have

$$[z^m]g(z) = \frac{1}{6p^3}\alpha_1 m^3 + \frac{1}{2p^3}(p\alpha_2 - q\alpha_1)m^2 + \frac{1}{6p^3}\left(p^2\alpha_3 - 6qp\alpha_2 + 3q^2\alpha_1\right)m + \frac{1}{6p^3}\left(6p^3 a_q - qp^2\alpha_3 + 3p^3\alpha_2 - q^3\alpha_1\right).$$

Using Lemma 3.9 to substitute in for $a_q, \alpha_1, \alpha_2, \alpha_3$ and simplifying, we obtain that $[z^m]f(z) - [z^m]g(z)$ is exactly the quasi-polynomial appearing in the statement of Theorem 1.3. The claim of the theorem thus follows from Lemma 3.6. □

4. Computation of χ^0

4.1. A combinatorial formula

Let $X = X_{\sigma_{A_n}}$ be the toric variety as described in Example 2.1, and let $Y = X_\Sigma$ with $\phi : Y \to X$ be the minimal resolution, where Σ is the fan defined in Example 2.3. The exceptional locus E consists exactly of torus-invariant divisors $E_1 = D_{\rho_1}, \ldots, E_n = D_{\rho_n}$. We shorten notation:

$$V_m^i(u) = V^{\rho_i}_{S^m\Omega^1_Y}(\rho_i(u)).$$

We use that

$$\chi^0\left(s_n, S^m\Omega^1_Y\right) = \dim \frac{\mathrm{H}^0\left(Y \setminus E, S^m\Omega^1_Y\right)}{\mathrm{H}^0\left(Y, S^m\Omega^1_Y\right)} = \sum_{u\in M} \dim \frac{\mathrm{H}^0\left(Y \setminus E, S^m\Omega^1_Y\right)_u}{\mathrm{H}^0\left(Y, S^m\Omega^1_Y\right)_u}.$$

By (2.7) we have

$$\mathrm{H}^0\left(Y \setminus E, S^m\Omega^1_Y\right)_u = V_m^0(u) \cap V_m^{n+1}(u),$$

$$\mathrm{H}^0\left(Y, S^m\Omega^1_Y\right)_u = \bigcap_{i=0}^{n+1} V_m^i(u).$$

Recall that for $u = (u_1, u_2)$ we have $\rho_i(u) = \rho_i(u_1, u_2) = iu_1 + u_2$. We adapt some notation from Section 3.2.

Lemma 4.1. *Let*

$$\lambda_m(i) = \begin{cases} 0 & i \le -m, \\ i+m & -m \le i \le 1, \\ m+1 & i \ge 1. \end{cases}$$

Then $\dim V_m^i(u) = m + 1 - \lambda_m(iu_1 + u_2)$. *Furthermore, these spaces are maximally independent, so for* $I \subset \{0, \ldots, n+1\}$ *we have*

$$\dim \bigcap_{i\in I} V_m^i(u) = \max\left\{0, m+1-\sum_{i\in I}\lambda_m(iu_1+u_2)\right\}.$$

Proof. The dimension result follows from Example 2.5. Furthermore, $\bigoplus_{m=0}^\infty V_{S^m\Omega^1_Y}$ is isomorphic to a bivariate polynomial ring in two variables, and the $\rho_i^\perp$ consist of linear forms that are pairwise coprime for different i. Hence, if the intersection of several of these spaces is not zero, then the codimension of the intersection is the sum of the codimensions of the spaces. □

We use Lemma 4.1 to write

$$\chi^0\left(s_n, S^m\Omega^1_Y\right) = \sum_{u\in M} z_m(u)$$

with

$$z_m(u) = \dim \frac{\mathrm{H}^0\left(Y \setminus E, S^m\Omega^1_Y\right)_u}{\mathrm{H}^0\left(Y, S^m\Omega^1_Y\right)_u}$$
$$= \min\left\{\max\{0,(m+1-\lambda_m(u_2)-\lambda_m((n+1)u_1+u_2))\}, \sum_{i=1}^{n}\lambda_m(iu_1+u_2)\right\}.$$

Lemma 4.2. *With the definitions above, the set*

$$\mathcal{G}_{m,n} = \left\{(u_1,u_2,z) \in \mathbb{R}^3 : 0 < z \le z_m(u_1,u_2)\right\}$$

is a bounded half-open non-convex polytope and

$$\chi^0\left(s_n, S^m\Omega^1_Y\right) = \#\left(\mathcal{G}_{m,n} \cap \mathbb{Z}^3\right).$$

Furthermore, $\mathcal{G}_m$ is stable under the transformation $(u_1,u_2) \mapsto (-u_1,(n+1)u_1+u_2)$.

Proof. It is straightforward to check that $z_m(u_1,u_2)$ is only non-zero on a bounded region, so $\mathcal{G}_m$ is bounded. It is a (non-convex) polytope because $z_m(u_1,u_2)$ is piecewise linear. Since $z_m(u_1,u_2)$ takes integer values at $(u_1,u_2) \in \mathbb{Z}$, we have that the sum $\sum_{(u_1,u_2)\in\mathbb{Z}^2} z_m(u_1,u_2)$ is equal to the lattice point count given.

The symmetry is easily verified through the identity

$$z_m(u_1,u_2) = z_m(-u_1,(n+1)u_1+u_2)$$

□

In Section 4.2 we give an explicit description of the non-convex polytope $\mathcal{G}_{m,n}$ as a dilation of a fixed non-convex polytope $\mathcal{G}_{0,n}$ by a factor of $m+1$.

4.2. Explicit description of the non-convex polytope $\mathcal{G}_{m,n}$

As it turns out, we get a nicer description of $\mathcal{G}_{m,n}$ by shifting our coordinates: we set $(u_1,u_2) = (a,b+1)$. We absorb the shift in a new piecewise linear function λ'_{m+1} defined by

$$\lambda'_{m+1}(i) = \lambda_m(i+1) = \begin{cases} 0 & i \le -(m+1), \\ i+m+1 & -(m+1) \le i \le 0, \\ m+1 & i \ge 0. \end{cases}$$

We obtain descriptions

$$z_m(a,b) = \min\left\{\max\{0,(m+1-\lambda'_{m+1}(b)-\lambda'_{m+1}((n+1)a+b))\}, \sum_{i=1}^{n}\lambda'_{m+1}(ia+b)\right\}$$

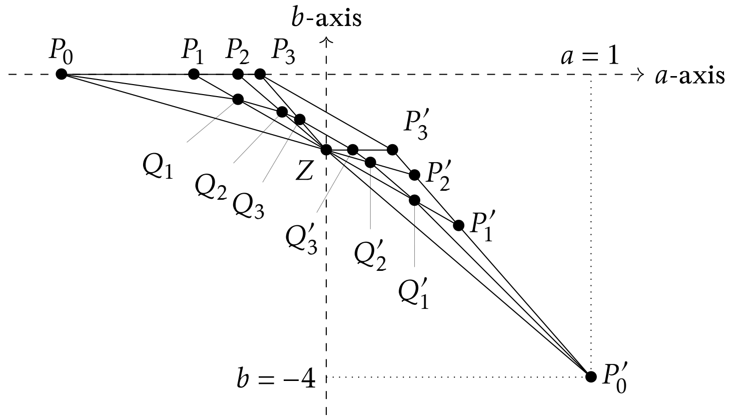
and

$$\mathcal{G}_{m,n} = \left\{(a,b,z) \in \mathbb{R}^3 : 0 < z \le z_m(a,b)\right\}.$$

The symmetry of the non-convex polytope $\mathcal{G}_{m,n}$ in these coordinates is under the same transformation $\tau_n = (a,b) \mapsto (-a,(n+1)a+b)$.

Recall that in Section 1.1 we defined the points

$$P_i = \left(-\frac{1}{i+1},0,0\right) \quad \text{for } i = 0,1,\dots,n,$$
$$Q_i = \left(-\frac{2}{(i+1)(i+2)},-\frac{i}{i+2},\frac{i}{i+2}\right) \quad \text{for } i = 0,1,\dots,n,$$
$$Z = (0,-1,0),$$



Note: The a-axis is stretched to ease viewing

Figure 4.1. Top view of $\mathcal{G}_{0,3}$

along with the half-open convex polytopes

$$\mathcal{P}_i = \mathrm{Conv}\{P_{i-1}, Q_{i-1}, P_i, Q_i, Z\} \setminus \mathrm{Conv}\{P_i, Q_i, Z\} \setminus \mathrm{Conv}\{P_{i-1}, P_i, Z\},$$
$$\mathcal{C}_n = \mathrm{Conv}\{P_n, \tau_n(P_n), Q_n, \tau_n(Q_n), Z\} \setminus \mathrm{Conv}\{P_n, \tau_n(P_n), Z\}$$

from (1.4). For reference we record

$$P_n' = \tau_n(P_n) = \left(\frac{1}{n+1}, -1, 0\right),$$
$$Q_n' = \tau_n(Q_n) = \left(\frac{2}{(n+1)(n+2)}, -1, \frac{n}{n+2}\right).$$

Lemma 4.3. *The non-convex polytope $\mathcal{G}_{m,n}$ is the dilation by $m+1$ of $\mathcal{G}_{0,n}$. Furthermore, we have*

$$\mathcal{G}_{0,n} = \mathcal{C}_n \cup \bigcup_{i=1}^{n} \mathcal{P}_i \cup \bigcup_{i=1}^{n} \tau_n(\mathcal{P}_i).$$

Proof. The first claim follows by inspecting the definition of $z_m(a,b)$ and the fact that

$$\lambda_{m+1}'((m+1)i) = (m+1)\lambda_1'(i).$$

It remains to describe $\mathcal{G}_{0,n}$. The faces spanned by $\{P_{i-1}, P_i, Q_{i-1}, Q_i\}$ and $\{Q_{i-1}, Q_i, Z\}$ for $i = 1, \ldots, n$ can be checked to be linear parts of the graph of $z_m(a,b)$. See Figure 4.1 for an illustration of the configuration for $n = 3$. We define the points $P_i' = \tau_n(P_i)$ and $Q_i' = \tau_n(Q_i)$.

By symmetry we get that $\{P_{i-1}', P_i', Q_{i-1}', Q_i'\}$ and $\{Q_{i-1}', Q_i', Z\}$ are also faces of the graph. We get two remaining faces $\{P_n, P_n', Q_n, Q_n'\}$ and $\{Q_n, Q_n', Z\}$, and outside these we have that $z_m(a,b)$ is identically zero. The description of $\mathcal{G}_{0,n}$ follows. ☐

4.3. Proof of Theorem 1.5

Lemma 4.2 expresses $\chi^0(s_n, S^m\Omega_Y^1)$ as a lattice point count in the dilation by $m+1$ of $\mathcal{G}_{0,n}$. Lemma 4.3 expresses $\mathcal{G}_{0,n}$ as a disjoint union of convex polytopes. The theorem follows directly from the volume and lattice point counts of those polytopes.

4.4. Proof of Proposition 1.6

We consider the half space $H = \{(a,b,z) : a \leq 0\}$. It is straightforward to verify that $\mathcal{C}_{n-1} \cap H \subset (\mathcal{P}_n \cup \mathcal{C}_n) \cap H$, so it follows that $\mathcal{G}_{0,n-1} \cap H \subset \mathcal{G}_{0,n} \cap H$.

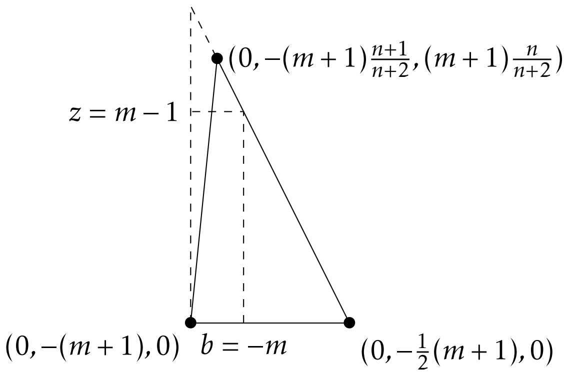


Figure 4.2. Intersection of $(m+1)\mathcal{C}_n$ with $a=0$

(1) First we note that lattice point counts are non-decreasing with increasing dilation, so $\chi^0(s_n, S^m\Omega^1_Y)$ is non-decreasing in m. Since $\mathcal{G}_{n,m} = (\mathcal{G}_{n,m}\cap H)\cup \tau_n(\mathcal{G}_{n,m}\cap H)$ and $\tau_n(\mathbb{Z}^3)=\mathbb{Z}^3$, we see from the observation above that the lattice point count is also non-decreasing in n.

(2) To establish that $\chi^0(s_n, S^m\Omega^1_Y)$ is constant in n for $n>m$, we observe that $(m+1)\mathcal{P}_n$ does not contain lattice points since any point $(a,b,z)\in(m+1)\mathcal{P}_n$ satisfies $\frac{m+1}{n+1}<a<0$. Similarly, any lattice points in $(m+1)\mathcal{C}_n$ must have $a=0$ and lie in the triangle with vertices

$$(0,-(m+1),0),\quad \left(0,-\frac{1}{2}(m+1),0\right),\quad \left(0,-(m+1)\frac{n+1}{n+2},(m+1)\frac{n}{n+2}\right).$$

See Figure 4.2. Only the third point depends on n. It lies on the line $z=-2b-(m+1)$ and tends to $(0,-(m+1),(m+1))$ as $n\to\infty$. Thus we see that the smallest b-coordinate of a lattice point has $b\geq -m$ and hence $z\leq m-1$. Such points are already contained in $\mathcal{C}_n$ for $n=m-1$, so as n grows beyond m, we see that $\mathcal{G}_{n,m}\cap H$ does not gain more lattice points, and therefore the lattice point count in $\mathcal{G}_{n,m}$ stabilizes for $n\geq m$ as well.

(3) A straightforward computation yields

$$\operatorname{vol}\mathcal{P}_n = \frac{n^2+3n-2}{6n(n+1)^2(n+2)}.$$

Since

$$\operatorname{vol}\mathcal{C}_n = \frac{n(n+4)}{6(n+1)(n+2)^2}$$

tends to 0 as $n\to\infty$, we see that the volume of $\mathcal{G}_{0,n}$ tends to

$$\lim_{n\to\infty}\mathcal{G}_{0,n} = 2\sum_{n=1}^{\infty}\operatorname{vol}\mathcal{P}_n = 2\sum_{n=1}^{\infty}\frac{n^2+3n-2}{6n(n+1)^2(n+2)} = \tfrac{2}{9}\pi^2-2.$$

By Ehrhart theory, we have that $\#(m+1)\mathcal{G}_{0,n}\cap\mathbb{Z}^3$ is a quasi-polynomial in m of degree equal to $\dim\mathcal{G}_{0,n}=3$, with leading coefficient equal to $\operatorname{vol}\mathcal{G}_{0,n}$. The argument above establishes that $\operatorname{vol}\mathcal{G}_{0,n}$ increases with n and tends to $\frac{2}{9}\pi^2-2$. The statement follows.

5. Explicit computation of regular differentials

5.1. Setup

Let Y be a normal surface over $\mathbf{k}$ with function field $\mathbf{k}(Y)$. We write $\Omega_{\mathbf{k}(Y)/\mathbf{k}}$ for the $\mathbf{k}(Y)$-module of Kähler differentials. For any open $U\subset Y$ we have an injection $\mathrm{H}^0(U,S^m\Omega^1_Y)\to S^m\Omega_{\mathbf{k}(Y)/\mathbf{k}}$. We represent a section by its corresponding Kähler differential.

The local rings $\mathcal{O}_{Y,D}$ of prime divisors D on Y give rise to discrete valuations ord_D on $\mathbf{k}(Y)$. In this section we use the following notation. Given a prime divisor D we choose non-constant functions $\pi, u \in \mathcal{O}_{Y,D}$ such that $\operatorname{ord}_D(\pi) = 1$ and $\operatorname{ord}_D(u) = 0$. The differentials $d\pi, du$ form an $\mathcal{O}_{Y,D}$-basis for $\Omega_{\mathcal{O}_{Y,D}/\mathbf{k}}$ and therefore also a $\mathbf{k}(Y)$-basis for $\Omega_{\mathbf{k}(Y)/\mathbf{k}}$.

The natural homomorphism $S^m\Omega_{\mathcal{O}_{Y,D}/\mathbf{k}} \to S^m\Omega_{\mathbf{k}(Y)/\mathbf{k}}$ is an injection, and its image is formed by the differentials that are regular at the generic point of D. We define $\operatorname{ord}_D(\omega)$ to be the largest integer n such that $\pi^{-n}\omega \in \Omega_{\mathcal{O}_{Y,D}/\mathbf{k}}$.

For $\omega \in S^m\Omega_{\mathbf{k}(Y)/\mathbf{k}}$ we have $\omega = f_0(du)^m + f_1(du)^{m-1}d\pi + \cdots + f_m(d\pi)^m$ and

$$\operatorname{ord}_D \omega = \min\{\operatorname{ord}_D(f_i) : i = 0, \ldots, m\}.$$

We furthermore have a reduction homomorphism $\rho_D \colon S^m\Omega_{\mathcal{O}_{Y,D}/\mathbf{k}} \to S^m\Omega_{\mathbf{k}(D)/\mathbf{k}}$ by reducing modulo π and sending $d\pi$ to 0.

Definition 5.1. Let Y be as above, and let $\omega \in S^m\Omega_{\mathbf{k}(Y)/k}$ be non-zero. We say that a prime divisor $D \subset Y$ is a *solution curve* to ω if $\rho(\pi^{-\operatorname{ord}_D(\omega)}\omega) = 0$.

In terms of the coordinates described above, D is a solution curve to ω if and only if $\operatorname{ord}_D(f_0) > \operatorname{ord}_D(\omega)$.

Proposition 5.2. *Let $\psi \colon Z \to Y$ be a finite morphism of normal surfaces. Let $D \subset Y$ be a prime divisor, and let $D' \subset Z$ be a prime divisor above D of ramification degree e. Suppose that D is a solution curve to $\omega \in S^m\Omega_{\mathbf{k}(Y)/\mathbf{k}}$. Then*

$$\operatorname{ord}_{D'} \psi^*\omega \geq e \operatorname{ord}_D \omega + (e-1).$$

Proof. The inequality is preserved under scaling ω by a non-zero element of $\mathbf{k}(Y)$, so it suffices to deal with the case $\operatorname{ord}_D(\omega) = 0$. Let us take a uniformizer π at D. Identifying $\mathbf{k}(Y) \subset \mathbf{k}(Z)$ via the pull-back ψ^*, we have that a uniformizer π' at D' is of the form $\pi = v(\pi')^e$ for some $v \in \mathbf{k}(Z)$ with $\operatorname{ord}_{D'}(v) = 0$. We also choose a non-constant $u \in \mathbf{k}(Y)$ so that we have

$$\omega = f_0(du)^m + f_1(du)^{m-1}d\pi + \cdots + f_m(d\pi)^m$$

with $\min \operatorname{ord}_D(f_i) = 0$. The fact that D is a solution curve means that $\operatorname{ord}_D(f_0) \geq 1$ and therefore $\operatorname{ord}_{D'}(f_0) \geq e$.

Note that $d\pi = d(v(\pi')^e) = (\pi')^e dv + ev(\pi')^{e-1}d\pi'$, so we have

$$\operatorname{ord}_{D'}\left(f_i(du)^{m-i}(d\pi)^i\right) \geq e \operatorname{ord}_D(f_i) + e - 1 \quad \text{for } i = 1, \ldots, m.$$

This implies that $\operatorname{ord}_{D'}(\psi^*\omega) \geq e-1$, as required. □

Let us now consider a normal surface X, with singular locus S and minimal resolution Y. Then $\mathbf{k}(X)$ and $\mathbf{k}(Y)$ are canonically isomorphic. Let $E \subset Y$ be the locus of Y mapping to S on X. Then $X \setminus S$ is isomorphic to $Y \setminus E$.

Because S is of codimension 2 in X, we can extend the sheaf $S^m\Omega^1_{X\setminus S}$ uniquely to a reflexive sheaf $\hat{S}^m\Omega^1_X$ on X, and its sections are completely determined by their behaviour on $X \setminus S$. Here too, we represent sections by the corresponding Kähler differentials: a section is regular on X if it is regular at all divisors on Y that are not contained in E.

Now suppose that we have a differential ω that is regular on $X \setminus S$. If $s \in S$ is an A_n-singularity, then we can bound $\operatorname{ord}_{E_i} \omega$ at components above s as well.

Proposition 5.3. *Let X be a surface with an A_n-singularity s, let Y be a minimal resolution of X, and let $E_i \subset Y$ be a prime divisor of Y above s. Suppose that $\omega \in \mathrm{H}^0(X, \hat{S}^m\Omega^1_X)$. Then*

$$\operatorname{ord}_{E_i} \omega \geq \left\lceil \frac{-mn}{n+1} \right\rceil.$$

Proof. We use the notation from Example 2.3. Then $E_i = D_{\rho_i}$ for some $i \in \{1,\dots,n\}$. It suffices to show the claim for torus semi-invariant symmetric differentials, so let $\omega \in \mathrm{H}^0(X, \hat{S}^m\Omega^1_X)_u$ for some weight $u \in \mathbb{Z}^2$. In the notation of Section 4.1, we have $V^0_m(u) \cap V^{n+1}_m(u) \neq 0$. By Lemma 4.1 we have

$$\rho_0(u) \le 0, \quad \rho_{n+1}(u) \le 0, \quad \rho_0(u) + \rho_{n+1}(u) \le -m.$$

For the ray ρ_i, $1 \le i \le n$, we have

$$\rho_i(u) = \frac{1}{n+1}\left((n+1-i)\rho_0(u) + i\rho_{n+1}(u)\right).$$

Since both $(n+1-i)$ and i are at least 1, it follows that $\rho_i(u) \le -m/(n+1)$, or equivalently,

$$j = \rho_i(u) + \left\lceil \frac{m\cdot(-n)}{n+1} \right\rceil \le -m.$$

From Example 2.5 it follows that $\omega \in x^u V^{\rho_i}(j)$ and hence by (2.6) that

$$\mathrm{ord}_{D_{\rho_i}}(\omega) \ge j - \rho_i(u) = \left\lceil \frac{m\cdot(-n)}{n+1} \right\rceil. \qquad \square$$

5.2. Labs surfaces

We consider surfaces in $\mathbb{P}^3$ from Labs [Lab06, Corollary A1] of degree $d = 2k$, constructed by taking a smooth plane quadric $X_1 \subset \mathbb{P}^3$ tangent to the four coordinate planes and pulling it back to a surface X_k under the map $(\xi_0 : \xi_1 : \xi_2 : \xi_3) \mapsto (\xi_0^k : \xi_1^k : \xi_2^k : \xi_3^k)$. An explicit choice of model yields

$$X_k\colon \xi_0^{2k} + \xi_1^{2k} + \xi_2^{2k} + \xi_3^{2k} - \xi_0^k\xi_1^k - \xi_0^k\xi_2^k - \xi_0^k\xi_3^k - \xi_1^k\xi_2^k - \xi_1^k\xi_3^k - \xi_2^k\xi_3^k = 0.$$

We write S for the singular locus of X_k. It consists of $4k^2$ isolated singularities of type A_{k-1}. The singularities lie in the four planes $\xi_0 = 0, \dots, \xi_3 = 0$. Writing ζ_k for a primitive k^{th} root of unity, the singularities with $\xi_3 = 0$ have coordinates

$$(1 : \zeta_k^i : \zeta_k^j : 0) \quad \text{for } i, j = 0, \dots, k-1.$$

We observe that X_1 is a non-singular quadric and that we have a finite morphism $\phi_k\colon X_k \to X_1$ defined by $(\xi_0 : \xi_1 : \xi_2 : \xi_3) \mapsto (\xi_0^k : \xi_1^k : \xi_2^k : \xi_3^k)$ of degree k^3 and branch locus $\xi_0\xi_1\xi_2\xi_3 = 0$, of ramification degree k over each of those plane sections.

Writing ζ_3 for a primitive third root of unity, we have that X_1 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over a field containing ζ_3. In terms of affine coordinates $(s_0 : s_1) \times (t_0 : t_1)$, we can express the isomorphism as

$$\begin{aligned}
\xi_0 &= 3s_0t_0,\\
\xi_1 &= s_1t_1 + (\zeta_3+2)s_1t_0 - (\zeta_3-1)s_0t_1 + 3s_0t_0,\\
\xi_2 &= s_1t_1 + (2\zeta_3+1)s_1t_0 - (2\zeta_3+1)s_0t_1 + 3s_0t_0,\\
\xi_3 &= s_1t_1.
\end{aligned}$$

We note that the plane $\xi_3 = 0$ is tangent to X_1 and hence that it intersects X_1 in two lines $L_{3,1}, L_{3,2}$. By symmetry, the same holds for the other coordinate planes $\xi_0 = 0$, $\xi_1 = 0$, $\xi_2 = 0$, for which we adopt the same notation.

We pass to affine coordinates $(x_1, x_2, x_3) = (\xi_1/\xi_0, \xi_2/\xi_0, \xi_3/\xi_0)$ on $\mathbb{P}^3$ and $(s,t) = (s_1/s_0, t_1/t_0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We obtain

$$\begin{aligned}
3dx_1 &= (t+\zeta_3+2)ds + (s-\zeta_3+1)dt,\\
3dx_2 &= (t+2\zeta_3+1)ds + (s-2\zeta_3-1)dt,\\
3dx_3 &= tds + sdt.
\end{aligned}$$

We consider the degree 2 differential $dsdt$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which under the isomorphism above yields

$$\omega_1 = \frac{-3}{x_1^2 + x_2^2 + 1 - 2x_1x_2 - 2x_1 - 2x_2}\left(x_2(dx_1)^2 + (1 - x_1 - x_2)dx_1dx_2 + x_1(dx_2)^2\right).$$

We record a few facts about ω_1.

Lemma 5.4. *The differential ω_1 has $\operatorname{ord}_{L_{0,1}} \omega_1 = \operatorname{ord}_{L_{0,2}} \omega_1 = -2$ and is regular elsewhere. Furthermore, the solution curves to ω_1 are exactly the lines constituting the two rulings on X_1.*

Proof. On $\mathbb{P}^1 \times \mathbb{P}^1$ we easily see that $dsdt$ has double poles at $s = \infty$ and $t = \infty$ and nowhere else. Furthermore, it is straightforward to check that the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ form exactly the solution curves of $dsdt = 0$. The statement on X_1 follows simply by applying the isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to X_1$. □

We next consider $\phi_k\colon X_k \to X_1$. The inverse images of the lines $L_{0,i}, L_{1,i}, L_{2,i}, L_{3,i}$ are prime divisors $D_{0,i}, D_{1,i}, D_{2,i}, D_{3,i}$. These are degree k Fermat curves, as one can see from the factorization

$$\xi_0^{2k} + \xi_1^{2k} + \xi_2^{2k} - \xi_0^k\xi_1^k - \xi_0^k\xi_2^k - \xi_1^k\xi_2^k = \left(\xi_0^k + \zeta_3\xi_1^k + \zeta_3^2\xi_2^k\right)\left(\xi_0^k + \zeta_3^2\xi_1^k + \zeta_3\xi_2^k\right).$$

We consider the pull-back $\omega_k = \phi_k^*\omega_1$ to X_k.

Lemma 5.5. *For $D = D_{0,i}$ we have $\operatorname{ord}_D \omega_k \geq -k-1$, and ω_k, as a section of $\hat{S}^2\Omega^1_{X_k}$, is regular elsewhere. Furthermore, for $D = D_{1,i}, D_{2,i}, D_{3,i}$ we have $\operatorname{ord}_D \omega_k \geq k-1$.*

As a result, for $k \geq 2$ we have that $(x_1x_2x_3)^{1-k}\omega_k$ is a global section of $\hat{S}^2\Omega^1_{X_k}$, and for $k \geq 4$ we have that $\tilde{\omega}_k = (x_1x_2x_3)^{2-k}\omega_k$ is a global section that vanishes identically on $\xi_0\xi_1\xi_2\xi_3 = 0$.

Proof. The curves mentioned in the lemma lie over solution curves for ω_1 with ramification degree k. The first claims are a direct application of Proposition 5.2.

The second part is just the observation that $\xi_0, \xi_1, \xi_2, \xi_3$ vanish to the first order on their respective curves. □

Lemma 5.6. *The solution curves of $\tilde{\omega}_k$ contained in $\xi_0\xi_1\xi_2\xi_3 = 0$ are degree k non-singular plane curves and hence of genus $(k-1)(k-2)/2$. The other solution curves are non-singular complete intersections of two degree k surfaces and hence curves of genus $k^3 - 2k^2 + 1$.*

Proof. By Lemma 5.4 we see that the solution curves arise as fibres of the composition $X^k \to X_1 \to \mathbb{P}^1$ induced by the projections from $X_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ onto either of the factors.

Let us first consider the projection onto the first factor. The fibre over the point $(1:s)$ can be expressed as an intersection of planes on X_1. Computation shows it is the kernel of the matrix

$$A = \begin{pmatrix} -(\frac{1}{3}(\zeta_3 + 2)s^2 + s) & s & 0 & -(s - \zeta_3 + 1) \\ -(\frac{1}{3}(2\zeta_3 + 1)s^2 + s) & 0 & s & -(s - 2\zeta_3 - 1) \end{pmatrix},$$

so the corresponding solution curve on X_k is described by

$$\begin{cases} A_{1,0}\xi_0^k + A_{1,1}\xi_1^k + A_{1,2}\xi_2^k + A_{1,3}\xi_3^k = 0, \\ A_{2,0}\xi_0^k + A_{2,1}\xi_1^k + A_{2,2}\xi_2^k + A_{2,3}\xi_3^k = 0. \end{cases}$$

Using the Jacobian criterion, any singular point must have an appropriate mixture of vanishing of homogeneous coordinates and minors of A. However, those minors only consist of factors $s, (s - \zeta_3 + 1), (s - 2\zeta_3 - 1)$, which lead to the curves $D_{1,i}, D_{2,i}, D_{3,i}$ contained in $\xi_1\xi_2\xi_3 = 0$. For $s = \infty$ we obtain the curves $D_{0,i}$ contained in $\xi_0 = 0$.

The other ruling consists of fibres over points $(1:t)$ on the second factor and behaves symmetrically to this one. □

5.3. Proof of Theorem 1.8

Let $Y_k \to X_k$ be a minimal desingularization of X_k. By Lemma 5.5 we have that $\tilde{\omega}_k$ is a regular section of $\hat{S}^2\Omega^1_X$. We consider the pull-back of $\tilde{\omega}_k$ to Y_k, and we also denote it by $\tilde{\omega}_k$. From Lemma 5.5 we obtain that $\tilde{\omega}_k$ is regular outside any prime divisor D of Y_k above a singularity s of X_k. In fact, since s lies on $\xi_0\xi_1\xi_2\xi_3$, which is contained in the vanishing locus of $\tilde{\omega}_k$, we see that $\tilde{\omega}_k = f\omega'$ for some differential ω' regular around s and function f vanishing at s. We identify f with its pull-back to Y_k and conclude that $\mathrm{ord}_D(f) \geq 1$.

The singularities on X_k are of type A_{k-1}, so Proposition 5.3 yields that $\mathrm{ord}_D\, \omega' \geq \lceil \frac{2(1-k)}{k} \rceil = -1$. It follows that $\mathrm{ord}_D(\tilde{\omega}_k) = \mathrm{ord}_D(f) + \mathrm{ord}_D(\omega') \geq 0$, so $\tilde{\omega}_k$ is regular everywhere on Y_k.

For a curve $D \subset Y_k$ we obtain a pull-back $S^m\Omega^1_Y \to S^m\Omega^1_D$ that preserves regularity. Hence, $\tilde{\omega}_k$ pulls back to a regular form on D. On a genus 0 curve such forms must vanish, so such a D must be a solution curve to $\tilde{\omega}_k$. Apart from the exceptional components above singularities, any such curve is a pull-back of a solution curve to $\tilde{\omega}_k$ on X_k. By Lemma 5.6 none of these are of genus 0.

For $k > 5$ we have more freedom: from valuations we can conclude that

$$\eta = (a_0 + a_1x_1 + a_2x_2 + a_3x_3)\tilde{\omega}_k$$

represents a regular differential on Y_k that vanishes on $a_0\xi_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = 0$.

For a putative genus 1 curve C on X_k, one can then choose a plane $a_0\xi_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = 0$ that intersects C transversally. Reducing to C would yield a regular degree 2 differential on C that additionally has zeros on C. But then η must reduce to 0 on C; ***i.e.*** C is a solution curve to η. Since η is a scaling of $\tilde{\omega}_k$ by an element in $\mathbf{k}(Y)$, the two differentials have the same solution curves. No curves on Y_k above singularities of X_k can be of genus 1, so C would need to be a pull-back of a solution curve to $\tilde{\omega}_k$ on X_k. Again, by Lemma 5.6 such curves do not have genus 1.

Appendix. Ehrhardt generating functions

We give the generating functions of the lattice point counts in dilations $(m+1)\mathcal{P}_n$ for $n = 1, \ldots, 5$.

n	Generating function $\sum_{t=0}^{\infty} L(\mathcal{P}_n, m+1)t^m$
1	$\frac{t^3}{(t^2+t+1)(t+1)(t-1)^4}$
2	$\frac{\left(t^4+t^2-t+1\right)t^2}{(t^2+t+1)^2(t^2-t+1)(t+1)(t-1)^4}$
3	$\frac{\left(t^{11}+t^9+t^8+t^7+t^6+t^4+t^2+1\right)t^3}{(t^4+t^3+t^2+t+1)(t^4-t^3+t^2-t+1)(t^2+t+1)(t^2-t+1)(t^2+1)(t+1)^2(t-1)^4}$
4	$\frac{\left(t^{18}+t^{16}+t^{14}+t^{13}+t^{12}+t^{11}+t^{10}+t^9+t^7+t^5+t^4+t^2+1\right)t^4}{(t^8-t^7+t^5-t^4+t^3-t+1)(t^4+t^3+t^2+t+1)^2(t^4-t^3+t^2-t+1)(t^2+t+1)(t^2+1)(t+1)(t-1)^4}$
5	$\frac{\left(t^{28}+t^{25}+t^{23}+t^{22}+t^{20}+t^{19}+t^{18}+t^{17}+t^{16}+t^{15}+t^{14}+t^{12}+t^{11}+t^9+t^8+t^6+t^5+t^3+1\right)t^5}{(t^{12}-t^{11}+t^9-t^8+t^6-t^4+t^3-t+1)(t^8-t^7+t^5-t^4+t^3-t+1)(t^6+t^5+t^4+t^3+t^2+t+1)(t^4+t^3+t^2+t+1)(t^2+t+1)^2(t^2-t+1)(t+1)(t-1)^4}$

We also give the generating functions of the lattice point counts in dilations $(m+1)\mathcal{C}_n$:

n	Generating function $\sum_{t=0}^{\infty} L(\mathcal{C}_n, m+1)t^m$
1	$\frac{\left(t^4+t^3+2t^2+3t+3\right)t^2}{(t^2+t+1)^2(t+1)^2(t-1)^4}$
2	$\frac{\left(t^4-t^2+2t+1\right)t^2}{(t^2+t+1)(t^2-t+1)(t+1)^2(t-1)^4}$
3	$\frac{\left(t^{12}+t^{10}+2t^8+2t^6+2t^5+2t^4+3t^2+1\right)\left(t^2+t+1\right)t^2}{(t^4+t^3+t^2+t+1)^2(t^4-t^3+t^2-t+1)(t^2+1)(t+1)^2(t-1)^4}$
4	$\frac{\left(t^9+t^7-t^6+t^3+t^2+1\right)\left(t^4-t^3+t^2-t+1\right)(t+1)t^2}{(t^8-t^7+t^5-t^4+t^3-t+1)(t^4+t^3+t^2+t+1)(t^2+t+1)^2(t-1)^4}$
5	$\frac{\left(t^{24}+t^{21}+2t^{18}+2t^{15}+2t^{13}+2t^{12}-2t^{11}+2t^{10}+2t^9+2t^7+2t^4+t^3+1\right)\left(t^4+t^3+t^2+t+1\right)t^2}{(t^{12}-t^{11}+t^9-t^8+t^6-t^4+t^3-t+1)(t^6+t^5+t^4+t^3+t^2+t+1)^2(t^2+t+1)(t^2-t+1)(t+1)^2(t-1)^4}$

As an example, for $n=2$ we get the generating function

$$\begin{aligned}\sum_{m=1}^{\infty}\chi^0(s_2, S^m\Omega_Y)t^m &= 2\left(\frac{t^3}{(t^2+t+1)(t+1)(t-1)^4}+\frac{\left(t^4+t^2-t+1\right)t^2}{(t^2+t+1)^2(t^2-t+1)(t+1)(t-1)^4}\right)\\ &\quad+\frac{\left(t^4-t^2+2t+1\right)t^2}{(t^2+t+1)(t^2-t+1)(t+1)^2(t-1)^4}\\ &= 3t^2+8t^3+15t^4+28t^5+O(t^6).\end{aligned}$$

See the ancillary files [BIX23] provided with this article for machine-readable representations of the corresponding quasi-polynomials, as well as sample Sage code for generating this data.

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