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Witt groups of Severi–Brauer varieties and of function fields of conics

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Abstract. The Witt group of skew-hermitian forms over a division algebra D with symplectic involution is shown to be canonically isomorphic to the Witt group of symmetric bilinear forms over the Severi–Brauer variety of D with values in a suitable invertible sheaf. In the special case where D is a quaternion algebra, we extend previous work by Pfister and by Parimala on the Witt group of conics to set up two five-terms exact sequences relating the Witt groups of hermitian or skew-hermitian forms over D with the Witt groups of the center, of the function field of the Severi–Brauer conic of D, and of the residue fields at each closed point of the conic.

Keywords. Locally free sheaves, Azumaya algebras, Morita equivalence, quaternionic hermitian form

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Contents

1. Introduction

This paper consists of two parts. In the first part, comprising Sections 2 and 3, we consider a central division algebra D with a symplectic involution σ over an arbitrary field k of characteristic different from 2. We make no restriction on the degree of D, which may be an arbitrary even power of 2. To the involution σ, we associate an invertible sheaf $\mathcal{L}_\sigma$ on the Severi–Brauer variety X of D, whose class generates $\mathrm{Pic}(X)$. We relate skew-hermitian spaces over (D,σ) and symmetric bilinear spaces over X with values in $\mathcal{L}_\sigma$ by a canonical isomorphism of Witt groups

$$M\colon W^-(D,\sigma)\xrightarrow{\sim} W(X,\mathcal{L}_\sigma);$$

see Theorem 3.3. The map M is defined as the composition of the scalar extension map $\mathsf{ext}_X\colon W^-(D,\sigma)\to W^-(\mathcal{D},\sigma)$ from D to the Azumaya algebra $\mathcal{D}=D\otimes_k\mathcal{O}_X$ over X, and a Morita isomorphism $\mathsf{Mor}\colon W^-(\mathcal{D},\sigma)\to W(X,\mathcal{L}_\sigma)$. The injectivity of M is obtained as a consequence of a theorem of Karpenko [Kar10], and the surjectivity is derived from Pumplün's description of $W(X,\mathcal{L}_\sigma)$ in [Pum99].

In the second part of the paper, we specialize our discussion to the case where D is a quaternion algebra. The involution σ is then the canonical involution, and X is a smooth projective conic without rational points. The Witt groups $W(X)$ and $W(X,\mathcal{L}_\sigma)$ satisfy the purity property (see [BW02, Definition 8.2 and Corollary 10.3]): They embed in the Witt group $W(F)$ of the function field of X, and their images are the kernels of suitable residue maps. We thus have exact sequences involving the residue fields $k(\mathfrak{p})$ at closed points $\mathfrak{p}\in X^{(1)}$:

$$(1.1)\qquad 0\longrightarrow W(X)\longrightarrow W(F)\xrightarrow{\delta}\bigoplus_{\mathfrak{p}}W(k(\mathfrak{p}))\quad\text{and}\quad 0\longrightarrow W(X,\mathcal{L}_\sigma)\longrightarrow W(F)\xrightarrow{\delta'}\bigoplus_{\mathfrak{p}}W(k(\mathfrak{p})).$$

We compute the cokernels of δ and δ' in terms of the Witt groups $W^+(D,\sigma)$ and $W^-(D,\sigma)$ of hermitian and skew-hermitian forms over (D,σ):

$$\operatorname{coker}\delta\simeq W^-(D,\sigma)\quad\text{and}\quad\operatorname{coker}\delta'\simeq\ker\big(W(k)\longrightarrow W^+(D,\sigma)\big).$$

These isomorphisms can be interpreted in terms of Witt groups of triangulated categories. Indeed, by [Bal05, Corollary 92] (see also [BW02, Section 8]), we have

$$\operatorname{coker}\delta\simeq W^1(X)\quad\text{and}\quad\operatorname{coker}\delta'\simeq W^1(X,\mathcal{L}_\sigma).$$

Hence, we get isomorphisms

$$W^1(X) \simeq W^-(D,\sigma) \quad \text{and} \quad W^1(X,\mathcal{L}_\sigma) \simeq \ker\big(W(k) \longrightarrow W^+(D,\sigma)\big).$$

Since $W^1(k) = W^2(k) = 0$ and $W^-(D,\sigma) = W^2(D,\sigma)$, the first isomorphism also follows from Xie's exact sequence, see [Xie19, Theorem 1.2],

$$\cdots \longrightarrow W^1(k) \longrightarrow W^1(X) \longrightarrow W^2(D,\sigma) \longrightarrow W^2(k) \longrightarrow \cdots.$$

As $W(X)$ can be described as the cokernel of an injective transfer map $W^+(D,\sigma) \to W(k)$ (see Proposition 6.3), the description of $\operatorname{coker}\delta$ and $\operatorname{coker}\delta'$, together with the isomorphism $M\colon W^-(D,\sigma) \simeq W(X,\mathcal{L}_\sigma)$, leads to two strikingly similar exact sequences

$$0 \longrightarrow W^+(D,\sigma) \longrightarrow W(k) \longrightarrow W(F) \xrightarrow{\delta} \bigoplus_{\mathfrak{p}} W(k(\mathfrak{p})) \longrightarrow W^-(D,\sigma) \longrightarrow 0 \tag{1.2}$$

and

$$0 \longrightarrow W^-(D,\sigma) \longrightarrow W(F) \xrightarrow{\delta'} \bigoplus_{\mathfrak{p}} W(k(\mathfrak{p})) \longrightarrow W(k) \longrightarrow W^+(D,\sigma) \longrightarrow 0. \tag{1.3}$$

In substance, sequences (1.2) and (1.3) are due to Pfister [Pfi93, Section 7], although Pfister does not consider forms over D: He substitutes for $W^+(D,\sigma)$ and $W^-(D,\sigma)$ in (1.2) and (1.3) groups that he defines specifically for this purpose.

The exactness of (1.2) and (1.3) is proved in Section 6. (The exactness of (1.3) at the middle term has been established by Parimala [Par88, Theorem 5.1], who also has an *ad hoc* description of the kernel of δ' in [Par88, Theorem 5.3].) A delicate part of the argument is to coherently choose uniformizers and transfer maps $k(\mathfrak{p}) \to k$ at each closed point $\mathfrak{p}$. This issue is addressed in Section 5. In Section 4, we set up an exact octagon relating the Witt groups $W^+(D,\sigma)$ and $W^-(D,\sigma)$ to the Witt groups of quadratic or hermitian forms over a maximal subfield of D. This exact octagon, due to Lewis [Lew82], is a key technical tool to show that (1.2) is exact at the next-to-last term.

For a suitable identification of $\mathcal{L}_\sigma$ with an ideal sheaf, it turns out that the residue maps δ and δ' only differ in one point of degree 2, which we designate by ∞. As a result, quadratic forms over k that are split by $k(\infty)$ map in $W(F)$ to forms that lie in the kernel of δ'; hence these forms can be used to describe skew-hermitian forms over D. This idea is a key ingredient in Becher's proof of the Pfister factor conjecture; see [Bec08]. It was also used in [QMT18, Proposition 3.4] to give examples of non-similar skew-hermitian forms over a quaternion algebra that become similar over the function field of its Severi–Brauer variety. Berhuy uses it in [Ber07] to define higher cohomological invariants of quaternionic skew-hermitian forms. Note that Berhuy's discussion at the top of p. 442 is flawed: The correspondence between skew-hermitian forms after scalar extension to the function field and quadratic forms *does* depend on the choice of splitting. However, Garrel [Gar18, Section 3.1.3] has shown how the exact sequences (1.2) and (1.3) can be used to amend Berhuy's arguments and expand his result, providing a general method that produces cohomological invariants of skew-hermitian forms that depend only on their similarity class.

Notation

Throughout the paper, we let D denote a central division algebra of 2-power degree $n = 2^d \geq 2$ over an arbitrary field k of characteristic different from 2. We assume D has exponent 2 and fix some symplectic involution σ of D. Let X be the Severi–Brauer variety of n-dimensional left ideals in D. Write $F = k(X)$ for its function field and $\mathcal{O}_X$ for its structure sheaf. For each point $\mathfrak{p}$ on X, we write $\mathcal{O}_\mathfrak{p}$ for the local ring at $\mathfrak{p}$ and $k(\mathfrak{p})$ for its residue field. We let $\mathcal{D} = D \otimes_k \mathcal{O}_X$ denote the Azumaya algebra over X obtained by scalar extension to $\mathcal{O}_X$. Its stalk and fiber at a point $\mathfrak{p}$ are

$$\mathcal{D}_\mathfrak{p} = D \otimes_k \mathcal{O}_\mathfrak{p} \quad \text{and} \quad D(\mathfrak{p}) = D \otimes_k k(\mathfrak{p}).$$

From Section 4 onward, D is assumed to be a quaternion algebra. The involution σ is therefore the canonical conjugation involution $\overline{}$; we often omit it from the notation and write simply $W(D)$, $W^-(D)$ for $W^+(D,\sigma)$, $W^-(D,\sigma)$.

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2. Locally free sheaves on Severi–Brauer varieties

In this section, we define on the Severi–Brauer variety X a locally free sheaf $\mathcal{T}$ of rank n, which is the main tool for the Morita equivalence developed in the next section. We use it to associate to the symplectic involution σ the generator $\mathcal{L}_\sigma$ of $\operatorname{Pic}(X)$ in which the symmetric bilinear spaces over X we consider take their values.

Let T be the generic point of X, which is an n-dimensional left ideal in the split algebra D_F obtained from D by scalar extension to the function field F of X. The sheaf of $\mathcal{O}_X$-modules $\mathcal{T}$ is defined as the intersection of $\mathcal{D}$ with T in D_F (viewing T and D_F as constant sheaves):

$$\mathcal{T} = T \cap \mathcal{D} \subset D_F. \tag{2.1}$$

Since T is a left ideal in D_F, it is clear that $\mathcal{T}$ is a sheaf of left $\mathcal{D}$-modules. The main properties of the sheaf $\mathcal{T}$ are given in the next proposition, using the following notation: For ℓ an arbitrary field extension of k, let $X_\ell = X \times \operatorname{Spec}(\ell)$ be the ℓ-variety obtained from X by base change, and let $p\colon X_\ell \to X$ be the projection map. For any $\mathcal{O}_X$-module $\mathcal{M}$, we let $\mathcal{M}_\ell = p^*(\mathcal{M})$ be the inverse image of $\mathcal{M}$; if ℓ is a finite extension of k and $\mathcal{N}$ is an $\mathcal{O}_{X_\ell}$-module, we let $\operatorname{tr}_{\ell/k}(\mathcal{N}) = p_*(\mathcal{N})$ be the direct image of $\mathcal{N}$.

Proposition 2.1.

(a) *The sheaf $\mathcal{T}$ is a locally free $\mathcal{O}_X$-module of rank n.*

(b) *If ℓ is a splitting field of D, every ℓ-algebra isomorphism $D_\ell \simeq \operatorname{End}_\ell V$ with V an n-dimensional ℓ-vector space induces an isomorphism of sheaves*

$$\mathcal{T}_\ell \simeq V \otimes_\ell \mathcal{O}_{X_\ell}(-1) \simeq \mathcal{O}_{X_\ell}(-1)^{\oplus n}. \tag{2.2}$$

(c) *The canonical homomorphism $D \to \operatorname{End}\mathcal{T}$ arising from the left $\mathcal{D}$-module structure on $\mathcal{T}$ yields an identification $D = \operatorname{End}\mathcal{T}$; hence $\mathcal{T}$ is an indecomposable locally free sheaf.*

(d) *For every maximal subfield ℓ of D, there is an isomorphism of sheaves*

$$\mathcal{T} \simeq \operatorname{tr}_{\ell/k}\Big(\mathcal{O}_{X_\ell}(-1)\Big).$$

Proof. We first prove[(1)] (b), as (a) follows by base change. Let ℓ be a splitting field of D, and fix an ℓ-algebra isomorphism to identify $D_\ell = \operatorname{End}_\ell(V) = V \otimes_\ell V^*$ for some n-dimensional ℓ-vector space V. Then X_ℓ is identified with the projective space $\mathbb{P}(V^*)$, viewing each line $d \subset V^*$ as the n-dimensional left ideal $V \otimes_\ell d$. Pick an ℓ-base $v_1, \ldots, v_n$ of V, so $X_\ell = \operatorname{Proj}(\ell[v_1,\ldots,v_n])$, and let $U \subset X_\ell$ be the open subscheme defined by $v_n \neq 0$, so $U = \operatorname{Spec}(\ell[v_1v_n^{-1},\ldots,v_{n-1}v_n^{-1}])$. The field $\ell(X_\ell) = \ell(U)$ is the rational function field $\ell(v_1v_n^{-1},\ldots,v_{n-1}v_n^{-1})$, and the module of sections of $\mathcal{D}_\ell$ over U is

$$\mathcal{D}_\ell(U) = V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell}(U) = V \otimes_\ell V^* \otimes_\ell \ell[v_1v_n^{-1},\ldots,v_{n-1}v_n^{-1}] \subset V \otimes_\ell V^* \otimes_\ell \ell(X_\ell) = D_{\ell(X_\ell)}.$$

(1) We are indebted to A. Merkurjev for suggesting this proof to us.

On the other hand, the $\ell(X_\ell)$-rational point induced by base change from the generic point of X is the line $S = \chi \cdot \ell(X_\ell) \subset V^* \otimes_\ell \ell(X_\ell)$, where

$$\chi = \sum_{i=1}^{n} v_i^* \otimes v_i v_n^{-1} \in V^* \otimes_\ell \ell(X_\ell).$$

Viewed as a left ideal in $D_{\ell(X_\ell)}$, this point is $T_\ell = V \otimes_\ell S$. Since $S \cap (V^* \otimes_\ell \mathcal{O}_{X_\ell}(U))$ is the $\mathcal{O}_{X_\ell}(U)$-span of χ, it follows that

$$\mathcal{T}_\ell(U) = T_\ell \cap \mathcal{D}_\ell(U) = V \otimes_\ell \chi \cdot \mathcal{O}_{X_\ell}(U). \tag{2.3}$$

Now, there is a canonical embedding $\mathcal{O}_{X_\ell}(-1) \to V^* \otimes_\ell \mathcal{O}_{X_\ell}$ which on U maps v_n^{-1} to χ. Tensoring with V yields an embedding $V \otimes_\ell \mathcal{O}_{X_\ell}(-1) \to V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell} = \mathcal{D}_\ell$. The module of sections over U of the image of this embedding is exactly $\mathcal{T}_\ell(U)$. The same holds for every open subscheme in the standard affine cover of X_ℓ; hence we may identify $\mathcal{T}_\ell = V \otimes_\ell \mathcal{O}_{X_\ell}(-1)$, proving (2.2).

(c) Continuing with the same notation, consider $\mathcal{T}_\ell^\vee = \mathcal{H}om(\mathcal{T}_\ell, \mathcal{O}_{X_\ell})$, the dual sheaf of $\mathcal{T}_\ell$. From (2.2) it follows that $\mathcal{T}_\ell^\vee \simeq V^* \otimes_\ell \mathcal{O}_{X_\ell}(1)$, hence

$$\mathcal{E}nd\,\mathcal{T}_\ell = \mathcal{T}_\ell \otimes \mathcal{T}_\ell^\vee \simeq \mathrm{End}_\ell(V) \otimes_\ell \mathcal{O}_{X_\ell}(0).$$

This shows that $\dim_\ell(\mathrm{End}\,\mathcal{T}_\ell) = n^2$, hence $\dim_k(\mathrm{End}\,\mathcal{T}) = n^2$. The canonical map $D \to \mathrm{End}\,\mathcal{T}$ is injective since D is a division algebra; hence it is an isomorphism by dimension count.

(d) Now, let ℓ be a maximal subfield of D. Since $D = \mathrm{End}\,\mathcal{T}$, the locally free $\mathcal{O}_X$-module $\mathcal{T}$ has an ℓ-structure, and $\mathcal{T} \simeq \mathrm{tr}_{\ell/k}(\mathcal{N})$ for some irreducible locally free $\mathcal{O}_{X_\ell}$-module $\mathcal{N}$ by [AEJ92, Theorem 1.8]. Lemma 1.4 of [AEJ92] shows that the $\mathcal{O}_{X_\ell}$-module $\mathcal{N}$ is a direct summand of $\mathcal{T}_\ell$. By (2.2) it follows that $\mathcal{N} \simeq \mathcal{O}_{X_\ell}(-1)$. □

Remark. For ℓ a Galois extension of k that splits D, it follows from Proposition 2.1 that $\mathcal{T}$ can be obtained by Galois descent from $V \otimes_\ell \mathcal{O}_{X_\ell}(-1)$ by using the cocycle with values in $\mathrm{PGL}(V)$ that twists $\mathrm{End}_\ell(V)$ into D. Therefore, $\mathcal{T}$ can be identified with the sheaf J defined by Quillen [Qui73, Section 8.4] in his computation of the K-theory of Severi–Brauer varieties.

In order to define the invertible $\mathcal{O}_X$-module $\mathcal{L}_\sigma$ attached to the involution σ, we start with some observations on split central simple algebras, which will be applied to the scalar extension of D to the residue fields at points of X.

Let A be a split central simple algebra of even degree $n = 2m$ over an arbitrary field E, and let σ_A be a symplectic involution on A. We let

$$\mathrm{Skew}(\sigma_A) = \{x \in A \mid \sigma_A(x) = -x\}$$

and write $\mathrm{Trd}\colon A \to E$ for the reduced trace map. Recall that the bilinear form $\mathrm{Trd}(xy)$ is nonsingular; hence for every nonzero $x \in A$, there exists a $y \in A$ such that $\mathrm{Trd}(xy) = 1$.

Lemma 2.2. *Let $I \subset A$ be an n-dimensional left ideal, let $\sigma_A(I) = \{\sigma_A(\xi) \mid \xi \in I\}$ be its conjugate n-dimensional right ideal, and denote by J the intersection $J = I \cap \sigma_A(I)$. Then $\dim_E J = 1$ and $J = I \cap \mathrm{Skew}(\sigma_A)$; moreover, for $\lambda \in J$ and $\mu \in A$, we have*

$$\sigma_A(\lambda) = -\lambda, \quad \lambda^2 = 0, \quad \textit{and} \quad \lambda\mu\lambda = \mathrm{Trd}(\lambda\mu)\lambda. \tag{2.4}$$

Multiplication in A defines an isomorphism of E-vector spaces

$$\mathsf{mult}\colon \sigma_A(I) \otimes_A I \longrightarrow J, \quad \sigma_A(\xi) \otimes \eta \longmapsto \sigma_A(\xi)\eta.$$

Moreover, there is a canonical isomorphism of A-bimodules

$$\mathsf{can}\colon I \otimes_E \sigma_A(I) \xrightarrow{\sim} J \otimes_E A$$

defined as follows: Pick $\lambda \in J$ *and* $\mu \in A$ *such that* $\operatorname{Trd}(\lambda\mu) = 1$*, and let*

$$\mathsf{can}(\xi \otimes \sigma_A(\eta)) = \lambda \otimes \xi\mu\sigma_A(\eta) \quad \textit{for } \xi,\, \eta \in I.$$

Proof. Fix some representation $A = \operatorname{End}_E V$ for some n-dimensional E-vector space V. The involution σ_A is then adjoint to some alternating bilinear form b on V, and the ideal I is the set of linear operators that vanish on some hyperplane H; its conjugate $\sigma_A(I)$ is the set of operators that map V into $H^\perp$, the orthogonal of H for the form b. Therefore, an operator lies in J if and only if its kernel contains H and its image is in the 1-dimensional subspace $H^\perp$. It follows that $\dim J = 1$. Moreover, the image of each $\lambda \in J$ lies in $H^\perp$, hence in H since b is alternating, and therefore $\lambda^2 = 0$.

To complete the proof of (2.4), pick $v \in V \setminus H$. Every vector $x \in V$ has the form $x = v\alpha + u$ for some $\alpha \in E$ and $u \in H$. For $\lambda \in J$, we have $\lambda(u) = 0$ and $\lambda(v) \in H^\perp$; hence for $x' = v\alpha' + u'$ with $\alpha' \in E$ and $u' \in H$,

$$b(\lambda(x), x') = b(\lambda(v)\alpha, v\alpha') \quad \text{and} \quad b(x, \lambda(x')) = b(v\alpha, \lambda(v)\alpha').$$

Since b is alternating, $b(v, \lambda(v)) = -b(\lambda(v), v)$, and it follows that $b(\lambda(x), x') = -b(x, \lambda(x'))$ for all $x,\, x' \in V$, hence $\sigma_A(\lambda) = -\lambda$.

Now, take $\mu \in A$. As $\mu\lambda$ vanishes on H, it follows that $\mu\lambda(v) = v\operatorname{Trd}(\mu\lambda) + u$ for some $u \in H$, hence $\lambda\mu\lambda(v) = \lambda(v)\operatorname{Trd}(\mu\lambda)$. Moreover, $\lambda\mu\lambda(u) = \lambda(u) = 0$ for all $u \in H$; hence $\lambda\mu\lambda = \operatorname{Trd}(\lambda\mu)\lambda$ since V is spanned by v and H.

It follows from (2.4) that $J \subset \operatorname{Skew}(\sigma_A)$, hence $J \subset I \cap \operatorname{Skew}(\sigma_A)$. For the reverse inclusion, it suffices to observe that if $\lambda \in I \cap \operatorname{Skew}(\sigma_A)$, then $\lambda = -\sigma_A(\lambda) \in \sigma_A(I)$, hence $\lambda \in I \cap \sigma_A(I)$. Therefore, $J = I \cap \operatorname{Skew}(\sigma_A)$.

Because I is a left ideal and $\sigma_A(I)$ is a right ideal, we have $\sigma_A(I) \cdot I \subset I \cap \sigma_A(I)$; hence multiplication defines an E-linear map $\mathsf{mult}\colon \sigma_A(I) \otimes_A I \to J$. To show that this map is onto, note that for any nonzero $\lambda \in J$, there exists a $\mu \in A$ such that $\operatorname{Trd}(\lambda\mu) = 1$. By (2.4), it follows that

$$\lambda = \operatorname{Trd}(\lambda\mu)\lambda = \lambda\mu\lambda. \tag{2.5}$$

Since $\lambda \in I \cap \sigma_A(I)$, this equation shows that $\lambda \in \sigma_A(I) \cdot I$; hence mult is surjective. To see that it is injective, pick $\lambda \in J$ and $\mu \in A$ as above. Since λ is nonzero and lies in I, we have $I = A\lambda$; hence every element in $\sigma_A(I) \otimes_A I$ can be written in the form $\xi \otimes \lambda$ for some $\xi \in \sigma_A(I)$. If $\xi\lambda = 0$, then using (2.5), we get

$$\xi \otimes \lambda = \xi \otimes \lambda\mu\lambda = \xi\lambda\mu \otimes \lambda = 0.$$

Therefore, mult is injective.

We next consider the map can, which is clearly a homomorphism of A-bimodules. We first show that it is canonical, *i.e.*, that it does not depend on the choice of λ and μ. Suppose $\lambda,\, \lambda' \in J$ and $\mu,\, \mu' \in A$ are such that $\operatorname{Trd}(\lambda\mu) = \operatorname{Trd}(\lambda'\mu') = 1$. Because $\dim J = 1$, there exists an $\alpha \in E^\times$ such that $\lambda' = \alpha\lambda$, hence $\operatorname{Trd}(\lambda\mu') = \alpha^{-1}$, so (2.4) and (2.5) yield

$$\lambda\mu'\lambda = \alpha^{-1}\lambda = \alpha^{-1}\lambda\mu\lambda. \tag{2.6}$$

Since $I = A\lambda$, for all $\xi,\, \eta \in I$, we may find $\xi_1,\, \eta_1 \in A$ such that

$$\xi = \xi_1\lambda \quad \text{and} \quad \eta = \eta_1\lambda.$$

Then, by (2.5),

$$\xi\mu'\sigma_A(\eta) = -\xi_1\lambda\mu'\lambda\sigma_A(\eta_1) = -\alpha^{-1}\xi_1\lambda\mu\lambda\sigma_A(\eta_1) = \alpha^{-1}\xi\mu\sigma_A(\eta).$$

Therefore,

$$\lambda' \otimes \xi\mu'\sigma_A(\eta) = \lambda \otimes \xi\mu\sigma_A(\eta).$$

It follows that can is canonical, and it remains to prove that it is bijective. Since A is a simple algebra, we have $A\lambda A = A$; hence to prove surjectivity, it suffices to show that $\lambda \otimes \xi\lambda\eta$ lies in the image of can for all $\xi,\, \eta \in A$. By (2.5), we have

$$\lambda \otimes \xi\lambda\eta = \lambda \otimes \xi\lambda\mu\lambda\eta = \mathsf{can}(\xi\lambda \otimes \lambda\eta);$$

hence can is surjective. It is therefore also injective by dimension count. □

Our first application of Lemma 2.2 is to $A = D_F$, the split algebra obtained from D by scalar extension to the function field of X. Taking for I the generic point T of X, we let

$$L_\sigma = T \cap \sigma(T) = T \cap \mathrm{Skew}(\sigma).$$

Lemma 2.2 shows that L_σ is an F-vector space of dimension 1 and yields canonical isomorphisms

$$\mathsf{mult}\colon \sigma(T) \otimes_{D_F} T \xrightarrow{\sim} L_\sigma \quad \text{and} \quad \mathsf{can}\colon T \otimes_F \sigma(T) \xrightarrow{\sim} L_\sigma \otimes_F D_F. \tag{2.7}$$

The sheaf of $\mathcal{O}_X$-modules $\mathcal{L}_\sigma$ is defined as the intersection of $\mathcal{D}$ with L_σ in D_F, viewing L_σ and D_F as constant sheaves:

$$\mathcal{L}_\sigma = L_\sigma \cap \mathcal{D} \subset D_F.$$

Proposition 2.3. *The sheaf $\mathcal{L}_\sigma$ is an invertible $\mathcal{O}_X$-module such that $(\mathcal{L}_\sigma)_\ell \simeq \mathcal{O}_{X_\ell}(-2)$ for every splitting field ℓ of D; hence $\mathcal{L}_\sigma$ generates the Picard group* $\mathrm{Pic}(X)$. *Moreover, there exist isomorphisms of $\mathcal{O}_X$-modules*

$$\mathsf{mult}_X\colon \sigma(\mathcal{T}) \otimes_{\mathcal{D}} \mathcal{T} \xrightarrow{\sim} \mathcal{L}_\sigma \quad \textit{and} \quad \mathsf{can}_X\colon \mathcal{T} \otimes_{\mathcal{O}_X} \sigma(\mathcal{T}) \xrightarrow{\sim} \mathcal{L}_\sigma \otimes_{\mathcal{O}_X} \mathcal{D}$$

that restrict on the generic fiber to the isomorphisms of (2.7).

Proof. Let ℓ be a splitting field of D. As in the proof of Proposition 2.1, we identify $D_\ell = \mathrm{End}_\ell V = V \otimes_\ell V^*$ for some n-dimensional ℓ-vector space V, hence also $X_\ell = \mathbb{P}(V^*)$. Writing again σ for the scalar extension of σ to D_ℓ, we know from [KMR+98, Equation (4.2)] that σ is the adjoint involution of a nonsingular alternating bilinear form b on V. Let $m = \frac{n}{2}$ and fix a symplectic base $(u_i, w_i)_{i=1}^m$ of V; thus, for i, $j = 1, \ldots, m$,

$$b(u_i, w_i) = 1 = -b(w_i, u_i), \quad b(u_i, u_j) = b(w_i, w_j) = 0,$$

and

$$b(u_i, w_j) = 0 = b(w_j, u_i) \quad \text{if } i \neq j.$$

It follows that for i, $j = 1, \ldots, m$,

$$\begin{aligned} \sigma(u_i \otimes u_j^*) &= w_j \otimes w_i^*, & \sigma(u_i \otimes w_j^*) &= -u_j \otimes w_i^*, \\ \sigma(w_i \otimes u_j^*) &= -w_j \otimes u_i^*, & \sigma(w_i \otimes w_j^*) &= u_j \otimes u_i^*. \end{aligned} \tag{2.8}$$

Let $U \subset X_\ell$ be the open subscheme defined by $w_m \neq 0$; hence

$$\mathcal{O}_{X_\ell}(U) = \ell\big[u_1 w_m^{-1}, \ldots, u_m w_m^{-1}, w_1 w_m^{-1}, \ldots, w_{m-1} w_m^{-1}\big].$$

As in the proof of Proposition 2.1, consider

$$\chi = \sum_{i=1}^m \big(u_i^* \otimes u_i w_m^{-1} + w_i^* \otimes w_i w_m^{-1}\big) \in V^* \otimes_\ell \mathcal{O}_{X_\ell}(U),$$

which has the property that $\chi \cdot \ell(X_\ell)$ is the $\ell(X_\ell)$-rational point induced by base change from the generic point of X. We saw in the proof of Proposition 2.1 (see (2.3)) that

$$\mathcal{T}_\ell(U) = V \otimes_\ell \chi \cdot \mathcal{O}_{X_\ell}(U) \subset V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell}(U);$$

hence every element $t \in \mathcal{T}_\ell(U)$ can be written in the form

$$\begin{aligned} t &= \sum_{i=1}^m (u_i \otimes \chi f_i + w_i \otimes \chi g_i) \\ &= \sum_{i,j=1}^m \big(u_i \otimes u_j^* \otimes u_j w_m^{-1} f_i + u_i \otimes w_j^* \otimes w_j w_m^{-1} f_i \\ &\qquad + w_i \otimes u_j^* \otimes u_j w_m^{-1} g_i + w_i \otimes w_j^* \otimes w_j w_m^{-1} g_i\big) \end{aligned}$$

for some $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathcal{O}_{X_\ell}(U)$. A straightforward computation using (2.8) shows that $\sigma(t) = -t$ holds if and only if for all $i, j = 1, \ldots, m$,

$$f_i = w_i w_m^{-1} f_m \quad \text{and} \quad g_i = -u_i w_m^{-1} f_m.$$

Therefore, $(\mathcal{L}_\sigma)_\ell(U) = \mathcal{T}_\ell(U) \cap \operatorname{Skew}(\sigma)$ is the $\mathcal{O}_{X_\ell}(U)$-span of the following element:

$$\zeta = \sum_{i,j=1}^{m} \Big(u_i \otimes u_j^* \otimes u_j w_i w_m^{-2} + u_i \otimes w_j^* \otimes w_i w_j w_m^{-2} \\ - w_i \otimes u_j^* \otimes u_i u_j w_m^{-2} - w_i \otimes w_j^* \otimes u_i w_j w_m^{-2} \Big) \in V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell}(U).$$

Under the identification $(V \otimes_\ell \mathcal{O}_{X_\ell}(U)) \otimes_{\mathcal{O}_{X_\ell}(U)} (V^* \otimes_\ell \mathcal{O}_{X_\ell}(U)) = V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell}(U)$, this element ζ is the tensor product $\zeta = \theta \otimes \chi$, where $\theta = \sum_{i=1}^m (u_i \otimes w_i w_m^{-1} - w_i \otimes u_i w_m^{-1}) \in V \otimes_\ell \mathcal{O}_{X_\ell}(U)$ is the element such that $b(\theta, \rho) = \chi(\rho)$ for all $\rho \in V \otimes_\ell \mathcal{O}_{X_\ell}(U)$.

Now, there is a canonical embedding $\mathcal{O}_{X_\ell}(-2) \to V \otimes_\ell V^* \otimes_\ell \mathcal{O}_{X_\ell}$ that on U maps w_m^{-2} to ζ, and the computation above shows that the module of sections over U of the image of this embedding is exactly $(\mathcal{L}_\sigma)_\ell(U)$. The same holds for every open subscheme in the standard affine cover of X_ℓ; hence $(\mathcal{L}_\sigma)_\ell \simeq \mathcal{O}_{X_\ell}(-2)$. By base change, it follows that $\mathcal{L}_\sigma$ is an invertible $\mathcal{O}_X$-module.

We next define the morphism can_X. Let $U \subset X$ be an affine open subscheme on which $\mathcal{L}_\sigma(U)$ is a free $\mathcal{O}_X(U)$-module, and let $\lambda \in \mathcal{L}_\sigma(U)$ be a base of $\mathcal{L}_\sigma(U)$. For each point $\mathfrak{p}$ of U, the germ $\lambda_\mathfrak{p}$ is an $\mathcal{O}_\mathfrak{p}$-base of the stalk $(\mathcal{L}_\sigma)_\mathfrak{p}$, and its image $\overline{\lambda}_\mathfrak{p}$ in $D(\mathfrak{p})$ is a $k(\mathfrak{p})$-base of the fiber of $\mathcal{L}_\sigma$ at $\mathfrak{p}$. Since the bilinear form $\operatorname{Trd}(xy)$ on $D(\mathfrak{p})$ is nonsingular, the linear form $\operatorname{Trd}(\overline{\lambda}_\mathfrak{p}\,\sqcup)\colon D(\mathfrak{p}) \to k(\mathfrak{p})$ is surjective. From Nakayama's lemma, it follows that the linear form $\operatorname{Trd}(\lambda_\mathfrak{p}\,\sqcup)\colon \mathcal{D}_\mathfrak{p} \to \mathcal{O}_\mathfrak{p}$ is surjective. This holds for every point $\mathfrak{p}$ of U; hence the linear form $\operatorname{Trd}(\lambda\,\sqcup)\colon \mathcal{D}(U) \to \mathcal{O}_X(U)$ is surjective. It follows that there exists a $\mu \in \mathcal{D}(U)$ such that $\operatorname{Trd}(\lambda\mu) = 1$. We may then define the $\mathcal{O}_X(U)$-module homomorphism $\mathsf{can}_U\colon \mathcal{T}(U) \otimes_{\mathcal{O}_X(U)} \sigma(\mathcal{T})(U) \to \mathcal{L}_\sigma(U) \otimes_{\mathcal{O}_X(U)} \mathcal{D}(U)$ by mapping $t \otimes \sigma(t')$ to $\lambda \otimes t\mu\sigma(t')$ for $t, t' \in \mathcal{T}(U)$.

If $\lambda' \in \mathcal{L}_\sigma(U)$ is another $\mathcal{O}_X(U)$-base of $\mathcal{L}_\sigma(U)$ and $\mu' \in \mathcal{D}(U)$ is such that $\operatorname{Trd}(\lambda'\mu') = 1$, then $\lambda' = \alpha\lambda$ for some $\alpha \in \mathcal{O}_X(U)^\times$, and the same arguments as in the proof of Lemma 2.2 show that $\lambda \otimes t\mu\sigma(t') = \lambda' \otimes t\mu'\sigma(t')$ for $t, t' \in \mathcal{T}(U)$. Therefore, the map can_U does not depend on the choice of λ, μ. Gluing the maps can_U for the subschemes U in an open cover of X yields a morphism of $\mathcal{O}_X$-modules $\mathsf{can}_X\colon \mathcal{T} \otimes_{\mathcal{O}_X} \sigma(\mathcal{T}) \to \mathcal{L}_\sigma \otimes_{\mathcal{O}_X} \mathcal{D}$.

On the other hand, it is clear that for every affine open subscheme $U \subset X$, the multiplication in $\mathcal{D}(U)$ yields a map $\sigma(\mathcal{T})(U) \otimes_{\mathcal{D}(U)} \mathcal{T}(U) \to \mathcal{L}_\sigma(U)$ since $L_\sigma = \mathsf{mult}(\sigma(T) \otimes_{D_F} T)$. Therefore, there is a morphism of $\mathcal{O}_X$-modules $\mathsf{mult}_X\colon \sigma(\mathcal{T}) \otimes_{\mathcal{D}} \mathcal{T} \to \mathcal{L}_\sigma$.

The morphisms can_X and mult_X are injective since their restrictions to the generic fiber are injective by Lemma 2.2. Therefore, it only remains to prove that the maps induced by can_X and mult_X are surjective on the stalks at each point, or (by Nakayama's lemma) on the fibers at each point. For each point $\mathfrak{p}$ of X, the fiber $T(\mathfrak{p})$ of $\mathcal{T}$ at $\mathfrak{p}$ is a left ideal of dimension n in $D(\mathfrak{p})$ since $\mathcal{T}$ is a locally free $\mathcal{O}_X$-module of rank n, and the fiber $L_\sigma(\mathfrak{p})$ of $\mathcal{L}_\sigma$ is $T(\mathfrak{p}) \cap \sigma(T(\mathfrak{p}))$. Lemma 2.2 with $A = D(\mathfrak{p})$ and $I = T(\mathfrak{p})$ shows that the maps mult and can yield isomorphisms

$$\sigma(T(\mathfrak{p})) \otimes_{D(\mathfrak{p})} T(\mathfrak{p}) \xrightarrow{\sim} L_\sigma(\mathfrak{p}) \quad \text{and} \quad T(\mathfrak{p}) \otimes_{k(\mathfrak{p})} \sigma(T(\mathfrak{p})) \xrightarrow{\sim} L_\sigma(\mathfrak{p}) \otimes_{k(\mathfrak{p})} D(\mathfrak{p}).$$

The proof is thus complete. □

3. Symmetric spaces over Severi–Brauer varieties

We use the same notation as in the preceding section: X is the Severi–Brauer variety of the division algebra D with symplectic involution σ, and $\mathcal{L}_\sigma = \mathcal{T} \cap \sigma(\mathcal{T})$ is the invertible $\mathcal{O}_X$-module obtained by intersecting the sheaf $\mathcal{T}$ defined in (2.1) and its conjugate $\sigma(\mathcal{T})$. Throughout this section, unadorned tensor products and $\mathcal{H}om$ of $\mathcal{O}_X$-modules are over $\mathcal{O}_X$.

Our goal in this section is to define a canonical isomorphism between the Witt groups $W^-(D,\sigma)$ and $W(X,\mathcal{L}_\sigma)$; see Theorem 3.3. The construction involves the Witt group $W^-(\mathcal{D},\sigma)$, which is shown in Proposition 3.2 to be canonically isomorphic to $W(X,\mathcal{L}_\sigma)$.

The Witt groups $W(X,\mathcal{L}_\sigma)$ and $W^-(\mathcal{D},\sigma)$ are obtained by Knebusch's construction (see [Bal05, Definition 27]) from categories with duality: Let Mod_X be the category of locally free $\mathcal{O}_X$-modules of finite rank and $\mathsf{Mod}_\mathcal{D}$ the category of locally free $\mathcal{O}_X$-modules with an action of D on the right, in other words, right $\mathcal{D}$-modules that are locally free of finite rank as $\mathcal{O}_X$-modules. Tensoring with $\mathcal{T}$ (resp. $\sigma(\mathcal{T})$) yields functors

$$\Theta\colon \mathsf{Mod}_\mathcal{D} \longrightarrow \mathsf{Mod}_X, \quad \mathcal{V} \longmapsto \mathcal{V}\otimes_\mathcal{D}\mathcal{T},$$
$$\Psi\colon \mathsf{Mod}_X \longrightarrow \mathsf{Mod}_\mathcal{D}, \quad \mathcal{M} \longmapsto \mathcal{M}\otimes\sigma(\mathcal{T}).$$

Since $\mathcal{T}\otimes\sigma(\mathcal{T})$ is an invertible $\mathcal{D}$-bimodule and $\sigma(\mathcal{T})\otimes_\mathcal{D}\mathcal{T}$ is an invertible $\mathcal{O}_X$-module (see Proposition 2.3), $\Psi\circ\Theta$ and $\Theta\circ\Psi$ are naturally equivalent to the identity on $\mathsf{Mod}_\mathcal{D}$ and Mod_X, respectively; hence Θ and Ψ are equivalences of categories.

By definition, $W(X,\mathcal{L}_\sigma) = W(\mathsf{Mod}_X,*,\varpi)$, where $(*,\varpi)$ is the duality defined on Mod_X by $\mathcal{M}^* = \mathcal{H}om(\mathcal{M},\mathcal{L}_\sigma)$ for every locally free $\mathcal{O}_X$-module of finite rank $\mathcal{M}$, with $\varpi_\mathcal{M}\colon \mathcal{M}\xrightarrow{\sim}\mathcal{M}^{**}$ the usual identification.

On the other hand, a duality $(\sharp,\pi)$ is defined on $\mathsf{Mod}_\mathcal{D}$ by

$$\mathcal{V}^\sharp = \mathcal{H}om_\mathcal{D}(\mathcal{V},\mathcal{D}) \quad \text{for every object } \mathcal{V} \text{ in } \mathsf{Mod}_\mathcal{D}$$

(where the left $\mathcal{D}$-module structure on $\mathcal{H}om_\mathcal{D}(\mathcal{V},\mathcal{D})$ is twisted by σ into a right $\mathcal{D}$-module structure), and $\pi_\mathcal{V}\colon \mathcal{V}\xrightarrow{\sim}\mathcal{V}^{\sharp\sharp}$ is the usual identification. By definition,

$$W(\mathcal{D},\sigma) = W(\mathsf{Mod}_\mathcal{D},\sharp,\pi) \quad\text{and}\quad W^-(\mathcal{D},\sigma) = W(\mathsf{Mod}_\mathcal{D},\sharp,-\pi).$$

In order to obtain a canonical isomorphism $W^-(\mathcal{D},\sigma)\to W(X,\mathcal{L}_\sigma)$, we first define a natural transformation $\theta\colon \Theta\circ\sharp\to *\circ\Theta$ and then deduce a morphism of categories with duality

$$(\Theta,\theta)\colon (\mathsf{Mod}_\mathcal{D},\sharp,-\pi)\longrightarrow(\mathsf{Mod}_X,*,\varpi).$$

Lemma 3.1. *For every object $\mathcal{V}$ in $\mathsf{Mod}_\mathcal{D}$, there is a canonical isomorphism of $\mathcal{O}_X$-modules*

$$\theta_\mathcal{V}\colon \mathcal{V}^\sharp\otimes_\mathcal{D}\mathcal{T}\xrightarrow{\sim}(\mathcal{V}\otimes_\mathcal{D}\mathcal{T})^*.$$

It is determined on the stalks at any point $\mathfrak{p}$ by

$$\Big\langle\theta_{\mathcal{V}_\mathfrak{p}}\big(x^\sharp\otimes t\big), y\otimes t'\Big\rangle_{\mathcal{L}_\sigma} = \mathsf{mult}_\mathfrak{p}\Big(\sigma(t)\big\langle x^\sharp,y\big\rangle_\mathcal{D}\otimes t'\Big) \quad \textit{for } x^\sharp\in\mathcal{V}_\mathfrak{p}^\sharp,\ y\in\mathcal{V}_\mathfrak{p},\ \textit{and } t,\ t'\in\mathcal{T}_\mathfrak{p},$$

where $\langle\sqcup,\sqcup\rangle_\mathcal{D}\colon \mathcal{V}_\mathfrak{p}^\sharp\times\mathcal{V}_\mathfrak{p}\to\mathcal{D}$ and $\langle\sqcup,\sqcup\rangle_{\mathcal{L}_\sigma}\colon (\mathcal{V}_\mathfrak{p}\otimes_\mathcal{D}\mathcal{T}_\mathfrak{p})^*\times(\mathcal{V}_\mathfrak{p}\otimes_\mathcal{D}\mathcal{T}_\mathfrak{p})\to\mathcal{L}_\sigma$ are the canonical bilinear maps.

Proof. The switch map is an isomorphism $\mathcal{V}^\sharp\otimes_\mathcal{D}\mathcal{T}\xrightarrow{\sim}\sigma(\mathcal{T})\otimes_\mathcal{D}\mathcal{H}om_\mathcal{D}(\mathcal{V},\mathcal{D})$. We first prove that the latter tensor product is isomorphic to $\mathcal{H}om_\mathcal{D}(\mathcal{V},\sigma(\mathcal{T}))$.

Let R be an arbitrary commutative k-algebra. Let $D_R = D\otimes_k R$, and let M be a right D_R-module. Write M_0 for the right R-module obtained from M by forgetting the D-action. By [MT16, Proposition 2.1], M is a direct summand of the right D_R-module $M_0\otimes_k D$. We recall the argument for the convenience of the reader: The Goldman element $g=\sum a_i\otimes b_i\in D\otimes_k D$ is defined by the condition that $\sum a_i x b_i = \mathrm{Trd}_D(x)$ for all $x\in D$; see [KMR+98, Equation (3.5)]. It satisfies the property that $(a\otimes b)g = g(b\otimes a)$ for every a, $b\in D$; see [KMR+98, Equation (3.6)]. If $u\in D$ is such that $\mathrm{Trd}_D(u)=1$, then $\sum a a_i u\otimes b_i = \sum a_i u\otimes b_i a$ for all $a\in D$, and the map $M\to M_0\otimes_k D$ which carries $m\in M$ to $\sum(m a_i u)\otimes b_i$ is an injective D_R-module homomorphism split by the multiplication map $M_0\otimes_k D\to M$.

If M_0 is a projective R-module, then $M_0 \otimes_k D$ is a projective D_R-module, so M also is a projective D_R-module. For every D_R-module N, the canonical homomorphism

$$N \otimes_{D_R} \mathrm{Hom}_{D_R}(M, D_R) \longrightarrow \mathrm{Hom}_{D_R}(M, N)$$

is then an isomorphism; see [Bou70, Section II.4.2, p. II.75]. This applies in particular with M and N the modules of sections of $\mathcal{V}$ and $\sigma(\mathcal{T})$ over any affine open subscheme of X, or the stalks of $\mathcal{V}$ and $\sigma(\mathcal{T})$ at any point of X, and yields an isomorphism

$$\tau_{\mathcal{V}} \colon \mathcal{V}^\sharp \otimes_{\mathcal{D}} \mathcal{T} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}}(\mathcal{V}, \sigma(\mathcal{T})),$$

which is given on the stalk at any point $\mathfrak{p}$ by

$$\tau_{\mathcal{V}_\mathfrak{p}}(x^\sharp \otimes t) \colon y \longmapsto \sigma(t)\langle x^\sharp, y\rangle_{\mathcal{D}} \quad \text{for } x^\sharp \in \mathcal{V}_\mathfrak{p}^\sharp,\ y \in \mathcal{V}_\mathfrak{p}, \text{ and } t \in \mathcal{T}_\mathfrak{p}.$$

The isomorphism $\theta_{\mathcal{V}}$ is obtained by composing $\tau_{\mathcal{V}}$ with the isomorphisms

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{V}, \sigma(\mathcal{T})) \xrightarrow{\sim} \mathcal{H}om(\mathcal{V} \otimes_{\mathcal{D}} \mathcal{T}, \sigma(\mathcal{T}) \otimes_{\mathcal{D}} \mathcal{T}) \xrightarrow{\sim} \mathcal{H}om(\mathcal{V} \otimes_{\mathcal{D}} \mathcal{T}, \mathcal{L}_\sigma)$$

that arise from the equivalence of categories Θ and the isomorphism mult_X of Proposition 2.3. □

The isomorphisms $\theta_{\mathcal{V}}$ of Lemma 3.1 define a natural transformation $\theta \colon \Theta \circ \sharp \xrightarrow{\sim} * \circ \Theta$. We next show that the pair (Θ, θ) is a morphism of categories with duality as in [Bal05, Definition 5] (a "duality-preserving functor" in the terminology of [Knu91, Section II(2.6)]).

Proposition 3.2. *The pair (Θ, θ) induces an isomorphism of Witt groups*

$$\mathsf{Mor} \colon W^-(\mathcal{D}, \sigma) \longrightarrow W(X, \mathcal{L}_\sigma)$$

by mapping the Witt class of every skew-hermitian space $(\mathcal{V}, \varphi)$ over $(\mathcal{D}, \sigma)$ to the Witt class of the symmetric bilinear space $(\mathcal{V} \otimes_{\mathcal{D}} \mathcal{T}, \theta_{\mathcal{V}} \circ (\varphi \otimes \mathrm{Id}_{\mathcal{T}}))$.

Proof. To see that (Θ, θ) is a morphism of categories with duality, it remains to prove that the following diagram commutes for every object $\mathcal{V}$ in $\mathsf{Mod}_{\mathcal{D}}$:

$$\begin{array}{ccc} \mathcal{V} \otimes_{\mathcal{D}} \mathcal{T} & \xrightarrow{-\pi_{\mathcal{V}} \otimes \mathrm{Id}_{\mathcal{T}}} & \mathcal{V}^{\sharp\sharp} \otimes_{\mathcal{D}} \mathcal{T} \\ {\scriptstyle \varpi_{\mathcal{V}\otimes_{\mathcal{D}}\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{V}^\sharp}} \\ (\mathcal{V} \otimes_{\mathcal{D}} \mathcal{T})^{**} & \xrightarrow{\theta_{\mathcal{V}}^*} & (\mathcal{V}^\sharp \otimes_{\mathcal{D}} \mathcal{T})^*. \end{array} \tag{3.1}$$

We compute on the stalks at any point $\mathfrak{p}$: For $x \in \mathcal{V}_\mathfrak{p}$, $y^\sharp \in \mathcal{V}_\mathfrak{p}^\sharp$, and t, $t' \in \mathcal{T}_\mathfrak{p}$,

$$\left\langle \theta_{\mathcal{V}_\mathfrak{p}^\sharp} \circ \pi_{\mathcal{V}}(x \otimes t), y^\sharp \otimes t' \right\rangle_{\mathcal{L}_\sigma} = \mathsf{mult}_\mathfrak{p}\left(\sigma(t)\left\langle \pi_{\mathcal{V}}(x), y^\sharp\right\rangle_{\mathcal{D}} \otimes t'\right) = \mathsf{mult}_\mathfrak{p}\left(\sigma(t)\sigma\left(\left\langle y^\sharp, x\right\rangle_{\mathcal{D}}\right) \otimes t'\right).$$

On the other hand,

$$\begin{aligned} \left\langle \theta_{\mathcal{V}}^* \circ \varpi_{\mathcal{V}\otimes_{\mathcal{D}}\mathcal{T}}(x \otimes t), y^\sharp \otimes t' \right\rangle_{\mathcal{L}_\sigma} &= \left\langle \varpi_{\mathcal{V}\otimes_{\mathcal{D}}\mathcal{T}}(x \otimes t), \theta_{\mathcal{V}}\left(y^\sharp \otimes t'\right)\right\rangle_{\mathcal{L}_\sigma} \\ &= \left\langle \theta_{\mathcal{V}}\left(y^\sharp \otimes t'\right), x \otimes t\right\rangle_{\mathcal{L}_\sigma} = \mathsf{mult}_\mathfrak{p}\left(\sigma(t')\left\langle y^\sharp, x\right\rangle_{\mathcal{D}} \otimes t\right). \end{aligned}$$

Since $\mathcal{L}_\sigma \subset \mathrm{Skew}(\sigma)$, it follows that $\mathsf{mult}_\mathfrak{p}(\sigma(t_1) \otimes t_2) = -\mathsf{mult}_\mathfrak{p}(\sigma(t_2) \otimes t_1)$ for all t_1, $t_2 \in \mathcal{T}_\mathfrak{p}$; hence the computation above yields

$$\theta_{\mathcal{V}}^* \circ \varpi_{\mathcal{V}\otimes_{\mathcal{D}}\mathcal{T}}(x \otimes t) = -\theta_{\mathcal{V}_\mathfrak{p}^\sharp} \circ \pi_{\mathcal{V}}(x \otimes t) \quad \text{for all } x \in \mathcal{V}_\mathfrak{p},\ t \in \mathcal{T}_\mathfrak{p}.$$

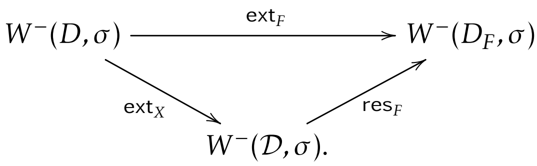
Therefore, the diagram (3.1) commutes, and (Θ, θ) is a morphism of categories with duality. The induced homomorphism of Witt groups $\mathsf{Mor} \colon W^-(\mathcal{D}, \sigma) \to W(X, \mathcal{L}_\sigma)$ is an isomorphism because Θ is an equivalence of categories. □

The main theorem of this section follows.

Theorem 3.3. *The composition of the scalar extension map* $\mathsf{ext}_X\colon W^-(D,\sigma)\to W^-(\mathcal{D},\sigma)$ *with the map* $\mathsf{Mor}\colon W^-(\mathcal{D},\sigma)\xrightarrow{\sim} W(X,\mathcal{L}_\sigma)$ *is an isomorphism*

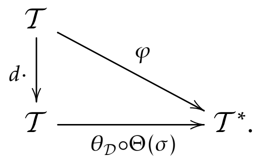
$$M\colon W^-(D,\sigma)\xrightarrow{\sim} W(X,\mathcal{L}_\sigma).$$

Proof. We first show that ext_X is injective. By a theorem of Karpenko [Kar10], the scalar extension map $\mathsf{ext}_F\colon W^-(D,\sigma)\to W^-(D_F,\sigma)$ is injective. The injectivity of ext_X then follows from the commutativity of the following diagram, where res_F is the restriction to the generic fiber:



As Mor is bijective, to complete the proof, it suffices to show M is onto. For this, we use Pumplün's results in [Pum99] (keeping in mind that Pumplün chooses as a generator for $\mathrm{Pic}(X)$ an invertible $\mathcal{O}_X$-module isomorphic to $\mathcal{H}om(\mathcal{L}_\sigma,\mathcal{O}_X)$ instead of $\mathcal{L}_\sigma$).

According to [Pum99, Theorem 4.3], $W(X,\mathcal{L}_\sigma)$ is generated by the Witt classes of symmetric bilinear spaces with underlying $\mathcal{O}_X$-module $\mathrm{tr}_{\ell/k}(\mathcal{N})$, where ℓ is a maximal separable subfield of D and $\mathcal{N}$ is a self-dual invertible $\mathcal{O}_{X_\ell}$-module. Since by Proposition 2.3, $(\mathcal{L}_\sigma)_\ell\simeq\mathcal{O}_{X_\ell}(-2)$, self-dual invertible $\mathcal{O}_{X_\ell}$-modules $\mathcal{N}$ for the duality $*$ are isomorphic to $\mathcal{O}_{X_\ell}(-1)$; hence $\mathrm{tr}_{\ell/k}(\mathcal{N})\simeq\mathcal{T}$ by Proposition 2.1. We can compare every isomorphism $\varphi\colon\mathcal{T}\to\mathcal{T}^*$ to the canonical isomorphism $\theta_{\mathcal{D}}\circ\Theta(\sigma)\colon\mathcal{T}\to\mathcal{T}^*$, viewing σ as an isomorphism $\mathcal{D}\to\mathcal{D}^\sharp$. Since $\mathrm{End}\,\mathcal{T}=D$ by Proposition 2.1, for every φ, there exists a $d\in D^\times$ such that the following diagram commutes:



On each stalk $\mathcal{T}_\mathfrak{p}$, the canonical isomorphism $\theta_{\mathcal{D}}\circ\Theta(\sigma)$ maps $t\in\mathcal{T}_\mathfrak{p}$ to the linear map $\mathcal{T}_\mathfrak{p}\to(\mathcal{L}_\sigma)_\mathfrak{p}$ that carries t' to $\sigma(t)t'$; hence $\varphi(t)$ maps t' to $\sigma(dt)t'$. The element d satisfies $\sigma(d)=-d$ since φ is skew-hermitian; hence the Witt class of $(\mathcal{T},\varphi)$ is the image under M of the Witt class of the skew-hermitian form $\langle -d\rangle$ over (D,σ). Therefore, the map M is onto. □

4. An octagon of Witt groups

Henceforth, we assume D is a quaternion division algebra; hence σ is the canonical conjugation involution $\bar{\ }$. We write simply $W^+(D)$ (resp. $W^-(D)$) for the Witt group of hermitian (resp. skew-hermitian) forms over D.

Let i, $j\in D$ be nonzero anticommuting quaternions, and let $K=k(i)\subset D$. We have $D=K\oplus jK$; hence for every ε-hermitian form $h\colon V\times V\to D$ (with $\varepsilon=\pm1$) on a right D-vector space V, we may define an ε-hermitian form $f\colon V\times V\to K$ (for the nontrivial automorphism $\bar{\ }$ on K) and a $(-\varepsilon)$-symmetric bilinear form $g\colon V\times V\to K$ by the equation

$$h(v,v')=f(v,v')+jg(v,v')\quad\text{for } v,\ v'\in V.$$

We thus obtain Witt group homomorphisms

$$\pi_1\colon W^\varepsilon(D)\longrightarrow W^\varepsilon(K,\bar{\ }),\quad h\longmapsto f\quad\text{and}\quad \pi_2\colon W^\varepsilon(D)\longrightarrow W^{-\varepsilon}(K),\quad h\longmapsto g;$$

see [Sch85, Lemma 10.3.1].(2) Computation yields an explicit description of $\pi_2\colon W^-(D)\to W(K)$ $(= W^+(K))$: For $h=\langle q\rangle$, with $q=iq_0+jq_1$, $q_0\in k$, and $q_1\in K$, we have

$$\pi_2(\langle q\rangle)=\begin{cases}\langle q_1\rangle\langle 1,-q^2\rangle & \text{if } q_1\neq 0,\\ 0 & \text{if } q_1=0.\end{cases} \tag{4.1}$$

Indeed, for all $\lambda\in K$ and $\mu\in K$, we have $g(\lambda+j\mu)=q_1\lambda^2-2iq_0\lambda\mu-b\bar{q}_1\mu^2$. Hence this quadratic form represents q_1 and has discriminant q^2.

We may also define maps in the opposite direction using scaled base change. More precisely, for every ε-hermitian form $f\colon U\times U\to K$ on a K-vector space U, there is a unique $(-\varepsilon)$-hermitian form

$$h\colon (U\otimes_K D)\times(U\otimes_K D)\longrightarrow D$$

such that $h(u,u')=f(u,u')i$ for u, $u'\in U$. Similarly, for every ε-symmetric bilinear form $g\colon U\times U\to K$ on a K-vector space U, there is a unique $(-\varepsilon)$-hermitian form

$$h'\colon (U\otimes_K D)\times(U\otimes_K D)\longrightarrow D$$

such that $h'(u,u')=ijg(u,u')$ for u, $u'\in U$. Thus, for $\varepsilon=\pm1$, we obtain Witt group homomorphisms

$$\sigma_1\colon W^{\varepsilon}(K,{}^-)\longrightarrow W^{-\varepsilon}(D),\quad f\longmapsto h\quad\text{and}\quad \sigma_2\colon W^{\varepsilon}(K)\longrightarrow W^{-\varepsilon}(D),\quad g\longmapsto h'.$$

(Of course, $W^-(K)=0$.)

Theorem 4.1. *The following octagon is exact:*

$$\begin{array}{ccccc}
W^+(D) & \xrightarrow{\pi_2} & W^-(K) & \xrightarrow{\sigma_2} & W^+(D)\\
\uparrow{\scriptstyle\sigma_1} & & & & \downarrow{\scriptstyle\pi_1}\\
W^-(K,{}^-) & & & & W^+(K,{}^-)\\
\uparrow{\scriptstyle\pi_1} & & & & \downarrow{\scriptstyle\sigma_1}\\
W^-(D) & \xleftarrow{\sigma_2} & W^+(K) & \xleftarrow{\pi_2} & W^-(D).
\end{array}$$

Proof. The exactness of the five-term sequence from $W^-(K)$ to $W^+(K)$ is proved in [Sch85, Theorem 10.3.2]. The same arguments can be used to prove the exactness of the other half; see also [Lew82, Proposition 2] or [GBM05, Section 6]. □

From here on, we omit the superscripts $+$. To define the first map in (1.2) and the last map in (1.3), note that every hermitian form on D has a diagonalization with coefficients in k; hence scalar extension yields a surjective group homomorphism

$$\mathrm{ext}_D\colon W(k)\longrightarrow W(D).$$

On the other hand, for every hermitian form h on D, the map $q_h\colon v\mapsto h(v,v)$ is a quadratic form on k, and mapping h to q_h yields a group homomorphism

$$s_D\colon W(D)\longrightarrow W(k).$$

The following result is proved in [Sch85, Theorem 10.1.7].

(2) There are several typos on p. 359 of [Sch85].

Lemma 4.2. *The following diagram, where n_D denotes multiplication by the norm form of D, commutes and has exact row and column:*

$$\begin{array}{ccccc} & & W(k) & & \\ & & \downarrow{\scriptstyle \mathrm{ext}_D} & \searrow^{n_D} & \\ 0 & \longrightarrow & W(D) & \xrightarrow{s_D} & W(k) \\ & & \downarrow & & \\ & & 0. & & \end{array}$$

5. Residues and transfers

Recall that D is now assumed to be a quaternion division algebra; hence its Severi–Brauer variety X is a smooth projective conic. The maps in the exact sequences (1.2) and (1.3) depend on the choice of uniformizers $\pi_\mathfrak{p}$ and linear functionals $s_\mathfrak{p}$ at each closed point $\mathfrak{p} \in X^{(1)}$. In order to make suitable choices, we first introduce coordinates, which will allow us to write an equation for X and to identify the invertible $\mathcal{O}_X$-module $\mathcal{L}_\sigma$ with the ideal sheaf of a point $\infty \in X^{(1)}$. Since there is a unique symplectic involution σ on D, namely, the conjugation involution $\bar{\ }$, we simplify the notation by writing $\mathcal{L}$ for $\mathcal{L}_\sigma$.

5.1. Coordinatization

Let D^0 be the 3-dimensional k-vector space of pure quaternions in D. We have $d^2 \in k$ for every $d \in D^0$; hence the map $q\colon D^0 \to k$ defined by $q(d) = d^2$ is a quadratic form. Since every 2-dimensional left ideal of D over a splitting field intersects D^0 in a line (see Lemma 2.2), we may identify X with the conic in the projective plane $\mathbb{P}(D^0)$ given by the equation $q = 0$. Under this identification, every line in D^0 corresponds to the left ideal of D that it generates.

Let $i, j \in D$ be two nonzero anticommuting pure quaternions, and let $i^2 = a$, $j^2 = b$, so that

$$D = (a,b)_k.$$

The elements aj, ij, bi form a k-base of D^0. If ξ, η, ζ denotes the dual base, the conic X is given by the equation

$$(aj\xi + ij\eta + bi\zeta)^2 = 0, \quad \textit{i.e.,} \quad a\xi^2 - \eta^2 + b\zeta^2 = 0.$$

Let $\infty \in X^{(1)}$ be the closed point given by the equation $\zeta = 0$; the residue field $k(\infty)$ at ∞ is canonically isomorphic to $k(i)$ by a map that carries the value $\frac{\eta}{\xi}(\infty)$ of the function $\frac{\eta}{\xi}$ at ∞ to i; see [EKM08, Proposition 45.12]. Let also $X_{\mathrm{af}} \subset X$ be the open subscheme defined by $\zeta \neq 0$, which is an affine conic, and let $\mathcal{O}_{\mathrm{af}}$ be the affine ring of X_{af}; then, writing $x = \frac{\xi}{\zeta}$ and $y = \frac{\eta}{\zeta}$, we have

$$\mathcal{O}_{\mathrm{af}} = k[x,y] \subset k(x,y) = F \quad \text{with } y^2 = ax^2 + b.$$

The point with coordinates (x,y) on the affine conic $X_{\mathrm{af}}(F)$ is $ajx + ijy + bi$; under the identification of the conic with the Severi–Brauer variety of D, it is the F-rational point obtained by base change from the generic point T of X. Therefore, the generic fiber of the sheaf $\mathcal{T}$ defined in (2.1) is

$$T = D_F e, \quad \text{where } e = bi + axj + yij \in D_F.$$

Clearly, $e \in T \cap \overline{T}$ since $\bar{e} = -e$; hence the generic fiber $T \cap \overline{T}$ of $\mathcal{L}$ is $L = eF$. We next describe the module of affine sections and the stalk at ∞ of $\mathcal{T}$ and $\mathcal{L}$, for which we use the notation $\mathcal{T}_{\mathrm{af}}$, $\mathcal{T}_\infty$, $\mathcal{L}_{\mathrm{af}}$, $\mathcal{L}_\infty$. We write $\mathcal{D}_{\mathrm{af}} = D \otimes \mathcal{O}_{\mathrm{af}}$ for the module of sections of $\mathcal{D}$ over X_{af} and $\mathcal{D}_\infty = D \otimes \mathcal{O}_\infty$ for the stalk of $\mathcal{D}$ at ∞.

Proposition 5.1. *We have $T = eF + jeF = jeF + ijeF$ and*

$$\mathcal{T}_{\mathrm{af}} = \mathcal{D}_{\mathrm{af}} e, \quad \mathcal{T}_\infty = \mathcal{D}_\infty e x^{-1}, \quad \mathcal{L}_{\mathrm{af}} = e\mathcal{O}_{\mathrm{af}}, \quad \mathcal{L}_\infty = ex^{-1}\mathcal{O}_\infty.$$

Proof. Because $e^2 = 0$, we have

$$bie + axje + yije = 0. \tag{5.1}$$

Multiplying on the left by i^{-1}, we obtain

$$be + xije + yje = 0.$$

Since $T = D_F e = eF + ieF + jeF + ijeF$, the first assertion follows from these equations.

By definition,

$$\mathcal{T}_{\mathrm{af}} = T \cap \mathcal{D}_{\mathrm{af}} = (D_F e) \cap \mathcal{D}_{\mathrm{af}} \quad \text{and} \quad \mathcal{L}_{\mathrm{af}} = L \cap \mathcal{D}_{\mathrm{af}} = (eF) \cap \mathcal{D}_{\mathrm{af}}.$$

Since $e \in \mathcal{D}_{\mathrm{af}}$, the inclusions $\mathcal{D}_{\mathrm{af}} e \subset \mathcal{T}_{\mathrm{af}}$ and $e\mathcal{O}_{\mathrm{af}} \subset \mathcal{L}_{\mathrm{af}}$ are clear. If $\lambda \in F$ is such that $e\lambda \in \mathcal{D}_{\mathrm{af}}$, then by looking at the coefficient of i in $e\lambda$, we see that $\lambda \in \mathcal{O}_{\mathrm{af}}$. Therefore, $\mathcal{L}_{\mathrm{af}} = e\mathcal{O}_{\mathrm{af}}$. Now, the first part of the proof shows that every element in T can be written as $je\lambda + ije\mu$ for some λ, $\mu \in F$. If λ, μ are such that $je\lambda + ije\mu \in \mathcal{D}_{\mathrm{af}}$, then inspection of the coefficients of j and ij shows that λ, $\mu \in \mathcal{O}_{\mathrm{af}}$. Therefore, $je\lambda + ije\mu \in \mathcal{D}_{\mathrm{af}} e$, and it follows that $\mathcal{T}_{\mathrm{af}} = \mathcal{D}_{\mathrm{af}} e$.

Next, we consider the stalks at ∞. By definition,

$$\mathcal{T}_\infty = T \cap \mathcal{D}_\infty = (D_F e) \cap \mathcal{D}_\infty \quad \text{and} \quad \mathcal{L}_\infty = L \cap \mathcal{D}_\infty = (eF) \cap \mathcal{D}_\infty.$$

Since $ex^{-1} \in \mathcal{D}_\infty$, we have $ex^{-1} \in \mathcal{T}_\infty$ and $ex^{-1} \in \mathcal{L}_\infty$, so the inclusions $\mathcal{D}_\infty ex^{-1} \subset \mathcal{T}_\infty$ and $ex^{-1}\mathcal{O}_\infty \subset \mathcal{L}_\infty$ are clear. If $\lambda \in F$ is such that $e\lambda \in \mathcal{D}_\infty$, then the coefficient of j shows that $x\lambda \in \mathcal{O}_\infty$, hence $e\lambda \in ex^{-1}\mathcal{O}_\infty$. To complete the description of $\mathcal{T}_\infty$, we use the equation $T = eF + jeF$ proven above. If $e\lambda + je\mu \in \mathcal{D}_\infty$ for some λ, $\mu \in F$, then by looking at the coefficients of 1 and j, we see that $x\lambda$, $x\mu \in \mathcal{O}_\infty$. Therefore, $e\lambda + je\mu \in \mathcal{D}_\infty ex^{-1}$. So we get $\mathcal{L}_\infty = ex^{-1}\mathcal{O}_\infty$ and $\mathcal{T}_\infty = \mathcal{D}_\infty ex^{-1}$. □

From the descriptions of $\mathcal{L}_{\mathrm{af}}$ and $\mathcal{L}_\infty$ above, it follows that mapping $e \in L$ to $1 \in F$ defines an isomorphism of invertible $\mathcal{O}_X$-modules

$$\varepsilon\colon \mathcal{L} \xrightarrow{\sim} \mathcal{I}(\infty), \tag{5.2}$$

where $\mathcal{I}(\infty)$ is the ideal sheaf of ∞, *i.e.*, the subsheaf of $\mathcal{O}_X$ whose module of affine sections is $\mathcal{O}_{\mathrm{af}}$ and whose stalk at ∞ is the maximal ideal $\mathfrak{m}_\infty = x^{-1}\mathcal{O}_\infty$ of $\mathcal{O}_\infty$.

5.2. Coherent choices

For each point $\mathfrak{p} \in X^{(1)}$, let $\pi_\mathfrak{p}$ be a uniformizer of the local ring $\mathcal{O}_\mathfrak{p}$. Two residue maps

$$\partial^1_\mathfrak{p},\ \partial^2_\mathfrak{p}\colon W(F) \longrightarrow W(k(\mathfrak{p}))$$

are defined as follows: Select in every symmetric bilinear space (W, b) over F an $\mathcal{O}_\mathfrak{p}$-lattice of the form $W_1 \perp W_2$, such that the restrictions b_1 of b to W_1 and b_2 of $\langle \pi_\mathfrak{p}^{-1}\rangle b$ to W_2 are nonsingular (as $\mathcal{O}_\mathfrak{p}$-bilinear forms). Then $\partial^i_\mathfrak{p}$ maps (W, b) to the Witt class of $(W_i \otimes_{\mathcal{O}_\mathfrak{p}} k(\mathfrak{p}), (b_i)_{k(\mathfrak{p})})$. Thus, $\partial^1_\mathfrak{p}$ does not depend on the choice of $\pi_\mathfrak{p}$, but $\partial^2_\mathfrak{p}$ does. If the bilinear space (W, b) is the generic fiber of a symmetric bilinear space $(\mathcal{W}, b)$ over X with values in $\mathcal{O}_X$, then by definition[(3)]

$$\partial^1_\mathfrak{p}(W, b) = \big(\mathcal{W}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} k(\mathfrak{p}), b_{k(\mathfrak{p})}\big) \quad \text{and} \quad \partial^2_\mathfrak{p}(W, b) = 0 \quad \text{for all } \mathfrak{p} \in X^{(1)}.$$

In contrast, if (W, b) is the generic fiber of a symmetric bilinear space $(\mathcal{W}, b)$ with values in $\mathcal{I}(\infty)$, then

$$\partial^1_\mathfrak{p}(W, b) = \begin{cases} (\mathcal{W}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} k(\mathfrak{p}), b_{k(\mathfrak{p})}) & \text{if } \mathfrak{p} \neq \infty, \\ 0 & \text{if } \mathfrak{p} = \infty, \end{cases}$$

(3) We abuse notation by not distinguishing between a space and its Witt class.

and there exists an $\alpha \in k(\infty)^\times$ depending on the choice of π_∞ such that

$$\partial^2_{\mathfrak{p}}(W,b) = \begin{cases} 0 & \text{if } \mathfrak{p} \neq \infty, \\ (\mathcal{W}_\infty \otimes_{\mathcal{O}_\infty} k(\infty), \langle \alpha \rangle b_{k(\infty)}) & \text{if } \mathfrak{p} = \infty. \end{cases}$$

It follows that the maps

$$\delta = \oplus_{\mathfrak{p}} \partial^2_{\mathfrak{p}} \colon W(F) \longrightarrow \bigoplus_{\mathfrak{p} \in X^{(1)}} W(k(\mathfrak{p})) \quad \text{and} \quad \delta' = \left(\oplus_{\mathfrak{p}\neq\infty} \partial^2_{\mathfrak{p}}\right) \oplus \partial^1_\infty \colon W(F) \longrightarrow \bigoplus_{\mathfrak{p} \in X^{(1)}} W(k(\mathfrak{p}))$$

vanish on the images of $W(X)$ and $W(X, \mathcal{I}(\infty))$, respectively. These maps fit in the exact sequences (1.1).

To extend these exact sequences further, we use transfer maps $W(k(\mathfrak{p})) \to W(k)$. Since δ and δ' depend on the choice of $\pi_{\mathfrak{p}}$, we need to make a coherent choice for these transfers. With this in mind, we consider the Weil differential $\omega = \frac{dx}{2y}$, which is uniquely determined up to a factor in $k^\times$ by the condition that its divisor is $-\infty$ (see [Che51, Section II.5]). Thus, for $\mathfrak{p} \in X^{(1)}$, the $\mathfrak{p}$-component $\omega_{\mathfrak{p}}$ is a linear map $F \to k$ that vanishes on $\mathcal{O}_{\mathfrak{p}}$ if $\mathfrak{p} \neq \infty$ and on $\mathfrak{m}_\infty$ if $\mathfrak{p} = \infty$. Abusing notation, we again write $\omega_{\mathfrak{p}}$ for the following k-linear maps induced by the local components of ω:

$$\omega_{\mathfrak{p}} \colon \mathfrak{m}_{\mathfrak{p}}^{-1}/\mathcal{O}_{\mathfrak{p}} \longrightarrow k \quad \text{for } \mathfrak{p} \neq \infty, \quad \text{and} \quad \omega_\infty \colon \mathcal{O}_\infty/\mathfrak{m}_\infty = k(\infty) \longrightarrow k.$$

For $\omega = \frac{dx}{2y}$, the computation in [Che51, Section VI.3] shows that ω_∞ is defined by

$$\omega_\infty(1) = 0 \quad \text{and} \quad \omega_\infty\left(\frac{y}{x}(\infty)\right) = -1, \tag{5.3}$$

where $\frac{y}{x}(\infty)$ is the image of $x^{-1}y \in \mathcal{O}_\infty$ in $k(\infty)$. In the following description of the maps $\omega_{\mathfrak{p}}$ for $\mathfrak{p} \neq \infty$, we write $v_{\mathfrak{q}}$ for the (normalized) $\mathfrak{q}$-adic valuation on F, for every $\mathfrak{q} \in X^{(1)}$.

Proposition 5.2. *For $\mathfrak{p} \neq \infty$, every element in $\mathfrak{m}_{\mathfrak{p}}^{-1}/\mathcal{O}_{\mathfrak{p}}$ can be represented in the form $f + \mathcal{O}_{\mathfrak{p}}$ for some $f \in \mathfrak{m}_{\mathfrak{p}}^{-1}$ such that $v_{\mathfrak{q}}(f) \geq 0$ for all $\mathfrak{q} \neq \mathfrak{p}$. For such f,*

$$\omega_{\mathfrak{p}}(f) = -\omega_\infty(f(\infty)).$$

Proof. As in [Che51, Section II.1], let[4]

$$\mathfrak{L}(-\mathfrak{p}) = \{f \in F \mid v_{\mathfrak{p}}(f) \geq -1 \text{ and } v_{\mathfrak{q}}(f) \geq 0 \text{ for } \mathfrak{q} \neq \mathfrak{p}\}.$$

Since F is an algebraic function field of genus zero, it follows from the Riemann–Roch theorem that $\mathfrak{L}(-\mathfrak{p})$ is a k-vector space of dimension $1 + \deg \mathfrak{p}$; see [Che51, Corollary, p. 32]. Consider the k-linear map

$$\varphi \colon \mathfrak{L}(-\mathfrak{p}) \to \mathfrak{m}_{\mathfrak{p}}^{-1}/\mathcal{O}_{\mathfrak{p}} \quad \text{given by } \varphi(f) = f + \mathcal{O}_{\mathfrak{p}}.$$

Its kernel consists of elements $f \in F$ such that $v_{\mathfrak{q}}(f) \geq 0$ for all $\mathfrak{q} \in X^{(1)}$, *i.e.*, $\ker \varphi = k$. Since $\dim(\mathfrak{m}_{\mathfrak{p}}^{-1}/\mathcal{O}_{\mathfrak{p}}) = \deg \mathfrak{p}$, dimension count shows that φ is onto, which proves the first statement. For $f \in \mathfrak{L}(-\mathfrak{p})$, we have $f \in \mathcal{O}_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \mathfrak{p}$, hence $\omega_{\mathfrak{q}}(f) = 0$ for all $\mathfrak{q} \neq \mathfrak{p}, \infty$. Since Weil differentials vanish on F, it follows that

$$\omega_{\mathfrak{p}}(f) + \omega_\infty(f) = 0,$$

which completes the proof. □

If $\pi_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ is a uniformizer at $\mathfrak{p}$, then $\mathfrak{m}_{\mathfrak{p}}^{-1} = \pi_{\mathfrak{p}}^{-1}\mathcal{O}_{\mathfrak{p}}$, and multiplication by $\pi_{\mathfrak{p}}$ defines an isomorphism of $k(\mathfrak{p})$-vector spaces

$$\mu_{\pi_{\mathfrak{p}}} \colon \mathfrak{m}_{\mathfrak{p}}^{-1}/\mathcal{O}_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}} = k(\mathfrak{p}).$$

[4] We use Chevalley's notation from [Che51]. Most references use the notation $L(\mathfrak{p})$ for Chevalley's $\mathfrak{L}(-\mathfrak{p})$.

Definition 5.3. A choice of uniformizer $\pi_\mathfrak{p}$ and of k-linear functional $s_\mathfrak{p}\colon k(\mathfrak{p}) \to k$ is said to be *coherent* at $\mathfrak{p} \in X_{\mathrm{af}}^{(1)}$ if the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{m}_\mathfrak{p}^{-1}/\mathcal{O}_\mathfrak{p} & \xrightarrow{\;\omega_\mathfrak{p}\;} & k \\ & \searrow^{\mu_{\pi_\mathfrak{p}}} \qquad \nearrow_{s_\mathfrak{p}} & \\ & k(\mathfrak{p}). & \end{array}$$

Proposition 5.4. *Let $\pi_\mathfrak{p}$, $s_\mathfrak{p}$ and $\pi'_\mathfrak{p}$, $s'_\mathfrak{p}$ be coherent choices of uniformizer and linear functional at $\mathfrak{p} \in X_{\mathrm{af}}^{(1)}$. The corresponding residue maps $\partial_\mathfrak{p}^2$, ${\partial'_\mathfrak{p}}^2\colon W(F) \to W(k(\mathfrak{p}))$ and transfer maps $(s_\mathfrak{p})_*$, $(s'_\mathfrak{p})_*\colon W(k(\mathfrak{p})) \to W(k)$ make the following diagram commute:*

$$\begin{CD} W(F) @>{\partial_\mathfrak{p}^2}>> W(k(\mathfrak{p})) \\ @V{{\partial'_\mathfrak{p}}^2}VV @VV{(s_\mathfrak{p})_*}V \\ W(k(\mathfrak{p})) @>{(s'_\mathfrak{p})_*}>> W(k). \end{CD}$$

Proof. Let $u = \pi'_\mathfrak{p}\pi_\mathfrak{p}^{-1} \in \mathcal{O}_\mathfrak{p}^\times$, and let $u(\mathfrak{p})$ be the image of u in $k(\mathfrak{p})^\times$. Then $\partial_\mathfrak{p}^2(\langle f\rangle) = \langle u(\mathfrak{p})\rangle {\partial'_\mathfrak{p}}^2(\langle f\rangle)$ for all $f \in F^\times$. On the other hand, $\mu_{\pi'_\mathfrak{p}} = u(\mathfrak{p})\mu_{\pi_\mathfrak{p}}$; hence $s_\mathfrak{p}(g) = s'_\mathfrak{p}(u(\mathfrak{p})g)$ for all $g \in k(\mathfrak{p})$ because $s_\mathfrak{p} \circ \mu_{\pi_\mathfrak{p}} = s'_\mathfrak{p} \circ \mu_{\pi'_\mathfrak{p}}$ as $\pi_\mathfrak{p}$, $s_\mathfrak{p}$ and $\pi'_\mathfrak{p}$, $s'_\mathfrak{p}$ are coherent choices. Therefore, the following diagrams commute:

$$\begin{array}{ccc} W(F) & \xrightarrow{\;\partial_\mathfrak{p}^2\;} & W(k(\mathfrak{p})) \\ {\scriptstyle {\partial'_\mathfrak{p}}^2}\big\downarrow & \nearrow_{\langle u(\mathfrak{p})\rangle} & \\ W(k(\mathfrak{p})), & & \end{array} \qquad\qquad \begin{array}{ccc} & & W(k(\mathfrak{p})) \\ & \swarrow^{\langle u(\mathfrak{p})\rangle} & \big\downarrow{\scriptstyle (s_\mathfrak{p})_*} \\ W(k(\mathfrak{p})) & \xrightarrow{\;(s'_\mathfrak{p})_*\;} & W(k). \end{array}$$

The proposition follows. □

5.3. Transfer maps

Besides the transfer maps $(s_\mathfrak{p})_*\colon W(k(\mathfrak{p})) \to W(k)$, which fit in the exact sequence (1.3), we also need transfer maps $W(k(\mathfrak{p})) \to W^-(D)$ to complete the sequence (1.2). For any nonzero linear functional $s_\mathfrak{p}\colon k(\mathfrak{p}) \to k$, the map $s_{D(\mathfrak{p})} = \mathrm{Id}_D \otimes s_\mathfrak{p}\colon D(\mathfrak{p}) \to D$ is D-linear for the left and right D-vector space structures on $D(\mathfrak{p})$, and commutes with quaternion conjugation. Moreover, if $\xi \in D(\mathfrak{p})$ is such that $s_{D(\mathfrak{p})}(\overline{\xi}\eta) = 0$ for all $\eta \in D(\mathfrak{p})$, then writing $\xi = 1 \otimes \xi_0 + i \otimes \xi_1 + j \otimes \xi_2 + ij \otimes \xi_3$ with $\xi_0, \ldots, \xi_3 \in k(\mathfrak{p})$, we get

$$s_{D(\mathfrak{p})}(\overline{\xi} \cdot 1 \otimes \zeta) = s_\mathfrak{p}(\xi_0\zeta) - is_\mathfrak{p}(\xi_1\zeta) - js_\mathfrak{p}(\xi_2\zeta) - ijs_\mathfrak{p}(\xi_3\zeta) = 0 \quad \text{for all } \zeta \in k(\mathfrak{p}),$$

hence $\xi_0 = \cdots = \xi_3 = 0$. It follows that $s_{D(\mathfrak{p})}$ is an involution trace in the sense of [Knu91, Section I.7.2, p. 40]. It induces a homomorphism of Witt groups (see [Knu91, Section I.10.3, p. 62])

$$(s_{D(\mathfrak{p})})_*\colon W^-(D(\mathfrak{p})) \longrightarrow W^-(D).$$

On the other hand, restricting the canonical isomorphism Mor of Proposition 3.2 to the fiber at $\mathfrak{p}$, we obtain an isomorphism

$$\mathsf{Mor}_\mathfrak{p}\colon W^-(D(\mathfrak{p})) \xrightarrow{\sim} W(k(\mathfrak{p}), L(\mathfrak{p})).$$

Next, consider the restriction of the isomorphism $\varepsilon\colon \mathcal{L} \to \mathcal{I}(\infty)$ of (5.2) to the fiber at $\mathfrak{p}$. If $\mathfrak{p} \neq \infty$, the fiber of $\mathcal{I}(\infty)$ at $\mathfrak{p}$ is $k(\mathfrak{p})$; hence $\varepsilon_\mathfrak{p}$ yields an isomorphism

$$(\varepsilon_\mathfrak{p})_*\colon W(k(\mathfrak{p}), L(\mathfrak{p})) \xrightarrow{\sim} W(k(\mathfrak{p})).$$

We let $t_{\mathfrak{p}}$ denote the composition (which depends on the choice of the linear functional $s_{\mathfrak{p}}$)

$$t_{\mathfrak{p}} = (s_{D(\mathfrak{p})})_* \circ \mathsf{Mor}_{\mathfrak{p}}^{-1} \circ (\varepsilon_{\mathfrak{p}})_*^{-1} : W(k(\mathfrak{p})) \longrightarrow W^-(D) \quad \text{for } \mathfrak{p} \in X_{\mathrm{af}}^{(1)}.$$

The fiber of $\mathcal{I}(\infty)$ at ∞ is $\mathfrak{m}_\infty/\mathfrak{m}_\infty^2$; hence the fiber of ε at ∞ yields an isomorphism

$$(\varepsilon_\infty)_* : W(k(\infty), L(\infty)) \xrightarrow{\sim} W\Big(k(\infty), \mathfrak{m}_\infty/\mathfrak{m}_\infty^2\Big).$$

Choosing a uniformizer π_∞ at ∞, we define a $k(\infty)$-linear isomorphism

$$\mu_{\pi_\infty} : \mathfrak{m}_\infty/\mathfrak{m}_\infty^2 \xrightarrow{\sim} k(\infty) \quad \text{by} \quad \mu_{\pi_\infty}\Big(f + \mathfrak{m}_\infty^2\Big) = \frac{f}{\pi_\infty}(\infty) \quad \text{for } f \in \mathfrak{m}_\infty,$$

hence an isomorphism

$$\Big(\mu_{\pi_\infty}\Big)_* : W\Big(k(\infty), \mathfrak{m}_\infty/\mathfrak{m}_\infty^2\Big) \xrightarrow{\sim} W(k(\infty)).$$

Choosing $s_\infty = \omega_\infty$ (defined in (5.3)), we mimic the definition of $t_{\mathfrak{p}}$ for $\mathfrak{p} \in X_{\mathrm{af}}^{(1)}$ and set

$$t_\infty = \Big(s_{D_\infty}\Big)_* \circ \mathsf{Mor}_\infty^{-1} \circ (\varepsilon_\infty)_*^{-1} \circ \Big(\mu_{\pi_\infty}\Big)_*^{-1} : W(k(\infty)) \longrightarrow W^-(D).$$

Note that the map t_∞ depends on the choice of uniformizer π_∞ via μ_{π_∞}.

We next give an explicit description of the transfer maps $t_{\mathfrak{p}}$. Recall that for every $\mathfrak{p} \in X^{(1)}$, we write $D(\mathfrak{p}) = D \otimes_k k(\mathfrak{p})$ for the fiber of $\mathcal{D}$ at $\mathfrak{p}$. We also let $T(\mathfrak{p}) = \mathcal{T}_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} k(\mathfrak{p})$ denote the fiber of $\mathcal{T}$ at $\mathfrak{p}$. If $\mathfrak{p} \neq \infty$, we write $e_{\mathfrak{p}}$ for the image of $e = bi + axj + yij$ in $T(\mathfrak{p})$; similarly, we let e_∞ denote the image of ex^{-1} in $T(\infty)$. Thus, from Proposition 5.1 it follows that

$$T(\mathfrak{p}) = D(\mathfrak{p})e_{\mathfrak{p}}, \quad \text{hence} \quad \overline{T(\mathfrak{p})} = e_{\mathfrak{p}}D(\mathfrak{p}) \quad \text{for all } \mathfrak{p} \in X^{(1)}.$$

Proposition 5.5. *Let $\mathfrak{p} \in X^{(1)}$ and $f \in k(\mathfrak{p})^\times$. If $\mathfrak{p} = \infty$, define t_∞ by selecting x^{-1} as a uniformizer. For all $\mathfrak{p} \in X^{(1)}$, the Witt class of $t_{\mathfrak{p}}(\langle f\rangle)$ is represented by the transfer along $s_{D(\mathfrak{p})}$ of the skew-hermitian form*

$$h_f : \overline{T(\mathfrak{p})} \times \overline{T(\mathfrak{p})} \longrightarrow D(\mathfrak{p}) \quad \textit{defined by} \quad h_f(e_{\mathfrak{p}}\xi, e_{\mathfrak{p}}\eta) = f\,\overline{\xi}\,e_{\mathfrak{p}}\eta \quad \textit{for } \xi, \eta \in D(\mathfrak{p}).$$

Proof. First, suppose $\mathfrak{p} \neq \infty$. It suffices to show that the skew-hermitian form h_f satisfies $(\varepsilon_{\mathfrak{p}})_*\Big(\mathsf{Mor}_{\mathfrak{p}}(h_f)\Big) = \langle f\rangle$. By definition,

$$\mathsf{Mor}_{\mathfrak{p}}(h_f) : \Big(\overline{T(\mathfrak{p})} \otimes_{D(\mathfrak{p})} T(\mathfrak{p})\Big) \times \Big(\overline{T(\mathfrak{p})} \otimes_{D(\mathfrak{p})} T(\mathfrak{p})\Big) \longrightarrow \overline{T(\mathfrak{p})} \cdot T(\mathfrak{p}) = L(\mathfrak{p})$$

carries $(e_{\mathfrak{p}}\xi_1 \otimes \xi_2 e_{\mathfrak{p}}, e_{\mathfrak{p}}\eta_1 \otimes \eta_2 e_{\mathfrak{p}})$ to $-f\,e_{\mathfrak{p}}\overline{\xi_2\xi_1}e_{\mathfrak{p}}\eta_1\eta_2 e_{\mathfrak{p}}$ for ξ_1, ξ_2, η_1, $\eta_2 \in D(\mathfrak{p})$. Note that $\overline{T(\mathfrak{p})} \otimes_{D(\mathfrak{p})} T(\mathfrak{p})$ is a $k(\mathfrak{p})$-vector space of dimension 1, isomorphic to $k(\mathfrak{p})$ under the composition

$$\overline{T(\mathfrak{p})} \otimes_{D(\mathfrak{p})} T(\mathfrak{p}) \xrightarrow{m_{\mathfrak{p}}} L(\mathfrak{p}) \xrightarrow{\varepsilon_{\mathfrak{p}}} k(\mathfrak{p}),$$

which maps $e_{\mathfrak{p}}\xi_1 \otimes \xi_2 e_{\mathfrak{p}}$ to $\mathrm{Trd}(\xi_1\xi_2 e_{\mathfrak{p}})$ (see (5.2)). If $\xi \in D(\mathfrak{p})$ is such that $\mathrm{Trd}(\xi e_{\mathfrak{p}}) = 1$, then $e_{\mathfrak{p}}\xi \otimes e_{\mathfrak{p}}$ is a $k(\mathfrak{p})$-base of $\overline{T(\mathfrak{p})} \otimes_{D(\mathfrak{p})} T(\mathfrak{p})$, and

$$\varepsilon_{\mathfrak{p}}\Big(\mathsf{Mor}_{\mathfrak{p}}\Big(h_f\Big)\Big(e_{\mathfrak{p}}\xi \otimes e_{\mathfrak{p}}, e_{\mathfrak{p}}\xi \otimes e_{\mathfrak{p}}\Big)\Big) = -f\,\mathrm{Trd}\Big(\overline{\xi}e_{\mathfrak{p}}\xi e_{\mathfrak{p}}\Big).$$

Since $e_{\mathfrak{p}}\xi e_{\mathfrak{p}} = e_{\mathfrak{p}}(\xi e_{\mathfrak{p}} - e_{\mathfrak{p}}\overline{\xi}) = e_{\mathfrak{p}}\,\mathrm{Trd}(\xi e_{\mathfrak{p}}) = e_{\mathfrak{p}}$, we have

$$-f\,\mathrm{Trd}\Big(\overline{\xi}e_{\mathfrak{p}}\xi e_{\mathfrak{p}}\Big) = -f\,\mathrm{Trd}\Big(\overline{\xi}e_{\mathfrak{p}}\Big) = f\,\mathrm{Trd}\Big(\overline{e_{\mathfrak{p}}}\overline{\xi}\Big) = f.$$

The proposition is thus proved for $\mathfrak{p} \neq \infty$.

For $\mathfrak{p} = \infty$, the map $\varepsilon_\infty : L(\infty) \to \mathfrak{m}_\infty/\mathfrak{m}_\infty^2$ carries e_∞ to $x^{-1} + \mathfrak{m}_\infty^2$; hence $\mu_{x^{-1}} \circ \varepsilon_\infty$ maps e_∞ to 1. The same arguments as in the case where $\mathfrak{p} \neq \infty$ yield the proof. $\square$

Proposition 5.6. *For $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$, let $\pi_{\mathfrak{p}}$, $s_{\mathfrak{p}}$ and $\pi'_{\mathfrak{p}}$, $s'_{\mathfrak{p}}$ be coherent choices of uniformizer and linear functional, and let $\partial^2_{\mathfrak{p}}$, ${\partial'_{\mathfrak{p}}}^2 \colon W(F) \to W(k(\mathfrak{p}))$ and $t_{\mathfrak{p}}$, $t'_{\mathfrak{p}} \colon W(k(\mathfrak{p})) \to W^-(D)$ be the corresponding residue and transfer maps. Similarly, let π_∞, π'_∞ be uniformizers at ∞, and let ∂^2_∞, ${\partial'_\infty}^2 \colon W(F) \to W(k(\infty))$ and t_∞, $t'_\infty \colon W(k(\infty)) \to W^-(D)$ be the corresponding residue and transfer maps. For every $\mathfrak{q} \in X^{(1)}$, the following diagram commutes:*

$$\begin{array}{ccc} W(F) & \xrightarrow{\partial^2_{\mathfrak{q}}} & W(k(\mathfrak{q})) \\ {\scriptstyle {\partial'_{\mathfrak{q}}}^2}\downarrow & & \downarrow{\scriptstyle t_{\mathfrak{q}}} \\ W(k(\mathfrak{q})) & \xrightarrow{t'_{\mathfrak{q}}} & W^-(D). \end{array}$$

Proof. For $\mathfrak{q} \neq \infty$, the same arguments as in the proof of Proposition 5.4 yield the proof. For $\mathfrak{q} = \infty$, it is readily verified that $(\mu_{\pi_\infty})^{-1}_* \circ \partial^2_\infty = (\mu_{\pi'_\infty})^{-1}_* \circ {\partial'_\infty}^2$. The proposition follows. □

6. Exactness of the sequences

In this section, we define the maps in the sequences (1.2) and (1.3), and prove their exactness. We start with the sequence (1.3). The isomorphism $\varepsilon \colon \mathcal{L} \xrightarrow{\sim} \mathcal{I}(\infty)$ (see (5.2)) yields a Witt group isomorphism

$$\varepsilon_* \colon W(X,\mathcal{L}) \xrightarrow{\sim} W(X,\mathcal{I}(\infty)).$$

Recall the isomorphism $M \colon W^-(D) \xrightarrow{\sim} W(X,\mathcal{L})$ of Theorem 3.3. Let $\rho \colon W^-(D) \to W(F)$ be the composition

$$W^-(D) \xrightarrow{M} W(X,\mathcal{L}) \xrightarrow{\varepsilon_*} W(X,\mathcal{I}(\infty)) \xrightarrow{\mathrm{res}_F} W(F).$$

Theorem 6.1. *Let $\pi_{\mathfrak{p}}$, $s_{\mathfrak{p}}$ be a coherent choice of uniformizer and linear functional at each point $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$, and let $s_\infty = \omega_\infty$. The following sequence is exact:*

$$0 \longrightarrow W^-(D) \xrightarrow{\rho} W(F) \xrightarrow{\delta'} \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})) \xrightarrow{\sum (s_{\mathfrak{p}})_*} W(k) \xrightarrow{\mathrm{ext}_D} W(D) \longrightarrow 0. \tag{6.1}$$

Proof. By the definition of ρ, the following diagram commutes:

$$\begin{array}{ccc} W^-(D) & \xrightarrow{\quad M \quad} & W(X,\mathcal{L}) \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle \mathrm{res}_F \circ \varepsilon_*} & \\ & W(F). & \end{array}$$

The exactness of the following sequence is the purity property of $W(X,\mathcal{I}(\infty))$ established in [BW02, Corollary 10.3]:

$$0 \longrightarrow W(X,\mathcal{I}(\infty)) \xrightarrow{\mathrm{res}_F} W(F) \xrightarrow{\delta'} \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})).$$

Since M and ε_* are isomorphisms, it follows that the sequence (6.1) is exact at $W^-(D)$ and $W(F)$. The exactness at $\bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p}))$ was proved by Parimala [Par88, Theorem 5.1]. (It is straightforward to check that the particular choice of uniformizers and linear functionals in [Par88, Section 4] is coherent, and Proposition 5.4 shows that the exactness of the sequence does not depend on this choice.) In [Pfi93, Theorem 6a], Pfister shows that the image of $\sum (s_{\mathfrak{p}})_*$ is the kernel of n_D (multiplication by the norm form of D). Therefore, the exactness at $W(k)$ and $W(D)$ follows from Lemma 4.2. □

For the rest of this section, we focus on the sequence (1.2). Our goal is to prove the following.

Theorem 6.2. *Let π_∞ be a uniformizer at ∞, and let ∂^2_∞ and t_∞ be the corresponding residue and transfer maps. For all $\mathfrak{q} \in X^{(1)}_{\mathrm{af}}$, let $\pi_\mathfrak{q}$, $s_\mathfrak{q}$ be a coherent choice of uniformizer and linear functional, and let $\partial^2_\mathfrak{q}$ and $t_\mathfrak{q}$ be the corresponding residue and transfer maps. The following sequence is exact:*

$$0 \longrightarrow W(D) \xrightarrow{s_D} W(k) \xrightarrow{\mathsf{ext}_F} W(F) \xrightarrow{\delta} \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})) \xrightarrow{\sum t_\mathfrak{p}} W^-(D) \longrightarrow 0. \tag{6.2}$$

We break the proof into several steps.

6.1. Exactness at the first three terms

The exactness of the sequence

$$0 \longrightarrow W(X) \xrightarrow{\mathsf{res}_F} W(F) \xrightarrow{\delta} \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})) \tag{6.3}$$

is the purity property of $W(X)$. It follows from Knebusch's general result [Kne70, Satz 13.3.6]. The first terms of the exact sequence (6.2) are obtained by pasting this sequence with the following.

Proposition 6.3. *The following sequence is exact:*

$$0 \longrightarrow W(D) \xrightarrow{s_D} W(k) \xrightarrow{\mathsf{ext}_X} W(X) \longrightarrow 0.$$

Proof. The scalar extension map $\mathsf{ext}_X \colon W(k) \to W(X)$ is known to be surjective; see [Pum98, Section 5]. (Pumplün's general result holds for arbitrary Severi–Brauer varieties. The case of conics is simpler; see [Pum98, Proposition 5.3] or [PSS01, Proposition 2.1].) Since restriction to the generic point is an injective map $W(X) \to W(F)$ (see (6.3)), the kernel of ext_X is also the kernel of the scalar extension map $W(k) \to W(F)$. The latter is the ideal generated by the norm form of D, see [Lam05, Corollary X.4.28], which by Lemma 4.2 can also be described as the image of the injective map s_D. The proposition follows. □

Note that the exactness of (6.2) at $W(k)$ and $W(F)$ has already been observed by Pfister [Pfi93, Theorem 4]. The rest of this section deals with the last two terms of this sequence.

6.2. Choice of uniformizers

Proposition 5.6 shows that for Theorem 6.2 the coherent choice of uniformizers and linear functionals is irrelevant. We make a specific choice as follows. At ∞, we choose x^{-1} as a uniformizer. To choose uniformizers at the points $\mathfrak{p} \neq \infty$, recall from [Pfi93, Proposition 1] or [MT16, Lemma A.9] that the affine ring $\mathcal{O}_{\mathrm{af}}$ is a principal ideal domain. Therefore, for each $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$, we may pick an irreducible element $\pi_\mathfrak{p} \in \mathcal{O}_{\mathrm{af}}$ generating the prime ideal $\mathcal{O}_{\mathrm{af}} \cap \mathfrak{m}_\mathfrak{p}$. The divisor of $\pi_\mathfrak{p}$ is thus $\mathfrak{p} + v_\infty(\pi_\mathfrak{p})\infty$; hence

$$\deg \mathfrak{p} = -2v_\infty(\pi_\mathfrak{p}) \tag{6.4}$$

because the degree of every principal divisor is 0. The element $\pi_\mathfrak{p}$ is a uniformizer at $\mathfrak{p}$, and the linear functional $s_\mathfrak{p} \colon k(\mathfrak{p}) \to k$ such that $\pi_\mathfrak{p}$, $s_\mathfrak{p}$ is coherent is uniquely determined. Moreover, the inclusion $\mathcal{O}_{\mathrm{af}} \subset \mathcal{O}_\mathfrak{p}$ induces a canonical isomorphism $\mathcal{O}_{\mathrm{af}}/\pi_\mathfrak{p}\mathcal{O}_{\mathrm{af}} = k(\mathfrak{p})$.

6.3. Nullity

We next show that the sequence in Theorem 6.2 is a zero sequence. Since we already know that it is exact at $W(D)$, $W(k)$, and $W(F)$, it suffices to prove

$$\sum_{\mathfrak{p}} t_\mathfrak{p}\Big(\partial^2_\mathfrak{p}(\langle f \rangle)\Big) = 0 \quad \text{for all } f \in F^\times.$$

We may assume $f \in \mathcal{O}_{\mathrm{af}}$ is square-free; hence $f = c\pi_1 \cdots \pi_n$ for some $c \in k^\times$ and some pairwise distinct irreducible elements $\pi_1, \ldots, \pi_n$ of $\mathcal{O}_{\mathrm{af}}$ selected in Section 6.2. Since $t_{\mathfrak{p}}$ and $\partial^2_{\mathfrak{p}}$ are $W(k)$-linear, we may moreover assume $c = 1$. Thus, for the rest of this subsection we fix

$$f = \pi_1 \cdots \pi_n \in \mathcal{O}_{\mathrm{af}}.$$

We let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in X^{(1)}_{\mathrm{af}}$ be the closed points corresponding to $\pi_1, \ldots, \pi_n$.

The primary decomposition of the $\mathcal{O}_{\mathrm{af}}$-module $f^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}$ is

$$f^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}} = (\pi_1^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}) \oplus \cdots \oplus (\pi_n^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}). \tag{6.5}$$

Multiplication by f defines an isomorphism $f^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}} \simeq \mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}}$; likewise, multiplication by π_α defines an isomorphism $\pi_\alpha^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}} \simeq \mathcal{O}_{\mathrm{af}}/\pi_\alpha\mathcal{O}_{\mathrm{af}} = k(\mathfrak{p}_\alpha)$ for $\alpha = 1, \ldots, n$. Hence we have an isomorphism of $\mathcal{O}_{\mathrm{af}}$-modules Φ which makes the following diagram commute:

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}} & \xrightarrow{\ \sim\ } & (\pi_1^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}) \oplus \cdots \oplus (\pi_n^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}) \\ {\scriptstyle f\cdot}\downarrow & & \downarrow{\scriptstyle (\pi_1\cdot)\oplus\cdots\oplus(\pi_n\cdot)} \\ \mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}} & \xrightarrow{\ \Phi\ } & k(\mathfrak{p}_1) \oplus \cdots \oplus k(\mathfrak{p}_n). \end{array}$$

In contrast with the isomorphism provided by the Chinese remainder theorem, the map Φ is *not* a ring homomorphism; it is readily verified that for $f_1, \ldots, f_n \in \mathcal{O}_{\mathrm{af}}$,

$$\Phi^{-1}\left(f_1(\mathfrak{p}_1), \ldots, f_n(\mathfrak{p}_n)\right) = (f_1\pi_2 \cdots \pi_n) + \cdots + (\pi_1 \cdots \pi_{n-1} f_n) + f\mathcal{O}_{\mathrm{af}}. \tag{6.6}$$

Recall that v_∞ denotes the (normalized) valuation at ∞ of F. We have $v_\infty(g) \le 0$ for all $g \in \mathcal{O}_{\mathrm{af}}$.

Lemma 6.4. *Every element in $\mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}}$ can be represented in the form $g + f\mathcal{O}_{\mathrm{af}}$ with $g \in \mathcal{O}_{\mathrm{af}}$ such that $v_\infty(g) \ge v_\infty(f)$. There is a well-defined k-linear map*

$$S\colon \mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}} \longrightarrow k$$

such that

$$S(g + f\mathcal{O}_{\mathrm{af}}) = -\omega_\infty\left(\frac{g}{f}(\infty)\right) \quad \text{for } g \in \mathcal{O}_{\mathrm{af}} \text{ with } v_\infty(g) \ge v_\infty(f).$$

The following diagram, where $s_{\mathfrak{p}_1}, \ldots, s_{\mathfrak{p}_n}$ are the linear functionals coherently chosen with the uniformizers $\pi_1, \ldots, \pi_n$, commutes:

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}} & \xrightarrow{\ \Phi\ } & k(\mathfrak{p}_1) \oplus \cdots \oplus k(\mathfrak{p}_n) \\ & {\scriptstyle S}\searrow \quad \swarrow{\scriptstyle s_{\mathfrak{p}_1}+\cdots+s_{\mathfrak{p}_n}} & \\ & k. & \end{array} \tag{6.7}$$

Proof. Proposition 5.2 shows that every element in $\pi_\alpha^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}$ can be represented in the form $g_\alpha + \mathcal{O}_{\mathrm{af}}$ for some $g_\alpha \in \pi_\alpha^{-1}\mathcal{O}_{\mathrm{af}}$ such that $v_\infty(g_\alpha) \ge 0$. Therefore, by (6.5), every element in $\mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}}$ has the form $(\sum_\alpha f g_\alpha) + f\mathcal{O}_{\mathrm{af}}$, where $f g_\alpha \in \mathcal{O}_{\mathrm{af}}$ and $v_\infty(f g_\alpha) \ge v_\infty(f)$ for all $\alpha = 1, \ldots, n$. This proves the first statement.

Note that the representation as $g + f\mathcal{O}_{\mathrm{af}}$ with $g \in \mathcal{O}_{\mathrm{af}}$ such that $v_\infty(g) \ge v_\infty(f)$ is not unique, but if $g_1, g_2 \in \mathcal{O}_{\mathrm{af}}$ are such that $v_\infty(g_1), v_\infty(g_2) \ge v_\infty(f)$ and $g_1 + f\mathcal{O}_{\mathrm{af}} = g_2 + f\mathcal{O}_{\mathrm{af}}$, then $f^{-1}g_1 - f^{-1}g_2 \in \mathcal{O}_{\mathrm{af}}$ and $v_\infty(f^{-1}g_1 - f^{-1}g_2) \ge 0$, so $f^{-1}g_1 - f^{-1}g_2 \in k$. Since ω_∞ vanishes on k, it follows that $\omega_\infty(\frac{g_1}{f}(\infty)) = \omega_\infty(\frac{g_2}{f}(\infty))$; hence the map S is well defined.

Since the choice of the functionals $s_{\mathfrak{p}_1}, \dots, s_{\mathfrak{p}_n}$ is coherent with the choice of uniformizers $\pi_1, \dots, \pi_n$, commutativity of (6.7) amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}} & \xrightarrow{\ \sim\ } & \bigoplus_{\alpha=1}^{n}\left(\pi_\alpha^{-1}\mathcal{O}_{\mathrm{af}}/\mathcal{O}_{\mathrm{af}}\right) \\ \downarrow & & \downarrow {\scriptstyle \sum_\alpha \omega_{\mathfrak{p}_\alpha}} \\ \mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}} & \xrightarrow{\quad S \quad} & k. \end{array}$$

This readily follows from the description of the maps $\omega_{\mathfrak{p}_\alpha}$ in Proposition 5.2. □

Tensoring Φ with the identity on the $\mathcal{O}_{\mathrm{af}}$-module $\overline{\mathcal{T}_{\mathrm{af}}}$, we obtain an isomorphism of right $\mathcal{D}_{\mathrm{af}}$-modules

$$\Phi_{\mathcal{T}}\colon \overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}} \xrightarrow{\sim} \overline{T(\mathfrak{p}_1)} \oplus \dots \oplus \overline{T(\mathfrak{p}_n)}.$$

On the other hand, tensoring S with the identity on D, we obtain a map

$$S_D\colon \mathcal{D}_{\mathrm{af}}/f\mathcal{D}_{\mathrm{af}} \longrightarrow D.$$

Recall from Proposition 5.1 that $\overline{\mathcal{T}_{\mathrm{af}}} = e\mathcal{D}_{\mathrm{af}}$. Define a skew-hermitian form

$$H\colon \left(\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}\right) \times \left(\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}\right) \longrightarrow \mathcal{D}_{\mathrm{af}}/f\mathcal{D}_{\mathrm{af}}$$

by

$$H\left(e\xi + f\overline{\mathcal{T}_{\mathrm{af}}}, e\eta + f\overline{\mathcal{T}_{\mathrm{af}}}\right) = \overline{\xi}e\eta + f\mathcal{D}_{\mathrm{af}} \quad \text{for } \xi, \eta \in \mathcal{D}_{\mathrm{af}}.$$

Proposition 6.5. *The map $\Phi_{\mathcal{T}}$ is an isometry of skew-hermitian D-modules*

$$(S_D)_*(H) \simeq t_{\mathfrak{p}_1}(\langle \pi_2 \cdots \pi_n(\mathfrak{p}_1)\rangle) \perp \dots \perp t_{\mathfrak{p}_n}(\langle \pi_1 \cdots \pi_{n-1}(\mathfrak{p}_n)\rangle).$$

Proof. This follows by a straightforward calculation, using (6.6) and Lemma 6.4. □

Next, we determine a base of $\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}$ as a right D-vector space.

Lemma 6.6. *If $v_\infty(f) = -n$, then $(ex^\alpha + f\overline{\mathcal{T}_{\mathrm{af}}})_{\alpha=0}^{n-1}$ is a D-base of $\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}$.*

Proof. For every $\mathfrak{p} \in X^{(1)}$, the fiber $\overline{T(\mathfrak{p})}$ is a 2-dimensional right ideal of $D(\mathfrak{p})$; hence $\dim_{k(\mathfrak{p})} \overline{T(\mathfrak{p})} = 2$, and therefore $\dim_k \overline{T(\mathfrak{p})} = 2\deg\mathfrak{p}$. By (6.4) it follows that $\dim_k \overline{T(\mathfrak{p})} = -4v_\infty(\pi_{\mathfrak{p}})$ for all $\mathfrak{p} \in X_{\mathrm{af}}^{(1)}$. Since $\Phi_{\mathcal{T}}$ is an isomorphism of k-vector spaces, we get

$$\dim_k\left(\overline{\mathcal{T}}_{\mathrm{af}}/f\overline{\mathcal{T}}_{\mathrm{af}}\right) = -4\left(v_\infty(\pi_1) + \dots + v_\infty(\pi_n)\right) = -4v_\infty(f),$$

hence $\dim_D(\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}) = n$. Therefore, to prove the lemma, it suffices to show that the sequence $(ex^\alpha + f\overline{\mathcal{T}_{\mathrm{af}}})_{\alpha=0}^{n-1}$ spans $\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}$.

From the description of $\mathcal{O}_{\mathrm{af}}$ as $k[x,y]$ with $y^2 = ax^2 + b$, we know that $(x^\alpha, x^\alpha y)_{\alpha=0}^{\infty}$ is a k-base of $\mathcal{O}_{\mathrm{af}}$. As $v_\infty(x^\alpha) = v_\infty(x^{\alpha-1}y) = -\alpha$, the elements $g \in \mathcal{O}_{\mathrm{af}}$ such that $v_\infty(g) \geq -n$ are linear combinations of $(x^\alpha)_{\alpha=0}^{n}$ and $(x^\beta y)_{\beta=0}^{n-1}$. Therefore, Lemma 6.4 shows that $\mathcal{O}_{\mathrm{af}}/f\mathcal{O}_{\mathrm{af}}$ is k-spanned by the images of $(x^\alpha)_{\alpha=0}^{n}$ and $(x^\beta y)_{\beta=0}^{n-1}$. It follows that $\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}$ is D-spanned by the images of $(ex^\alpha)_{\alpha=0}^{n}$ and $(ex^\beta y)_{\beta=0}^{n-1}$. By multiplying (5.1) on the left by $(ij)^{-1}$, we get $je - xie + ye = 0$; hence after conjugation

$$ey = ej - eix \quad \text{in } \overline{\mathcal{T}_{\mathrm{af}}}. \tag{6.8}$$

Therefore, the elements $ex^\beta y$ for $\beta = 0, \dots, n-1$ are in the D-span of $(ex^\alpha)_{\alpha=0}^{n}$. Thus, it only remains to see that the image of ex^n in $\overline{\mathcal{T}_{\mathrm{af}}}/f\overline{\mathcal{T}_{\mathrm{af}}}$ lies in the D-span of the image of $(ex^\alpha)_{\alpha=0}^{n-1}$.

For this, note that since $v_\infty(f) = -n$, we have

$$f = c_1 x^n + c_2 x^{n-1} y + f_0$$

for some $f_0 \in \mathcal{O}_{\rm af}$ such that $v_\infty(f_0) \geq -n+1$ and some $c_1, c_2 \in k$ not both 0, hence

$$ec_1x^n + ef_0 \equiv -ec_2x^{n-1}y \mod f\overline{\mathcal{T}_{\rm af}}.$$

Comparing with (6.8), we obtain

$$ec_1x^n + ef_0 \equiv -ejc_2x^{n-1} + eic_2x^n \mod f\overline{\mathcal{T}_{\rm af}},$$

hence

$$e(c_1 - ic_2)x^n \equiv -ejc_2x^{n-1} - ef_0 \mod f\overline{\mathcal{T}_{\rm af}}.$$

Note that since $v_\infty(f_0) \geq -n+1$, the arguments above show that the image of ef_0 in $\overline{\mathcal{T}_{\rm af}}/f\overline{\mathcal{T}_{\rm af}}$ lies in the D-span of $(ex^\alpha)_{\alpha=0}^{n-1}$. Since the quaternion $c_1 - ic_2$ is invertible in D, the proof is complete. □

We may now prove that the sequence in Theorem 6.2 is a zero sequence.

Proposition 6.7. *For f as above, $\sum_{\mathfrak{p}\in X^{(1)}} t_\mathfrak{p}\left(\partial^2_\mathfrak{p}(\langle f\rangle)\right) = 0$.*

Proof. Let $v_\infty(f) = -n$. For $\alpha, \beta = 0, \dots, n-1$, we have

$$H\left(ex^\alpha + f\overline{\mathcal{T}_{\rm af}}, ex^\beta + f\overline{\mathcal{T}_{\rm af}}\right) = ex^{\alpha+\beta} + f\mathcal{D}_{\rm af};$$

hence, as $e = bi + axj + yij$,

$$(S_D)_*(H)\left(ex^\alpha + f\overline{\mathcal{T}_{\rm af}}, ex^\beta + f\overline{\mathcal{T}_{\rm af}}\right) = bS\left(x^{\alpha+\beta} + f\mathcal{O}_{\rm af}\right)i + aS\left(x^{\alpha+\beta+1} + f\mathcal{O}_{\rm af}]\right)j + S\left(x^{\alpha+\beta}y + f\mathcal{O}_{\rm af}\right)ij.$$

Now, for $g \in \mathcal{O}_{\rm af}$ such that $v_\infty(g) > -n$,

$$S(g + f\mathcal{O}_{\rm af}) = -\omega_\infty\left(\frac{g}{f}(\infty)\right) = 0;$$

hence $ex^\alpha + f\overline{\mathcal{T}_{\rm af}}$ and $ex^\beta + f\overline{\mathcal{T}_{\rm af}}$ are orthogonal for $(S_D)_*(H)$ when $\alpha+\beta+1 < n$. In particular, the images of ex^α for $\alpha < \frac{n-1}{2}$ span a totally isotropic subspace of $\overline{\mathcal{T}_{\rm af}}/f\overline{\mathcal{T}_{\rm af}}$. If n is even, this totally isotropic subspace has D-dimension $\frac{n}{2} = \frac{1}{2}\dim_D(\overline{\mathcal{T}_{\rm af}}/f\overline{\mathcal{T}_{\rm af}})$, so $(S_D)_*(H)$ is hyperbolic. By Proposition 6.5, it follows that $\sum_{\mathfrak{p}\in X^{(1)}_{\rm af}} t_\mathfrak{p}\left(\partial^2_\mathfrak{p}(\langle f\rangle)\right) = 0$. Since $\partial^2_\infty(\langle f\rangle) = 0$, the proposition follows.

Now suppose $n = 2m+1$ for some integer m. Then the image of $(ex^\alpha)_{\alpha=0}^{m-1}$ spans a totally isotropic subspace in the orthogonal complement of $ex^m + f\overline{\mathcal{T}_{\rm af}}$, so $(S_D)_*(H)$ is Witt-equivalent to its restriction to the span of $ex^m + f\overline{\mathcal{T}_{\rm af}}$. Computation shows

$$(S_D)_*(H)\left(ex^m + f\overline{\mathcal{T}_{\rm af}}, ex^m + f\overline{\mathcal{T}_{\rm af}}\right) = -a\omega_\infty\left(\frac{x^n}{f}(\infty)\right)j - \omega_\infty\left(\frac{x^{n-1}y}{f}(\infty)\right)ij;$$

hence by Proposition 6.5

$$\sum_{\mathfrak{p}\in X^{(1)}_{\rm af}} t_\mathfrak{p}\left(\partial^2_\mathfrak{p}(\langle f\rangle)\right) = \left\langle -a\omega_\infty\left(\frac{x^n}{f}(\infty)\right)j - \omega_\infty\left(\frac{x^{n-1}y}{f}(\infty)\right)ij\right\rangle \quad \text{in } W^-(D). \tag{6.9}$$

On the other hand, since n is odd, we have

$$\langle f\rangle = \left\langle x^{-n-1}f\right\rangle = \left\langle x^{-1}\right\rangle\left\langle f^{-1}x^n\right\rangle,$$

hence $\partial^2_\infty(\langle f\rangle) = \langle\frac{x^n}{f}(\infty)\rangle$ in $W(k(\infty))$. By Proposition 5.5, the Witt class $t_\infty\left(\partial^2_\infty(\langle f\rangle)\right)$ is represented by the transfer along s_{D_∞} of the skew-hermitian form h_f on $\overline{T(\infty)}$ such that

$$h_f(e_\infty, e_\infty) = \frac{x^n}{f}(\infty)e_\infty.$$

As observed in the proof of Lemma 6.6, $\overline{T(\infty)}$ is a D-vector space of dimension 1. Taking e_∞ as a base of $\overline{T(\infty)}$, we obtain

$$t_\infty\left(\partial^2_\infty(\langle f\rangle)\right) = \left\langle s_{D_\infty}\left(\frac{x^n}{f}(\infty)e_\infty\right)\right\rangle. \tag{6.10}$$

Recall that e_∞ is the image in $T(\infty)$ of ex^{-1}, so $e_\infty = aj + \frac{y}{x}(\infty)ij$. Therefore,

$$s_{D_\infty}\left(\frac{x^n}{f}(\infty)e_\infty\right) = a\omega_\infty\left(\frac{x^n}{f}(\infty)\right)j + \omega_\infty\left(\frac{x^{n-1}y}{f}(\infty)\right)ij. \tag{6.11}$$

The proof follows by comparing (6.9), (6.10), and (6.11). □

6.4. Exactness at $\bigoplus_{\mathfrak{p}} W(k(\mathfrak{p}))$

We prove the exactness of the sequence in Theorem 6.2 by relating it to the following exact sequence due to Pfister [Pfi93, Theorem 5]:

$$W(F) \xrightarrow{\delta''} \bigoplus_{\mathfrak{p}\in X^{(1)}_{\mathrm{af}}} W(k(\mathfrak{p})) \xrightarrow{\sum (s_{\mathfrak{p}})_*} W(k)/J, \tag{6.12}$$

where $J = \{\varphi \in W(k) \mid \langle 1, -a\rangle\varphi = 0\}$ is the subgroup annihilated by the norm form of $k(\infty)$ and δ'' is the map whose $\mathfrak{p}$-component is $\partial^2_{\mathfrak{p}}$ for all $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$, for a coherent choice of uniformizer and linear functional at each $\mathfrak{p}$. For this, we use the canonical isomorphism $k(\infty) \simeq k(i) \subset D$ of [EKM08, Proposition 45.12]. Let $K = k(i)$, and let γ be the canonical isomorphism

$$\gamma\colon k(\infty) \xrightarrow{\sim} K, \quad \frac{y}{x}(\infty) \longmapsto i.$$

Theorem 4.1 yields an exact sequence

$$W^-(D) \xrightarrow{\pi_2} W(K) \xrightarrow{\sigma_2} W^-(D) \xrightarrow{\pi_1} W^-(K, {}^-).$$

Lemma 6.8. *Let $\Psi\colon W(k(\infty)) \to W(K)$ be the isomorphism that maps every quadratic form $\langle g\rangle$ to $\langle -\overline{\gamma(g)}\rangle$, for $g \in k(\infty)^\times$. The following diagram commutes:*

$$\begin{array}{ccc} W(k(\infty)) & \xrightarrow{\quad\Psi\quad} & W(K) \\ & {\scriptstyle t_\infty}\searrow \quad \swarrow{\scriptstyle \sigma_2} & \\ & W^-(D). & \end{array}$$

Proof. For $g \in k(\infty)^\times$, the definitions of σ_2 and Ψ yield

$$\sigma_2(\Psi(\langle g\rangle)) = \left\langle -ij\overline{\gamma(g)}\right\rangle = \langle -\gamma(g)ij\rangle,$$

whereas

$$t_\infty(\langle g\rangle) = \langle s_{D_\infty}(ge_\infty)\rangle = \left\langle a\omega_\infty(g)j + \omega_\infty\left(g\frac{y}{x}(\infty)\right)ij\right\rangle$$

(see (6.10) and (6.11)). Computation yields

$$a\omega_\infty(g) + \omega_\infty\left(g\frac{y}{x}(\infty)\right)i = -\gamma(g)i,$$

hence

$$t_\infty(\langle g\rangle) = \langle -\gamma(g)ij\rangle = \sigma_2 \circ \Psi(\langle g\rangle).$$ □

Lemma 6.9. *The map that carries every symmetric bilinear form $\varphi\colon U\times U\to k$ to the skew-hermitian form $\langle bi\rangle\varphi_{(K,\overline{\ })}\colon (U\otimes_k K)\times(U\otimes_k K)\to K$ such that for u, $u'\in U$ and α, $\alpha'\in K$,*

$$\langle bi\rangle\varphi_{(K,\overline{\ })}(u\otimes\alpha, u'\otimes\alpha') = bi\overline{\alpha}\,\varphi(u,u')\alpha'$$

induces a group isomorphism

$$\Theta\colon W(k)/J \xrightarrow{\sim} W^-(K,\overline{\ }).$$

This map makes the following diagram commute for every $\mathfrak{p}\in X^{(1)}_{\mathrm{af}}$:

$$\begin{array}{ccc} W(k(\mathfrak{p})) & \xrightarrow{t_{\mathfrak{p}}} & W^-(D) \\ \Big\downarrow{\scriptstyle (s_{\mathfrak{p}})_*} & & \Big\downarrow{\scriptstyle \pi_1} \\ W(k)/J & \xrightarrow{\Theta} & W^-(K,\overline{\ }). \end{array}$$

Proof. Multiplication by bi defines an isomorphism $W^-(K,\overline{\ })\simeq W(K,\overline{\ })$; hence for every symmetric bilinear form $\varphi\colon U\times U\to k$, the skew-hermitian form $\langle bi\rangle\varphi_{(K,\overline{\ })}$ is hyperbolic if and only if the hermitian form $\varphi_{(K,\overline{\ })}$ is hyperbolic. This occurs if and only if the quadratic form $s_K(\varphi_{(K,\overline{\ })})\colon U\otimes_k K\to k$ defined by $s_K(\varphi_{(K,\overline{\ })})(u\otimes\alpha)=\overline{\alpha}\,\varphi(u,u)\alpha$ is hyperbolic; see [Sch85, Section 10.1, p. 348]. Since $s_K(\varphi_{(K,\overline{\ })})=\langle 1,-a\rangle\varphi$, it follows that the map Θ is well defined and injective. It is also surjective because every skew-hermitian form over $(K,\overline{\ })$ has a diagonalization $\langle\alpha_1 i,\dots,\alpha_n i\rangle$ with $\alpha_1,\dots,\alpha_n\in k^\times$.

Now, let $f\in k(\mathfrak{p})^\times$. We know from Proposition 5.5 that the Witt class of $t_{\mathfrak{p}}(\langle f\rangle)$ is represented by the skew-hermitian form

$$h\colon \overline{T(\mathfrak{p})}\times\overline{T(\mathfrak{p})}\longrightarrow D \quad\text{defined by}\quad h(e_{\mathfrak{p}}\xi, e_{\mathfrak{p}}\eta) = s_{D(\mathfrak{p})}\big(f\overline{\xi}e_{\mathfrak{p}}\eta\big) \quad \text{for } \xi,\eta\in D(\mathfrak{p}).$$

By multiplying (5.1) on the left by $(ij)^{-1}$ and by j^{-1}, we obtain

$$je - xie + ye = 0 \quad\text{and}\quad -ije + axe - yie = 0;$$

hence, after conjugation,

$$ej = ey + eix \quad\text{and}\quad eij = -eax - eiy \quad\text{in } \overline{T_{\mathrm{af}}}.$$

Therefore, $e_{\mathfrak{p}}j$ and $e_{\mathfrak{p}}ij$ are in the $k(\mathfrak{p})$-span of $e_{\mathfrak{p}}$ and $e_{\mathfrak{p}}i$ in $\overline{T(\mathfrak{p})}$; hence $(e_{\mathfrak{p}}, e_{\mathfrak{p}}i)$ is a $k(\mathfrak{p})$-base of $\overline{T(\mathfrak{p})}$. Let $(c_\alpha)_{\alpha=1}^{\deg\mathfrak{p}}$ be a k-base of $k(\mathfrak{p})$. Then $(e_{\mathfrak{p}}c_\alpha, e_{\mathfrak{p}}ic_\alpha)_{\alpha=1}^{\deg\mathfrak{p}}$ is a k-base of $\overline{T(\mathfrak{p})}$; hence $(e_{\mathfrak{p}}c_\alpha)_{\alpha=1}^{\deg\mathfrak{p}}$ is a K-base of $\overline{T(\mathfrak{p})}$. Since $e_{\mathfrak{p}} = bi + ax(\mathfrak{p})j + y(\mathfrak{p})ij$, we have

$$s_{D(\mathfrak{p})}\big(fc_\alpha e_{\mathfrak{p}}c_\beta\big) = s_{\mathfrak{p}}\big(fc_\alpha c_\beta\big)bi + s_{\mathfrak{p}}\big(fc_\alpha x(\mathfrak{p})c_\beta\big)aj + s_{\mathfrak{p}}\big(fc_\alpha y(\mathfrak{p})c_\beta\big)ij;$$

hence the matrix of $\pi_1(h)$ in the base $(e_{\mathfrak{p}}c_\alpha)_{\alpha=1}^{\deg\mathfrak{p}}$ is

$$\Big(s_{\mathfrak{p}}\big(fc_\alpha c_\beta\big)bi\Big)_{\alpha,\beta=1}^{\deg\mathfrak{p}}.$$

The skew-hermitian form $\langle bi\rangle(s_{\mathfrak{p}})_*(\langle f\rangle)_{(K,\overline{\ })}$ has the same matrix. □

In the next lemma, we use the following notation: We write

$$\varpi\colon \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})) \longrightarrow \bigoplus_{\mathfrak{p}\in X^{(1)}_{\mathrm{af}}} W(k(\mathfrak{p}))$$

for the map that "forgets" the component at ∞.

Lemma 6.10. *We have* $\ker\varpi\cap\ker(\sum t_{\mathfrak{p}})\subset\operatorname{image}(\delta)$.

Proof. Let $(\varphi_\mathfrak{p})_{\mathfrak{p}\in X^{(1)}} \in \ker\varpi \cap \ker(\sum t_\mathfrak{p})$. Thus, $\varphi_\mathfrak{p} = 0$ for all $\mathfrak{p} \neq \infty$ and $t_\infty(\varphi_\infty) = 0$. Lemma 6.8 yields $\sigma_2(\Psi(\varphi_\infty)) = 0$; hence by Theorem 4.1, we may find an $h \in W^-(D)$ such that $\pi_2(h) = \Psi(\varphi_\infty)$. From the description of π_2 in (4.1), it follows that φ_∞ is a sum of Witt classes represented by quadratic forms of the type $\langle u\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle$, for $u \in K^\times$ and $\lambda \in k$. To complete the proof, it suffices to show that every element in $\bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p}))$ whose $\mathfrak{p}$-components are 0 for all $\mathfrak{p} \neq \infty$ and whose ∞-component is represented by a form of the type above is in the image of δ.

Fix $\lambda \in k$ and $u = u_1 + u_2\frac{y}{x}(\infty) \in K^\times$ (with u_1, $u_2 \in k$), and consider

$$f = \lambda + u_1 x + u_2 y \in \mathcal{O}_{\mathrm{af}}.$$

Since u_1 and u_2 are not both zero, we have $v_\infty(f) = -1$; hence f is irreducible in $\mathcal{O}_{\mathrm{af}}$. Let $\mathfrak{q} \in X^{(1)}_{\mathrm{af}}$ be the point such that $f \in \mathfrak{m}_\mathfrak{q} \cap \mathcal{O}_{\mathrm{af}}$. Then

$$k(\mathfrak{q}) \simeq k\left(\sqrt{a\lambda^2 + bu_1^2 - abu_2^2}\right);$$

hence (for any choice of uniformizer at $\mathfrak{q}$)

$$\partial^2_\mathfrak{q}\left(\langle f\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle\right) = 0 \quad \text{in } W(k(\mathfrak{q})).$$

Moreover, for every $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$ with $\mathfrak{p} \neq \mathfrak{q}$,

$$\partial^2_\mathfrak{p}\left(\langle f\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle\right) = 0 \quad \text{in } W(k(\mathfrak{p}))$$

because $v_\mathfrak{p}(f) = 0$. Furthermore, with x^{-1} as uniformizer at ∞,

$$\partial^2_\infty\left(\langle f\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle\right) = \langle u\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle \quad \text{in } W(k(\infty)).$$

Thus, the element in $\bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p}))$ whose $\mathfrak{p}$-components are all 0 for $\mathfrak{p} \neq \infty$ and whose ∞-component is represented by $\langle u\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle$ is the image of $\langle f\rangle\langle 1, -a\lambda^2 - bN_{k(\infty)/k}(u)\rangle$ under δ. □

Proposition 6.11. *The sequence* (6.2) *is exact at* $\bigoplus_\mathfrak{p} W(k(\mathfrak{p}))$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
W(F) & \xrightarrow{\ \delta\ } & \bigoplus_{\mathfrak{p}\in X^{(1)}} W(k(\mathfrak{p})) & \xrightarrow{\ \sum t_\mathfrak{p}\ } & W^-(D) \\
\Vert & & \downarrow \varpi & & \downarrow \Theta^{-1}\circ\pi_1 \\
W(F) & \xrightarrow{\ \delta''\ } & \bigoplus_{\mathfrak{p}\in X^{(1)}_{\mathrm{af}}} W(k(\mathfrak{p})) & \xrightarrow{\ \sum (s_\mathfrak{p})_*\ } & W(k)/J.
\end{array}$$

The left square commutes by the definition of the maps, and the right square commutes by Lemma 6.9. The upper sequence is a zero sequence by Proposition 6.7, and the lower sequence is exact by Pfister's theorem [Pfi93, Theorem 5]. Therefore, a diagram chase yields for every $u \in \ker(\sum t_\mathfrak{p})$ an element $v \in W(F)$ such that $\delta''(v) = \varpi(u)$. Then $u - \delta(v) \in \ker\varpi$, and $u - \delta(v) \in \ker(\sum t_\mathfrak{p})$ because the upper sequence is a zero sequence. Therefore, Lemma 6.10 shows that $u - \delta(v) \in \mathrm{image}(\delta)$, hence $u \in \mathrm{image}(\delta)$. □

6.5. Exactness at $W^-(D)$

To complete the proof of Theorem 6.2, we show that the map $\sum t_\mathfrak{p}$ is onto. It suffices to prove that 1-dimensional skew-hermitian forms over D are in the image of $\sum t_\mathfrak{p}$.

Proposition 6.12. *For every nonzero* $q \in D^0$*, there exist a* $\mathfrak{p} \in X^{(1)}_{\mathrm{af}}$ *of degree* 2 *and an* $f \in k(\mathfrak{p})^\times$ *such that* $t_\mathfrak{p}(\langle f\rangle) = \langle q\rangle$.

Proof. Let $q = \lambda_1 i + \lambda_2 j + \lambda_3 ij$ with $\lambda_1, \lambda_2, \lambda_3 \in k$. We may find $\alpha_1, \alpha_2, \alpha_3 \in k$ with α_2, α_3 not both zero such that

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 = 0.$$

Let $\mathfrak{p} \in X^{(1)}$ be the intersection of the conic with the line $\alpha_1 bZ + \alpha_2 aX + \alpha_3 Y = 0$ in the projective plane $\mathbb{P}(D^0)$. The point $\mathfrak{p}$ has degree 2, and $\mathfrak{p} \neq \infty$ since α_2 and α_3 are not both zero. In $k(\mathfrak{p})$, the following equation holds:

$$\alpha_1 b + \alpha_2 ax(\mathfrak{p}) + \alpha_3 y(\mathfrak{p}) = 0.$$

Therefore, there is a linear functional $r\colon k(\mathfrak{p}) \to k$ such that

$$r(b) = \lambda_1, \quad r(ax(\mathfrak{p})) = \lambda_2, \quad \text{and} \quad r(y(\mathfrak{p})) = \lambda_3.$$

Every k-linear functional on $k(\mathfrak{p})$ has the form $g \mapsto s_{\mathfrak{p}}(fg)$ for some $f \in k(\mathfrak{p})$; hence we may find an $f \in k(\mathfrak{p})^\times$ such that $r(g) = s_{\mathfrak{p}}(fg)$ for all $g \in k(\mathfrak{p})$. The element $e_{\mathfrak{p}} = bi + ax(\mathfrak{p})j + y(\mathfrak{p})ij$ is a D-base of $\overline{T(\mathfrak{p})}$, and Proposition 5.5 shows that in this base

$$t_{\mathfrak{p}}(\langle f\rangle) = \langle s_{D(\mathfrak{p})}(fe_{\mathfrak{p}})\rangle = \langle s_{\mathfrak{p}}(fb)i + s_{\mathfrak{p}}(fax(\mathfrak{p}))j + s_{\mathfrak{p}}(fy(\mathfrak{p}))ij\rangle = \langle q\rangle. \qquad \square$$

The proof of Theorem 6.2 is thus complete.

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