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# Finiteness of the space of $n$-cycles for a reduced $(n-2)$-concave complex space

**Daniel Barlet**

*pour Yum-Tong Siu, avec mon amitié*

**Abstract.** We show that for $n \geq 2$ the space of closed $n$-cycles in a strongly $(n-2)$-concave complex space has a natural structure of reduced complex space locally of finite dimension and represents the functor "analytic family of $n$-cycles" parametrized by Banach analytic sets.

**Keywords.** Closed $n$-cycles; strongly $q$-concave space; Hartogs figure; $\mathsf{f}$-analytic family of cycles

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**[Français]**

**Titre. Finitude de l'espace des $n$-cycles pour un espace complexe $(n-2)$-concave réduit**

**Résumé.** Nous montrons que, pour $n \geq 2$, l'espace des $n$-cycles fermés dans un espace complexe fortement $(n-2)$-concave a une structure naturelle d'espace complexe réduit localement de dimension finie et que cet espace représente le foncteur "famille analytique de $n$-cycles" paramétrée par des ensembles analytiques banachiques.

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# Contents



# 1. Introduction

The aim of this article is to show that in a reduced strongly $(n-2)$-concave[1] complex space $Z$ with $n \geq 2$, the space of closed $n$-cycles is in a natural way endowed with a structure of a reduced complex space locally of finite dimension. With its tautological family of $n$-cycles it represents the functor "analytic family of $n$-cycles in $Z$" and also the functor "$\mathsf{f}$-analytic family of $n$-cycles in $Z$" introduced in [Bar08] (see also [Bar13] and [Bar15]) parametrized by a Banach analytic set.

This answers a question asked to me by Y-T. Siu forty years ago.

I was able to solve this question thanks to the notion of $\mathsf{f}$-analytic family introduced in *loc. cit.* and using the space $\mathcal{C}_n^{\mathsf{f}}(Z)$ of finite type cycles with its natural topology.

We obtain the following results.

**Theorem 1.1.** *Let $n \geq 2$ be an integer. Let $Z$ be a strongly $(n-2)$-concave reduced complex space of pure dimension $n+p$ that is to say admitting a $\mathscr{C}^2$ exhaustion function $\varphi : Z \to ]0,2]$ which is strongly $(n-2)$-convex outside the compact set $K := \varphi^{-1}([1,2])$. For any $\alpha \in ]0,1[$ and any $n$-cycle $X_0$ in an open neighbourhood of the compact set $\varphi^{-1}([\alpha,2])$ there exists $\beta \in ]0,\alpha[$ such that, if $Z_\beta := \{z \in Z \ / \ \varphi(z) > \beta\}$, the cycle $X_0$ extends in a unique way to the open set $Z_\beta$ and admits an open neighbourhood $\mathcal{U}$ in the space $\mathcal{C}_n^{\mathsf{f}}(Z_\beta)$ such that the ringed space defined by $\mathcal{U}$ and the sheaf of holomorphic functions on $\mathcal{U}$ is a (reduced) complex space locally of finite dimension.*

Recall that a holomorphic function $h : \mathcal{U} \to \mathbb{C}$ on an open set in $\mathcal{C}_n^{\mathsf{f}}(Z_\beta)$ is a continuous function on $\mathcal{U}$ such that for any holomorphic map $f : S \to \mathcal{U}$ (corresponding to an $\mathsf{f}$-analytic family of $n$-cycles in $Z$, see *loc. cit.*) of a Banach analytic set $S$ to $\mathcal{U}$ the composed function $h \circ f$ is holomorphic.

[1] ↑ Our conventions will be precised below.

**Theorem 1.2.** *Consider the same situation as in the previous theorem, and let now $X_0 \in \mathcal{C}_n^{\mathsf{f}}(Z)$ be a finite type $n$-cycle in $Z$. Then there exists $\beta \in ]0,1[$ and open neighbourhoods $\mathcal{V}$ and $\mathcal{U}$ respectively of $X_0$ in $\mathcal{C}_n^{\mathsf{f}}(Z)$ and of $X_0 \cap Z_\beta$ in $\mathcal{C}_n^{\mathsf{f}}(Z_\beta)$ such that the restriction map*

$$r : \mathcal{V} \to \mathcal{U}$$

*is well defined and bi-holomorphic.*

We are going to recall briefly the notion of $\mathsf{f}$-analytic family of finite type $n$-cycles in a complex space $Z$.

Firstly the notion of an analytic family of $n$-cycles in a reduced complex space $Z$ parametrized by a reduced complex space $S$ is defined as follows (see[2] [Bar75, Chapter I, p. 33] or [BM14, Chapter IV, Section 3]), using the following notion of an adapted scale (see [BM14, Chapter IV, Section 2.1]).

**Definition 1.3.** We call $E := (U, B, j)$ a **$n$-scale on a complex space** $Z$ when $U$ and $B$ are open relatively compact polydiscs respectively in $\mathbb{C}^n$ and $\mathbb{C}^p$ and where $j : Z_E \to W$ is a closed embedding of an open set $Z_E$ in $Z$ into an open neighbourhood $W$ of $\bar{U} \times \bar{B}$ in $\mathbb{C}^{n+p}$. The open set $Z_E$ is called the **domain** of $E$ and the open set $j^{-1}(U \times B)$ the **center** of $E$. When $X$ is a $n$-cycle in $Z$, the $n$-scale $E$ is **adapted** to $X$ when $|X| \cap j^{-1}(\bar{U} \times \partial B) = \emptyset$.

Note that when $E$ is a $n$-scale adapted to a $n$-cycle $X$ in $Z$, the projection of $U \times B$ on $U$ restricted to $j_*(X \cap j^{-1}(U \times B))$ gives a finite proper map of degree $k \geq 0$ and the fibers of this map are classified by a holomorphic map $f : U \to \mathrm{Sym}^k(B)$. In this case we shall say that $f$ is the **classifying map** of the cycle $X$ in the adapted scale $E$.

**Definition 1.4.** Let $Z$ be a complex space and let $(X_s)_{s \in S}$ be a family of $n$-cycles in $Z$ parametrized by a reduced complex space $S$. We shall say that this family is **analytic at a point** $s_0 \in S$ if for any $n$-scale $E := (U, B, j)$ on $Z$ which is adapted to the cycle $X_{s_0}$ there exists an open neighbourhood $S_0$ of $s_0$ in $S$ satisfying the following properties:

i) For each $s \in S_0$ the scale $E$ is adapted to $X_s$.

ii) Assume that $k := \deg_E(X_{s_0})$. Then for each $s \in S_0$ we have $\deg_E(X_s) = k$.

iii) There exists a holomorphic map $f : S_0 \times U \to \mathrm{Sym}^k(B)$ such that for each $s \in S_0$ the restriction of $f$ to $\{s_0\} \times U$ classifies the cycle $X_s$ in the scale $E$.

It is easy to see that an analytic family of cycles has a "set theoretic" graph

$$|G| := \{(s, x) \in S \times Z \ / \ x \in |X_s|\}$$

which is a closed analytic subset in $S \times Z$ and that its projection on $S$ has pure $n$-dimensional fibers (which are the supports of the cycles). When we have an analytic family $(X_s)_{s \in S}$ and when the projection of its graph $pr : |G| \to S$ is quasi-proper[3] we shall say that $(X_s)_{s \in S}$ is a **f-analytic family** of (finite type) $n$-cycles in $Z$ . Of course this condition implies that each cycle $X_s$ is a finite type $n$-cycle (it means that each cycle admits only finitely many irreducible components) but this condition contains this fact in a local uniform manner on $S$.

In an analogous way, when we have an analytic family $(X_s)_{s \in S}$ and when the projection of its graph $pr : |G| \to S$ is proper, we shall say that it is a **proper analytic family of compact cycles in** $Z$.

The following corollary is of course the main result.

[2] ↑ See Chapter 3 Section 4 in [Bar75] for the case when $S$ is a Banach analytic set.

[3] ↑ This means, by definition, that for any $s_0 \in S$ there exists an open neighbourhood $S_1$ of $s_0$ in $S$ and a compact set $K$ in $|G|$ such that any irreducible component of any fiber $pr^{-1}(s)$ for any $s \in S_1$ meets $K$.

**Corollary 1.5.** *Consider the same situation as in the previous theorems. Then the ringed space given by the sheaf of holomorphic functions on $\mathcal{C}_n^{\mathsf{f}}(Z)$ is a reduced complex space locally of finite dimension. Moreover, endowed with its tautological family of (finite type) $n$-cycles it represents the functor*

$$\begin{array}{rcl}\text{(reduced complex spaces)} & \to & \text{(sets)} \\ S & \mapsto & \{\mathsf{f}\text{-analytic family of } n\text{-cycles in } Z \text{ parametrized by } S\}.\end{array}$$

**Remark.** Let $X$ be a (non empty) irreducible analytic subset of dimension $n$ in a strongly $(n-2)$-concave complex space $Z$ as in Theorem 1.1. Let $x_0 \in X$ be a point in $X$ where the supremum of the restriction of the exhaustion function $\varphi$ to $X$ is obtained. Then the point $x_0$ is in the compact set $K = \varphi^{-1}([1,2])$ because the Levi form of $\varphi$ at the point $x_0$ has at least $n$ non positive eigenvalues. So any (non empty) irreducible analytic subset of dimension $n$ in $Z$ has to meet the compact set $K$. Then any $n$-cycle in $Z$ is of finite type and any analytic family of $n$-cycles in $Z$ has a quasi-proper graph so is a $\mathsf{f}$-analytic family. This implies that, in the previous corollary, the obvious map[4] $\mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{loc}(Z)$ is an isomorphism of ringed spaces and $\mathcal{C}_n^{\mathsf{f}}(Z)$ represents also the functor "analytic family of $n$-cycles in $Z$".

**Question.** As it appears in the previous remark, we may expect the same result for $(n-1)$-cycles under our assumption of strong $(n-2)$-concavity. But our way to use Hartogs figures in the present article needs one more positive eigenvalue than one can expect. Is the result also true for $(n-1)$-cycles under our hypothesis? It would probably be interesting, for instance, to have this kind of result for 1-cycles in a strongly 0-concave complex space.

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# 2. Hartogs figures

## 2.A. Banachization

For the analytic extension via Hartogs figures, the use of the Banach spaces $H(\bar{U}, \mathbb{C})$ of continuous functions, holomorphic inside on a compact polydisc $\bar{U}$ is not well adapted. We shall use the Banach space $\mathcal{B}(U, \mathbb{C})$ of bounded holomorphic functions on $U$ with the "sup" norm on $U$. Of course, $H(\bar{U}, \mathbb{C})$ is a closed Banach subspace in $\mathcal{B}(U, \mathbb{C})$.

**Proposition 2.1.** *Let $U$ be a relatively compact polydisc in $\mathbb{C}^n$ and let $S$ be a Banach analytic set. Let $F : S \to \mathcal{B}(U, \mathbb{C})$ be a holomorphic map. Then the corresponding function $f : S \times U \to \mathbb{C}$ defined by $f(s,t) := F(s)[t]$ for $(s,t)$ in $S \times U$ is holomorphic (and locally on $S$ uniformely bounded on $U$).*

*Conversely, if we have a holomorphic function $f : S \times U \to \mathbb{C}$ and an open polydisc $U'' \subset\subset U$, the associated map $F : S \to \mathcal{B}(U'', \mathbb{C})$, defined for $(s,t)$ in $S \times U''$ by $F(s)[t] := f(s,t)$, is holomorphic.*

*Proof.* The evaluation function $ev : \mathcal{B}(U, \mathbb{C}) \times U \to \mathbb{C}$ is holomorphic as one may easily see by differentiation. Then the function $f$ associated to $F$ is the composition of the holomorphic maps $F \times id_U$ and $ev$. So it is holomorphic.

The converse is consequence of the linear isometric inclusion of $H(\bar{U}'', \mathbb{C})$ in $\mathcal{B}(U'', \mathbb{C})$ as $F'' : S \to H(\bar{U}'', \mathbb{C})$ is holomorphic as soon as $f$ is holomorphic (see [Bar75] or [BM]). □

[4] ↑ The set $\mathcal{C}_n^{loc}(Z)$ is the set of all closed cycles of dimension $n$ in the complex space $Z$; its natural topology which is associated to the adapted scales is described in [BM14, Chapter IV]. In general, the inclusion map $\mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{loc}(Z)$ is continuous but is not a homeomorphism onto its image. See [Bar15] for the comparison between compact sets in these two spaces.

## 2.B. $n$-Hartogs figure on a complex space

For $\alpha \in (\mathbb{R}_+^*)^2$ let $M(\alpha)$ be the open set in $\mathbb{C}^2$ defined by

$$\begin{aligned} M(\alpha) &:= M^P(\alpha) \cup M^C(\alpha) \quad \text{with} \\ M^P(\alpha) &:= \big\{|t_1 - \alpha_1/2| < \alpha_1/4, \quad |t_2| < \alpha_2\big\} \\ M^C(\alpha) &:= \big\{|t_1| < \alpha_1, \quad \alpha_2/2 < |t_2| < \alpha_2\big\} \end{aligned}$$

and let also $\mathcal{M}(\alpha) := \big\{(t_1,t_2) \in \mathbb{C}^2 \ / \ |t_i| < \alpha_i, \ i = 1,2\big\}$.

For $\varepsilon > 0$ small enough, we define

$$\begin{aligned} M(\alpha)^\varepsilon &:= M^P(\alpha)^\varepsilon \cup M^C(\alpha)^\varepsilon \quad \text{with} \\ M^P(\alpha)^\varepsilon &:= \big\{|t_1 - \alpha_1/2| < \alpha_1/4 - \varepsilon/4, \quad |t_2| < \alpha_2 - \varepsilon\big\} \\ M^C(\alpha)^\varepsilon &:= \big\{|t_1| < \alpha_1 - \varepsilon, \quad \alpha_2/2 + \varepsilon/2 < |t_2| < \alpha_2 - \varepsilon\big\} \end{aligned}$$

and also $\mathcal{M}(\alpha)^\varepsilon := \big\{(t_1,t_2) \in \mathbb{C}^2 \ / \ |t_i| < \alpha_i - \varepsilon, \ i = 1,2\big\}$.

For a polydisc of radius $R$ in $\mathbb{C}^m$ we shall denote by $P^\varepsilon$ the polydisc with same center and radius $R - \varepsilon$ for $0 < \varepsilon < R$.

**Definition 2.2.** Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space $Z$. We shall call $\mathcal{H} := (\mathcal{M}, M, B, j)$ a **$n$-Hartogs figure in $Z$ relative to the boundary of $\Delta$**, the following data

- an embedding $j$ of an open set $Z'$ in $\Delta'$ into an open set in $\mathbb{C}^{n+p}$,
- open sets $\mathcal{M} \subset\subset \Delta'$ and $M \subset\subset \Delta$ relatively compact in $Z'$,
- a relatively compact polydisc $B$ in $\mathbb{C}^p$

such that there exists $\alpha \in (\mathbb{R}_+^*)^2$ and a relatively compact polydisc $V$ in $\mathbb{C}^{n-2}$ with the following conditions:

i) The map $j$ induces a closed embedding of $\mathcal{M}$ in $\mathcal{M}(\alpha) \times V \times B$.

ii) The map $j$ induces a closed embedding of $M$ in $M(\alpha) \times V \times B$.

iii) We have $j^{-1}\big(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B\big) \subset \Delta$.

**Definition 2.3.** Let $n \geq 2$ and $p \geq 1$ two integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space $Z$. Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ be a $n$-Hartogs figure in $Z$ relative to the boundary of $\Delta$ and let $X_0$ be a $n$-cycle in $\Delta$. We shall say that $\mathcal{H}$ **is adapted to** $X_0$ when the following condition is satisfied :

$$j^{-1}(\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B) \cap |X_0| = \emptyset. \qquad (@)$$

**Remarks.**

1. Let $\mathcal{H}$ be a $n$-Hartogs figure in $Z$ relative to the boundary of $\Delta$. If the open set $\Delta_1 \subset\subset \Delta'$ has a boundary $\partial\Delta_1$ near enough to $\partial\Delta$, then $\mathcal{H}$ is again a $n$-Hartogs figure relative to the boundary of $\Delta_1$. For instance, if $\Delta := \{\varphi > 0\}$ where $\varphi$ is a continuous proper function on $Z$, we may choose $\Delta_1 := \{\varphi > \varepsilon\}$ for $\varepsilon > 0$ small enough.
2. Note that the $n$-scale $E_{\mathcal{H}} := (M(\alpha) \times V, B, j)$ on $\Delta$ associated to $\mathcal{H}$ is adapted to $X_0$ as soon as the $n$-Hartogs figure $\mathcal{H}$ is adapted to $X_0$.

3. If $\tilde{X}_0$ is a $n$-cycle in $\Delta'$ such that its restriction to $\Delta$ is equal to $X_0$, the $n$-scale $E_{\tilde{\mathcal{H}}} := (\mathcal{M}(\alpha) \times V, B, j)$ on $\Delta'$ is adapted to $\tilde{X}_0$ if and only if the $n$-Hartogs figure $\mathcal{H}$ is adapted to $X_0$.

Note that the $n$-scale $E_{\tilde{\mathcal{H}}}$ is not a $n$-scale on $\Delta$ although the subset $\bar{\mathcal{M}}(\alpha) \times \bar{V} \times \partial B$ is contained in $\Delta$.

**Definition 2.4.** In the situation above we define, for $\varepsilon > 0$ small enough, the $n$-Hartogs figure $\mathcal{H}^\varepsilon$ on $\Delta$ as follows :

$$\mathcal{H}^\varepsilon := (\mathcal{M}^\varepsilon, M^\varepsilon, B, j)$$

where we use the notations

$$\mathcal{M}^\varepsilon := j^{-1}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon \times B) \quad \text{and also} \quad M^\varepsilon := j^{-1}(M(\alpha)^\varepsilon \times V^\varepsilon \times B).$$

It is obvious to see that when $\mathcal{H}$ is a $n$-Hartogs figure relative to the boundary of $\Delta$, then $\mathcal{H}^\varepsilon$ is again a $n$-Hartogs figure relative to the boundary of $\Delta$ for all $\varepsilon > 0$ small enough.

Moreover, if $\mathcal{H}$ is adapted to the $n$-cycle $X_0$ of $\Delta$, the same is true for $\mathcal{H}^\varepsilon$ for all $\varepsilon > 0$ small enough.

**Lemma 2.5.** *Let $V$ be a relatively compact open polydisc in $\mathbb{C}^q$. The restriction map*

$$res : \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C}) \to \mathcal{B}(M(\alpha) \times V, \mathbb{C})$$

*is a linear isometry of Banach spaces.*

Note that the restriction map

$$res : \mathcal{B}(\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C}) \to \mathcal{B}(M(\alpha)^\varepsilon \times V^\varepsilon, \mathbb{C})$$

induces also an isometry for all $\varepsilon > 0$ small enough.

*Proof.* Let $f(v, t_1, t_2) := \sum_{m \in \mathcal{Z}} a_m(v, t_1).t_2^m$ the Laurent expansion of the holomorphic function $f : M^C(\alpha) \times V \to \mathbb{C}$. The holomorphic functions $a_m$, $m \in \mathbb{Z}$ on the product of $V$ by the disc $\{|t_1| < \alpha_1\}$ are given by the formula

$$a_m(v, t_1) := \frac{1}{2i\pi}.\int_{|z|=r} f(v, t_1, z).\frac{dz}{z^{m+1}} \quad \text{with} \quad r \in ]\alpha_2/2, \alpha_2[.$$

As the holomorphy of $f$ on $M^P(\alpha) \times V$ implies that $a_m \equiv 0$ for each negative $m$ on the open set $\{|t_1 - \alpha_1/2| < \alpha_1/4\} \times V$, we conclude that the functions $a_m$ are identically zero for $m < 0$ and so $f$ is holomorphic on $\mathcal{M}(\alpha) \times V$. This shows that the restriction map $res$ is bijective (and also it is linear continuous) between the two Fréchet spaces $\mathcal{O}(\mathcal{M}(\alpha) \times V)$ and $\mathcal{O}(M(\alpha) \times V)$; so it is an isomorphism of Fréchet spaces.

Let us show that if $f$ is in $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathbb{C})$, then $res(f)$, which belongs to the space $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, has the same "sup" norm. For this purpose fix $\varepsilon > 0$ small enough. As $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ is a compact polydisc in $\mathcal{M}(\alpha) \times V$, the maximum of $f$ on this compact is obtained at some point $z$ in the distinguish boundary of it. But as $z$ is also in the boundary of $M^C(\alpha)^\varepsilon \times \bar{V}^\varepsilon$, the desired equality follows. Conversely, if $g$ is in $\mathcal{B}(M(\alpha) \times V, \mathbb{C})$, its analytic extension $f$ to $\mathcal{M}(\alpha) \times V$ will be bounded on the boundary of $\mathcal{M}(\alpha)^\varepsilon \times V^\varepsilon$ by the sup of $g$ on $M^C(\alpha)^\varepsilon \times V^\varepsilon$. So we obtain the equality of the "sup" norms for $g$ and $f$ respectively on $M(\alpha) \times V$ and $\mathcal{M}(\alpha) \times V$. □

**The Banach analytic set $\mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p))$.** Recall that if $p \geq 1$ and $k \geq 1$ are integers, there exists a closed embedding (in fact given by a polynomial map) of $\mathrm{Sym}^k(\mathbb{C}^p) := (\mathbb{C}^p)^k \big/ \mathfrak{S}_k$ into the vector space $E(k) := \bigoplus_{h=1}^k S^h(\mathbb{C}^p)$ given by the elementary tensorial symetric functions[5]. If $U$ is an open relatively compact polydisc in $\mathbb{C}^n$, the subset $\mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p))$ is closed and Banach analytic in the Banach space $\mathcal{B}(U, E(k))$. Indeed, if $Q : E(k) \to \mathbb{C}^N$ is a polynomial map such that $Q^{-1}(0) = \mathrm{Sym}^k(\mathbb{C}^p)$, then the holomorphic map

$$\mathcal{Q} : \mathcal{B}(U, E(k)) \to \mathcal{B}(U, \mathbb{C}^N) \quad \text{defined by} \quad f \mapsto Q \circ f$$

satisfies $\mathcal{Q}^{-1}(0) = \mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p))$.

Nevertheless, be aware that for an open set $\Omega$ in $E(k)$ the subset $\mathcal{B}(U, \Omega)$ of elements in $\mathcal{B}(U, E(k))$ taking their values in $\Omega$ is not, in general, open in $\mathcal{B}(U, E(k))$; so, for an open polydisc $B \subset\subset \mathbb{C}^p$, the subset $\mathcal{B}(U, \mathrm{Sym}^k(B))$ is not open in $\mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p))$ in general.

**Remark.** The obvious map $H(\bar{U}, E(k)) \to \mathcal{B}(U, E(k))$ is a closed (linear) isometry and induces a holomorphic inclusion map

$$i_U : H(\bar{U}, \mathrm{Sym}^k(\mathbb{C}^p)) \hookrightarrow \mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p))$$

and for all $\varepsilon > 0$ the restriction

$$r : \mathcal{B}(U, E(k)) \to H(\bar{U}^\varepsilon, E(k))$$

is a (linear and continuous) compact map which induces a holomorphic restriction map

$$\mathcal{B}(U, \mathrm{Sym}^k(\mathbb{C}^p)) \to H(\bar{U}^\varepsilon, \mathrm{Sym}^k(\mathbb{C}^p)).$$

This remark will allow us to use Lemma 2.5 with Banach analytic sets like $H(\bar{U}, \mathrm{Sym}^k(B))$.

**Notations.** Let $k \in \mathbb{N}$ and let $U' \subset\subset U \subset\subset \mathbb{C}^n$ and $B \subset\subset \mathbb{C}^p$ be polydiscs. We shall note $\Sigma_{U,U'}(k)$ the Banach analytic set classifying the couples of an element in $H(\bar{U}, \mathrm{Sym}^k(B))$ with its isotropy data on $\bar{U}'$. Recall that for a holomorphic map $f : U \to \mathrm{Sym}^k(B)$ the isotropy data on the polydisc $U'$ is the map

$$T(f) : U' \to F \otimes E'$$

where

$$F := \bigoplus_i \Big( L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \Big) \quad \text{and} \quad E' := \bigoplus_{m=0}^{k-1} S^m(\mathbb{C}^p).$$

It corresponds to the collection of holomorphic maps

$$T^i_m(f) : U' \to L(\Lambda^i(\mathbb{C}^n), \Lambda^i(\mathbb{C}^p)) \otimes S^m(\mathbb{C}^p)$$

for all $i \in [1, \min(n,p)]$ and $m \in [0, k-1]$ which are given, near a point in $U'$ where the multiform graph $X_f$ associated to $f$ has local branches $f_1, \dots, f_k$, by the formula

$$T^i_m(f)(t) := \sum_{j=1}^k \Lambda^i(Df_j(t)) \otimes f_j(t)^m.$$

These maps are always holomorphic on all $U$ and determine the trace map for the projection $\pi : X_f \to U$ of the holomorphic differential forms on $U \times B$.

The subset

$$\Sigma_{U,U'}(k) \subset H(\bar{U}, \mathrm{Sym}^k(B)) \times H(\bar{U}', F \otimes E')$$

[5] ↑ See for instance [BM14, Chapter I, §4].

is defined as the graph of the map $f \mapsto T(f) := \bigoplus_{i,m} T^i_m(f)$ which is not holomorphic in general. Nevertheless it is a Banach analytic subset and its natural projection

$$\Sigma_{U,U'}(k) \to H(\bar{U}, \mathrm{Sym}^k(B))$$

is a holomorphic homeomorphism (see [Bar75, Chapter III, Proposition 2, p. 81] or [BM, Chapter V]).

The important point which motivates the introduction of this Banach analytic subset is the fact that, when $S$ is a reduced complex space, an analytic family of multiform graphs given by a holomorphic map $f : S \times U \to \mathrm{Sym}^k(B)$ will give an analytic family of cycles in $U \times B$ parametrized by $S$ if and only if the corresponding isotropy data (given by the maps $T^i_m(f_{/S})$ on $S \times U$) are holomorphic on $S \times U$. This is the isotropy condition ; see [Bar75, Chapter II] or [BM14, Chapter IV, Section 5].

Our next result is the main tool for performing the analytic extension of $n$-cycles near a $(n-2)$-concave boundary.

**Proposition 2.6.** *Consider the open sets $\mathcal{M}(\alpha) \times V$ and $M(\alpha) \times V$ in $\mathbb{C}^n$. The inverse of the restriction map is a holomorphic isomorphism of analytic extension*

$$prlgt : \mathcal{B}(M(\alpha) \times V, \mathrm{Sym}^k(\mathbb{C}^p)) \to \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathrm{Sym}^k(\mathbb{C}^p)).$$

*Composed with the restriction to the compact set $\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon$ it sends the subset $H(\bar{M}(\alpha) \times \bar{V}, \mathrm{Sym}^k(B))$ into $H(\bar{\mathcal{M}}(\alpha)^\varepsilon \times \bar{V}^\varepsilon, \mathrm{Sym}^k(B))$ for $\varepsilon > 0$ small enough.*

*Moreover, this holomorphic map induces a holomorphic map, again for $\varepsilon > 0$ small enough*

$$\Sigma_{M(\alpha)\times V,\ M(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3}}(k) \longrightarrow \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3},\ \mathcal{M}(\alpha)^{2\varepsilon/3}\times V^{2\varepsilon/3}}(k)$$

*which factorizes the restriction map*

$$\Sigma_{\mathcal{M}(\alpha)\times V,\ \mathcal{M}(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3}}(k) \to \Sigma_{\mathcal{M}(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3},\ \mathcal{M}(\alpha)^{2\varepsilon/3}\times V^{2\varepsilon/3}}(k)$$

*through the restriction*

$$\Sigma_{\mathcal{M}(\alpha)\times V,\ \mathcal{M}(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3}}(k) \to \Sigma_{M(\alpha)\times V,\ M(\alpha)^{\varepsilon/3}\times V^{\varepsilon/3}}(k).$$

*Proof.* Lemma 2.5 gives that the map

$$prlgt : \mathcal{B}(M(\alpha) \times V, E(k)) \to \mathcal{B}(\mathcal{M}(\alpha) \times V, E(k))$$

is an isometry of Banach spaces. It is clear that its inverse sends $\mathcal{B}(\mathcal{M}(\alpha) \times V, \mathrm{Sym}^k(\mathbb{C}^p))$ into the Banach analytic subset $\mathcal{B}(M(\alpha) \times V, \mathrm{Sym}^k(\mathbb{C}^p))$ and that if the map $f \in \mathcal{B}(\mathcal{M}(\alpha) \times V, \mathrm{Sym}^k(\mathbb{C}^p))$ takes values in $\mathrm{Sym}^k(B)$, the same is true for its restriction. This proves the first part of the proposition.

In order to prove the second part, it is enough to show that a holomorphic map of a Banach analytic set $S$ with values in the subset

$$H(\bar{M}(\alpha) \times \bar{V}, \mathrm{Sym}^k(B))$$

which is isotropic on the product of $S$ with any relatively compact subset in the open set $M(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$ will have an analytic extension which will be isotropic on any relatively compact open set in $\mathcal{M}(\alpha)^{\varepsilon/3} \times V^{\varepsilon/3}$. So it will be isotropic on the closure of the open set $\mathcal{M}(\alpha)^{2\varepsilon/3} \times V^{2\varepsilon/3}$. □

**Proposition 2.7.** *Let $n \geq 2$ and $p \geq 1$ be integers and let $U_1 \times B_1$ the product of two polydiscs with centers $0$ respectively in $\mathbb{C}^n$ and $\mathbb{C}^p$. Denote by $(t_1, \cdots, t_n, x_1, \cdots, x_p)$ coordinates on $U_1 \times B_1$. Let $\varphi$ be a real valued function of class $\mathscr{C}^2$ on $U_1 \times B_1$, such that*

$$\varphi(t,x) = \mathrm{Re}(t_1) + \sum_{i=1}^{n} \rho_i.|t_i|^2 + \sum_{j=1}^{p} \sigma_j.|x_j|^2 + o(||(t,x)||^2) \qquad (@@)$$

*where the real numbers $\rho_2, \sigma_1, \cdots, \sigma_p$ are positive (so $\varphi$ is $(n-2)$-convex near $(0,0)$ and $d\varphi_{0,0} \neq 0$).*

*Let $\Delta$ be the open set $\{\varphi > 0\}$ in $U_1 \times B_1$ and let $\Delta'$ be an open neighbourhood of the compact set $\bar{\Delta}$ in $\mathbb{C}^{n+p}$. Let $X_0$ be a closed analytic subset of pure dimension $n$ in $\Delta'$ such that each irreducible component of $X_0$ meets $\Delta$ and such that*

$$|X_0| \cap \{t_1 = \cdots = t_n = 0\} \subset \{0\}. \qquad (*)$$

*Then there exists $\alpha \in (\mathbb{R}_+^*)^2$ and polydiscs $V$ and $B \subset\subset B_1$ with centers $0$ respectively in $\mathbb{C}^{n-2}$ and $\mathbb{C}^p$ such that the following conditions are satisfied :*

1. *$\mathcal{M}(\alpha) \times V \times B \subset\subset \Delta'$;*
2. *$M(\alpha) \times V \times B \subset\subset \Delta$;*
3. *$\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B \subset \Delta$;*
4. *$|X_0| \cap (\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B) = \emptyset$ (this implies $|X_0| \cap (\overline{M(\alpha)} \times \bar{V} \times \partial B) = \emptyset$).*

*Proof.* Choose the polydisc $B \subset\subset B_1$ small enough in order that we have

$$X_0 \cap (\{0\} \times \bar{B}) \subset \{0\} \quad \text{and} \quad \{0\} \times \partial B \subset \Delta.$$

This is possible as we have $|X_0| \cap \{t_1 = \cdots = t_n = 0\} \subset \{0\}$ and as $\varphi$ is positive on a small enough punctured neighbourhood of the origin in the $p$-plane $\{t_1 = \cdots = t_n = 0\} \times \mathbb{C}^p$. So we shall have

$$|X_0| \cap (\bar{W} \times \partial B) = \emptyset \quad \text{and} \quad \bar{W} \times \partial B \subset \Delta$$

for any small enough open neighbourhood $W$ of the origin in $\mathbb{C}^n$. A immediate consequence is that Conditions 1, 3 and 4 will be satisfied as soon as $\alpha$ and $V$ are small enough.

In order to check Condition 2, let us remark first that, up to choosing the real numbers $\rho'_1$ and $\rho'_2$ such that ${\rho'}_1 > |\rho_1|$, ${\rho'}_2 \in ]0, \rho_2[$ and $r > \sup_{i \geq 3} |\rho_i|$, we obtain on $W \times B$ chosen small enough

$$\varphi(t,x) \geq \mathrm{Re}(t_1) - \rho'_1.|t_1|^2 + \rho'_2.|t_2|^2 - r.(\sum_3^n |t_i|^2) \qquad (**)$$

with strict inequality as soon as $x \neq 0$. Then for

$$V_\varepsilon = \{(t_3, \cdots, t_n) \ / \ \sum_3^n |t_i|^2 < \varepsilon^2\}$$

the following inequalities hold

$$\varphi(t,x) \geq \frac{1}{4}\alpha_1 - {\rho'}_1.{\alpha_1}^2 - r.\varepsilon^2 \quad \text{on} \quad \overline{M^P(\alpha) \times V_\varepsilon \times B} \qquad (1)$$

$$\varphi(t,x) \geq -\alpha_1 - {\rho'}_1.{\alpha_1}^2 + \frac{1}{4}{\rho'}_2.{\alpha_2}^2 - r.\varepsilon^2 \quad \text{on} \quad \overline{M^C(\alpha) \times V_\varepsilon \times B} \qquad (2)$$

for $\alpha$ and $\varepsilon$ small enough in order that $\mathcal{M}(\alpha) \times V_\varepsilon$ is contained in $W$.

This allows to fix $\alpha_1, \alpha_2$ and $\varepsilon$.

Now we shall choose $\alpha_1$ and $\varepsilon$ smaller in order to satisfy the following conditions :

$$8\alpha_1 < {\rho'}_2.{\alpha_2}^2 , \quad \alpha_1 < \frac{1}{8{\rho'}_1} \quad \text{and} \quad \varepsilon^2 < \frac{1}{8r}\alpha_1. \qquad (3)$$

To obtain $M^P(\alpha)\times V_\varepsilon\times B\subset\subset\Delta$ it is enough to show that on $M^P(\alpha)\times V_\varepsilon\times B$ we have, if we let $\alpha_1=u.\alpha_2$ and $\varepsilon^2=v.\alpha_1=uv.\alpha_2$

$$\frac{1}{4}>{\rho'}_1.\alpha_1+r.v.$$

Indeed, as we assumed ${\rho'}_1.\alpha_1<\frac{1}{8}$ and $r.\varepsilon^2<\frac{1}{8}.\alpha_1$ (so $r.v<1/8$) the first condition holds.

In order to satisfy $M^C(\alpha)\times V_\varepsilon\times B\subset\subset\Delta$ it is enough to show that on $M^C(\alpha)\times V_\varepsilon\times B$ we have

$$\frac{1}{4}{\rho'}_2.\alpha_2>u+{\rho'}_1u^2\alpha_2+r.uv.$$

But our condition implies

$${\rho'}_1u^2\alpha_2<\frac{1}{8}.u \qquad r.uv<\frac{1}{8}.u$$

which gives $u+{\rho'}_1u^2.\alpha_2+r.uv<2.u$. The condition $8\alpha_1<{\rho'}_2.{\alpha_2}^2$ which implies $2u<\frac{1}{4}\rho'_2.\alpha_2$, allows to conclude. □

**Remarks.**

1. We only used Condition (*) for $X_0$ and Inequality (**) for $\varphi$ in a neighbourhood of the origin in the proof above.
2. Sufficient conditions on $\varphi\in\mathscr{C}^2$ to satisfy (@@) are:
   i) The origin is not a critical point of $\varphi$.
   ii) The Levi form of $\varphi$ at 0 has, at most, $(n-2)$ non positive eigenvalues in the complex tangent hyperplane to the real hypersurface $\{\varphi(z)=0\}$; the existence of real function $\varphi\in\mathscr{C}^2$ such that $\Delta=\{\varphi>0\}$ and satisfying these two conditions is equivalent to the fact that the open set $\Delta$ has a strongly $(n-2)$-concave smooth boundary near the origin (see Definition 2.10 given below). Indeed, if $\varphi$ is not critical at 0 and has a Levi form at 0 with, at most, $(n-2)$ non positive eigenvalues in the complex hyperplane tangent to the real hypersurface $\{\varphi(z)=0\}$, its order 2 Taylor expansion at the origin is written, in suitable local holomorphic coordinates $(\tau,x)$

   $$\varphi(\tau,x)=\mathrm{Re}(\tau_1)+\mathrm{Re}(Q(\tau,x))+\sum_{i=1}^{n}\rho_i.|\tau_i|^2+\sum_{j=1}^{p}\sigma_j.|x_j|^2+o(||(\tau,x)||^2)$$

   where $Q$ is a holomorphic homogeneous degree 2 polynomial and where the real numbers $\rho_2$ and $\sigma_j, j\in[1,p]$ are positive. Define new local holomorphic coordinates

   $$t_1:=\tau_1+Q(\tau,x), t_i:=\tau_i \quad \text{for } i\in[2,n] \quad \text{and} \quad x_j:=x_j \quad \text{for } j\in[1,p].$$

   Then we obtain (@@).
3. Condition (*) implies that $X_0$ has no local irreducible component at 0 contained in the hyperplane $\{t_1=0\}$. In fact, as the coordinate $t_1$ is choosen in order to suppress the real part of the holomorphic homogeneous degree 2 term in the order 2 Taylor expansion of $\varphi$ at 0 (see the previous remark), we want that no local irreducible component of $X_0$ at the origin is contained in the complex hypersurface $\tau_1+Q(\tau,x)=0$ locally defined near 0 for $\varphi$ given. Then, as soon as the restriction $\varphi_{|X_0}$ has not 0 as a critical point, Condition (*) could be realized when the Levi form of $\varphi$ has at most $(n-2)$ non positive eigenvalues on the complex tangent hyperplane at the origin of the hypersurface $\{\varphi=0\}$ .

4. One may easily see that under our hypothesis, the cycle $X_0$ meets the open set $\Delta$ when it contains 0. Indeed, the analytic subset
$$\{t_1 = t_3 = \cdots = t_n = 0\} \cap |X_0|$$
is nonempty, has dimension at least 1 and meets the complement of $\Delta$ only at the origin.
Of course, assuming that $0 \in X_0$, the proposition shows that, in fact, $X_0$ contains a branched covering of degree $k \geq 1$ of $M^P(\alpha) \times V_\varepsilon$ inside
$$M^P(\alpha) \times V_\varepsilon \times B \subset\subset \Delta.$$

5. In the situation of Proposition 2.7, for any continuous family $(X_s)_{s\in S}$ of $n$-cycles in $\Delta$ parametrized by a Banach analytic set $S$ such that $X_{s_0} = X_0 \cap \Delta$, there exists an open neighbourhood $S'$ of $s_0$ in $S$, such that for each $s \in S'$ Condition 4 remains true after analytic extension of the cycles (see Proposition 2.6), because, thanks to Condition 3, $\overline{\mathcal{M}(\alpha)} \times \bar{V} \times \partial B$ is a compact subset in $\Delta$.

## 2.C. $q$-concave open sets.

**Definition 2.8.** Let $\varphi : U \to \mathbb{R}$ be real valued $\mathscr{C}^2$ function on an open set $U$ in $\mathbb{C}^N$. We shall say that $\varphi$ is **strongly $q$-convex** when its Levi form at each point of $U$ has at most $q$ non positive eigenvalues.

So, with this definition a strongly 0-convex function is a strongly plurisubharmonic function.

**Definition 2.9.** *Let $\varphi : Z \to \mathbb{R}$ a real valued $\mathscr{C}^2$ function on a reduced complex space $Z$. We shall say that $\varphi$ is* **strongly $q$-convex** *if locally near each point of $Z$ it can be induced by a $\mathscr{C}^2$ strongly $q$-convex function in a local embedding in an open set of an affine space.*

Remark that a strongly $q$-convex function on an irreducible complex space of dimension at least equal to $q + 1$ has no local maximum because there exists at any point a germ of curve on which the restriction of $\varphi$ is strongly p.s.h.

**Definition 2.10.** Let $Z$ be a reduced complex space and let $\Delta$ be a relatively compact open set in $Z$. We shall say that $\Delta$ has a **smooth $\mathscr{C}^2$ boundary** when for each point $z$ in $\partial\Delta$ there exists a local holomorphic embedding $j : W \to U$ of an open neighbourhood $W$ of $z$ in an open set $U$ of the Zariski tangent space of $Z$ at $z$ and an open set $D$ with smooth $\mathscr{C}^2$ boundary in $U$ such that $W \cap j^{-1}(\partial D) = W \cap \partial\Delta$.

We shall say that the open set $\Delta \subset Z$ with smooth $\mathscr{C}^2$ boundary is **strongly $q$-concave at a point** $z \in \partial\Delta$ if, in some local holomorphic embedding $j : W \to U$ of $Z$ around $z$ as above, one can define $\Delta$ in $W$ as the subset $\{j \circ \varphi > 0\} \cap W$ where $\varphi$ is a real valued $\mathscr{C}^2$ function on $U$ such that

1. $d\varphi_{j(z)} \neq 0$ on the tangent space $T_{U,j(z)}$ of $U$ at $j(z)$.

2. The restriction of the Levi form at $j(z)$ of $\varphi$ to the complex hyperplane tangent at $j(z)$ to the real hypersurface $\{\varphi(x) = \varphi(j(z))\}$ in $U$ has at most $q$ non positive eigenvalues.

We shall say that $\Delta$ is **strongly $q$-concave** if $\Delta$ is strongly $q$-concave near each point in $\partial\Delta$.

**Remark.** Assume that $Z$ is of pure dimension $q+p$. If the defining function $\varphi$ of $\Delta$ satisfies Conditions 1 and 2 above, we can compose $\varphi$ with a real strictly increasing (non critical) convex $\mathscr{C}^2$ function (this does not change the level sets $\{\varphi = \text{constant}\}$), in order that $c \circ \varphi$ is $\mathscr{C}^2$ strongly $q$-convex (and non critical) near $z$.

Conversely, if $\varphi$ is a real valued $\mathscr{C}^2$ function which is strongly $q$-convex and not critical near at a point $z \in Z$, the open set $\{\varphi(x) > \varphi(z)\}$ has a strongly $q$-concave boundary in a neighbourhood of $z$.

With this terminology, using the remarks above, we may give the following reformulation of Proposition 2.7:

**Corollary 2.11.** *Let $n \geq 2$ and $p \geq 1$ be integers, let $Z$ be a reduced complex space of pure dimension $n+p$ and let $\Delta := \{\varphi > 0\}$ be an open set with $\mathscr{C}^2$ smooth boundary in $Z$. Let $X_0$ be a $n$-cycle in an open neighbourhood of a point $z \in \partial\Delta$ such that the function $\varphi_{|X_0}$ is not critical at $z$.*

*Assume that $\Delta$ is strongly $(n-2)$-concave near $z$ ; then there exists a $n$-Hartogs figure $\mathcal{H} := (\mathcal{M}, M, B, j)$ relative to the boundary of $\Delta$, adapted to $X_0$, and such that the point $z$ lies in $\mathcal{M}$.*

*Proof.* Using a local embedding of an open neighbourhood of $z$ in an open set of the Zariski tangent space $T_{Z,z}$, it is enough to prove the corollary in the case where $Z$ is an open set in $\mathbb{C}^{n+p'}$, with $p' \geq p$ an integer. As we may choose the function $\varphi$ strongly $(n-2)$-convex such that $d\varphi_z \neq 0$ thanks to the previous remarks, we can choose local coordinates near $z$ in order to be in the situation of Proposition 2.7 in the case $z \in X_0$, as we assumed that $\varphi_{|X_0}$ is not critical at $z$. In this case the proposition gives the result.

If $z$ is not in $X_0$, the same construction in an open neighbourhood of $z$ with no limit point in $X_0$ allows to conclude, and in this case the degree of $X_0$ in the (adapted) scale $E_{\mathcal{H}}$ will be zero. $\square$

## 2.D. Convexity–concavity

In this paragraph we want to have a brief discussion about $q$-convexity and $q$-concavity.

Let us consider in an open set $U$ of $\mathbb{C}^{n+p}$ a $\mathscr{C}^2$ function $\varphi : U \to \mathbb{R}$ and a non critical zero $z_0$ on $\varphi$. So $\varphi(z_0) = 0$ and $d\varphi_{z_0} \neq 0$. Let $D := \{z \in U \ / \ \varphi(z) < 0\}$ and let $H$ be the complex hyperplane tangent at $z_0$ to the real hypersurface $\{\varphi = 0\}$ which is smooth near $z_0$.

Our terminology (Norguet–Siu convention, see [NS77]) is to say that the open set $D$ is strongly $q$-convex near $z_0 \in \partial D$ if the restriction to $H$ of the Levi form of $\varphi$ at $z_0$ has at most $q$ non positive eigenvalues.

Looking now at the same open set $D$ but asking for some strong concavity condition, we write $D := \{z \in U \ / \ -\varphi(z) > 0\}$. Then we shall say that $D$ is strongly $q$-concave at the point $z_0$ if the restriction to $H$ of Levi form of $-\varphi$ at $z_0$ has at most $q$ non positive eigenvalues.

If the signature of the restriction to $H$ of the Levi form of $\varphi$ at $z_0$ is given by $(p-1)$ "plus" and $n$ "minus" we see that that near $z_0$ our open set $D$ will be strongly $n$-convex near $z_0$ and strongly $(p-1)$-concave near $z_0$. So $D$ will be strongly $(p-1)$-concave near $z_0$ if the function $-\varphi$ is strongly $(p-1)$-convex at the point $z_0$.

In order that a $\mathscr{C}^2$ exhaustion function $\varphi : Z \to ]0,2]$ on a reduced complex space $Z$ gives relatively compact $q$-concave subsets $Z_\alpha := \{\varphi(z) > \alpha\}$ for each $\alpha \in ]0,1[$ which is not critical for $\varphi$, we see that it is enough that the Levi form of $\varphi$ at each point in $\varphi^{-1}(]0,1[)$ has at most $q$ non positive eigenvalues. That is to say that $\varphi$ is strongly $q$-convex on this open set.

In order to reach the key situation given in Proposition 2.7 with a $n$-cycle, we need to dispose of a $\mathscr{C}^2$-exhaustion $\varphi : Z \to ]0,2]$ which is $(n-2)$-strongly convex on the open set $\varphi^{-1}(]0,1[)$ . So we need to assume that $n \geq 2$.

## 2.E. Boxed Hartogs figures

**Definition 2.12.** Let $n \geq 2$ and $p \geq 1$ be integers and let $\Delta \subset\subset \Delta'$ be two open sets in a reduced complex space $Z$. Let $\mathcal{H} = (\mathcal{M}, M, B, j)$ and $\mathcal{H}' := (\mathcal{H}', M', B, j)$ be two $n$-Hartogs figures in $Z$ relative to the boundary of $\Delta$ given by the same (local) embedding $j$ and having the same polydisc $B \subset\subset \mathbb{C}^p$. We shall say that these two $n$-Hartogs figures are **boxed** when we can choose $\alpha, \alpha', V, V'$ in Definition 2.2 in order to have

- $\mathcal{M}'(\alpha') \subset\subset \mathcal{M}(\alpha)$,
- $M'(\alpha') \subset\subset M(\alpha)$,
- $V' \subset\subset V$.

For instance, if $\varepsilon > 0$ is small enough, the $n$-Hartogs figures $(\mathcal{H}, \mathcal{H}^\varepsilon)$ are boxed (see Definition 2.4).

**Proposition 2.13.** *Let $n \geq 2$ and $p \geq 1$ be integers, let $Z$ be a reduced complex space of pure dimension $n+p$ and let $\Delta \subset\subset Z$ be an open set with smooth $\mathscr{C}^2$ boundary in $Z$ which is strongly $(n-2)$-concave. Assume that $\Delta := \{\varphi > 0\}$ and let $X_0$ be a $n$-cycle in an open neighbourhood $\Delta'$ of the compact set $\bar{\Delta}$, such that any irreducible component of $X_0$ meets $\Delta$. Then there exists a finite family of boxed $n$-Hartogs figures $(\mathcal{H}'_a, \mathcal{H}_a)_{a\in A}$ relative to the boundary of $\Delta$, such that the following conditions hold:*

1. *The open sets $\mathcal{M}'_a$ for $a \in A$ cover the boundary $\partial\Delta$.*
2. *For each $a \in A$ the Hartogs figures $\mathcal{H}_a$ and $\mathcal{H}'_a$ are adapted to $X_0$.*
3. *For each $a \in A$ any irreducible component of $X_0$ meeting $\mathcal{M}_a$ meets the open set $M'_a$.*
4. *No* **compact** *irreducible component of $X_0 \cap \Delta$ meets the union of the compact sets $\bar{\mathcal{M}}_a$, $a \in A$.*

**Remark.** Let $X_0$ be any $n$-cycle in $\Delta'$. Choosing the open set $\Delta'$ small enough around the compact set $\bar{\Delta}$, we can assume that the cycle $X_0$ has only finitely many irreducible components in $\Delta'$ and that each of them which is not compact meets $\partial\Delta$ (see Remark 3 following Proposition 2.7).

**Corollary 2.14.** *In the situation of Proposition 2.13, if we assume that the open set $\Delta'$ containing $\bar{\Delta}$ is small enough, any irreducible component $\Gamma$ of the cycle $X_0$ in $\Delta'$ satisfies for all $a \in A$ and all $\eta > 0$ small enough:*

$$\Gamma \cap (\mathcal{M}_a^\eta \times V_a^\eta \times B_a) = prlgt_a\big[\Gamma \cap (M_a \times V_a \times B_a)\big]$$

*where $prlgt_a : H(\bar{M}_a \times \bar{V}_a, \mathrm{Sym}^k(B_a)) \to H(\overline{\mathcal{M}}_a^\eta \times \bar{V}_a^\eta, \mathrm{Sym}^k(B_a))$ is the holomorphic map of analytic extension built in Proposition 2.6.*

**Remark.** In the situation of the previous corollary, choosing $\varepsilon > 0$ small enough, there exists, for each $a \in A$, a holomorphic extension map which lifts the map $prlgt_a$:

$$iprlgt_a : \Sigma_{M_a, M_a^\varepsilon}(k) \longrightarrow \Sigma_{\mathcal{M}_a^\eta, \mathcal{M}_a^{\eta+\varepsilon}}(k).$$

It allows to extend in this setting an analytic family of branched coverings in $M_a$ parametrized by a Banach analytic set $S$ and which is isotropic on $S \times M_a^\varepsilon$ to an analytic family of branched coverings in $\mathcal{M}_a^\eta$ which is isotropic on $S \times \mathcal{M}_a^{\eta+\varepsilon}$.

*Proof of Proposition 2.13.* Corollary 2.11 and the remark following it implies the existence, for each $z \in \partial\Delta$ of a $n$-Hartogs figure $\mathcal{H}_z$ relative to the boundary of $\Delta$, contained in $\Delta'$ and satisfying the following properties:

i) $z \in \mathcal{M}_z$;

ii) $\mathcal{H}_z$ is adapted to $X_0$ ;

iii) each irreducible component of $X_0$ meeting $\mathcal{M}_z$ meets the open set $M_z$;

iv) No compact irreducible component of $X_0 \cap \Delta$ meets $\bar{\mathcal{M}}_z$.

As the open sets $\mathcal{M}_z$ cover the compact set $\partial\Delta$ we can find a finite sub-covering. Then the properties 1, 2, 3 and 4 are consequences of i), ii), iii) and iv) by letting $\mathcal{H}'_a := \mathcal{H}^\varepsilon_a$ and choosing $\varepsilon > 0$ small enough. □

*Proof of Corollary 2.14.* Let $\Gamma$ be an irreducible component of $X_0$ meeting $\mathcal{M}_a$ for some $a \in A$. Then $\Gamma$ meets $M_a$. As $\Gamma$ does not meet $\bar{M}(\alpha)_a \times \bar{V}_a \times \partial B_a$ because $\mathcal{H}_a$ is adapted to $X_0$, the intersection $\Gamma \cap \bar{M}_a$ is the graph of an element $\gamma \in H(\bar{M}(\alpha)_a \times \bar{V}_a, \mathrm{Sym}^{k_a}(B_a))$ with $k_a \in \mathbb{N}^*$ [6].

The closed analytic subset $Y$ of the open set $\mathcal{M}^\eta_a$ defined by $Y := prlgt_a\big[\Gamma \cap M_a\big]$ is not empty, of pure dimension $n$ and is contained in $\Gamma$. So it is a union of irreducible components of $\Gamma \cap \mathcal{M}^\eta_a$. But it contains a non empty open set in each irreducible component of this branched covering. So these two analytic subsets coincide.

If an irreducible component of $X_0$ does not meet any $\bar{\mathcal{M}}_a$ it has to be compact and contained in $\Delta$. In this case the desired equality is obvious. □

# 3. The extension and finiteness theorem

## 3.A. Some useful lemmas

The version below of Sard's lemma is more or less classical.

**Lemma 3.1.** *Let $Z$ be a reduced complex space and let $\varphi : Z \to \mathbb{R}$ be a real valued $\mathscr{C}^1$ function. Then the set of critical values of $\varphi$ has Lebesgue measure* 0.

*Proof.* Firstly note that a point $z \in Z$ is critical for $\varphi$ if, by definition, the differential of $\varphi$ vanishes on $T_{Z,z}$, the Zariski tangent space of $Z$ at $z$. Remember also that a complex space is, by definition, countable at infinity ; so $Z$ and its singular locus have only countably many irreducible components. As a countable union of sets of measure 0 is again of measure 0, it is enough to prove the lemma when $Z$ is irreducible. We shall prove the lemma by induction on the integer $\dim Z$. The case $\dim Z = 0$ is obvious. Assume the lemma true for $\dim Z \leq n-1$ for some integer $n \geq 1$ and take an irreducible complex space $Z$ of dimension $n$. The singular set $S$ of $Z$ has dimension at most $(n-1)$, and for each irreducible component $S_i$ of $S$ the image of the critical set of $\varphi_{|S_i}$ has measure 0. So the critical set of $\varphi_{|S}$ is again of measure 0. But a critical point of $\varphi$ which belongs to $S$ is a critical point of $\varphi_{|S}$. So it is enough to show that the set of critical values of $\varphi$ restricted to the complex connected manifold $Z \setminus S$ has measure 0. This is the classical Sard's lemma. □

**Lemma 3.2.** *Let $V$ be an open set and $K$ be a compact set in $\bar{U} \times \bar{B}$. The subset $\mathcal{V}$ in $H(\bar{U}, \mathrm{Sym}^k(B))$ consisting of the $X$ such that any irreducible component meeting $K$ meets $V$ is an open set in $H(\bar{U}, \mathrm{Sym}^k(B))$.*

*Proof.* Let us clarify the meaning of an irreducible component of an element $X$ in $H(\bar{U}, \mathrm{Sym}^k(B))$: we call irreducible component of such a $X$ the closure in $\bar{U} \times B$ of an irreducible component of the branched covering of $U$ defined by the projection of $X \cap (U \times B)$ on $U$.

[6] ↑ as it exists some $(m, v) \in \bar{M}(\alpha)_a \times \bar{V}_a$ such that $\Gamma \cap (\{m, v\} \times B_a) \neq \emptyset$.

Let $X_0$ be such that any irreducible component of $X_0$ which meets $K$ meets $V$, and assume that $(X_\nu)_{\nu\geq 1}$ is a sequence converging to $X_0$ such that for each $\nu \geq 1$ there exists an irreducible component $\Gamma_\nu$ of $X_\nu$ meeting $K$ but not $V$. Passing to a subsequence, we may assume that the sequence $(\Gamma_\nu)_{\nu\geq 1}$ converges uniformely on any compact of $U \times B$ to a non empty $n$-cycle $\Gamma$ with closure contained in $X_0$ and which is a branched covering of $U$. Then $\bar\Gamma$ meets $K$ and not $V$. Indeed, if $(t_0, x_0)$ would be in $\bar\Gamma \cap V$, there exists open neighbourhoods $U_1$ and $B_1$ of $t_0$ and $x_0$ respectively in $\bar U$ and $\bar B$ such that $U_1 \times B_1$ is contained in $V$. But then, as $U_2 := U_1 \cap U$ and $B_2 := B_1 \cap B$ are non empty open sets, for $t_2$ in $U_2$ the fibers of the $\Gamma_\nu$ at $t_2$ for $\nu$ big enough will meet $\{t_2\} \times B_2$ and so $V$. As at least one irreducible component of $\Gamma$ meets $K$ without meeting $V$ and as its closure is an irreducible component of $X_0$, this gives a contradiction. □

Of course, in the case $V = \emptyset$, we get back the fact that the subset in $H(\bar U, \mathrm{Sym}^k(B))$ of elements which do not meet $K$ is open.

**Lemma 3.3.** *Let $Z$ be a complex space and let $(\mathcal{U}_i)_{i\in I}$ be an open covering of $Z$. Assume that for each $i \in I$ a closed $n$-cycle $X_i$ is given in $\mathcal{U}_i$. Assume that the following patching condition holds:*

$$\forall (i,j) \in I^2 \quad X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$$

*as an equality of cycles in $\mathcal{U}_i \cap \mathcal{U}_j$. Then there exists a unique closed $n$-cycle $X$ in $Z$ such that for each $i \in I$ we have $X \cap \mathcal{U}_i = X_i$.*

For the easy proof see [BM14, Chapter IV, Proposition 1.3.1].

The following variant will be used.

**Lemma 3.4. (Variant)** *In the situation of the previous lemma replace the patching condition by the following two conditions:*

1. *For each couple $(i,j) \in I^2$ an open subset $W_{i,j} \subset\subset \mathcal{U}_i \cap \mathcal{U}_j$ is given and we ask that $X_i \cap W_{i,j} = X_j \cap W_{i,j}$.*
2. *For each couple $(i,j) \in I^2$ we ask that any irreducible component of the cycle $X_i \cap \mathcal{U}_j$ meets the open set $W_{i,j}$.*

*Then the conclusion is the same.*

*Proof.* Let $\Gamma$ be an irreducible component with multiplicity $\delta$ in the cycle $X_i \cap \mathcal{U}_j$. Let $\Gamma'$ be the irreducible component of $X_i$ which contains $\Gamma$, and put $X_i = X'_i + \delta.\Gamma'$. Then $\Gamma'$ meets $W_{i,j}$ and there exists a closed analytic subset $Y$ of pure dimension $n$ in $|X_j|$ such that its restriction to $W_{i,j}$ is equal to $\Gamma' \cap W_{i,j}$: indeed, $Y$ is the union of the irreducible components of $X_j$ containing a non empty open set in $\Gamma' \cap W_{i,j}$. Note that each of these irreducible components of $X_j$ has multiplicity $\delta$ in the cycle $X_j$. Then put $X_j = X'_j + \delta.Y$. We see that the cycles $X'_i$ and $X'_j$ respectively in $\mathcal{U}_i$ and $\mathcal{U}_j$ satisfy again the patching condition $X'_i \cap W_{i,j} = X'_j \cap W_{i,j}$.

This allows, for fixed $(i,j)$, to make a descending induction on the number (necessarily finite as $W_{i,j}$ is relatively compact) of irreducible components of $X_i \cap \mathcal{U}_j$, to show that the condition $X_i \cap \mathcal{U}_j = X_j \cap \mathcal{U}_i$ holds. This reduces this lemma to the previous one. □

## 3.B. Adjusted scales.

The definition of a scale adapted to a cycle is recalled in Definition 1.3.

**Definition 3.5.** 1. Let $Z$ be a complex space. We shall call **adjusted $n$-scale on $Z$**, written down $\mathbb{E} := (U, U', U'', B, B'', j)$, the data of a $n$-scale on $Z$, $E := (U, B, j)$, with additional polydiscs $U'' \subset\subset U' \subset\subset U$ and $B'' \subset\subset B$. We call $E$ the **underlying scale of the adjusted scale $\mathbb{E}$**.

2. We shall say that **the adjusted scale $\mathbb{E}$ is adapted to a $n$-cycle** $X$ in $Z$ when we have
$$j^{-1}(\bar{U} \times (\bar{B} \setminus B'')) \cap |X| = \emptyset.$$
Note that this implies that the underlying scale $E$ is adapted to $X$, but this condition is more restrictive.

3. When the adjusted scale $\mathbb{E}$ is adapted to the $n$-cycle $X$, we shall call **degree of** $X$ **in** $\mathbb{E}$ the degree of $X$ in $E$.

4. We shall call **center of the adjusted scale**, written down $D(\mathbb{E})$, or more simply, $D(E)$, the open set $j^{-1}(U \times B)$ in $Z$ which is also the center of the scale $E$.

5. We shall call **domain of isotropy of the adjusted scale**, written down $D'(\mathbb{E})$, the open set $j^{-1}(U' \times B)$ in $Z$.

6. We shall call **domain of patching of the adjusted scale**, written down $D''(\mathbb{E})$, the open set $j^{-1}(U'' \times B'')$ in $Z$.

**Remarks.**

1. The open set $D''(\mathbb{E})$ is relatively compact in $D'(\mathbb{E})$.

2. When a $n$-scale $E$ is given, for any compact set $K$ in $D(E)$, there exists an adjusted $n$-scale $\mathbb{E}$ such that $E$ is the underlying scale of $\mathbb{E}$ and with $K \subset D''(\mathbb{E})$. Moreover, if $E$ is adapted to a $n$-cycle $X_0$ in $Z$, we may choose $\mathbb{E}$ in order that it is adapted to $X_0$.

3. As for $X \in \mathcal{C}_n^{loc}(Z)$ the condition to avoid a given compact subset is open in $\mathcal{C}_n^{loc}(Z)$, when the adjusted scale $\mathbb{E}$ is adapted to a cycle $X_0$ there exists an open neighbourhood, written down $\Omega_k(\mathbb{E})$, of $X_0$ in $\mathcal{C}_n^{loc}(Z)$ such that $\Omega_k(\mathbb{E})$ is the subset of all $n$-cycles $X$ in $Z$ for which $\mathbb{E}$ is adapted and $\deg_{\mathbb{E}}(X) = k$ where $k := \deg_{\mathbb{E}}(X_0)$.

Let $Z$ be a reduced complex space and let $\mathbb{E} = (U, U', U'', B, B'', j)$ be an adjusted scale on $Z$. For a given integer $k$ consider the continuous map sending a branched covering in $H(\bar{U}, \mathrm{Sym}^k(B''))$ to its isotropy data on $\bar{U}'$ (for the notations see what follows Lemma 2.5)
$$T : H(\bar{U}, \mathrm{Sym}^k(B'')) \to H(\bar{U}', F \otimes E').$$
The graph $\Sigma_{U,U'}(k)$ of this map is a Banach analytic set[7], thanks to [Bar75, Propositon 2, p. 81] (see also [BM]).

The set of couples $(f, T(f))$ in $\Sigma_{U,U'}(k)$ for which the associated branched covering is contained in $j(Z \cap \bar{D}(\mathbb{E}))$ is a closed Banach analytic subset of $\Sigma_{U,U'}(k)$ being the pull-back by the projection of the subset of elements in $H(\bar{U}, \mathrm{Sym}^k(B''))$ contained in $j(Z \cap \bar{D}(\mathbb{E}))$ which is a closed Banach analytic subset of $H(\bar{U}, \mathrm{Sym}^k(B''))$ by Proposition 4, p. 27 of [Bar75] (see also [BM, Chapter V]).

**Definition 3.6.** We shall denote $\mathcal{G}_k(\mathbb{E})$ this Banach analytic set and we shall call it the **the $k$-th classifying space of the adjusted $n$-scale** $\mathbb{E}$ on $Z$.

We have then a tautological family of $n$-cycles in the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$. It is an analytic family of cycles in the open set $D'(\mathbb{E})$, in the sense of [Bar75], and the fact that, for $k \geq 1$, locally on $\mathcal{G}_k(\mathbb{E})$, any irreducible component of a branched covering in this family meets $\bar{U}'' \times \bar{B}''$ implies that we have a $\mathsf{f}$-analytic family of cycles in $D'(\mathbb{E})$.

[7] ↑ homeomorphic to $H(\bar{U}, \mathrm{Sym}^k(B''))$ via the projection!

Be careful that the tautological family of cycles on the open set $D(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ is not, in general, an analytic family of $n$-cycles ; see the example of [Bar75, p. 83] (and also [BM14, Chapter IV]).

The next lemma is an obvious consequence of *loc. cit.*

**Lemma 3.7.** *Let $\mathbb{E}$ be an adjusted $n$-scale on a reduced complex space $Z$ and let $k$ be an integer. The tautological family of $n$-cycles in the open set $D'(\mathbb{E})$ parametrized by $\mathcal{G}_k(\mathbb{E})$ has the following "almost universal" property:*

> *For any analytic family of $n$-cycles $(X_s)_{s\in S}$ in $Z$ parametrized by a Banach analytic set $S$ such that for each $s \in S$ the adjusted scale $\mathbb{E}$ is adapted to $X_s$ with $\deg_E(X_s) = k$, there exists an unique holomorphic map*
> $$f : S \to \mathcal{G}_k(\mathbb{E})$$
> *such that the pull-back by $f$ of the tautological family is the restriction to the open set $D'(\mathbb{E})$ of the given family.*

Of course, conversely, such a holomorphic map gives a f-analytic family of $n$-cycles on the open set $D'(\mathbb{E})$.

Note that the pull-back family is in fact defined on the open set $D(\mathbb{E})$ but, as already noticed above, it may not be analytic outside $D'(\mathbb{E})$.

As a consequence of this "almost universal" property, we obtain that for any analytic family $(X_s)_{s\in S}$ of $n$-cycles in $Z$ such that for a point $s_0 \in S$ the adjusted $n$-scale $\mathbb{E}$ is adapted to the cycle $X_{s_0}$ with $\deg_{\mathbb{E}}(X_{s_0}) = k$, there exists an open neighbourhood $S'$ of $s_0$ in $S$ such that the previous lemma applies for the family parametrized by $S'$. So we shall have a holomorphic classifying map $f : S' \to \mathcal{G}_k(\mathbb{E})$ in this situation.

We shall generalize now the concept of classifying space to the case of a finite family of adjusted $n$-scales.

**Definition 3.8.** Consider a reduced complex space $Z$ and a finite family of adjusted $n$-scales $(\mathbb{E}_i)_{i\in I}$ on $Z$. Assume that they are adapted to a given finite type $n$-cycle $\hat{X}_0$ in $Z$. Assume that any irreducible component of $\hat{X}_0$ meets the open set $W'' := \bigcup_{i\in I} D''(\mathbb{E}_i)$.

We shall call **patching data for $\hat{X}_0$ associated to the family** $(\mathbb{E}_i)_{i\in I}$, written $\mathcal{R}((\mathbb{E}_i)_{i\in I}, F)$ or more simply $\mathcal{R}$ when there is no ambiguity, a finite collection of $n$-scales $(F_{i,j,h})$ for $(i,j) \in I^2$, $i \neq j$, where $h$ belongs to a finite set $H(i,j)$ for each couple $(i,j) \in I^2$, $i \neq j$, such that the following properties hold:

i) $F_{i,j,h}$ is a $n$-scale on the open set $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

ii) The $n$-scales $F_{i,j,h}$ are adapted to $\hat{X}_0$.

We shall say that the patching data $\mathcal{R}$ are **complete** when the following condition also holds:

iii) For each $i \neq j$ given, the union of domains of the scales $F_{i,j,h}$, $h \in H(i,j)$, covers the compact subset $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ of $D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

**Notations.**

1. In the sequel, when we shall consider a reduced complex space $Z$ and a finite family of adjusted $n$-scales $(\mathbb{E}_i)_{i\in I}$, adapted to a finite type $n$-cycle $\hat{X}_0$ in $Z$, such that any irreducible component of $\hat{X}_0$ meets the open set
$$W'' := \bigcup_{i\in I} D''(\mathbb{E}_i),$$
we shall say that the family $(\mathbb{E}_i)_{i\in I}$ is **convenient** for $\hat{X}_0$.

2. In this setting we shall use the following definitions :
   - $W := \bigcup_{i\in I} D(\mathbb{E}_i)$ ;
   - $W' := \bigcup_{i\in I} D'(\mathbb{E}_i)$ ;
   - $W'' := \bigcup_{i\in I} D''(\mathbb{E}_i)$ ;
   - $K := \bigcup_{i\in I} j_i^{-1}(\bar{U}_i \times (\bar{B}_i \setminus B_i''))$.
   - When the family $(\mathbb{E}_i)_{i\in I}$ is convenient for a finite type $n$-cycle $\hat{X}_0$, $\mathcal{K}$ will be a compact neighbourhood of $K$ disjoint from $\hat{X}_0$.
3. For $\tilde{X} \in \prod_{i\in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ we shall denote by $X^i$ the closed cycle in $D(\mathbb{E}_i)$ associated to the $i$-th component of $\tilde{X}$.

**Lemma 3.9.** *Let $(\mathbb{E}_i)_{i\in I}$ be a finite family of adjusted $n$-scales on $Z$, convenient for a $n$-cycle $\hat{X}_0$ of finite type in $Z$, and let $\mathcal{R}$ be some corresponding complete patching data. There exists an open neighbourhood $\mathcal{V}$ of the image $\tilde{X}_0$ of $\hat{X}_0$ in the product $\prod_{i\in I} \mathcal{G}_{k_i}(\mathbb{E}_i)$ such that for each $\tilde{X} \in \mathcal{V}$ we have the following properties:*

1. *No $X^i$ meets the compact set $\mathcal{K}$.*
2. *For each $i \in I$, any irreducible component of $X^i$ meeting $\overline{D''(\mathbb{E}_i)} \cap \overline{D''(\mathbb{E}_j)}$ with $j \neq i$, meets the open set $\bigcup_h D(F_{i,j,h})$.*
3. *For each $(i, j, h)$ the scale $F_{i,j,h}$ is adapted to $X^i$ and $X^j$.*
4. *For each $(i, j, h)$ we have $\deg_{F_{i,j,h}}(X^i) = \deg_{F_{i,j,h}}(X^j) = \deg_{F_{i,j,h}}(\hat{X}_0) = k_{i,j,h}$.*

*Proof.* Conditions 1, 3 and 4 are clearly open. An easy consequence of Condition 1, of the inclusion ii) of Definition 3.8 and of Lemma 3.2 is that Condition 2 is also open. □

For a $n$-scale $E := (U, B, j)$ on $Z$ we shall abbreviate $H(\bar{U}, \mathrm{Sym}^k(B))$ in $G_k(E)$.

When we consider a cycle $X_0$ in an adapted scale $E := (U, B, j)$ and when we dispose of a $n$-scale $F := (V, C, h)$ on $U \times B$, adapted to $X_0$, where $h$ is given by an isomorphism of an open set in $U \times B$ into some open neighbourhood of $\bar{V} \times \bar{C}$ in $\mathbb{C}^n \times \mathbb{C}^p$, we have a well-defined map of an open neighbourhood $\mathcal{U}$ of $X_0$ is $H(\bar{U}, \mathrm{Sym}^k(B))$ into $H(\bar{V}, \mathrm{Sym}^l(C))$, where $l := \deg_F(X_0)$, sending $X \in \mathcal{U}$ to the multiform graph associated to $h_*(X)$ in the scale $F$. This is a consequence of the fact that the condition $X_0 \cap h^{-1}(\bar{V} \times \partial C) = \emptyset$ is open in $H(\bar{U}, \mathrm{Sym}^k(B))$ and that the degree of $X$ near enough $X_0$ in the adapted scale $F$ will be equal to $l$.

Such a map, which will be called a **change of scale**, is not holomorphic in general but becomes holomorphic when we add the isotropy condition:

precisely, if $U' \subset\subset U$ and if $h^{-1}(\bar{V} \times \bar{C}) \subset U' \times B$, then the change of scale map

$$\Sigma_{U,U'} \to H(\bar{V}, \mathrm{Sym}^l(C))$$

will be holomorphic (see Theorem 4, p. 66 in [Bar75]).

**Definition 3.10.** Let $(\mathbb{E}_i)_{i\in I}$ a finite family of adjusted $n$-scales on $Z$, convenient for a $n$-cycle $\hat{X}_0$ of finite type in $Z$, and let $\mathcal{R}$ be some corresponding patching data. Let $k_i$ be the degree of $\hat{X}_0$ in the adjusted scale $\mathbb{E}_i$. For each $(i, j, h)$, $i \neq j$ we have a couple of holomorphic maps

$$\prod_{\alpha\in I} \mathcal{G}_{k_\alpha}(\mathbb{E}_\alpha) \rightrightarrows G_{k_{i,j,h}}(F_{i,j,h})$$

obtained by the changes of scales $\mathbb{E}_i \to F_{i,j,h}$ and $\mathbb{E}_j \to F_{i,j,h}$, because, by construction, we have $D(F_{i,j,h}) \subset\subset D'(\mathbb{E}_i) \cap D'(\mathbb{E}_j)$.

We shall denote by $S(\mathcal{R})$ the intersection of the kernels of these double maps[8] with the open set $\mathcal{V}$ built in Lemma 3.9. It is a Banach analytic set and we shall call it the **classifying space associated to** $(\mathbb{E}_i)_{i\in I}, \hat{X}_0$ **and** $\mathcal{R}$.

Remark that the patching data $\mathcal{R}$ are not assumed to be complete in the previous definition.

**Proposition 3.11.** *Consider a finite family of adjusted $n$-scales which is convenient for the $n$-cycle $X_0$ in $Z$ and let $\mathcal{R}$ be some* **complete** *patching data associated. Keeping the previous notations we have for each $(X^i)_{i\in I} \in S(\mathcal{R})$ a unique $n$-cycle $X \in \mathcal{C}_n^{\mathsf{f}}(W')$ such that $X \cap D'(\mathbb{E}_i) = X^i \cap D'(\mathbb{E}_i)$, $\forall i \in I$.*

*Moreover, this defines a tautological family of cycles in $W'$ which is a $\mathsf{f}$-analytic family of cycles satisfying the following "almost universal" property:*

> *For any analytic family of $n$-cycles $(X_s)_{s\in S}$ in an open neighbourhood of $\bar{W}$ parametrized by a Banach analytic set $S$[9] such that for $s_0 \in S$ we have $X_{s_0} = X_0$ in a neighbourhood of $\bar{W}$, there exists an open neighbourhood $S'$ of $s_0$ in $S$ and a unique holomorphic map*
>
> $$f : S' \to S(\mathcal{R})$$
>
> *such that the pull-back by $f$ of the tautological family parametrized by $S(\mathcal{R})$ is the restriction to the open set $W'$ of the family $(X_s)_{s\in S'}$.*

*Proof.* For each $\tilde{X} \in \mathcal{V}$ any $X^i$ does not meet $\mathcal{K}$. As $\mathcal{R}$ is complete, we may use Lemma 3.4 with $\mathcal{U}_i := D'(\mathbb{E}_i)$ and $W_{i,j} = \bigcup_h D(F_{i,j,h})$ to associate to $\tilde{X}$ a finite type $n$-cycle $X$ of the open set $W'$. The $\mathsf{f}$-analyticity of the so defined family is obvious. The "almost universal" property is then clear. □

Note that this proposition implies that the map $S(\mathcal{R}) \to \mathcal{C}_n^{\mathsf{f}}(W')$ classifying the tautological family of $n$-cycles in $W'$ is a holomorphic map.

## 3.C. Shrinkage.

**Definition 3.12.** Let $Z$ be a reduced complex space and let $\mathbb{E} := (U, U', U'', B, B'', j)$ be an adjusted $n$-scale on $Z$. For any real $\tau > 0$ small enough we shall denote by $\mathbb{E}^\tau$ the adjusted $n$-scale on $Z$ defined as $\mathbb{E}^\tau := (U^\tau, U'^\tau, U'', B, B'', j)$. We shall call $\mathbb{E}^\tau$ the $\tau$-shrinkage of $\mathbb{E}$.

Recall that for a polydisc $P$ of radius $R$, $P^\tau$ is the polydisc with same center and radius $R - \tau$. The definitions of $\mathcal{M}(\alpha)^\tau$, $M(\alpha)^\tau$ are given in the section 2.B.

**Remarks.**

1. By definition, the shrinkage of $\mathbb{E}$ does not change the embedding $j$ and the polydiscs $U'', B, B''$.
2. As $j$ is a closed embedding of an open set in $Z$ in an open neighbourhood of the compact set $\bar{U} \times \bar{B}$, it is clear that for any given adjusted $n$-scale $\mathbb{E}$ on $Z$, there exists a real $\varepsilon > 0$ (depending on $\mathbb{E}$) such that for any $\tau \in ]0, \varepsilon[$, $\mathbb{E}^\tau$ is again an adjusted $n$-scale on $Z$.
3. If $\mathbb{E}$ is an adjusted scale adapted to the $n$-cycle $X_0$ in $Z$, for $\tau$ small enough (depending on $\mathbb{E}$ and $X_0$), the adjusted $n$-scale $\mathbb{E}^\tau$ remains adapted to $X_0$ and we shall have also $\deg_{E^\tau}(X_0) = \deg_E(X_0)$.

---

[8] ↑ The kernel of a double map $f, g : A \to B$ is the pull-back of the diagonal in $B \times B$ by the map $(f, g) : A \to B \times B$.

[9] ↑ Recall that, by definition, this means that for any $n$-scale $E := (U, B, j)$ on $Z$, adapted to some $X_{s_0}$, $s_0 \in S$, with $\deg_E(X_{s_0}) = k$, we have an open neighbourhood $S_0$ of $s_0$ in $S$ and a classifying map for the corresponding family of branched coverings $f : S_0 \times U \to \mathrm{Sym}^k(B)$ which is **isotropic**.

4. If $\mathbb{E}$ is an adjusted scale adapted to the $n$-cycle $X_0$ in $Z$, there exists an open neighbourhood $\mathcal{U}$ of $X_0$ in $\mathcal{C}_n^{loc}(Z)$ and a real $\varepsilon > 0$ such that for any $X \in \mathcal{U}$ and any $\tau \in ]0, \varepsilon[$, the adjusted scale $\mathbb{E}^\tau$ remains adapted to $X$ with again $\deg_{E^\tau}(X) = \deg_E(X_0)$.

5. In the situation of the previous proposition 3.11, we may, for $\tau > 0$ small enough, keep the same patching data $\mathcal{R}$ on the finite family $(\mathbb{E}_i^\tau)_{i \in I}$ of adjusted $n$-scales; if it was complete, it remains complete and if it was convenient[10] for the $n$-cycle $X_0$, it remains convenient for the $n$-cycle $X_0$. Then we have a holomorphic restriction map
$$S(\mathcal{R}) \to S^\tau(\mathcal{R})$$
where $S^\tau(\mathcal{R})$ is the classifying space associated to the family $(\mathbb{E}^\tau)_{i \in I}$, and this map is induced by a **finite product of linear (continuous) compact maps**. This last point is crucial for the finiteness results.

## 3.D. Excellent family.

In order to avoid that our notations become too heavy we shall introduce the following conventions when $\mathcal{H}$ is a $n$-Hartogs figure in $\mathbb{C}^{n+p}$ ; we shall put, using the notations introduced above
$$U := M(\alpha) \times V, \quad U' := M(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad U'' := M(\alpha)^{\varepsilon''} \times V^{\varepsilon''}, \quad B'' := B^{\varepsilon'}$$
where $\varepsilon' > 0$ is small enough, and where $0 < \varepsilon'' < \varepsilon'$. The choices of $\varepsilon'$ and $\varepsilon''$ will be precised when they are useful. We shall associate to $\mathcal{H}$ the adjusted $n$-scale on $\Delta$ given by:
$$\mathbb{E}_{\mathcal{H}} := (U, U', U'', B, B'', j).$$
We shall put also
$$\tilde{U} := \mathcal{M}(\alpha) \times V, \quad \tilde{U}' := \mathcal{M}(\alpha)^{\varepsilon'} \times V^{\varepsilon'}, \quad \tilde{U}'' := \mathcal{M}(\alpha)^{\varepsilon''} \times V^{\varepsilon''}$$
the holomorphy envelopes respectively of $U$, $U'$ and $U''$. Then we shall have a family of adjusted $n$-scales on $\Delta'$, written down:
$$\mathbb{E}^\eta_{\tilde{\mathcal{H}}} := (\tilde{U}^\eta, \tilde{U'}^\eta, \tilde{U}'', B, B'', j) \quad \text{with} \quad \tilde{U}^\eta := \mathcal{M}(\alpha)^\eta \times V^\eta, \ \tilde{U'}^\eta := \mathcal{M}(\alpha)^{\varepsilon'+\eta} \times V^{\varepsilon'+\eta},$$
where $\eta$ is a non negative real number, small enough (for $\eta = 0$ we shall simply write $\mathbb{E}_{\tilde{\mathcal{H}}}$).

Then we shall have the isotropic classifying spaces
$$\mathcal{G}_k(\mathcal{H}) := \Sigma_{U,U'}(k) \quad \text{and also} \quad \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) := \Sigma_{\tilde{U}^\eta, \tilde{U'}^\eta}(k).$$
Proposition 2.6 gives a holomorphic analytic extension map $prlgt^\eta$ such that the following diagram commutes:
$$\begin{array}{ccc} \mathcal{G}_k(\tilde{\mathcal{H}}) & \xrightarrow{res^\eta} & \mathcal{G}_k^\eta(\tilde{\mathcal{H}}) \\ {\scriptstyle res}\downarrow & \nearrow{\scriptstyle prlgt^\eta} & \\ \mathcal{G}_k(\mathcal{H}) & & \end{array}$$

**Definition 3.13.** Let $Z$ be a reduced complex space of pure dimension $n + p$, let $\Delta \subset\subset Z$ be a strongly $(n-2)$-concave open set in $Z$ and $\tilde{X}_0$ a $n$-cycle in an open neighbourhood $\Delta'$ of $\bar{\Delta}$ in $Z$. We shall say that a finite family $(\mathcal{H}_a)_{a \in A}$ of $n$-Hartogs figures relative to the boundary of $\Delta$ is **excellent** for the cycle $\tilde{X}_0$ when the following conditions hold, where we write $\tilde{\mathbb{E}}_a$ and $\mathbb{E}_a$ the adjusted scales respectively on $\Delta'$ and $\Delta$ associated to the $n$-Hartogs figure $\mathcal{H}_a$ :

[10] ↑ See the notations following Definition 3.8.

1. The adjusted scales $\tilde{\mathbb{E}}_a$ and $\mathbb{E}_a$ are adapted to the cycle $\tilde{X}_0$.

2. We may choose the patching domains of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a\in A}$ in order that the union

$$\tilde{D}''(A) := \bigcup_{a\in A} j_a^{-1}(\tilde{U''}_a \times B''_a)$$

contains the compact set $\partial\Delta$.

3. There exists a finite family of adjusted $n$-scales $(\mathbb{E}_b)_{b\in B}$ on $\Delta$, adapted to $\tilde{X}_0$, such that the finite families $(\mathbb{E}_c)_{c\in A\cup B}$ and $(\tilde{\mathbb{E}}_c)_{c\in A\cup B}$ are convenient for $\tilde{X}_0$, where we put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$.

   Moreover we ask that the union $D''(B)$ of the patching domains of the $(\mathbb{E}_b)_{b\in B}$ covers the compact set $\Delta \setminus \tilde{D}''(A)$ of $\Delta$; so $\tilde{D}''(A) \cup D''(B)$ in an open set containing $\bar{\Delta}$.

As a consequence, the union $\tilde{D}'(A)^\eta \cup D'(B)$ of the isotropy domains will cover $\bar{\Delta}$ for $\eta > 0$ small enough.

**Proposition 3.14. (Existence of excellent families)** *Let $Z$ be a reduced complex space of pure dimension $n+p$, $\Delta \subset\subset Z$ be a strongly $(n-2)$-concave open set with smooth boundary and $\tilde{X}_0$ a $n$-cycle in an open neighbourhood $\Delta' \subset Z$ of $\bar{\Delta}$ such that any irreducible component of $\tilde{X}_0$ meets $\Delta$.*

*Then there exists a finite family $(\mathcal{H}_a)_{a\in A}$ of $n$-Hartogs figures relative to the boundary of $\Delta$ which is excellent for the cycle $\tilde{X}_0$.*

*Proof.* First we use Proposition 2.13 to cover $\partial\Delta$ by a finite family of $n$-Hartogs figures relative to the boundary of $\Delta$ adapted to the cycle $\tilde{X}_0$ such that Conditions 1 and 2 hold. Then we build a finite family of adjusted $n$-scales $(\mathbb{E}_b)_{b\in B}$ on $\Delta$, adapted to $\tilde{X}_0$ in order that Condition 3 holds. □

## 3.E. The extension and finiteness theorem.

The next theorem will be crucial in the proof of Theorem 1.1.

**Theorem 3.15.** *Let $Z$ be a reduced complex space of pure dimension $n+p$, where $n \geq 2$, $p \geq 1$. Assume that there exists a $\mathscr{C}^2$ exhaustion $\varphi : Z \to ]0,2]$ which is strongly $(n-2)$-convex on the open set $\varphi^{-1}(]0,1[)$ and let $\Delta := \{x \in Z \ / \ \varphi(x) > \alpha\}$ for some $\alpha \in ]0,1[$ which is not a critical value for $\varphi$. Let $\tilde{X}_0$ a closed $n$-cycle of an open neighbourhood $\Delta'$ of $\bar{\Delta}$ in $Z$ such that any irreducible component of $\tilde{X}_0$ meets $\Delta$. Then there exists an open neighbourhood $\Delta''$ of $\bar{\Delta}$ in $\Delta'$ and a $\mathsf{f}$-analytic family $(\tilde{X}_\xi)_{\xi\in\Xi}$ of $n$-cycles in $\Delta''$ parametrized by a reduced complex space $\Xi$ (of finite dimension) such that $X_{\xi_0} = \tilde{X}_0 \cap \Delta''$ and such that $\Xi$ is isomorphic to an open neighbourhood of $\tilde{X}_0 \cap \Delta''$ in $\mathcal{C}_n^{\mathsf{f}}(\Delta'')$. It satisfies the following universal property:*

> *For any $\mathsf{f}$-analytic family $(X_s)_{s\in S}$ of $n$-cycles in $\Delta$ parametrized by a Banach analytic set $S$ and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$, there exists an open neighbourhood $S'$ of $s_0$ in $S$ and an unique holomorphic map*
>
> $$h : S' \to \Xi$$
>
> *satisfying the equality $X_s = \tilde{X}_{h(s)} \cap \Delta$ for each $s \in S'$.*

*Proof.* Note that $\Delta$ is a relatively compact open set in $Z$ with a $\mathscr{C}^2$ boundary which is strongly $(n-2)$-concave.

Begin by covering the compact set $\partial\Delta$ by an excellent finite family $(\mathcal{H}_a)_{a\in A}$ of $n$-Hartogs figures for $\Delta$ adapted to the cycle $\tilde{X}_0$. Choose then open sets $\Delta_1 \subset\subset \Delta \subset\subset \Delta'' \subset\subset \Delta'$, such that the following properties hold, where we use the notations introduced above for the finite family of the adjusted scales $(\tilde{\mathbb{E}}_a)_{a\in A}$ associated to $(\mathcal{H}_a)_{a\in A}$:

i) $\bar{\Delta}'' \setminus \Delta_1 \subset \bigcup_{a\in A} W''(\tilde{\mathbb{E}}_a)$.

ii) $\bigcup_{a\in A} W(\tilde{\mathbb{E}}_a) \subset\subset \Delta'$.

iii) $\bigcup_{a\in A} \bar{W}(\mathbb{E}_a) \subset \Delta$.

iv) $K := \bigcup_{a\in A} j_a^{-1}(\bar{U}_a \times (\bar{B}_a \setminus B''_a)) \subset \Delta_1$

It is easy to fulfill these conditions for $\Delta_1$ and $\Delta''$ near enough to $\Delta$, as, by assumption, the subsets $\bar{W}(\mathbb{E}_a)$ and $j_a^{-1}(\bar{U}_a \times \bar{B}_a \setminus B''_a)$ are compact in $\Delta$, and as the union of the open sets $W''(\tilde{\mathbb{E}}_a)$ contains $\partial\Delta$.

Note that Condition iv) allows to choose the compact neighbourhood $\mathcal{K}$ of $K$ inside $\Delta_1$.

Choose now a convenient finite family $(\mathbb{E}_b)_{b\in B}$ of adjusted $n$-scales on $\Delta$, adapted to $\tilde{X}_0$ in order that the open set $\bigcup_{b\in B} W''(\mathbb{E}_b)$ contains $\bar{\Delta}_1$.

Put $\tilde{\mathbb{E}}_b = \mathbb{E}_b$ for $b \in B$, and define $C := A \cup B$. The family of adjusted scales $(\mathbb{E}_c)_{c\in C}$ in $\Delta$ is then convenient for $\tilde{X}_0 \cap \Delta_1$, up to choosing the patching domains big enough. Fix some complete patching data $\mathcal{R}$ associated to the family $(\mathbb{E}_c)_{c\in C}$.

The family $(\tilde{\mathbb{E}}_c)_{c\in C}$ of adjusted scales in $\Delta'$ is convenient for $\tilde{X}_0 \cap \Delta''$ if we choose the patching domains big enough. Consider now some complete patching data for this family of the form $\mathcal{R} \cup \mathcal{R}''$, that is to say containing the patching scales already in $\mathcal{R}$. Define the following Banach analytic sets:

1. $S_0$ is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c\in C}$, the degrees being these of $\tilde{X}_0$ in the various scales adapted to the cycle $\tilde{X}_0$, with the patching conditions defined by $\mathcal{R}$. Note that $\mathcal{R}$ is not complete in general.

2. $S_+$ is the classifying space of the family $(\tilde{\mathbb{E}}_c)_{c\in C}$ with the patching conditions defined by $\mathcal{R}\cup\mathcal{R}''$.

3. $S_-$ is the classifying space of the family $(\mathbb{E}_c)_{c\in C}$, with the (complete) patching conditions defined by $\mathcal{R}$.

Then we get a holomorphic extension map

$$\alpha : S_- \to S_0$$

deduced from the extension maps in the $n$-Hartogs figures $(\mathcal{H}_a)_{a\in A}$.

By definition $S_+$ is a closed Banach analytic subset of $S_0$ as it is defined in $S_0$ by the patching conditions given by $\mathcal{R}''$. Then put $\Xi := \alpha^{-1}(S_+)$. So we have a holomorphic map $\alpha : \Xi \to S_+$. We want to show the following claim:

**Claim.** There exists a holomorphic map $\beta : S_+ \to \Xi$ satisfying the two properties:

1. We have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ in a neighbourhood of the points defined by $\tilde{X}_0$ respectively in $S_+$ and $\Xi$.

2. The holomorphic map $\beta$ is the composition of a holomorphic map induced by a linear (continuous) compact map and a holomorphic map.

As the open set $\bigcup_{c\in C} W''(\tilde{\mathbb{E}}_c)$ contains $\Delta''$, there is, on $S_+$, a tautological family of $n$-cycles which is f-analytic on $\Delta''$. Then the "almost universal" property of $S_-$ gives a holomorphic map $\tilde{\beta} : S_+ \to S_-$ which factorizes via the closed Banach analytic subset $\Xi \subset S_-$. Let us show that the holomorphic map $\beta : S_+ \to \Xi$ deduced from this factorization satisfies the two properties of the claim.

First we have $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$ respectively in $S_+$ and $\Xi$, by definition of $\Xi$ and $\beta$ (see Proposition 2.6).

To see the second property, consider $\tau > 0$ small enough and let us show that the holomorphic map induced by the linear compact (by Vitali's theorem) restriction $r^\tau : S_+ \to S_+^\tau$ factorizes $\beta$, where

$S_+^\tau$ is the classifying space corresponding to the $\tau$-shrinkage $(\tilde{\mathbb{E}}_c^\tau)_{c\in C}$ of the family $(\tilde{\mathbb{E}}_c)_{c\in C}$ with the patching data $\mathcal{R}\cup\mathcal{R}''$. Indeed, for $\tau$ small enough, the tautological family of $S_+^\tau$ is still f-analytic on $\Delta$ and the "almost universal" property of $S_-$ gives again a holomorphic map $\tilde{\beta}^\tau : S_+^\tau \to S_-$. And we have $\beta = \tilde{\beta}^\tau \circ r^\tau$ proving our assertion. In fact, the map $\tilde{\beta}^\tau$ takes its values in $\Xi$ because the $\tau$-shrinkage does not change the patching data deduced from $\mathcal{R}\cup\mathcal{R}''$ for $\tau$ small enough.

We conclude that the Banach analytic set $\Xi$ has finite dimension thanks to the finiteness lemma of [Bar75, p. 8] (see also [BM, Chapter V]). Moreover, it is isomorphic to $S_+$ which is also of finite dimension.

The holomorphic isomorphism $\beta : S_+ \to \Xi$ factorizes via an open neighbourhood of $X_0 \cap \Delta''$ in $\mathcal{C}_n^{\mathsf{f}}(\Delta'')$ because $S_+$ parametrizes a f-analytic family of cycles in $\Delta''$, and because a cycle $X$ near enough to $X_0\cap\Delta''$ defines an element in $\Xi \subset S_-$ as it satisfies the patching conditions $\mathcal{R}\cup\mathcal{R}''$. This shows that we can identify the Banach analytic set $\Xi$ (which is a reduced finite dimensional complex space) with an open neighbourhood of $X_0\cap\Delta''$ in $\mathcal{C}_n^{\mathsf{f}}(\Delta'')$.

The universal property is obvious. □

Note that it is not restrictive to choose $\Delta'' := \{\varphi > \alpha_1\}$ for some $\alpha_1 \in ]0,\alpha[$, near enough to $\alpha$.

**Remarks.**

1. The reduced complex space (of finite dimension) $\Xi$ built in the previous theorem parametrizes a f-analytic family of $n$-cycles in the open set $\Delta''$ which is an open neighbourhood of $\bar{\Delta}$. So, when we have a f-analytic family $(X_s)_{s\in S}$ of $n$-cycles in $\Delta$ parametrized by a Banach analytic set $S$ and such that its value at some $s_0 \in S$ is equal to $\tilde{X}_0 \cap \Delta$ we can extend each cycle $X_s$, $s \in S'$, to a $n$-cycle $\tilde{X}_s$ in $\Delta''$ in order that the family $(\tilde{X}_s)_{s\in S'}$ is f-analytic in $\Delta''$, with the condition that each irreducible component of $\tilde{X}_s$ meets $\Delta$ and with the equality $\tilde{X}_s \cap \Delta = X_s$ for each $s \in S'$.

2. We shall see later on that $\Xi$ is also (isomorphic to) an open neighbourhood of the point $\tilde{X}_0 \cap \Delta$ in $\mathcal{C}_n^{\mathsf{f}}(\Delta)$.

# 4. Finiteness of the space of $n$-cycles of a reduced strongly $(n-2)$-concave space $(n \geq 2)$.

## 4.A. The global extension theorem.

First we have to complete our terminology.

**Definition 4.1.** We shall say that a reduced complex space $Z$ is **strongly $q$-concave**, where $q \geq 0$ is a natural integer, if there exists a real valued $\mathscr{C}^2$ exhaustion function on $Z$, $\varphi : Z \to ]0,2]$, which is strongly $q$-convex outside the compact set $K := \varphi^{-1}([1,2])$.

In the sequel, when we consider a reduced complex space $Z$ which is assumed to be $q$-concave, we shall always assume implicitly that we have chosen such an exhaustion $\varphi$. For instance, any reduced compact complex space is strongly $q$-concave for any $q \geq 0$.

When $Z$ is strongly $q$-concave irreducible and non compact of dimension at least $q+1$, the function $\varphi$ achieves its maximum at a point in $K$. So we shall have $\varphi(Z) = ]0,u]$ with $u \geq 1$.

**Theorem 4.2.** *Let $n \geq 2$ be an integer. Let $Z$ be a reduced complex space which is strongly $(n-2)$-concave. Let $\alpha \in ]0,1[$ and let $X$ be a finite type $n$-cycle in the open set $Z_\alpha := \{z \in Z \ / \ \varphi(z) > \alpha\}$. Then there exists a unique $n$-cycle $\tilde{X}$ in $\mathcal{C}_n^{\mathsf{f}}(Z)$ such that $\tilde{X} \cap Z_\alpha = X$.*

The proof will use the following remark.

**Remark.** Consider a closed irreducible analytic subset $\Gamma$ of dimension $n$ in $Z$. As the restriction $\varphi_{|\Gamma}$ of the exhaustion $\varphi$ to $\Gamma$ must reach its maximum (as $\varphi$ is continuous and proper), this maximum cannot be obtained at a point in which $\varphi$ is strongly $(n-2)$-convex. So we have $\Gamma \cap \varphi^{-1}[1,2]) \neq \emptyset$.

*Proof.* The uniqueness of $\tilde{X}$ is a consequence of the previous remark: if $\tilde{X}_i$, $i = 1, 2$, are in $\mathcal{C}_n^{\mathsf{f}}(Z)$ and satisfy $\tilde{X}_i \cap Z_\alpha = X$, then any irreducible component $\Gamma_1$ of $\tilde{X}_1$ has to meet $Z_\alpha$ and so has to contain an open set in $\tilde{X}_2$. Then it has to be an irreducible component of $\tilde{X}_2$ and its multiplicity in $\tilde{X}_1$ and $\tilde{X}_2$ must coincide. So $\tilde{X}_1 = \tilde{X}_2$.

To show that this cycle exists, consider first the case where $X$ is compact in $Z_\alpha$. Then $\tilde{X} := X$ is a solution. So it enough to consider the case where $X$ is irreducible and non compact. Thanks to [ST71, Theorem 8.3], for each $z \in \partial Z_\alpha$ there exists an open set $U_z$ and an unique closed analytic set $X_z$ in $Z_\alpha \cup U_z$ of pure dimension $n$ such that $X_z \cap Z_\alpha = X$. Choose a finite set of points $z_1, \dots, z_N$ and open sets $U'_i \subset\subset U_i := U_{z_i}$ such that the union of the $U'_i$ covers the compact set $\partial Z_\alpha$. Let $\Omega := Z_\alpha \cup \big(\bigcup_{i=1}^N U'_i\big)$ and put

$$X_1 := (X \cup (\bigcup_{i=1}^{N} X_{z_i})) \cap \Omega.$$

Let us show that $X_1$ is a closed analytic subset in $\Omega$. Consider $z \in \Omega$. If $z$ is in $Z_\alpha$ we have $X_1 = X$ in a neighbourhood of $z$ and the assertion is clear. If not, either $z$ is not in any $\partial U'_i$ and $X_1$ is the union of the $X_{z_i}$ in a neighbourhood of $z$ and the assertion is clear, or $z$ is in $\partial U'_{j_1}, \dots, \partial U'_{j_k}$ for $j_1, \dots, j_k$ in $[1, N]$. As the set $X_{j_h}$ is closed and analytic in $Z_\alpha \cup U_{j_h}$, then $X_1$ is again the union of the $X_{z_i}$ near $z$ in $\Omega$, and the assertion is proved.

So in this situation there exists a real positive $\beta < \alpha$ such that $Z_\beta \subset \Omega$. Let $X_2$ be the irreducible component of $X_1 \cap Z_\beta$ which contains $X$; then $X_2$ is a closed irreducible analytic subset of $Z_\beta$ such that $X_2 \cap Z_\alpha = X$.

Now let

$$\gamma := \inf\{\beta \leq \alpha \ / \ \exists X_\beta \text{ irreducible } n\text{-cycle of } Z_\beta \text{ such that } X_\beta \cap Z_\alpha = X\}.$$

Then what we obtained above shows that we have $\gamma < \alpha$, and, applying the same arguments to the cycle $X_\gamma$ defined on $Z_\gamma$ via the cover of $Z_\gamma$ by the $Z_\beta, \beta > \gamma$ in which we already built an irreducible $n$-cycle $X_\beta$ extending $X$, we conclude that $\gamma = 0$ and that there exists an (unique) irreducible $n$-cycle $\tilde{X}$ in $Z$ extending $X$. □

## 4.B. Some consequences.

We shall give first some easy consequences of the fact that the reduced complex space $Z$ is strongly $n$-concave.

**Proposition 4.3.** *Let $n \geq 2$ be an integer and let $Z$ be a reduced complex space which is strongly $(n-2)$-concave. Then the natural map $j : \mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{loc}(Z)$ is a homeomorphism. Moreover, for each $\alpha \in ]0,1[$ the restriction map*

$$res_\alpha : \mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{\mathsf{f}}(Z_\alpha)$$

*is well defined and is also a homeomorphism.*

*Proof.* Let us prove first that any $n$-cycle $X$ in $Z$ has finitely many irreducible components. As this implies the same result for each $Z_\alpha$ for $\alpha \in ]0,1[$, this will imply the fact that the restriction map $res_\alpha$ is well defined, and then bijective as a consequence of Theorem 4.2.

As the family of irreducible components of a $n$-cycle is locally finite, only finitely many irreducible components of $X$ can meet the compact set $K := \varphi^{-1}([1,2])$. But we have seen in the remark following the previous theorem that **any** irreducible component of $X$ must meet $K$. So $X$ is a finite type cycle.

To show the continuity of $res_\alpha^{-1}$ it is then enough to prove that $j$ is a homeomorphism which is an easy consequence of the lemma below.

**Lemma 4.4.** *Let $Z$ be a reduced complex space and let $(X_\nu)_{\nu\geq 0}$ be a sequence of $n$-cycles in $Z$ converging in $\mathcal{C}_n^{loc}(Z)$ to a cycle $Y$. Assume that there exists a relatively compact open set $\Omega$ in $Z$ such that any irreducible component of each $X_\nu$ and of $Y$ meets $\Omega$. Then all these cycles are of finite type and the sequence converges to $Y$ in $\mathcal{C}_n^{\mathsf{f}}(Z)$.*

*Proof of Lemma 4.4.* First if $Y = \emptyset$, for $\nu \gg 1$ the cycle $X_\nu$ will be disjoint from the compact set $K := \bar{\Omega}$, and this implies that $X_\nu$ is the empty cycle. So the conclusion holds in this case. If $Y$ is not empty, let $U$ be a relatively compact open set in $Z$ meeting each irreducible component of $Y$. We have to show, by definition of the topology of $\mathcal{C}_n^{\mathsf{f}}(Z)$, that for $\nu \gg 1$ each irreducible component of $X_\nu$ meets $U$. If it is not the case, passing to a subsequence, we may assume that for each $\nu$ there exists an irreducible component $\Gamma_\nu$ of $X_\nu$ disjoint from $U$. As, passing again to a subsequence, we may assume that the sequence $(\Gamma_\nu)$ converges in $\mathcal{C}_n^{loc}(Z)$ to a cycle $\Gamma$, we shall have $|\Gamma| \subset |Y|$ and $|\Gamma| \cap U = \emptyset$. To conclude, it is enough to show that $\Gamma$ is not the empty cycle, as any irreducible component of $\Gamma$ is also an irreducible component of $Y$ and then meets $U$ by hypothesis. As each $\Gamma_\nu$ is not empty, it has to meet $K = \bar{\Omega}$. This implies that $\Gamma$ also meets $K$ and so is not empty. This contradicts our assumption. □

*End of the proof of Proposition 4.3.* We have proved that $j$, and then also each $j_\alpha : \mathcal{C}_n^{\mathsf{f}}(Z_\alpha) \to \mathcal{C}_n^{loc}(Z_\alpha)$ for $\alpha \in ]0,1[$, is a holomorphic homeomorphism. To conlude the proof we have to show the continuity of $res_\alpha^{-1}$, and this reduces to prove that if the sequence $(X_\nu)$ of $\mathcal{C}_n^{loc}(Z)$ is such that the sequence $(X_\nu \cap Z_\alpha)$ converges in $\mathcal{C}_n^{loc}(Z_\alpha)$, then it converges in $\mathcal{C}_n^{loc}(Z)$. Let $Y_\alpha \in \mathcal{C}_n^{loc}(Z_\alpha)$ be the limit of this sequence in $\mathcal{C}_n^{loc}(Z_\alpha)$ and let $Y \in \mathcal{C}_n^{loc}(Z)$ be the cycle extending it. Let $A$ be the set of $\beta \in ]0,\alpha]$ such that the sequence $(X_\nu \cap Z_\beta)$ converges in $\mathcal{C}_n^{loc}(Z_\beta)$ to $Y \cap Z_\beta$. Then $\alpha$ is in $A$ so $A$ is not empty. Put $\gamma := \inf A$. Theorem 3.15 implies that $\gamma = 0$ and we obtain also the convergence in any $\mathcal{C}_n^{loc}(Z_\beta)$, for any $\beta > 0$; this gives the convergence in $\mathcal{C}_n^{loc}(Z)$, as, by definition, a $n$-scale on $Z$ is also a $n$-scale on $Z_\beta$ for $\beta > 0$ small enough. □

## 4.C. An analytic extension criterion.

The aim of this paragraph is to prove the following analytic extension result.

**Theorem 4.5.** *Let $Z$ be a complex space and $n$ an integer. Consider a $\mathsf{f}$-continuous*[11] *family $(X_s)_{s\in S}$ of $n$-cycles of finite type in $Z$ parametrized by a reduced complex space $S$. Fix a point $s_0$ in $S$ and assume that the open set $Z'$ in $Z$ meets all irreducible components of $X_{s_0}$ and such that the family of cycles $(X_s \cap Z')_{s\in S}$ is analytic at $s_0$. Then there exists an open neighbourhood $S_0$ of $s_0$ in $S$ such that the family $(X_s)_{s\in S_0}$ is $\mathsf{f}$-analytic.*

The hypotheses translated in terms of classifying maps means that we have a continuous map $\varphi : S \to \mathcal{C}_n^{\mathsf{f}}(Z)$ such that the composed map $r \circ \varphi$ is holomorphic at $s_0$, where $r : \mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{loc}(Z')$ is the restriction map.

Then the theorem says that there exists an open neighbourhood $S_0$ of $s_0$ in $S$ such that the map $\varphi$ is holomorphic on $S_0$. Note that, as $r$ is holomorphic[12], the hypothesis that $\varphi$ is holomorphic at $s_0$ is a necessary condition.

This result is not true in general if we take for $S$ a non smooth Banach analytic set which is not of finite dimension (locally). The reader may find a counter-exemple with an isolated singularity in [BM, Chapter V].

The key point for the proof of the previous theorem is the following analytic extension result.

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[11] ↑ This means that we have a continuous family of finite type $n$-cycles such that its graph is quasi-proper over $S$. This is equivalent to the continuity of the classifying map $\varphi : S \to \mathcal{C}_n^{\mathsf{f}}(Z)$ of this family.

[12] ↑ in the sense that for any holomorphic map $\psi : T \to \mathcal{C}_n^{\mathsf{f}}(Z)$ of a reduced complex space $T$ the composed map $r \circ \psi$ is holomorphic.

**Proposition 4.6.** *Let $S$ a reduced complex space and let $\emptyset \neq U_1 \subset U_2$ be two polydiscs in $\mathbb{C}^n$. Let $f : S \times U_2 \to \mathbb{C}$ a continuous function, holomorphic on $\{s\} \times U_2$ for each $s \in S$. Assume moreover that the restriction of $f$ to $S \times U_1$ is holomorphic. Then $f$ is holomorphic on $S \times U_2$.*

*Proof of Proposition 4.6.* First consider the case $S$ smooth. As the question is local on $S$ it is enough to consider the case where $S$ is an open set in $\mathbb{C}^m$. Fix an open relatively compact polydisc $P$ in $S$. The function $f$ defines a map $F : U_2 \to \mathscr{C}^0(\bar{P}, \mathbb{C})$, where we write down $\mathscr{C}^0(\bar{P}, \mathbb{C})$ the Banach space of continuous functions on $\bar{P}$, via the formula $F(t)[s] = f(s,t)$ for $t \in U_2$ and $s \in \bar{P}$. First we shall show that the map $F$ is holomorphic.

Let $U \subset\subset U_2$ be a polydisc. For any fix $s \in S$ we have

$$\frac{\partial f}{\partial t_i}(s,t) = \frac{1}{(2i\pi)^n}.\int_{\partial\partial U} f(s,\tau).\frac{d\tau_1 \wedge \dots \wedge d\tau_n}{(\tau_1 - t_1)\dots(\tau_i - t_i)^2 \dots (\tau_n - t_n)} \quad \forall t \in U \quad \forall i \in [1,n].$$

where $t := (t_1, \dots, t_n)$ are coordinates on $\mathbb{C}^n$. This Cauchy formula shows that $F$ is $\mathbb{C}$-differentiable and its differential at the point $t \in U$ is given by

$$h \mapsto \sum_{i=1}^{n} F_i(t).h_i, \quad h \in \mathbb{C}^n,$$

where $F_i$ is associated to the function

$$(s,t) \mapsto \frac{\partial f}{\partial t_i}(s,t) \quad i \in [1,n]$$

which is holomorphic for each fixed $s \in S$ thanks to the Cauchy formula above.

Let $H(\bar{P}, \mathbb{C})$ the closed subspace of $\mathscr{C}^0(\bar{P}, \mathbb{C})$ of functions which are holomorphic on $P$. Our hypothesis implies that the restriction of $F$ to the non empty open set $U_1$ takes its values in this subspace. Let us show that this is also true on $U_2$. Assume that there exists $t_0 \in U_2$ such that $F(t_0)$ is not in $H(\bar{P}, \mathbb{C})$. Thanks to the *Hahn-Banach theorem* we can find a continuous linear form $\lambda$ on $\mathscr{C}^0(\bar{P}, \mathbb{C})$, vanishing on $H(\bar{P}, \mathbb{C})$, and such that $\lambda(F(t_0)) \neq 0$. But the function $t \mapsto \lambda(F(t))$ is holomorphic on $U_2$ and vanishes on $U_1$; this contradicts $\lambda(F(t_0)) \neq 0$. So $F$ takes values in $H(\bar{P}, \mathbb{C})$ and $f$ is holomorphic on $S \times U_2$ when $S$ is a complex manifold.

The case where $S$ is a weakly normal complex space follows immediately.

When $S$ is a general reduced complex space, the function $f$ is meromorphic and continuous on $S \times U_2$ and holomorphic on $S \times U_1$. So the closed analytic subset $Y \subset S \times U_2$ along which $f$ may not be holomorphic has no interior point in each $\{s\} \times U_2$. The analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7] allows to conclude. □

*Proof of Theorem 4.5.* Let $|G| \subset S \times Z$ be the graph of the $f$-continuous family $(X_s)_{s \in S}$ and let $A$ be the set of points in $(\sigma, \zeta) \in |G|$ admitting an open neighbourhood $S_\sigma \times Z_\zeta$ in $S \times Z$ such that the family of cycles $(X_s \cap Z_\zeta)_{s \in S_\sigma}$ is analytic. Remark that, thanks to our hypothesis, the open set $A$ in $|G|$ meets every irreducible component of $\{s_0\} \times |X_{s_0}|$.

Assume to begin that there exists a smooth point $z_0$ of $|X_{s_0}|$ in the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$. Choose a $n$-scale $E := (U, B, j)$ on $Z$ which is adapted to $X_{s_0}$ and satisfying :

$$\deg_E(|X_{s_0}|) = 1, \quad j_*(X_{s_0}) = k.(U \times \{0\})$$
$$z_0 \in j^{-1}(U \times B), \quad j(z_0) := (t_0, 0).$$

It is clear that such a $n$-scale exists as $z_0$ is a smooth point in $|X_{s_0}|$. Let $S_1$ be a sufficiently small open neighbourhood of $s_0$ in $S$ and let $f : S_1 \times U \to \mathrm{Sym}^k(B)$ be the (continuous) classifying map for the family $(X_s)_{s \in S_1}$ in the scale $E$. As $j^{-1}(U \times \{0\})$ meets $A$, there exists a non empty polydisc $U_2 \subset U$ such that $U_2 \times \{0\}$ is contained in $A$. Then we may apply Proposition 4.6 to each scalar component of $f$ in order to obtain that $f$ is holomorphic on $S_1 \times U$. Moreover, as the same argument applies to any linear projection of $U \times B$ to $U$ near enough the vertical one; this implies that $f$ is an isotropic map, up to shrinking slightly $U$. This contradicts the fact that the point $(s_0, z_0)$ is in the boundary of the open set $A \cap (\{s_0\} \times |X_{s_0}|)$ of $|X_{s_0}|$.

If the boundary of $A \cap (\{s_0\} \times |X_{s_0}|)$ is contained in the singular set of $|X_{s_0}|$, we may apply the analytic extension criterion of [BM14, Chapter IV, Criterion 3.1.7], and we obtain directly that $A$ contains $|X_{s_0}|$. So in any case the family of cycles $(X_s)_{s\in S}$ is analytic at $s_0$. As the graph $|G|$ is, by assumption, quasi-proper on $S$, it is enough to use the next proposition (which is proved in [Bar15, Proposition 2.2.3]) to conclude.

**Proposition 4.7.** *Let $Z$ and $S$ be reduced complex spaces and let $(X_s)_{s\in S}$ be a f-continuous family of $n$-cycles in $Z$. Assume that this family is analytic in $s_0 \in S$. Then there exists an open neighbourhood $S'$ of $s_0$ in $S$ such that the family $(X_s)_{s\in S'}$ is a $\mathsf{f}$-analytic family of $n$-cycles in $Z$.*

## 4.D. Proof of Theorem 1.2 and its corollary

We shall begin by a lemma which will give the case where the $n$-cycle $X_0$ is compact.

**Lemma 4.8.** *Let $Z$ be a strongly $(n-2)$-concave reduced complex space. Then $\mathcal{C}_n(Z)$ is open in $\mathcal{C}_n^{\mathsf{f}}(Z)$.*

*Proof.* Let $X_0$ be a compact cycle in $Z$. There exists $\alpha \in ]0,1[$ such that $X_0$ is contained in $Z_\alpha = \{x \in Z \ / \ \varphi(x) > \alpha\}$. So $X_0$ does not meet the compact set $\varphi^{-1}(\{\alpha\})$. This is an open condition in $\mathcal{C}_n^{\mathsf{f}}(Z)$. And as any irreducible $n$-dimensional analytic subset in $Z$ has to meet $K$, if it does not meet $\varphi^{-1}(\{\alpha\})$ it is contained in $Z_\alpha$ (by connectedness). Then any $X \in \mathcal{C}_n^{\mathsf{f}}(Z)$ which is near enough $X_0$ is contained in $Z_\alpha$ so is compact. $\square$

Note that under the hypothesis of the previous lemma, $\mathcal{C}_n(Z)$ is not closed in $\mathcal{C}_n^{\mathsf{f}}(Z)$ in general, as one can see taking $Z := \mathbb{P}_N \setminus \{0\}$ and considering the set of hyperplanes in $\mathbb{P}_N$.

As we already know from [Bar75] that $\mathcal{C}_n(Z)$ is a reduced complex space, Theorem 1.2 and its corollary are proved near a compact $n$-cycle in $Z$.

*The case where $X_0$ is not compact.* Of course we are in the case where $Z$ is not compact. Fix $X_0$ a non compact $n$-cycle in $\mathcal{C}_n^{\mathsf{f}}(Z)$ and choose an $\alpha \in ]0,1[$ which is not a critical value of $\varphi$ and of the restriction of $\varphi$ to $|X_0|$; this is possible thanks to Sard's Lemma 3.1 and the fact that $\varphi(|X_0|)$ contains $]0,1[$.

Consider now the reduced complex space $\Xi$ constructed in Theorem 3.15. It is an open neighbourhood of $X_0 \cap \Delta''$ in $\mathcal{C}_n^{\mathsf{f}}(\Delta'')$ and we may assume that $\Delta'' := Z_\beta$ for some $\beta < \alpha$ very near $\alpha$. Due to Proposition 4.3 it is homeomorphic to an open neighbourhood $\mathcal{V}$ of $X_0$ in $\mathcal{C}_n^{\mathsf{f}}(Z)$. So the restriction map

$$res_0 : \mathcal{V} \to \Xi$$

is holomorphic, bijective and is a homeomorphism. Now the continuity of $res_0^{-1}$ and the finiteness of $\Xi$ allow to apply Theorem 4.5 because we already know that the tautological family of cycles parametrized by $\Xi$ is $\mathsf{f}$-analytic on the open set $\Delta''$ which is an open set which meets any irreducible component of each cycle in this family (because $K \subset Z_\beta$). Then $res_0^{-1}$ is holomorphic and so $res_0$ is an isomorphism of Banach analytic sets. $\square$

### 4.E. A compactness criterion for the connected components of the reduced complex space $\mathcal{C}_n^{\mathsf{f}}(Z)$ when $Z$ is strongly $(n-2)$-concave

For a reduced complex space $Z$ which is compact, the compactness of the connected components of $\mathcal{C}_n(Z)$ is a consequence of the existence of a $\mathscr{C}^1$ $2n$-differential form on $Z$ which is $d$-closed and such that its $(n,n)$ part is positive definite in the Lelong sense. Indeed, this gives that the volume (computed with this $(n,n)$ part) of the $n$-cycles is constant on connected components. The result follows then from Bishop's theorem [Bis64] (see [BM14, Chapter IV] for details).

In the case of a non compact strongly $(n-2)$-concave reduced complex space $Z$ we have the following analogous result :

**Proposition 4.9.** *Let $Z$ be a reduced complex space which is strongly $(n-2)$-concave. Assume that there exists on $Z$ a $\mathscr{C}^1$ $2n$-differential form $\omega$ which is $d$-closed with compact support and such that its $(n,n)$ part is positive definite in the Lelong sense in a neighbourhood of $K := \varphi^{-1}([1,2])$, and everywhere non negative in the Lelong sense. Then the connected components of $\mathcal{C}_n^{\mathsf{f}}(Z)$ are compact.*

*Proof.* For $\alpha < 1$ near enough to 1 and for any continuous hermitian metric $h$ on $Z$ there exists a constant $C$ such that the following inequality holds:

$$vol_h(X \cap Z_\alpha) \leq C.\int_X \omega \qquad \text{for any cycle } X \in \mathcal{C}_n^{\mathsf{f}}(Z).$$

As the function $X \mapsto \int_X \omega$ is locally constant on $\mathcal{C}_n^{\mathsf{f}}(Z)$ because $d\omega = 0$ (the direct image of $\omega$ as a current is $d$-closed, so locally constant at smooth points of $\mathcal{C}_n^{\mathsf{f}}(Z)$, and this current is a continuous function on $\mathcal{C}_n^{\mathsf{f}}(Z)$ thanks to Proposition IV 2.3.1 of *loc. cit.*), we have a uniform bound for the volume of $X \cap Z_\alpha$ for $X$ in a given connected component of $\mathcal{C}_n^{\mathsf{f}}(Z)$. This implies that the closure of the image of this connected component in $\mathcal{C}_n^{\mathsf{f}}(Z_\alpha)$ is compact, thanks to Bishop's theorem (see [BM14, Chapter IV, Theorem 2.7.20]). But the restriction map $\mathcal{C}_n^{\mathsf{f}}(Z) \to \mathcal{C}_n^{\mathsf{f}}(Z_\alpha)$ is a homeomorphism by Proposition 4.3, so the image of a connected component is closed and then compact. □

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