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Hyperelliptic Curves and Ulrich sheaves on the complete intersection of two quadrics

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For Claire Voisin on the occasion of her Birthday

Abstract. Using the connection between hyperelliptic curves, Clifford algebras, and smooth complete intersections X of two quadrics, we describe Ulrich bundles on X and construct some of minimal possible rank.

Keywords. Free resolutions, complete intersections, quadrics, Ulrich bundles, Ulrich modules, Clifford algebras

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Contents

1. Introduction

Let k be an algebraically closed field of characteristic not 2. The periodicity theorem of Knörrer [Knö87] shows that the indecomposable Ulrich bundles on a smooth quadric hypersurface in $\mathbb{P}^{2g+1}$ over k have rank 2^{g-1}. In this paper we construct Ulrich bundles of the same rank 2^{g-1} on every smooth complete intersection X of 2 quadrics in $\mathbb{P}^{2g+1}$, and we show that every Ulrich bundle has rank of the form $r2^{g-2}$ where $r \geq 2$ and rg is even. To prove this we use an equivalence of categories that extends Reid's famous description of the Jacobian of a hyperelliptic curve [Rei72].

Let $X \subset \mathbb{P}^n$ be a projective scheme with homogenous coordinate ring P_X. Recall that a sheaf $\mathcal{E}$ on X is called Ulrich if the graded module of twisted global sections $H^0_*(\mathcal{E})$ is a maximal Cohen–Macaulay P_X-module generated in degree 0 and having linear free resolution over the coordinate ring of $\mathbb{P}^n$, or equivalently if $H^i(\mathcal{E}(m)) = 0$ for all m with $-1 \geq m \geq -\dim X$ and all i. See [ES03] for further information and examples.

Let X be the smooth complete intersection defined by two quadratic forms q_1, q_2 on $\mathbb{P}^{2g+1}$ over an algebraically closed field k of characteristic not 2.

The pencil of quadrics $sq_1 + tq_2, (s,t) \in \mathbb{P}^1$ becomes singular at $2g+2$ points of $\mathbb{P}^1$. Let E be the hyperelliptic curve with homogeneous coordinate ring $k[s,t,y]/(y^2 - f)$ branched over these points, and let C be the $\mathbb{Z}$-graded Clifford algebra of the form $sq_1 + tq_2$ over $k[s,t]$.

We give two approaches to the construction of Ulrich sheaves on X. The first makes use of three categories:

(i) the category of coherent sheaves on E,
(ii) the category of graded C-modules, and
(iii) the category of coherent sheaves on X.

Categories (i) and (ii) are related by Morita equivalence, while categories (ii) and (iii) are related by a version of the Bernstein–Gel'fand–Gel'fand correspondence.

Composing these correspondences to go from (i) to (iii), we show that every Ulrich module on X has rank $r2^{g-2}$ for some integer $r \geq 2$.

Following [FL10] we say that a bundle $\mathcal{B}$ on E has the *Raynaud property* if $H^0(C,\mathcal{B}) = H^1(C,\mathcal{B}) = 0$. We use the fact that the center of the even Clifford algebra is the homogeneous coordinate ring of E, and that the category of coherent sheaves of modules over the sheafified even Clifford algebra $\mathcal{C}^{\mathrm{ev}} \cong \mathcal{E}nd_E(\mathcal{F}_U)$ is Morita equivalent to the category of coherent sheaves on E *via* an $\mathcal{O}_E - \mathcal{C}^{\mathrm{ev}}$ bundle $\mathcal{F}_U$ defined in Section 4. With this notation, our first main theorem is the following.

Theorem 1.1. *There is a 1-1 correspondence between Ulrich bundles on the smooth complete intersection of two quadrics $X \subset \mathbb{P}^{2g+1}$ and bundles of the form $\mathcal{G} \otimes \mathcal{F}_U$ with the Raynaud property on the corresponding hyperelliptic curve E. The Ulrich bundle corresponding to a rank r vector bundle $\mathcal{G}$ has rank $r2^{g-2}$.*

If $\mathcal{L}$ is a line bundle on E then $\mathcal{L} \otimes \mathcal{F}_U$ does not have the Raynaud property, so the minimal possible rank of an Ulrich sheaf on X is 2^{g-1}, and Ulrich bundles of rank 2^{g-1} exist.

The set of bundles $\mathcal{G}$ such that $\mathcal{G} \otimes \mathcal{F}_U$ has the Raynaud property forms a (possibly empty) open subset in any flat family of rank r vector bundles on E. Our second main theorem, the existence statement for $r = 2$ is proven using a previously undiscovered property of Knörrer's matrix factorizations to give a construction of an Ulrich sheaf of the minimal possible rank, 2^{g-1} on any smooth complete intersection of two quadrics in $\mathbb{P}^{2g+1}$ and in $\mathbb{P}^{2g+2}$.

Based on computed examples using our package [EKS22] with Yeongrak Kim, we conjecture the following.

Conjecture 1.2. *There exist indecomposable Ulrich bundles of rank $r2^{g-2}$ on every smooth complete intersection of two quadrics in $\mathbb{P}^{2g+1}$ for $g \geq 1$ and $r \geq 2$ if and only if $rg \equiv 0 \mod 2$.*

By Proposition 5.11 the condition is necessary.

In Section 2 we explain the description of vector bundles on E in terms of matrix factorizations. In the case of line bundles, this theory can be traced through Mumford's [Mum84] to work of Jacobi [Jac46].

In Section 3 we explain the relation of categories (ii) and (iii), a form of the Bernstein–Gel'fand–Gel'fand (BGG) correspondence that holds for all complete intersections of quadrics. As far as we know this correspondence was first introduced in [BEH87], and greatly extended in [Kap89]. For the reader's convenience we review the results that we will use.

In Section 4 we establish the Morita equivalence between categories (i) and (ii). In fact every maximal (simultaneous) isotropic plane U for q_1 and q_2 gives rise to a Morita bundle $\mathcal{F}_U$ and any two differ by the tensor product with a line bundle on E. This explains the well-known result of Miles Reid's thesis that the space of maximal (simultaneous) isotropic planes for q_1 and q_2 can be identified with the Jacobian of E.

In Section 5 we put these tools together with the theory of Tate resolutions and maximal Cohen–Macaulay approximations to establish the equivalence between Ulrich modules of rank $r2^{g-2}$ on X and vector bundles of rank r on E that satisfy certain cohomological conditions. We show that no line bundles on E satisfy the conditions, establishing the lower bound for the rank of Ulrich modules announced above. This section was inspired by Buchweitz's famously unpublished manuscript on Koszul duality from 1986, now available at [Buc21] and by the theory of Cohen–Macaulay approximations by Auslander and Buchweitz [AB89].

It is natural to look for Ulrich bundles on X using the shape of their Tate resolutions over P_X. Theorem 5.5 is analogous to the main result on Tate resolution of coherent sheaves on $\mathbb{P}^n$ in [EFS03]: the Betti table of the Tate resolution over the exterior algebra coincides with cohomology tables of the corresponding sheaf. In Theorem 5.5 the resolution over the exterior algebra is replaced by the Tate resolution over P_X.

In Section 6, which is independent of the rest of the paper, we give a direct construction of Ulrich modules of rank 2^{g-1} on any smooth complete intersection of quadrics in $\mathbb{P}^{2g+1}$ and $\mathbb{P}^{2g+2}$ with the minimal possible rank, 2^{g-1}. In the case $g = 2$ the existence and minimality was established by [CKL21] with a different method.

Historical remarks

The study of complete intersections of quadrics has a long history. The connection to vector bundles was discovered by Newstead [New68], Reid [Rei72] and Desale-Ramanan [DR76] in the 1970's. The connection with Clifford algebras and Koszul pairs was used in [BEH87] and more generally by Kapranov [Kap89] in the 1980's.

The first three sections of the paper, which take the point of view of matrix factorizations, have their roots in an unpublished manuscript by our dear friend Ragnar Buchweitz (1952–2017) and the second author in 90's, now lost. The referee kindly pointed out to us that parts of Theorem 5.10 can be deduced from Kuznetsov's work [Kuz08], which, like the work of Kapranov [Kap89] instead takes the point of view of derived categories.

The theory of quadratic complete intersections has many guises, and appears in descriptions of certain completely integrable systems, for example in the recent paper of Claire Voisin and her coauthors [BEH+24].

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2. Vector Bundles over a hyperelliptic curve *via* matrix factorizations

Let E be a hyperelliptic curve of genus g and let $\pi{:}E\to\mathbb{P}^1$ its double cover of $\mathbb{P}^1$. Let $\mathcal{H}=\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and let $f(s,t)$ be the homogeneous polynomial of degree $2g+2$ such that

$$R_E := k[s,t,y]/(y^2-f) = \bigoplus_n H^0(E,\mathcal{H}^{\otimes n}),$$

so that the roots of f are the ramification points of π and $y\in H^0(E,\mathcal{H}^{\otimes g+1})$.

For a coherent sheaf $\mathcal{G}$ on E we denote by

$$H^i_*(\mathcal{G}) = \bigoplus_n H^i(E,\mathcal{G}\otimes\mathcal{H}^{\otimes n}).$$

Thus $H^0_*(\mathcal{O}_E)=R_E$ and π_* corresponds to forgetting the y-action on $H^0_*(\mathcal{G})$.

Proposition 2.1. *If $\mathcal{L}$ is a vector bundle on E, then $B=H^0_*(\mathcal{L})$ is a graded free module over the homogeneous coordinate ring $k[s,t]$ of $\mathbb{P}^1$, and $y{:}\mathcal{L}\to\mathcal{L}(g+1)$ induces a map $\phi = H^0_*y{:}B\to B(g+1)$ such that ϕ^2 is multiplication by f; that is, a matrix factorization of f.*

Furthermore, given a graded free module B corresponding to the vector bundle $\mathcal{B}$ on $\mathbb{P}^1$, and a map $\phi{:}B\to B(g+1)$ with $\phi^2=f\cdot Id_B$, the sheaf

$$\mathcal{L} = \operatorname{coker}\left(y-\phi{:}\ \pi^*\mathcal{B}(-g-1)\longrightarrow\pi^*\mathcal{B}\right)$$

is a vector bundle on E whose pushforward is $\mathcal{B}$, and on which y induces the matrix factorization ϕ. We have

$$\chi(\mathcal{B})=\chi(\mathcal{L}),\quad \operatorname{rk}\mathcal{B}=2\operatorname{rk}\mathcal{L},\quad \textit{and}\quad \deg\mathcal{B}=\deg\mathcal{L}-(\operatorname{rk}\mathcal{L})(1+g).$$

The proof could be extended to show that the category of vector bundles on E is equivalent to the category of matrix factorizations of f over $k[s,t]$, *cf.* [Eis80].

Proof of Proposition 2.1. The equation $\phi^2=f$ follows from functoriality. Conversely, if a matrix factorization $\phi^2=f\cdot Id_B$ is given, then $(y-\phi,y+\phi)$ is a matrix factorization of y^2-f over $k[s,t,y]$. Thus the module $\operatorname{coker}(y-\phi)$ is a maximal Cohen–Macaulay R_E-module, and it follows that the sheaf associated to its cokernel is a vector bundle on E. □

The next Theorem reduces the computation of the tensor product of vector bundles on E to a syzygy computation, and will be used this way in the sequel.

Theorem 2.2. *If $\mathcal{L}_1,\mathcal{L}_2$ are vector bundles on E with matrix factorizations ϕ_i on the graded free $k[s,t]$-modules $B_i=H^0_*(\mathcal{L}_i)$, then*

$$H^0_*(\mathcal{L}_1\otimes\mathcal{L}_2)=\ker\left(\phi_1\otimes1-1\otimes\phi_2{:}\ B_1\otimes B_2(g+1)\longrightarrow B_1\otimes B_2(2g+2)\right)$$

and π_*y acts on $\pi_*(\mathcal{L}_1\otimes\mathcal{L}_2)$ with the common action of $\phi_1\otimes1$ and $1\otimes\phi_2$.

Proof. The following sequence of maps is a complex because $y^2 = f$:

$$(*)\qquad B_1 \otimes B_2(-g-1) \xrightarrow{\phi_1\otimes 1-1\otimes\phi_2} B_1 \otimes B_2 \xrightarrow{\phi_1\otimes 1+1\otimes\phi_2} B_1 \otimes B_2(g+1) \xrightarrow{\phi_1\otimes 1-1\otimes\phi_2} B_1 \otimes B_2(2g+2)$$

Since the $k[s,t]$-module

$$\ker\left(B_1 \otimes B_2(g+1) \xrightarrow{\phi_1\otimes 1-1\otimes\phi_2} B_1 \otimes B_2(2g+2)\right)$$

is a 2^{nd} syzygy, it is free. Thus, to prove the theorem, it suffices to show that the complex $(*)$ is locally exact and that the sheaf cokernel

$$\operatorname{coker}\left(\mathcal{B}_1 \otimes \mathcal{B}_2(-g-1) \xrightarrow{\phi_1\otimes 1-1\otimes\phi_2} \mathcal{B}_1 \otimes \mathcal{B}_2\right)$$

is $\pi_*(\mathcal{L}_1 \otimes_E \mathcal{L}_2)$.

For simplicity of notation we ignore the twists by powers of $\mathcal{H}$. Note that $\mathcal{B}_i := \pi_*(\mathcal{L}_i)$ is the sheafification of B_i. Since $\mathcal{L}_i$ is the cokernel of $y - \phi_i$ we see that $\mathcal{L}_1 \otimes_E \mathcal{L}_2$ is the cokernel of

$$(\pi^*\mathcal{B}_1 \otimes_E \pi^*\mathcal{B}_2) \oplus (\pi^*\mathcal{B}_2 \otimes_E \pi^*\mathcal{B}_1) \xrightarrow{(y\otimes 1-\phi_1\otimes 1,\, 1\otimes y-1\otimes\phi_2)} \pi^*\mathcal{B}_1 \otimes_E \pi^*\mathcal{B}_2.$$

Since the tensor products are over E, the maps $y \otimes 1$ and $1 \otimes y$ are equal, and are simply multiplication by y, so this says that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the universal quotient of $\pi^*\mathcal{B}_1 \otimes_E \pi^*\mathcal{B}_2$ on which the maps $y, \phi_1 \otimes 1, 1 \otimes \phi_2$ all agree. Furthermore,

$$\pi_*(\pi^*\mathcal{B}_1 \otimes_E \pi^*\mathcal{B}_2) = \pi_*\pi^*(\mathcal{B}_1 \otimes_{\mathbb{P}^1} \mathcal{B}_2) = \pi_*(\mathcal{O}_E) \otimes_{\mathbb{P}^1} \mathcal{B}_1 \otimes_{\mathbb{P}^1} \mathcal{B}_2.$$

where the action of y is on the first factor only. Thus $\pi_*(\mathcal{L}_1 \otimes \mathcal{L}_2)$ is the cokernel of

$$\phi_1 \otimes 1 - 1 \otimes \phi_2 : \mathcal{B}_1 \otimes \mathcal{B}_2 \longrightarrow \mathcal{B}_1 \otimes \mathcal{B}_2.$$

To complete the proof we must show that the sequence $(*)$ is locally exact. Choose a point $x \in \mathbb{P}^1$ and denote the local ring $\mathcal{O}_{\mathbb{P}^1,x}$ by A and the A-module $\mathcal{B}_{i,x}$ by $F_1 + yF_1$ where the F_i are free A-modules. The endomorphism ϕ_i takes F_i to yF_i by multiplying with y, and yF_i to F_i by sending y to $f \in A$. In this notation, the maps $\phi_1 \otimes 1 \pm 1 \otimes \phi_2$ may be written as block matrices of the form

$$\begin{array}{c} \\ F_1\otimes F_2 \\ F_1\otimes yF_2 \\ yF_1\otimes F_2 \\ yF_1\otimes yF_2 \end{array} \begin{array}{c} \begin{array}{cccc} F_1\otimes F_2 & F_1\otimes yF_2 & yF_1\otimes F_2 & yF_1\otimes yF_2 \end{array} \\ \begin{pmatrix} 0 & \pm f & f & 0 \\ \pm 1 & 0 & 0 & f \\ 1 & 0 & 0 & \pm f \\ 0 & 1 & \pm 1 & 0 \end{pmatrix} \end{array}$$

Modulo the maximal ideal of A both these maps have rank equal to twice the rank of $F_1 \otimes F_2$, so the sequence above is locally split exact, as required. $\square$

Definition 2.3. Let $f(s,t) = \prod_{i=1}^{2g+2} f_i$ be a factorization of f into (necessarily distinct) linear factors, and, for $I \subset \{1,\ldots,2g+2\}$, write $f_I := \prod_{i\in I} f_i$. We write ϕ_I for the matrix

$$\begin{pmatrix} 0 & f_{I^c} \\ f_I & 0 \end{pmatrix} : \mathcal{O}_{\mathbb{P}^1}(\lceil -|I|/2\rceil) \oplus \mathcal{O}_{\mathbb{P}^1}(\lceil -|I^c|/2\rceil) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(\lceil |I^c|/2\rceil) \oplus \mathcal{O}_{\mathbb{P}^1}(\lceil |I|/2\rceil)$$

on $\mathbb{P}^1$ where I^c denotes the complement of I. Note that (ϕ_I, ϕ_{I^c}) is a matrix factorization of f. Let $\mathcal{L}_I$ be the corresponding line bundle on E, as defined in Proposition 2.1. Note that $\mathcal{L}_I \cong \mathcal{L}_{I^c}$ and $\mathcal{L}_\varnothing \cong \mathcal{O}_E$. Write $I\Delta J = (I \smallsetminus J) \cup (J \smallsetminus I)$ for the symmetric difference of I and J.

Theorem 2.4. *For $I, J \subset \{1,\ldots,2g+2\}$*

$$\mathcal{L}_I \otimes \mathcal{L}_J \cong \begin{cases} \mathcal{L}_{I\Delta J} & \text{if } |I|\cdot|J| \equiv 0 \mod 2, \\ \mathcal{L}_{I\Delta J}(\mathcal{H}) & \text{else.} \end{cases}$$

Thus the line bundles $\mathcal{L}_I$ with $|I|$ even are the 2^{2g} two-torsion line bundles on E. The line bundles $\mathcal{L}_I$ with $|I|$ odd are the 2^{2g} square roots of $\mathcal{O}_E(\mathcal{H})$.

Proof. In this case the matrix $\phi_I \otimes 1 - 1 \otimes \phi_J$ has the form

$$\begin{pmatrix} 0 & f_{I^c} & -f_{J^c} & 0 \\ f_I & 0 & 0 & -f_{J^c} \\ -f_J & 0 & 0 & f_{I^c} \\ 0 & -f_J & f_I & 0 \end{pmatrix}.$$

By Theorem 2.2, its kernel is the free module $H^0_*(\mathcal{L}_I \otimes \mathcal{L}_J)$. Because $J^c \smallsetminus I^c = I \smallsetminus J$ and $I \smallsetminus J^c = J \smallsetminus I^c$ this kernel contains the free submodule B generated by the column vectors

$$\begin{pmatrix} 0 & f_{J^c \smallsetminus I} \\ f_{I \smallsetminus J} & 0 \\ f_{J \smallsetminus I} & 0 \\ 0 & f_{I \smallsetminus J^c} \end{pmatrix}.$$

These columns generate the kernel because the 2×2 minors of B have no common factor (see [BE73, Corollary 1]).

To show that $\mathcal{L}_I \otimes \mathcal{L}_J \cong \mathcal{L}_{I \Delta J}$ it now suffices to show that the matrix representing the action of $\phi_1 \otimes 1$ restricted to the columns of B is

$$\begin{pmatrix} 0 & f_{(I\Delta J)^c} \\ f_{I \Delta J} & 0 \end{pmatrix}.$$

This, in turn, follows at once from the identities

$$I^c \cup (I \smallsetminus J) = (I \Delta J) \cup (J^c \smallsetminus I), \quad I \cup (J \smallsetminus I) = (I \Delta J) \cup (I \smallsetminus J^c)$$

and similarly

$$I \cup (J^c \smallsetminus I) = (I \Delta J)^c \cup (I \smallsetminus J), \quad I^c \cup (I \smallsetminus J^c) = (I \Delta J)^c \cup (J \smallsetminus I).$$

To show that $\mathcal{L}_I \ncong \mathcal{L}_J$ for $J \notin \{I, I^c\}$ are non-isomorphic, we consider the ideals generated by the entries of

$$\begin{pmatrix} 0 & f_{I^c} \\ f_I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & f_{J^c} \\ f_J & 0 \end{pmatrix}.$$

By looking at the elements of smallest degree, we see that these ideals could not be equal unless $|I| = |J| = g+1$. Also, in case $|I| = |J| = g+1$, the intersection $I \cap J$ is non-empty since $J \neq I^c$ and for $i \in I \cap J$ we recover f_I as the smallest degree generator of $(f_i) \cap (f_I, f_{I^c})$.

There are $2^{2g+2}/4$ unordered pairs $\{I, I^c\}$ of even subsets of $\{1, \dots, 2g+2\}$. Thus we get all 2^{2g} different two-torsion line bundles $\mathcal{L}_I$ for even I. A similar argument applies to roots of $\mathcal{H}$. □

3. BGG for complete intersections of quadrics

This section provides what we need of the theory of [BEH87] and [Kap89]. Let $P_X := k[V^*]/(q_1, \dots, q_c)$ be the homogeneous coordinate ring of the complete intersection $X = Q_1 \cap \dots \cap Q_c \subset \mathbb{P}(V^*) = \mathbb{P}^{r-1}$ of c quadrics $Q_i = V(q_i)$ and choose a basis $x_1, \dots, x_r$ of V^*. Write B_ℓ for the symmetric matrix with i,j entry

$$b_{\ell,i,j} = \frac{1}{2}(q_\ell(x_i + x_j) - q_\ell(x_i) - q_\ell(x_j)).$$

Let $T = k[t_1, \dots, t_c]$ denote a polynomial ring in c variables each of degree 2 and let

$$q : \begin{cases} T \otimes V & \longrightarrow \quad T \\ 1 \otimes v & \longmapsto \quad t_1 q_1(v) + \dots + t_c q_c(v) \end{cases}$$

denote the corresponding family of quadratic forms over $\operatorname{Spec} T$. Let

$$C := \left(T \otimes \left(\bigoplus_n V^{\otimes n} \right) \right) \Big/ (v \otimes v - q(v) \mid v \in V)$$

denote the $\mathbb{Z}$-graded Clifford algebra of q, so that C is the quadratic dual of P_X in the sense of [PP05]. The algebra C is free as a T-module with basis

$$e_I = e_{i_1} e_{i_2} \cdots e_{i_k}$$

where $e_1, \dots, e_r$ is a basis of V dual to $x_1, \dots, x_r$ and $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \dots, r\}$ an ordered subset. See for example [Jac80, Section 4.8].

Theorem 3.1. *Let P_X be the homogeneous coordinate ring of a complete intersection of c quadrics, and let C denote the corresponding $\mathbb{Z}$-graded Clifford algebra. Then P_X and C are a pair of Koszul dual graded algebras. In particular*

$$\operatorname{Ext}_{P_X}(k,k) \cong C \textit{ and } \operatorname{Ext}_C(k,k) \cong P_X.$$

Proof. See [Sjö76], [Kap89, Section 1] and [PP05]. □

Corollary 3.2. *For any graded P_X-module M the module $\operatorname{Ext}_{P_X}(M,k)$ is a graded $C = \operatorname{Ext}_{P_X}(k,k)$-module.*

The main result of this section is that for a graded P_X-modules M with a linear resolution one can recover M from the graded C-module $\operatorname{Ext}_{P_X}(M,k)$.

If M is a (left) P_X-module and N is a right C-module then we define an endomorphism of left $P_X \otimes C$-modules

$$d : \operatorname{Hom}_k(N,M) \longrightarrow \operatorname{Hom}_k(N,M)$$

taking $\phi \in \operatorname{Hom}_k(N,M)$ to ψ, where $\psi(n) = \sum_i x_i \phi(n e_i)$.

Note that

$$\begin{aligned} d^2(\phi)(n) &= \sum_{i,j} x_i x_j \phi(n e_i e_j) = \sum_{i \le j} x_i x_j \phi(n(e_i e_j + e_j e_i)) = \sum_{i \le j} x_i x_j \phi(n \sum_{\ell} (t_\ell b_{\ell,i,j})) \\ &= \sum_{\ell} \sum_{i \le j} b_{\ell,i,j} x_i x_j \phi(n t_\ell) = \sum_{\ell} q_\ell(x) \phi(n t_\ell) = 0. \end{aligned}$$

Thus, when N is $\mathbb{Z}$-graded, $\operatorname{Hom}_k(N,M)$ may be regarded as a complex of P_X-modules

$$\operatorname{Hom}_k(N,M) : \cdots \longrightarrow \operatorname{Hom}_k(N_i, M) \longrightarrow \operatorname{Hom}_k(N_{i-1}, M) \longrightarrow \cdots.$$

When M is $\mathbb{Z}$-graded and N is a $C-C$-bimodule, then $\operatorname{Hom}_k(N,M)$ may also be regarded as a complex of right C-modules

$$\operatorname{Hom}_k(N,M) : \cdots \longrightarrow \operatorname{Hom}_k(N, M_i) \longrightarrow \operatorname{Hom}_k(N, M_{i+1}) \longrightarrow \cdots.$$

Similar statements hold for $\operatorname{Hom}_k(M,N)$.

Theorem 3.3. *If the graded P_X-module M has a linear free resolution, then the resolution may be written in the form*

$$\operatorname{Hom}_k(\operatorname{Ext}_{P_X}(M,k), P_X)$$

where we view $\operatorname{Ext}_{P_X}(M,k)$ as a graded $C = \operatorname{Ext}_{P_X}(k,k)$ module, and apply the construction above.

Example 3.4. The complex $\operatorname{Hom}_k(C, P_X)$,

$$0 \longleftarrow C_0^* \otimes_k P_X \longleftarrow C_1^* \otimes_k P_X \longleftarrow C_2^* \otimes_k P_X \longleftarrow \cdots$$

is isomorphic to the P_X-free resolution of k.

Note that this statement may be deduced from [PP05, Corollary 3.2(iiM)]. Since this result plays a crucial role in the proof of Proposition 5.6, we give a proof below. For our proof we need an explicit description of the action of $\operatorname{Ext}^1_{P_X}(k,k)$ on $\operatorname{Ext}_{P_X}(M,k)$.

To avoid keeping track of grading shifts we formulate this in case of a finitely generated module M over a Noetherian local ring R with maximal ideal $\mathfrak{m}$. Let $(x_1,\dots,x_r)$ denote minimal generators of $\mathfrak{m}$, and let $e_i \in \operatorname{Ext}^1_R(k,k)$ be the extension

$$e_i\colon 0 \longrightarrow k \xrightarrow{x_i} E_i \longrightarrow k \longrightarrow 0,$$

where $E_i = R/(x_1,\dots,x_{i-1},x_i^2,x_{i+1},\dots x_r)$. Let

$$\mathbb{F}\colon \cdots \xrightarrow{d} F_j \xrightarrow{d} \cdots \xrightarrow{d} F_0$$

be the minimal free resolution of a finitely generated R-module M. Since the resolution F is minimal the differential $d(f)$ of an element $f \in F_{j+1}$ can be written in the form $d(f) = \sum_{i=1}^r x_i f_i$ for $f_i \in F_j$.

Lemma 3.5. *Let $\alpha \in \operatorname{Ext}^j_R(M,k)$ be a class represented by a map $\alpha'\colon F_j \to k$. The element $\alpha e_i \in \operatorname{Ext}^{j+1}_R(M,k)$ is then represented by the map β_i with $\beta_i(f) = \alpha'(f_i)$ for $f \in F_{j+1}$ with differential $d(f) = \sum_{i=1}^r x_i f_i$.*

Proof. We compute the image of α under the connecting homomorphism δ

$$\operatorname{Ext}^j_R(M,E_i) \longrightarrow \operatorname{Ext}^j_R(M,k) \xrightarrow{\delta} \operatorname{Ext}^{j+1}_R(M,k) \longrightarrow \operatorname{Ext}^{j+1}_R(M,E_i)$$

associated to the sequence e_i above. Consider the diagram

$$\begin{array}{ccccc} F_{j+1} & \xrightarrow{d} & F_j & & \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \alpha''} & \searrow^{\alpha'} & \\ k & \xrightarrow{x_i} & E_i & \longrightarrow & k \end{array}$$

where α'' is a lift of α' to E_i. The composition $\alpha' \circ d$ is zero since $\alpha'(\mathfrak{m}F_j) = 0$. Thus $\alpha'' \circ d$ factors over the map

$$\beta_i\colon \begin{cases} F_{j+1} & \longrightarrow \quad k \\ \quad f & \longmapsto \quad \alpha'(f_i). \end{cases}$$

This map is well-defined, *i.e.* independent of the choice of f_i. Indeed, if $d(f) = \sum_{i=1}^r x_i f_i'$ is a different choice for the presentation of $d(f)$ then $x_i(f_i - f_i') \in (x_1,\dots,x_{i-1},x_{i+1},\dots x_r)F_j$ which maps to zero in E_i. □

Proof of Theorem 3.3. Let

$$\mathbb{F}\colon \cdots \xrightarrow{d} F_j \xrightarrow{d} \cdots \xrightarrow{d} F_0$$

be the minimal graded free resolution of M as a P_X-module. Then

$$F_j \cong \overline{F}_j \otimes_k P_X$$

where $\overline{F}_j \cong F_j/\mathfrak{m}F_j$. If M has a linear resolution then we claim that the isomorphisms

$$\begin{cases} F_j = \overline{F}_j \otimes P_X & \xrightarrow{\cong} \quad \operatorname{Hom}_k(\operatorname{Hom}_{P_X}(F_j,k),P_X) \\ \quad \overline{f} \otimes p & \longmapsto \quad \{\varphi\colon \alpha \mapsto \alpha(f) \otimes p\} \end{cases}$$

induce an isomorphism of complexes, *i.e.* we have to show that these maps commute with differentials of the two complexes. Let $b_1,\dots,b_\ell$ be a k-basis of $\overline{F}_j$ with dual basis $b_1^*,\dots,b_\ell^*$ of $\overline{F}_j^* = \operatorname{Hom}_k(\overline{F}_j,k) = \operatorname{Hom}_{P_X}(F_j,k)$.

Consider an element $f = \overline{f} \otimes 1 \in F_{j+1}$. Then

$$d(f) = \sum_{i=1}^{r} \sum_{\nu=1}^{\ell} c_{i\nu} b_\nu \otimes x_i \text{ with } c_{i\nu} \in k$$

and we can take $f_i = \sum_{\nu=1}^{\ell} c_{i\nu} b_\nu$ for the coefficient of x_i as in Lemma 3.5. The map

$$\{\varphi\colon \alpha \mapsto \alpha(f)\} \in \operatorname{Hom}_k\Big(\operatorname{Hom}_{P_X}(F_{j+1}, k), P_X\Big)$$

maps to

$$\{\alpha' \mapsto \sum_{i=1}^{r} x_i \varphi(\alpha' e_i)\} \in \operatorname{Hom}_k\Big(\operatorname{Hom}_{P_X}(F_j, k), P_X\Big)$$

by the definition of the differential of $\operatorname{Hom}_k(\operatorname{Ext}_{P_X}(M,k), P_X)$. We have

$$\begin{aligned} \sum_{i=1}^{r} x_i \varphi(\alpha' e_i) &= \sum_{i=1}^{r} x_i \alpha'(f_i) \qquad \text{(by Lemma 3.5)} \\ &= \sum_{i=1}^{r} x_i \alpha'(\sum_{\nu=1}^{\ell} c_{i\nu} b_\nu) \end{aligned}$$

In particular, for $\alpha' = b_\mu^*$ we obtain $b_\mu^* \mapsto \sum_{i=1}^{r} c_{i\mu} x_i$. These values coincide with the values of the image of

$$d(f) = \sum_{i=1}^{r} \sum_{\nu=1}^{\ell} c_{i\nu} b_\nu \otimes x_i$$

in $\operatorname{Hom}_k\Big(\operatorname{Hom}_{P_X}(F_j, k), P_X\Big)$, since $b_\mu^*(\sum_{\nu=1}^{\ell} c_{i\nu} b_\nu) = c_{i\mu}$. ☐

Corollary 3.6. *Let N be a graded left C-module. The complex $\operatorname{Hom}_k(N, P_X)$ is a resolution if and only if $N \cong \operatorname{Ext}_{P_X}(M,k)$ up to shift where M is a P_X-module with a linear resolution.*

Proof. If $N \cong \operatorname{Ext}_{P_X}(M,k)$ up to shift where M is a P_X-module with a linear resolution then by Theorem 2.4 the resolution of M is $\operatorname{Hom}_k(N, P_X)$. Conversely, if the complex $\operatorname{Hom}_k(N, P_X)$ is a resolution, then since it is linear we may take the module it resolves to be M. ☐

4. Pencils of quadrics and hyperelliptic curves

We now specialize to the case of a smooth intersection of two quadrics in $\mathbb{P}^{2g+1}$ with coordinate ring $P_X = k[x_1, \ldots, x_{2g+2}]/(q_1, q_2)$. To simplify notation we write s, t instead of t_1, t_2. Let $q = q(s,t) = sq_1 + tq_2$ and let $C = \operatorname{Cliff}(q)$ denote the $\mathbb{Z}$-graded Clifford algebra of q, so that $T = k[s,t] \subset C$.

As in Reid's thesis [Rei72] we note that none of the quadrics in the pencil can have corank 2: for, if one of the quadrics had singular locus L of dimension at least 2, then X would be singular at $L \cap X$. Further, by Bertini's Theorem the general linear combination of the two quadrics is non-singular outside the intersection. But if it were singular at a point of the intersection, then the intersection would be singular there too. Thus we may assume that one of the quadrics has full rank, and it follows that the two quadrics can be simultaneously diagonalized (see [Gan59, XII, Paragraph 6, Theorem 7]). Thus we may assume that the bilinear form $q(s,t) = sq_1 + tq_2$ is given by a diagonal matrix

$$\begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_{2g+2} \end{pmatrix}$$

with entries that are pairwise coprime linear polynomials $f_i \in k[s,t]$. As in Section 2 we denote by $f = \prod f_i$, and use the notation $f_I = \prod_{i \in I} f_i$.

We write
$$C = C^{\mathrm{ev}} \oplus C^{\mathrm{odd}}$$
for the decomposition of the Clifford algebra into its even and odd parts. As a $T = k[s,t]$-module, C is free with basis e_I and

$$e_I e_J = \epsilon(I,J) f_{I\cap J}\, e_{I\Delta J}. \tag{4.1}$$

with the sign $\epsilon(I,J) = (-1)^{\sum_{i\in I} |\{j\in J | j<i\}|}$.

Since
$$\sum_{i\in I} |\{j \in \{1,\dots,2g+2\} \mid j < i\}| = \sum_{i\in I}(i-1)$$
and
$$\sum_{j\in\{1,\dots,2g+2\}} |\{i \in I | i < j\}| = \sum_{i\in I}(2g+2-i) \equiv \sum_{i\in I}(i-1) \mod 2$$
for even I, we see that $e_{\{1,\dots,2g+2\}}$ lies in the center of the even Clifford algebra. Because
$$\sum_{i=1}^{2g+2}(i-1) = \binom{2g+2}{2} \equiv g+1 \mod 2,$$
the element $e_{\{1,\dots,2g+2\}}$ satisfies the equation
$$e^2_{\{1,\dots,2g+2\}} = (-1)^{g+1} f.$$
To adjust for the sign we take $y = (\sqrt{-1})^{g+1} e_{\{1,\dots,2g+2\}}$ as a generator of the center of the even Clifford algebra over $k[s,t]$ so that $y^2 = f$. Note that the formula above for the central element y is only correct in the case of diagonal quadrics; for the general case see [Haa91, Satz 1].

Furthermore, for any I,
$$e_I e_{\{1,\dots,2g+2\}} = (-1)^{\sum_{i\in I}(i-1)} f_I\, e_{I^c} \quad \text{and} \quad e_{I^c} e_{\{1,\dots,2g+2\}} = (-1)^{\sum_{i\in I^c}(i-1)} f_{I^c}\, e_I.$$
Note that the signs in the two formulas differ by $(-1)^{g+1}$. Thus with $R_E = k[s,t,y]/(y^2 - f)$ the coordinate ring of the corresponding hyperelliptic curve, the R_E-submodule of C generated by e_I and e_{I^c} coincides with $H^0_*(\mathcal{L}_I)$ from Definition 2.3.

Notice however, that here, differently than in section 1, R_E is $2\mathbb{Z}$-graded as a subring of C^{ev}. Thus, since the elements of C^{odd} have odd degree, we have to twist by an odd number to obtain a non-trivial sheaf of $\mathcal{O}_E$-modules.

We define $\mathcal{C}^{\mathrm{ev}} = \widetilde{C^{\mathrm{ev}}}$ and $\mathcal{C}^{\mathrm{odd}} = \widetilde{C^{\mathrm{odd}}(1)}$. Hence multiplication in C gives a map
$$C^{\mathrm{odd}}(1) \times C^{\mathrm{odd}}(1) \longrightarrow C^{\mathrm{ev}}(2)$$
which sheafifies to a map
$$\mathcal{C}^{\mathrm{odd}} \otimes_{\mathcal{O}_E} \mathcal{C}^{\mathrm{odd}} \longrightarrow \mathcal{C}^{\mathrm{ev}} \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{C}^{\mathrm{ev}} \otimes \mathcal{H}.$$

In summary, we get the following statement.

Proposition 4.1. *Let $y = (\sqrt{-1})^{g+1} e_{\{1,\dots,2g+2\}}$. The element y is in the center of C^{ev} and satisfies the equation $y^2 = f$, where $f = \prod_{i=1}^{2g+2} f_i$. If we write $R_E = k[s,t,y]/(y^2 - f)$ then the even Clifford algebra decomposes as an R_E-module as*
$$C^{\mathrm{ev}} = \bigoplus_{\substack{\{I,I^c\}\\ |I|\ \text{even}}} H^0_*(\mathcal{L}_I).$$
The odd part of the Clifford algebra decomposes as a right R_E-module as
$$C^{\mathrm{odd}}(1) = \bigoplus_{\substack{\{I,I^c\}\\ |I|\ \text{odd}}} H^0_*(\mathcal{L}_I).$$

Moreover,

$$\mathcal{C}^{\text{odd}} \cong \mathcal{O}_E(p) \otimes \mathcal{C}^{\text{ev}}$$

where p is any ramification point of $\pi: E \to \mathbb{P}^1$.

Proof. This follows from Theorem 2.4. Note that since p is a ramification point we have $\mathcal{O}_E(2p) \cong \mathcal{H}$ and the multiplication map $\mathcal{C}^{\text{odd}} \otimes \mathcal{C}^{\text{odd}} \to \mathcal{C}^{\text{ev}} \otimes \mathcal{H}$ is compatible with the map

$$\mathcal{O}_E(p) \otimes \mathcal{O}_E(p) \longrightarrow \mathcal{O}_E(2p) \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{H}. \qquad \square$$

Remark 4.2. Notice that y and the elements of C^{odd} anti-commute by equation (4.1) applied to the case when I is a singleton and $J = \{1, \dots, 2g+2\}$.

The following result was proven in [Rei72].

Lemma 4.3. *Let q_1, q_2 be two quadratic forms on a $2g+2$-dimensional vector space V over k. The set of g-dimensional common isotropic subspaces of q_1, q_2 is non-empty and has dimension $\geq g$ locally at every point.*

Proof. Let $\mathcal{U}$ be the universal sub-bundle on the Grassmannian $\mathbb{G} := \mathbb{G}(g, V)$. The forms q_i define homomorphisms $\text{Sym}^2 V^* \otimes_k \mathcal{O}_{\mathbb{G}} \to \mathcal{O}_{\mathbb{G}}$, and thus, by restriction, sections of $\text{Sym}^2(\mathcal{U}^*)$. The set of g-dimensional common isotropic subspaces is the common zero locus of these two sections. Computing the Chern class we see that the locus is non empty and, since

$$\dim \mathbb{G}(g, V) - 2\,\text{rk}\ \text{Sym}^2(\mathcal{U}^*) = g(g+2) - 2\binom{g+1}{2} = g,$$

the inequality on dimensions follows. $\square$

We return to the situation at the beginning of Section 3, with

$$P = k[x_1, \dots, x_{2g+2}] = \text{Sym}(V^*).$$

Let $U \subset V$ be a g-dimensional isotropic linear subspace and denote by $P_U = \text{Sym}(U^*) = P/(U^\perp)$ its coordinate ring, where $U^\perp \subset V^*$ is the space of linear equations of the isotropic space U.

Proposition 4.4. *Let $G = ks \oplus kt \cong k^2$ be the space of parameters for the family of quadratic forms $sq_1 + tq_2$. Considered as a P_X-module, P_U has a linear free resolution. Moreover*

$$\text{Ext}^{2p}_{P_X}(P_U, k) = \bigoplus_i \left(\Lambda^{2i} U^\perp \otimes_k (\text{Sym}_{p-i}\, G)^*\right)^* \tag{4.2}$$

and

$$\text{Ext}^{2p+1}_{P_X}(P_U, k) = \bigoplus_i \left(\Lambda^{2i+1} U^\perp \otimes_k (\text{Sym}_{p-i}\, G)^*\right)^*. \tag{4.3}$$

Proof. The ideal $(U^\perp)$ contains the 2-dimensional vector space $G := \langle q_1, q_2 \rangle$. This ideal is generated by a regular sequence of linear forms, and the P-free resolution of $P_U = P/(U^\perp)$ is thus a Koszul complex with underlying free module $P \otimes \Lambda U^\perp$. Let $\gamma : G \to P_1 \otimes U^\perp$ be a map of vector spaces such that the composition of γ with the multiplication map

$$G \longrightarrow P_1 \otimes_k U^\perp \longrightarrow P_2$$

is the inclusion of G in P_2.

By [Tat57, Theorem 4], the minimal P_X-free resolution of P_U is the differential graded R-algebra

$$P_X \otimes_k \Lambda U^\perp \otimes (\text{Sym}\, G^*)^*.$$

Here $U^\perp$ has internal degree 1 and homological degree 1, while G^* has internal degree 2 and homological degree 2, and the component of the differential $G = \left(\text{Sym}^1 G^*\right)^* \to P_X \otimes_k U^\perp$ is induced by γ.

This resolution is linear, and has degree j term

$$P_X \otimes_k \left(\bigoplus_{j=a+2b} (\Lambda^a U^\perp) \otimes_k (\mathrm{Sym}_b\, G)^* \right). \qquad \square$$

Let $T = \mathrm{Sym}\, G \cong k[s,t]$ and write

$$F_U = \mathrm{Ext}^{\mathrm{ev}}_{P_X}(P_U, k) = \bigoplus_i \left((\Lambda^{2i} U^\perp)^* \otimes_k T(-i) \right)$$

regarded as a module over $\mathrm{Ext}^{\mathrm{ev}}_{P_X}(k,k) = C^{\mathrm{ev}}$ *via* the Yoneda pairing.

Proposition 4.5. *The sheafification $\mathcal{F}_U$ of F_U as an $\mathcal{O}_E$-module is a vector bundle of* $\mathrm{rk}\,\mathcal{F}_U = 2^g$ *and degree* $\deg \mathcal{F}_U = g2^{g-1}$ *on E. Moreover,*

$$H^0_*(\mathcal{F}_U) = Ext^{\mathrm{ev}}(P_U, k),$$

and

$$H^0_*(\mathcal{F}_U(p)) = Ext^{\mathrm{odd}}(P_U, k).$$

Proof. It follows from the formulas above that the sheafification $\mathcal{F}_U$ of F_U as an $\mathcal{O}_E$-module is a vector bundle of rank equal to $\dim_k(\Lambda^{\mathrm{ev}} U^\perp)/2 = 2^g$. Moreover

$$\deg \pi_* \mathcal{F}_U = -\sum_{i \geq 0} i \binom{g+2}{2i} = -(g+2)2^{g-1}.$$

By Proposition 2.1, $\mathcal{F}_U$ has degree

$$\deg \mathcal{F}_U = (g+1)\,\mathrm{rk}\,\mathcal{F}_U + \deg \pi_* \mathcal{F}_U = (g+1)2^g - (g+2)2^{g-1} = g2^{g-1}.$$

The first displayed formula is immediate from the definition of F_U, while the second follows from the equality $\mathcal{C}^{\mathrm{odd}} = \mathcal{C}^{\mathrm{ev}}(p)$. $\square$

Theorem 4.6. *The endomorphism bundle of $\mathcal{F}_U$ is isomorphic as an $\mathcal{O}_E$-algebra to the sheafified even Clifford algebra $\mathcal{C}^{\mathrm{ev}}$; that is,*

$$\mathcal{E}nd_E(\mathcal{F}_U) \cong \mathcal{C}^{\mathrm{ev}}.$$

Proof. Let $(a,b) \in \mathbb{P}^1$ be a point that is not a branch point of π. The algebra $\pi_* \mathcal{C}^{\mathrm{ev}}$ is a sheaf of algebras whose fiber at (a,b) is isomorphic to the product of the fibers of $\mathcal{C}^{\mathrm{ev}}$ at the two preimages of (a,b) in E. On the other hand, the fiber of $\pi_* \mathcal{C}^{\mathrm{ev}}$ is the even Clifford algebra of the nonsingular quadratic form $aq_1 + bq_2$. Thus it is a semisimple algebra with 2-dimensional center generated over k by y. Since we have assumed that k is algebraically closed, this center is $k \times k$. The corresponding decomposition of the push forward of $\mathcal{C}^{\mathrm{ev}}$ as a direct product is the unique decomposition as the product of two algebras. Thus the fibers of $\mathcal{C}^{\mathrm{ev}}$ at points of E other than the ramification points are simple algebras by [Jac80, Theorem 4.13].

Since F_U is an $R_E - C^{\mathrm{ev}}$ bimodule we have an $\mathcal{O}_E$-algebra homomorphism

$$\phi : \mathcal{C}^{\mathrm{ev}} \longrightarrow \mathcal{E}nd_E(\mathcal{F}_U).$$

Since the general fiber of $\mathcal{C}^{\mathrm{ev}}$ is simple, the kernel of this homomorphism must be torsion, and thus 0. The source and target of ϕ are vector bundles of the same rank. By Proposition 4.1 the sheaf $\mathcal{C}^{\mathrm{ev}}$ is a sum of the degree 0 line bundles $\mathcal{L}_I$, and since the endomorphism bundle also has degree 0, the map is an isomorphism. $\square$

Corollary 4.7 (Morita equivalence, see [Bas68, Chapter 2]). *The $\mathcal{O}_E - \mathcal{C}^{\mathrm{ev}}$ bimodule $\mathcal{F}_U$ defines an equivalence of module categories*

$$\left\{ \begin{array}{ccc} \mathcal{O}_E - mod & \longleftrightarrow & mod - \mathcal{C}^{\mathrm{ev}} \\ \mathcal{L} & \longmapsto & \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{F}_U \\ \mathcal{G} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{F}_U^* & \longleftarrow & \mathcal{G} \end{array} \right.$$

where $\mathcal{F}_U^* = \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}_U, \mathcal{O}_E)$.

Corollary 4.8 (Reid, 1972 [Rei72]). *Let $X = Q_1 \cap Q_2 \subset \mathbb{P}^{2g+1}$ be a smooth intersection of two quadrics and let E be the corresponding hyperelliptic curve. Let $U_0 \subset V$ be a g-dimensional linear subspace such that $\mathbb{P}(U_0^*) \subset X$. Then the map*

$$\varphi\colon \begin{cases} \{U \in \mathbb{G}(g,V) \mid \mathbb{P}(U^*) \subset X\} & \longrightarrow \quad \mathrm{Pic}^0(E) \\ \qquad\qquad U & \longmapsto \quad \mathcal{F}_U \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{F}_{U_0}^* \end{cases}$$

is a bijection. If the ground field k has characteristic 0, it is an isomorphism.

Proof. By Lemma 4.3, a space U_0 of dimension g such that $\mathbb{P}(U_0^*) \subset X$ exists. We claim that $\mathcal{F}_U \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{F}_{U_0}^*$ is an element of $\mathrm{Pic}^0(E)$. We know by Corollary 4.7 that $\mathcal{F}_{U_0}$ and $\mathcal{F}_U$ both define Morita equivalences. Hence $\mathcal{L} := \mathcal{F}_U \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{F}_{U_0}^*$ must be an invertible object in $\mathcal{O}_E - mod$, hence a line bundle. This line bundle has degree 0 since $\mathcal{F}_U \cong \mathcal{L} \otimes \mathcal{F}_{U_0}$ and both vector bundles have the same degree.

The map φ is injective because we can recover U from $\mathcal{F}_U \cong \mathcal{L} \otimes \mathcal{F}_{U_0}$ as follows: by Proposition 5.6 (3) below, we can recover the $C = \mathrm{Ext}_{P_X}(k,k)$-module $\mathrm{Ext}_{P_X}(P_U,k)$ from $\mathcal{F}_U$. The free resolution of P_U, hence $U^\perp$, can be obtained from $\mathrm{Ext}_{P_X}(P_U,k)$ by Theorem 3.3.

Since the source and target of φ are projective and the target is connected, smooth, and of the same dimension as the source, the map is a surjective, hence a bijection. In case the ground field k has characteristic 0 φ is thus an isomorphism. If k has positive characteristic it could be a purely inseperable morphism. Miles Reid proved in [Rei72, Theorem 4.8] that $\{U \in \mathbb{G}(g,V) \mid \mathbb{P}(U^*) \subset X\}$ and $\mathrm{Pic}^0(E)$ are isomorphic for arbitrary characteristic. □

Remark 4.9. Our Macaulay2 package [EKS22] computes the action of $\mathrm{Pic}^0(E)$ on the space of maximal isotropic subspaces

$$\mathbb{G}(g,X) = \{U \in \mathbb{G}(g,V) | \mathbb{P}(U^*) \subset X\}.$$

For a different approach to the group law on $\mathrm{Pic}^0(E)$ in terms of $\mathbb{G}(g,X)$ see [Don80].

5. Tate resolutions of P_X-modules from Clifford modules

The constructions in this section are inspired by the theory of Cohen–Macaulay approximations of Auslander and Buchweitz [AB89] and the construction of Tate resolutions as in [ES21]. Let R be a Noetherian local or graded Gorenstein ring, and let M be a finitely generated R-module. Let F be the minimal R-free resolution of M:

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots.$$

We will use the notation $N^* = \mathrm{Hom}_R(N,R)$ for the dual of an R-module N. If N is a maximal Cohen–Macaulay (MCM) module, that is, an R-module of depth $\dim R$, then we have $(N^*)^* \cong N$, because R is Gorenstein.

The Tate resolution associated to M is a doubly infinite exact complex of free R-modules obtained as follows: The i^{th} syzygy module $M_i = \ker(F_{i-1} \to F_{i-2})$ is an MCM module when $i > \dim R$, so $M_i^* = \ker\left(F_i^* \to F_{i+1}^*\right)$ is also an MCM module.

Choose an integer $i > \dim R$, and let

$$\cdots \longrightarrow G_{i-2} \longrightarrow G_{i-1} \longrightarrow M_i^* \longrightarrow 0$$

be a minimal free resolution of M_i^*. The Tate resolution $\mathbb{T}(M)$ of M is obtained by splicing the dual complex G^* with the complex $F_i \longleftarrow F_{i+1} \longleftarrow \cdots$ to a doubly infinite complex

$$\mathbb{T}(M)\colon \cdots \longleftarrow G_{i-2}^* \longleftarrow G_{i-1}^* \longleftarrow F_i \longleftarrow F_{i+1} \longleftarrow \cdots$$

of free graded R-modules. This is an exact complex because both $M_i = \ker(F_{i-1} \to F_{i-2})$ and $M_i^* \cong \ker\left(F_i^* \to F_{i+1}^*\right)$ are MCM modules. Up to isomorphism this complex is independent of the choice of i and the choice of the minimal free resolutions. The dual complex $\mathbb{T}(M)^*$ is exact as well.

Example 5.1. In case of a hypersurface ring $R = P/(f)$ the Tate resolutions are the double infinite periodic complexes

$$\cdots \xleftarrow{\overline{\phi}} R^n \xleftarrow{\overline{\psi}} R^n \xleftarrow{\overline{\phi}} R^n \xleftarrow{\overline{\psi}} \cdots$$

obtained from matrix factorizations (ϕ, ψ) of f, *cf.* [Eis80].

Remark 5.2. Auslander and Buchweitz [AB89] used Tate resolutions to define the MCM approximation of M for arbitrary Cohen–Macaulay rings. When R is Gorenstein, as in our case, we set $M^{\mathrm{es}} = \operatorname{coker}\left(G_1^* \to G_0^*\right)$, the ***essential MCM approximation***, so that M^{es} is an MCM over R. By [AB89] there is an induced map $M^{\mathrm{es}} \to M$ and the modules M and M^{es} have free resolutions that differ in only finitely many terms: If $R^n \to M$ is a map from a graded free P_X module such that

$$0 \longleftarrow M \longleftarrow M^{\mathrm{es}} \oplus R^n$$

is a surjection, then the kernel of this homomorphism has a finite free resolution of length $\operatorname{codepth} M - 1$. Auslander–Buchweitz define this homomorphism to be the MCM approximation of M if n is taken to be minimal.

Proposition 5.3. *Let $P_X = P/(q_1, \dots, q_c)$ be the homogeneous coordinate ring of a complete intersection of quadrics. Let M be a P_X-module which has a linear resolution as a P-module. Then $\operatorname{Ext}_{P_X}(M,k)$ is a $C = \operatorname{Ext}_{P_X}(k,k)$-module which is free as a $k[t_1, \dots, t_c]$-module. If moreover M is a Cohen–Macaulay P_X-module of codimension ℓ then the Tate resolution of M has the form*

$$\begin{array}{ccccccccccccc}
\cdots & \longleftarrow & P_X^{b_{-2}}(3) & \longleftarrow & P_X^{b_{-1}}(2) & \longleftarrow & P_X^{b_0}(1) & \longleftarrow & \cdots & \longleftarrow & P_X^{b_\ell}(-\ell+2) & \longleftarrow & 0 \\
 & & & & \uparrow{\scriptstyle \phi_0} & & \uparrow{\scriptstyle \phi_1} & & & & \uparrow{\scriptstyle \phi_\ell} & & \\
 & & 0 & \longleftarrow & P_X^{a_0} & \longleftarrow & P_X^{a_1}(-1) & \longleftarrow & \cdots & \longleftarrow & P_X^{a_\ell}(-\ell) & \longleftarrow & \cdots
\end{array}$$

with $b_{\ell-i} = a_i$ with an overlap of length ℓ. The bottom row, which is a quotient complex, is the Eisenbud–Shamash resolution of M as a P_X-module, and the top row, a subcomplex, is its P_X dual.

Proof. As in the special case explained in the proof of Proposition 4.4, the Eisenbud–Shamash graded free resolution of M as a P_X module [Eis80, Theorem 7.2] can be constructed from a series of higher homotopies on a graded P-free resolution F of M. Because the q_i have degree 2, all the higher homotopies are linear maps, so the construction yields a minimal linear resolution of M whose underlying graded free module is a divided power algebra over P_X on c generators tensored with the underlying module of F, and this implies that $\operatorname{Ext}_{P_X}(M,k)$ is a free module over the dual algebra, $k[t_1, \dots, t_c]$.

If M is Cohen–Macaulay of codimension ℓ then the $(\ell+1)^{\text{th}}$ syzygy of M is a maximal Cohen–Macaulay module, and by [ES21] the Tate resolution of M has the given form. ☐

In [ES21] there is an explicit description of all maps in the Tate resolution in case of a nested pair of complete intersections such as the following.

Example 5.4. Consider the coordinate ring P_U of a g-dimensional isotropic subspace U in the complete intersection X of two quadrics as a P_X-module. The Tate resolution $\mathbb{T}(P_U)$ has an overlap of length $\ell = \operatorname{codim}_X \mathbb{P}(U^*) = 2g - 1 - (g-1) = g$. In case $g = 3$ it has Betti table

$$\begin{array}{cccccccccc}
\cdots & 28 & 20 & 12 & 5 & 1 & & & & \\
 & & & 1 & 5 & 12 & 20 & 28 & 36 & \cdots
\end{array}$$

The vertical maps in the display of $\mathbb{T}(P_U)$ are northwest diagonal maps in the Betti table, which are represented by matrices of quadratic forms. For example the map ϕ_0 as in Proposition 5.3 is given by a

20×1 matrix of quadrics, represented in the Betti table by the northwest map from the left-most 1 on the lower to the 20 in the upper row. For arbitrary g we obtain the formulas

$$a_{2p} = \sum_{i=0}^{p} (p-i+1) \binom{g+2}{2i} \quad \text{and} \quad a_{2p+1} = \sum_{i=0}^{p} (p-i+1) \binom{g+2}{2i+1}$$

for the ranks a_i in the lower row of the diagram above from the equations (4.2) and (4.3) in Section 4.

Theorem 5.5. *Let $C = \mathrm{Cliff}(q_1, q_2)$ be the Clifford algebra over $k[s,t]$ of a nonsingular complete intersection of two quadrics in $\mathbb{P}^{2g+1}$. Let N be a graded C-module that is free as a $k[s,t]$-module, and such that the corresponding vector bundles $\mathcal{N}^{\mathrm{ev}} = \widetilde{N^{\mathrm{ev}}}$ and $\mathcal{N}^{\mathrm{odd}} = \widetilde{N^{\mathrm{odd}}(1)}$ defined on the associated hyperelliptic curve E satisfies*

$$\mathcal{N}^{\mathrm{odd}} \cong \mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}}.$$

Let $p \in E$ be a ramification point. There is a doubly infinite exact complex

$$\mathbb{T}(N): \cdots \longrightarrow F_i \longrightarrow F_{i+1} \longrightarrow \cdots$$

of free modules $F_i = P_X^{a_i}(i) \oplus P_X^{b_i}(i+1)$ with Betti numbers $a_i = h^1(\mathcal{N}^{\mathrm{ev}}(ip))$ and $b_i = h^0(\mathcal{N}^{\mathrm{ev}}((i+1)p))$. In terms of this decomposition, the complex $\mathbb{T}(N)$ takes the form

$$\begin{array}{ccccccc}
\to & H^1(\mathcal{N}^{\mathrm{ev}}) \otimes_k P_X & \to & H^1(\mathcal{N}^{\mathrm{ev}}(p)) \otimes_k P_X(1) & \to & H^1(\mathcal{N}^{\mathrm{ev}}(2p)) \otimes_k P_X(2) & \to \\
\searrow & \oplus & \searrow & \oplus & \searrow & \oplus & \searrow \\
\to & H^0(\mathcal{N}^{\mathrm{ev}}(p)) \otimes_k P_X(p) & \to & H^0(\mathcal{N}^{\mathrm{ev}}(2p)) \otimes_k P_X(2) & \to & H^0(\mathcal{N}^{\mathrm{ev}}(3p)) \otimes_k P_X(3) & \to .
\end{array}$$

Proof. We will use the notations x_i, e_i as defined in Section 3. Consider the sequence of maps

$$\cdots \xrightarrow{d} N_{i-1} \otimes_k P \xrightarrow{d} N_i \otimes_k P \xrightarrow{d} N_{i+1} \otimes_k P \xrightarrow{d} \cdots$$

defined by $d(n \otimes_k r) = \sum_{i=1}^{2g+2} ne_i \otimes_k x_i r$.

Computations similar to that at the beginning of Section 3 show that

$$d^2(n \otimes_k r) = \sum_{i,j} (ne_i e_j) \otimes_k (x_i x_j r) = ns \otimes_k q_1(x) r + nt \otimes_k q_2(x) r = n \otimes_{k[s,t]} (sq_1(x) + tq_2(x)) r,$$

where the last step uses the identification $N \otimes_k P = N \otimes_{k[s,t]} P[s,t]$.

Set $A := N^{\mathrm{ev}} \otimes_k P$ and $B := N^{\mathrm{odd}} \otimes_k P$. The map d induces a matrix factorization

$$(A \longrightarrow B(0,1),\ B(0,1) \longrightarrow A(1,2))$$

of $sq_1 + tq_2$ over the bi-graded polynomial ring $k[s,t,x_1,\ldots,x_{2g+2}]$. As in Example 5.1, this matrix factorization induces a 2-periodic resolution

$$\cdots \longrightarrow \overline{B}(-1,-1) \longrightarrow \overline{A} \longrightarrow \overline{B}(0,1) \longrightarrow \overline{A}(1,2) \longrightarrow \cdots$$

where $\overline{A}$ and $\overline{B}$ are restrictions of A and B to $k[s,t,x_1,\ldots,x_{2g+2}]/(sq_1 + tq_2)$.

Sheafifying with respect to the variables (s,t) we get a doubly infinite exact complex

$$\cdots \longrightarrow \widetilde{B}(-1,-1) \longrightarrow \widetilde{A} \longrightarrow \widetilde{B}(0,1) \longrightarrow \widetilde{A}(1,2) \longrightarrow \cdots$$

of direct sums of line bundles on the hypersurface $V(sq_1 + tq_2) \subset \mathbb{P}^1 \times \mathbb{A}^{2g+2}$.

We define an exact complex of $\mathcal{O}_{\mathbb{P}^1} \otimes P_X$-modules by factoring out q_1 on the set $t \neq 0$ and q_2 on the set $s \neq 0$, identified on the set where neither s nor t is zero with $k[s/t, t/s] \otimes P/(q_1, q_2)$.

Since the central element y of the even Clifford algebra anti-commutes with the action of the e_i on N by Remark 4.2 we may regard this also as a complex of $\mathcal{O}_E \otimes P_X$-modules that are box products of locally free $\mathcal{O}_E$-modules with graded free P_X-modules,

$$\mathbb{T}: \cdots \longrightarrow \mathcal{A}_E \boxtimes P_X \longrightarrow \mathcal{B}_E \boxtimes P_X(1) \longrightarrow \mathcal{A}_E(1) \boxtimes P_X(2) \longrightarrow \cdots,$$

where use the fact that $\mathcal{O}_E(1) \cong \mathcal{O}_E(2p)$. Here $\mathcal{A}_E = \mathcal{N}^{\mathrm{ev}}$ and $\mathcal{B}_E$ is isomorphic to

$$\mathcal{N}^{\mathrm{odd}} = \mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}} = \mathcal{N}^{\mathrm{ev}}(p)$$

by Proposition 4.1, where the action of y on $\mathcal{B}_E$ is induced by the action of $-y$ on N^{odd}. Thus these are the vector bundles on E defined by the action of y or $-y$ on the even and odd part of N respectively. In other words, $\mathcal{B}_E \cong \iota^*\mathcal{N}^{\mathrm{odd}}$, where $\iota\colon E \to E$ denotes the covering involution of $E \to \mathbb{P}^1$.

Let $\rho\colon E \times \operatorname{Spec} P_X \to \operatorname{Spec} P_X$ denote the second projection. The desired Tate resolution $\mathbb{T}(N)$ associated to the Clifford module N is essentially $R\rho_*\mathbb{T}$. Since $\mathbb{T}$ is a complex, we get a spectral sequence, which we analyze as follows: truncate $\mathbb{T}$ on the left to obtain a left bounded complex

$$L_i \longrightarrow \mathcal{A}_E(i) \boxtimes P_X(2i) \longrightarrow \mathcal{B}_E(i) \boxtimes P_X(2i+1) \longrightarrow \mathcal{A}_E(i+1) \boxtimes P_X(2i+2) \longrightarrow \cdots,$$

and take a Čech resolution on E coming from a covering with two affine open subsets. We obtain a double complex:

$$(*_i)\qquad \begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
C^1(L_i) & & C^1(\mathcal{A}_E(i)) \boxtimes P_X(2i) & \longrightarrow & C^1(\mathcal{B}_E(i)) \boxtimes P_X(2i+1) \longrightarrow \cdots \\
\uparrow & & \uparrow & & \uparrow \\
C^0(L_i) & & C^0(\mathcal{A}_E(i)) \boxtimes P_X(2i) & \longrightarrow & C^0(\mathcal{B}_E(i)) \boxtimes P_X(2i+1) \longrightarrow \cdots \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

The vertical homology of this double complex is a box product with the cohomology of $\mathcal{A}_E$ and $\mathcal{B}_E$ and their twists. The E_2-differentials of the spectral sequence of the double complex can be lifted to maps of the form $H^1(\mathcal{A}_E) \otimes P_X \to H^0(\mathcal{A}_E(1)) \otimes P_X(2)$ on the E_1-page of the sequence. To do this, we choose k-vector space splittings h of the Čech sequence

$$(\alpha)\qquad 0 \longrightarrow H^0(\mathcal{A}_E) \longrightarrow C^0(\mathcal{A}_E) \longrightarrow C^1(\mathcal{A}_E) \longrightarrow H^1(\mathcal{A}_E) \to 0$$

and the corresponding sequences (α_i) and (β_i) for the sheaves $\mathcal{A}_E(i)$'s and $\mathcal{B}_E(i)$'s respectively. We define the map

$$H^1(\mathcal{A}_E) \otimes P_X \longrightarrow H^0(\mathcal{A}_E(1)) \otimes P_X(2)$$

as the composition

$$\begin{array}{llll}
H^1(\mathcal{A}_E) \otimes P_X & & & \\
\big\downarrow h\otimes \mathrm{id} & & & \\
C^1(\mathcal{A}_E) \boxtimes P_X \longrightarrow & C^1(\mathcal{B}_E) \boxtimes P_X(1) & & \\
 & \big\downarrow h\boxtimes \mathrm{id} & & \\
 & C^0(\mathcal{B}_E) \boxtimes P_X(1) \longrightarrow & C^0(\mathcal{A}_E(1)) \boxtimes P_X(2) & \\
 & & \big\downarrow h\otimes \mathrm{id} & \\
 & & H^0(\mathcal{A}_E(1)) \otimes P_X(2). &
\end{array}$$

Abusing notation we write $\tilde{h}$ for all south arrows, $\tilde{\partial}$ for all north arrows, and φ for all east arrows in the corresponding diagram

$$
\begin{array}{ccccc}
H^1(\mathcal{A}_E(i))\otimes P_X(2i) & \xrightarrow{\varphi} & H^1(\mathcal{B}_E(i))\otimes P_X(2i+1) & \xrightarrow{\varphi} & H^1(\mathcal{A}_E(i+1))\otimes P_X(2i+2) \\
\tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} \\
C^1(\mathcal{A}_E(i))\otimes P_X(2i) & \xrightarrow{\varphi} & C^1(\mathcal{B}_E(i))\otimes P_X(2i+1) & \xrightarrow{\varphi} & C^1(\mathcal{A}_E(i+1))\otimes P_X(2i+2) \\
\tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} \\
C^0(\mathcal{A}_E(i))\otimes P_X(2i) & \xrightarrow{\varphi} & C^0(\mathcal{B}_E(i))\otimes P_X(2i+1) & \xrightarrow{\varphi} & C^0(\mathcal{A}_E(i+1))\otimes P_X(2i+2) \\
\tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} & & \tilde{\partial}\uparrow\downarrow\tilde{h} \\
H^0(\mathcal{A}_E(i))\otimes P_X(2i) & \xrightarrow{\varphi} & H^1(\mathcal{B}_E(i))\otimes P_X(2i+1) & \xrightarrow{\varphi} & H^0(\mathcal{A}_E(i+1))\otimes P_X(2i+2)
\end{array} \tag{4}
$$

with four rows.

For $\alpha \in H^1(\mathcal{A}_E)\boxtimes P_X$ we have

$$
\begin{array}{llll}
 & \alpha = \tilde{\partial}\tilde{h}\alpha & & \text{since } \partial h = \mathrm{id}_{H^1} \\
\Rightarrow & \varphi\alpha = \tilde{\partial}\varphi\tilde{h}\alpha & & \text{since } [\varphi,\tilde{\partial}] = 0 \\
\Rightarrow & \tilde{h}\varphi\alpha = -\tilde{\partial}\tilde{h}\varphi\tilde{h}\alpha + \varphi\tilde{h}\alpha & & \text{since } \partial h + h\partial = \mathrm{id}_{C^1} \\
\Rightarrow & \varphi\tilde{h}\varphi\alpha = -\varphi\tilde{\partial}\tilde{h}\varphi\tilde{h}\alpha & & \text{since } \varphi^2 = 0 \\
\Rightarrow & \varphi\tilde{h}\varphi\alpha = -\tilde{\partial}\varphi\tilde{h}\varphi\tilde{h}\alpha & & \text{since } [\varphi,\tilde{\partial}] = 0 \\
\Rightarrow & \tilde{h}\varphi\tilde{h}\varphi\alpha = \tilde{\partial}\tilde{h}\varphi\tilde{h}\varphi\tilde{h}\alpha - \varphi\tilde{h}\varphi\tilde{h}\alpha & & \text{since } \partial h + h\partial = \mathrm{id}_{C^0} \\
\Rightarrow & \varphi\tilde{h}\varphi\tilde{h}\varphi\alpha = \tilde{\partial}\varphi\tilde{h}\varphi\tilde{h}\varphi\tilde{h}\alpha & & \text{since } \varphi^2 = 0 \text{ and } [\varphi,\tilde{\partial}] = 0 \\
\Rightarrow & \tilde{h}\varphi\tilde{h}\varphi\tilde{h}\varphi\alpha = \varphi\tilde{h}\varphi\tilde{h}\varphi\tilde{h}\alpha & & \text{since } h\partial = \mathrm{id}_{H^0} \\
\Rightarrow & (\tilde{h}\varphi\tilde{h}\varphi\tilde{h})\varphi = \varphi(\tilde{h}\varphi\tilde{h}\varphi\tilde{h}). & &
\end{array}
$$

Thus with the lifted maps we obtain a double complex, whose total complex is our desired complex $\mathbb{T}(N)$:

$$
\begin{array}{cccccc}
\to & H^1(\mathcal{A}_E)\otimes P_X & \to & H^1(\mathcal{B}_E)\otimes P_X(1) & \to & H^1(\mathcal{A}_E(1))\otimes P_X(2) \\
\searrow & \oplus & \searrow & \oplus & \searrow & \oplus \\
\to & H^0(\mathcal{B}_E)\otimes P_X(1) & \to & H^0(\mathcal{A}_E(1))\otimes P_X(2) & \to & H^0(\mathcal{B}_E(1))\otimes P_X(3).
\end{array}
$$

The right truncated complexes are exact except at the first two position since the spectral sequence of ($*_i$) converges to the cohomology of L_i. Since we can take i arbitrarily large negative, the complex $\mathbb{T}(N)$ is exact. ☐

Proposition 5.6. *Let M be a P_X-module with a linear resolution as an P-module. Then*

(1) $N = \mathrm{Ext}_{P_X}(M,k)$ is a $C = \mathrm{Ext}_{P_X}(k,k)$-module which is free as an $k[s,t]$-module.

(2) The sheafifications $\mathcal{N}^{\mathrm{ev}}$ and $\mathcal{N}^{\mathrm{odd}} = \widetilde{N^{\mathrm{odd}}(1)}$ satisfies

$$\mathcal{N}^{\mathrm{odd}} \cong \mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}}.$$

(3) $N = H^0_*(\mathcal{N}^{\mathrm{ev}}) \oplus H^0_*(\mathcal{N}^{\mathrm{odd}})(-1)$ and the C-module N is determined by the $\mathcal{C}^{\mathrm{ev}}$-module $\mathcal{N}^{\mathrm{ev}}$.

(4) The P_X-dual complex $\mathbb{T}(N)^*$ is the Tate resolution $\mathbb{T}(M)$ of M.

Proof. (1) Let $0 \to F_c \to \cdots \to F_1 \to F_0 \to M \to 0$ be the linear P-resolution of M. Then by the Eisenbud–Shamash construction [Eis80, Theorem 7.2], $\mathrm{Ext}_{P_X}(M,k) = N = N^{\mathrm{ev}} \oplus N^{\mathrm{odd}}$ is a free $k[s,t]$-module.

(2) We have

$$\mathrm{rk}_{k[s,t]} N^{\mathrm{ev}} = \sum_{i\geq 0} \mathrm{rk}_P F_{2i} \quad \text{and} \quad \mathrm{rk}_{k[s,t]} N^{\mathrm{odd}} = \sum_{i\geq 1} \mathrm{rk}_P F_{2i+1}.$$

Since $\sum_{i=0}^{c}(-1)^i \operatorname{rk}_P F_i = 0$ the $k[s,t]$-modules N^{ev} and N^{odd} have equal rank. Theorem 3.3 shows that the minimal free P_X-resolution of M is isomorphic to $\operatorname{Hom}_k(\operatorname{Ext}_{P_X}(M,k),P_X)$. From this construction we see that if one of the maps

$$\operatorname{Ext}^i_{P_X}(M,k)\times \operatorname{Ext}^1_{P_X}(k,k) \longrightarrow \operatorname{Ext}^{i+1}_{P_X}(M,k)$$

were not surjective, then there would be a generator of the module $\operatorname{Hom}_k(\operatorname{Ext}^{i+1}_{P_X}(M,k),k)$ which maps to zero in the complex. This is not possible because the complex is minimal. We conclude that the map

$$\mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}} \longrightarrow \mathcal{N}^{\mathrm{odd}}$$

is a surjective morphism of $\mathcal{O}_E$-vector bundles of the same rank and hence an isomorphism of $\mathcal{C}^{\mathrm{ev}}$ modules.

(3) It follows that

$$\mathcal{N}^{\mathrm{odd}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}} \cong \mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}} \longrightarrow \mathcal{N}^{\mathrm{ev}} \otimes \mathcal{H}$$

is also an isomorphism.

The formula for N follows because N is a free $k[s,t]$-module. Since $C^{\mathrm{ev}} = H^0_*(\mathcal{C}^{\mathrm{ev}})$ and $C^{\mathrm{odd}} = H^0_*(\mathcal{C}^{\mathrm{odd}})(-1)$ the maps above determine the maps $N^{\mathrm{ev}} \otimes_k C^{\mathrm{odd}} \to N^{\mathrm{odd}}$ and $N^{\mathrm{odd}} \otimes_k C^{\mathrm{odd}} \to N^{\mathrm{ev}}$, and thus the C-module structure on N.

(4) By parts (1) and (2) we can apply Theorem 5.5. The dual of the H^0-strand of $\mathbb{T}(N)$ coincides with $\operatorname{Hom}_k(\operatorname{Ext}_{P_X}(M,k),P_X)$ by construction. Since $\mathbb{T}(N)^*$ and $\mathbb{T}(M)$ are exact minimal complexes which coincide for large homological degree, they are isomorphic. □

Example 5.7. Thus in case $g = 3$ the Betti table

$$\begin{matrix} \cdots & 28 & 20 & 12 & 5 & 1 & & & & \\ & & & & 1 & 5 & 12 & 20 & 28 & 36 & \cdots \end{matrix}$$

of the Tate resolution of $M = \mathbb{T}(H^0_*(\mathcal{F}_U \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}))$ has a second interpretation. It is also the cohomology table

$$\big(h^i(\mathcal{F}_U((j+1-i)p))\big)_{\substack{i=0,1\\ j\in\mathbb{Z}}}$$

of $\mathcal{F}_U$ as a vector bundle on the hyperelliptic curve E.

Theorem 5.8. *Let N be a C-module which is free over $k[s,t]$ satisfying $\mathcal{N}^{\mathrm{odd}} \cong \mathcal{N}^{\mathrm{ev}} \otimes_{\mathcal{C}^{\mathrm{ev}}} \mathcal{C}^{\mathrm{odd}}$. Let $\mathbb{T}(N)$ be the complex constructed in Theorem 5.5 whose terms are described by cohomology groups of $\mathcal{A}_E = \mathcal{N}^{\mathrm{ev}}$ and $\mathcal{B} = \mathcal{N}^{\mathrm{odd}}$ and their twists. The cokernel G_X of the map*

$$H^1(\mathcal{B}_E(-1)) \otimes P_X(-1) \longrightarrow H^1(\mathcal{A}_E) \otimes P_X,$$

which is a component of the differential $F_{-1} \to F_0$ of $\mathbb{T}(N)$, is an Ulrich module if and only if $H^1(\mathcal{B}_E)$ and $H^0(\mathcal{B}_E)$ vanish.

Proof. If G_X is an Ulrich P_X-module, then it is its own MCM approximation. Hence the Tate resolution of G_X has non-overlapping strands so $H^1(\mathcal{B}_E)$ and $H^0(\mathcal{B}_E)$ vanish.

Conversely, if these groups vanish then G_X is a MCM module over P_X with a linear P_X-resolution, and from the form of the complex $\mathbb{T}(N)$ we see that $H^0(\mathcal{A}_E)$ and all terms to the left of it in the lower row must also vanish. To show that G_X is an Ulrich module we must prove that G_X has a linear resolution as a P-module.

We first make the form of the P_X-resolution more explicit. The cohomological vanishing $h^0(\mathcal{B}_E) = h^1(\mathcal{B}_E) = 0$ implies that $\pi_*\mathcal{B}_E = \mathcal{O}_{\mathbb{P}^1}(-1)^{2r}$, where $r = \operatorname{rk}\mathcal{B}_E = \operatorname{rk}\mathcal{A}_E$. Since $\mathcal{B}(-p) \cong \mathcal{A}$ we have $\deg\mathcal{A}_E = \deg\mathcal{B}_E - r$. Thus $H^0(\mathcal{A}_E) = 0$ and, by the Riemann-Roch formula, $h^1(\mathcal{A}_E) = r$. The form of the Tate resolution implies that the bundle $\pi_*\mathcal{A}_E$ splits into a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(-2)$. Indeed, there cannot be any summands of the form $\mathcal{O}_{\mathbb{P}^1}(-d)$ with $d \le -3$ because there are no nonzero maps to this sheaf from $\pi_*\mathcal{B}_E(-1) = \mathcal{O}_{\mathbb{P}^1}(-2)^{2r}$. Hence

$$\pi_*\mathcal{A}_E = \mathcal{O}_{\mathbb{P}^1}(-1)^r \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^r.$$

Since $\pi_*\mathcal{B}_E(-1) = \mathcal{O}_{\mathbb{P}^1}(-2)^{2\operatorname{rk}\mathcal{B}_E}$ we see that G_X is defined by an $r \times 2r$ matrix of linear forms and the P_X-free resolution of G_X has the form

$$\cdots \longrightarrow P_X^{(i+1)r}(-i) \longrightarrow \cdots \longrightarrow P_X^{2r}(-1) \xrightarrow{\phi_1} P_X^r \longrightarrow G_X \longrightarrow 0.$$

We can now show that G_X has linear resolution as a P-module. Since G_X is maximal Cohen–Macaulay module over P_X, this statement can be checked after factoring out a maximal P_X-regular sequence z of linear forms in P. Note that P_X/zP_X has Hilbert function $1,2,1$. The sequence z is also a regular sequence on G_X because G_X is a maximal Cohen–Macaulay module. From the resolution of G_X over P_X we see that the values of the Hilbert function of G_X/zG_X are $r,0,0,\ldots$; that is, $G_X/zG_X \cong k^r$. As a module over P/zP this has a linear resolution, and thus G_X has a linear resolution as a P-module. Thus G_X is an Ulrich P_X-module. □

Remark 5.9. The proof shows in particular that, the matrix

$$P^{2r}(-1) \xrightarrow{\phi_1} P^r$$

obtained by regarding the linear P_X-presentation of G_X as a matrix over P is a presentation matrix of G_X as a P-module.

Using the Morita equivalence between the hyperelliptic curve E and the Clifford algebra C we can make this more precise. Recall that a bundle $\mathcal{B}$ on E has the Raynaud property if $H^0(C,\mathcal{B}) = H^1(C,\mathcal{B}) = 0$. We are now ready to prove parts of Theorem 1.1 from the introduction, which we repeat for the reader's convenience.

Theorem 5.10. *There is a* $1-1$ *correspondence between Ulrich bundles on the smooth complete intersection of two quadrics* $X \subset \mathbb{P}^{2g+1}$ *and bundles with the Raynaud property on the corresponding hyperelliptic curve* E *of the form* $\mathcal{G} \otimes \mathcal{F}_U$. *The Ulrich bundle corresponding to a rank* r *vector bundle* $\mathcal{G}$ *has rank* $r2^{g-2}$.

If $\mathcal{L}$ *is a line bundle on* E *then* $\mathcal{L} \otimes \mathcal{F}_U$ *does not have the Raynaud property, so the minimal possible rank of an Ulrich sheaf on* X *is* 2^{g-1}, *and Ulrich bundles of rank* 2^{g-1} *exist.*

Proof. Let $p \in E$ be a ramification point. Consider $\mathcal{B} = \mathcal{G} \otimes \mathcal{F}_U$, $\mathcal{A} = \mathcal{G}(-p) \otimes \mathcal{F}_U$ and the Clifford module $N = \oplus_i H^0(\mathcal{A}(ip))$. By Theorem 5.8 $\mathbb{T}(N)$ is the Tate resolution of the Ulrich module $G_X = \operatorname{coker}\left(H^1(\mathcal{B}_E(-1)) \otimes P_X(-1) \to H^1(\mathcal{A}_E) \otimes P_X\right)$ if and only if $H^0(\mathcal{B}) = H^1(\mathcal{B}) = 0$. If $r = \operatorname{rk}\mathcal{G}$ and the condition is satisfied then the corresponding Ulrich module G_X on X has rank $G_X = r2^{g-2}$ since the number of generators of G_X is $\operatorname{rk}(\mathcal{G} \otimes \mathcal{F}_U) = r2^g$.

Conversely, suppose that M is an Ulrich module on P_X, and let $N = \operatorname{Ext}_{P_X}(M,k)$. This is a C-module, and thus an R_E-module which is a free $k[s,t]$-module by the Eisenbud–Shamash construction [Eis80, Theorem 7.2]. The odd part of its sheafification is thus of the form $\mathcal{N}^{\text{odd}} = \mathcal{G} \otimes_{\mathcal{O}_E} \mathcal{F}_U$ for some a vector bundle $\mathcal{G}$ by Corollary 4.7, the Morita theorem. By Theorem 5.8 $\mathcal{G} \otimes_{\mathcal{O}_E} \mathcal{F}_U$ has the Raynaud property.

An Ulrich module of rank 2^{g-2} would correspond to a line bundle $\mathcal{L}$ on E such that $\mathcal{L} \otimes \mathcal{F}_U$ has vanishing cohomology. By Corollary 4.8, $\mathcal{L} \otimes \mathcal{F}_U = \mathcal{F}_{U'}(mp)$ for some maximal isotropic plane U' and some integer m. Thus $\mathbb{T}(N)^*$ would be the Tate resolution of $P_{U'}$ up to shift. But $\mathbb{T}(P_{U'})$ has overlapping strands (in fact $P_{U'}$ is not a MCM P_X-module).

The existence of Ulrich bundles of rank 2^{g-1} is proven in Section 6. □

Proposition 5.11. *Ulrich bundles of rank* $r2^{g-2}$ *on a smooth complete intersections of two quadrics in* $\mathbb{P}^{2g+1}$ *do not exist if* $r \cdot g \equiv 1 \mod 2$.

Proof. If $\mathcal{G}$ is a vector bundle on E of rank r and degree d then

$$\deg(\mathcal{G} \otimes \mathcal{F}_U) = \deg\mathcal{G}\operatorname{rk}\mathcal{F}_U + \operatorname{rk}\mathcal{G}\deg\mathcal{F}_U = d2^g + rg2^{g-1}$$

by Proposition 4.5 and

$$\chi(\mathcal{G}\otimes\mathcal{F}_U)=\deg(\mathcal{G}\otimes\mathcal{F}_U)+\mathrm{rk}(\mathcal{G}\otimes\mathcal{F}_U)(1-g)=d2^g+rg2^{g-1}+r2^g(1-g)$$

by Riemann-Roch. Thus $\chi(\mathcal{G}\otimes\mathcal{F}_U)=0$ implies $r\cdot g\equiv 0 \mod 2$. □

For small g we constructed Ulrich bundles of rank 2^{g-1} from sufficiently general rank 2 bundles $\mathcal{G}$ on E with our Macaulay2 package [EKS22]. Consider the direct sum $\mathcal{G}_0=\mathcal{L}_0\oplus\mathcal{L}_g$ of two general line bundle $\mathcal{L}_i$ of degree i. In case of $g=3$ the cohomology table of the bundle $\mathcal{G}_0\otimes\mathcal{F}_U$ is the sum of two tables, one of which we displayed in Example 5.7 in case of $g=3$. The other is a shifted version of that table.

So in case of $g=3$ the cohomology table of $\mathcal{G}_0\otimes\mathcal{F}_U$ has shape

$$\begin{matrix}\cdots & 64 & 48 & 33 & 21 & 12 & 5 & 1 & & \\ & & 1 & 5 & 12 & 21 & 33 & 48 & 64 & \cdots.\end{matrix}$$

If for a general extension $0\to\mathcal{L}_0\to\mathcal{G}\to\mathcal{L}_3\to 0$ the connecting homomorphisms are of maximal rank, then the cohomology table of $\mathcal{G}\otimes\mathcal{F}_U$ has the form

$$\begin{matrix}\cdots & 64 & 48 & 32 & 16 & & & & & \\ & & & & & 16 & 32 & 48 & 64 & \cdots.\end{matrix}$$

and $\mathcal{G}$ gives rise to an Ulrich bundle of rank $2\cdot 2^{g-2}$. In special cases, for small g we verified that this does occur with Macaulay2 [GS] using our package [EKS22]. With the same idea we constructed Ulrich bundles of rank $3\cdot 2^{g-2}$ in special cases for $g=2$.

However we were not able to control the cohomology of $\mathcal{G}\otimes\mathcal{F}_U$ theoretically well enough to prove the existence of rank 2^{g-1} Ulrich bundle for every X.

6. Ulrich bundles of rank 2^{g-1}

In this section we prove that a smooth complete intersection of two quadrics in $\mathbb{P}^{2g+2}$, and therefore also in $\mathbb{P}^{2g+1}$, carries an Ulrich bundle of rank 2^{g-1}. Our construction uses the construction of Ulrich bundles on a single quadric by Knörrer, which we now review.

Theorem 6.1 (*cf.* [Knö87]). *The quadric $q_n=\sum_{i=0}^n x_iy_i$ has the matrix factorization (φ_n,ψ_n) of size 2^n defined recursively by $\varphi_0=(x_0),\psi_0=(y_0)$ and*

$$\varphi_n=\begin{pmatrix}x_n & \varphi_{n-1}\\ \psi_{n-1} & -y_n\end{pmatrix},\quad \psi_n=\begin{pmatrix}y_n & \varphi_{n-1}\\ \psi_{n-1} & -x_n\end{pmatrix}$$

for $n\geq 1$.

Let $(A,B)=(\varphi_n,\psi_n)$ and consider the matrix factorizations

$$(A(x,y),B(x,y)) \quad\text{and}\quad (A(v,w),B(v,w))$$

of $q(x,y)=\sum_{i=0}^n x_iy_i$ and $q(v,w)=\sum_{i=0}^n v_iw_i$ respectively over the ring $P:=k[x|y,v|w]$, where $x|y$ denotes the catenation $x_0,\dots,x_n,y_0,\dots,y_n$ and similarly for $v|w$.

Proposition 6.2. *Let*

$$\widetilde{q}(v,w,x,y)=\sum_{i=0}^n(x_iw_i+y_iv_i)=(v|w)\cdot(y|x).$$

There is an identity

$$\begin{pmatrix}A(x,y) & A(v,w)\end{pmatrix}\begin{pmatrix}B(v,w)\\ B(x,y)\end{pmatrix}=\widetilde{q}(v,w,x,y)\mathrm{id}_{2^n}.$$

Proof. Since $A(x,y)+A(v,w)=A(x+v,y+w)$ and $B(x,y)+B(v,w)=B(x+v,y+w)$ we have

$$A(x+v,y+w)B(x+v,y+v)=q(x+v,y+w)\mathrm{id}_{2^n}.$$

The mixed terms give

$$A(x,y)B(v,w)+A(v,w)B(x,y)=\widetilde{q}(v,w,x,y)\mathrm{id}_{2^n}.$$ □

Thus if we restrict the matrices in Proposition 6.2 to an isotropic subspace Σ of $\widetilde{q}$ we get a complex and we will see that, for a sufficiently general choice of the isotropic subspace, the restriction to Σ is a minimal free resolution of an Ulrich module over P_Σ.

To define the isotropic subspace, let Λ be a skew-symmetric $2(n+1)\times 2(n+1)$ matrix of scalars, and set

$$G_\Lambda=\begin{pmatrix} 0 & \mathrm{id}_{n+1}\\ \mathrm{id}_{n+1} & 0\end{pmatrix}\Lambda.$$

We have

$$(x|y)G_\Lambda\cdot(y|x)=(y|x)\Lambda\cdot(y|x)=0$$

and thus the equation $(v|w)=(x|y)G_\Lambda$ defines an isotropic subspace of $\widetilde{q}(v,w,x,y)$.

The matrices

$$A_1=A(x,y),\ B_1=B(x,y) \quad \text{and} \quad A_2=A((x|y)G_\Lambda),\ B_2=B((x|y)G_\Lambda)$$

define matrix factorizations of $q_1=q(x,y)$ and $q_2=q((x|y)G_\Lambda))$. Let

$$A_\Lambda=A_1|A_2$$

be the concatenation, which is a $2^n\times 2^{n+1}$ matrix in the $2n+2$ variables $x_0,\dots,y_n$.

Theorem 6.3. *For a general choice of Λ the ring $k[x_0,\dots,y_n]/(q_1,q_2)$ is a complete intersection with isolated singularities and*

$$M_\Lambda:=\operatorname{coker}A_\Lambda$$

is an Ulrich module of rank 2^{n-2} over this ring.

Proof. Set $P=k[x_0,\dots,y_n]$. For each Λ we have maps

$$0\longleftarrow M_\Lambda\longleftarrow P^{2^n}\xleftarrow{\begin{pmatrix}A_1 & A_2\end{pmatrix}}P^{2^{n+1}}(-1)\xleftarrow{\begin{pmatrix}B_2\\ B_1\end{pmatrix}}P^{2^n}(-2)\longleftarrow 0.$$

By our choice of A_2 and B_2 this is a complex.

We claim that for a general choice of Λ the ideal (q_1,q_2) is a prime ideal of codimension 2 with isolated singularities. It suffices to prove this for a particular choice of Λ.

We will actually prove the result for matrices Λ of the form

$$\Lambda=\begin{pmatrix}0 & D\\ -D & 0\end{pmatrix}$$

where D is a diagonal matrix with entries d_i such that

$$d_0,\dots,d_n,-d_0,\dots,-d_n$$

are $2(n+1)$ different values. In this case

$$G_\Lambda=\begin{pmatrix}-D & 0\\ 0 & D\end{pmatrix},\quad A_\Lambda=(A(x_0,\dots,x_n,y_0,\dots,y_n)|A(-d_0x_0,\dots-d_nx_n,d_0y_0,\dots,d_ny_n),$$

and

$$q_2=q_1(-d_0x_0,\dots,-d_nx_n,d_0y_0,\dots,d_ny_n)=-\sum_{i=0}^n d_i^2x_iy_i.$$

We will now show that $V(q_1,q_2)$ is singular precisely at the coordinate points. The jacobian matrix of $(q_1,-q_2)$ is

$$\begin{pmatrix} y_0 & y_1 & \dots & y_n & x_0 & \dots & x_n \\ d_0^2 y_0 & d_1^2 y_1 & \dots & d_n^2 y_n & d_0^2 x_0 & \dots & d_n^2 x_n \end{pmatrix}$$

The squares $d_0^2,\dots,d_n^2$ are pairwise distinct, since $d_0,\dots,d_n,-d_0,\dots,-d_n$ are $2(n+1)$ distinct values by assumption. Thus the zero locus of the ideal of 2×2 minors of the jacobian matrix is the union of the $n+1$ lines $L_i = V(\bigcup_{j\neq i}\{x_j,y_j\})$ defined by those linear combinations of the two rows that do not consist of independent linear forms. These lines intersect $V(q_1,q_2)$ in the $2n+2$ coordinate points. It follows that (q_1,q_2) has codimension 2 and isolated singularities, and thus is prime.

Since each q_i is prime and A_i is part of a matrix factorization of q_i, the determinant of A_i is a power of q_i. Thus if Λ is general, the maximal minors of A_Λ generate an ideal of codimension at least 2, and similarly for B_Λ so the complex is exact by [BE73].

We conclude that

$$\operatorname{ann} M_\Lambda = (q_1,q_2)$$

since any element of $\operatorname{ann} M_\Lambda \setminus (q_1,q_2)$ would lead to a support of codimension at least 3. Thus M_Λ is an Ulrich module over the ring $P/(q_1,q_2)$ and the degree of M_Λ is 2^n, so the rank of M_Λ as an $P/(q_1,q_2)$-module is 2^{n-2}. ☐

Theorem 6.4. *Let k be an algebraically closed field of* $\operatorname{char} k \neq 2$, *and $X \subset \mathbb{P}^{2n}$ be a smooth complete intersection of two quadrics. Then X carries an Ulrich bundle of rank 2^{n-2}.*

Corollary 6.5. *Let k be an algebraically closed field of* $\operatorname{char} k \neq 2$, *and $X \subset \mathbb{P}^{2g+1}$ be a smooth complete intersection of two quadrics. Then X carries an Ulrich bundle of rank 2^{g-1}.*

Proof of Corollary 6.5. Any smooth complete intersection in $\mathbb{P}^{2g+1}$ is a hyperplane section of a smooth complete intersection in $\mathbb{P}^{2g+2}$. Taking $n = g+1$, the restriction of the Ulrich module constructed in Theorem 6.4 is an Ulrich module of rank 2^{g-1}. ☐

Proof of Theorem 6.4. We obtain an Ulrich module on some smooth complete intersection by restricting M_Λ from above to a general hyperplane $H = \mathbb{P}^{2n} \subset \mathbb{P}^{2n+1}$. The intersection will be smooth because $V(q_1,q_2)$ has only isolated singularities. To prove that every smooth complete intersection carries an Ulrich module we need additional arguments. The complete intersection $V(q_1',q_2')$ of two quadrics in $\mathbb{P}^{2n}$ is smooth if and only if the discriminant

$$f = \det \operatorname{hess}(sq_1' + q_2') \in k[s]$$

of the pencil has $2n+1$ distinct roots, and in that case q_1' and q_2' can be simultaneously diagonalized by the argument given at the beginning of Section 4. Thus it suffices to construct an Ulrich module M' on a the complete intersection $V(q_1',q_2')$ whose discriminant has any given set of $2n+1$ distinct roots. In the proof of Theorem 6.3 we constructed an Ulrich module for $q_1 = \sum_{i=0}^n x_i y_i$ and $q_2 = -\sum_{i=0}^n d_i^2 x_i y_i$ for distinct values $d_0^2,\dots,d_n^2$. Since k is algebraically closed there exists an Ulrich module for $V(\sum_{i=0}^n x_i y_i, \sum_{i=0}^n a_i x_i y_i)$ for every tuple of distinct values $a_0,\dots,a_n$. The corresponding Hessian is

$$H = \begin{pmatrix} 0 & D' \\ D' & 0 \end{pmatrix} \text{ with a diagonal matrix } D' = \begin{pmatrix} s+a_0 & & \\ & \ddots & \\ & & s+a_n \end{pmatrix}.$$

We restrict the quadrics to the subspace generated by the columns of the $(2n+2)\times(2n+1)$ matrix of

$$B = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ b_0 & \dots & b_{2n} \end{pmatrix}.$$

Setting $\ell_i = s + a_i$ the Hessian of the restricted pencil is

$$B^t HB = \begin{pmatrix} & & & \ell_n b_0 & \ell_0 & & \\ & 0 & & \vdots & & \ddots & \\ & & & \ell_n b_{n-1} & & & \ell_{n-1} \\ \ell_n b_0 & \dots & \ell_n b_{n-1} & 2\ell_n b_n & \ell_n b_{n+1} & \dots & \ell_n b_{2n} \\ \ell_0 & & & \ell_n b_{n+1} & & & \\ & \ddots & & \vdots & & 0 & \\ & & \ell_{n-1} & \ell_n b_{2n} & & & \end{pmatrix}.$$

Direct computation shows that the determinant of this matrix is

$$f = (-1)^n 2h \prod_{i=0}^{n} \ell_i = (-1)^n 2h \prod_{i=0}^{n} (s + a_i)$$

with

$$h = \sum_{i=0}^{n-1} (b_i b_{i+n+1} \prod_{j \neq i} (s + a_j)) - b_n \prod_{j \neq n} (s + a_j).$$

Since the coefficients of $\prod_{j\neq i}(s+a_j)$ are the elementary symmetric functions $e_{i,k}$ on $\{a_0,\dots,a_n\} \setminus \{a_i\}$, we obtain

$$h = (b_0 b_{n+1}, \dots, b_{n-1} b_{2n}, -b_n) E \begin{pmatrix} s^n \\ \vdots \\ s \\ 1 \end{pmatrix} \tag{6.1}$$

where $E = (e_{i,k})_{\substack{i=0,\dots,n \\ k=0,\dots,n}}$. We claim that

$$\det E = \prod_{0 \le i < j \le n} (a_i - a_j).$$

Regarding the a_i's as variables, we see that $\det E \in k[a_0,\dots,a_n]$ is not identically zero, because the term $\prod_{i=0}^{n-1} a_i^{n-i}$ occurs precisely once in the determinant as the product of the leading terms $1, a_0, a_0 a_1, \dots, a_0 a_1 \dots a_{n-1}$ of the diagonal entries. On the other hand $(a_i - a_j)$ is a factor of $\det E \in k[a_0,\dots,a_n]$ because if $a_i = a_j$ then the matrix E has two equal rows. So these linear forms are factors of $\det E \in k[a_0,\dots,a_n]$, and their product coincides with $\det E$ for degree reasons and by comparing the leading term.

Thus if the a_i are distinct, then E is invertible, and every polynomial h of degree n in $k[s]$ can be represented in the form (6.1). In particular, we can choose $b_0,\dots,b_{2n} \in k$ such that the discriminant f is equal to $\prod_{i=0}^{n}(s+a_i)\prod_{i=1}^{n}(s+c_i)$ for any $2n+1$ distinct non-zero values $a_0,\dots,a_n,c_1,\dots,c_n \in k$. A smooth complete intersection of 2 quadrics in $\mathbb{P}^{2n}$ is determined up to projective equivalence by the $2n+1$ distinct roots of its discriminant, this concludes the proof. □

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