*Épijournal de Géométrie Algébrique*
epiga.episciences.org
Volume 9 (2025), Article No. 10



# On $G$-birational rigidity of del Pezzo surfaces

**Egor Yasinsky**

**Abstract**. Let $G$ be a finite group and $H \subseteq G$ be a subgroup. We prove that if a smooth del Pezzo surface over an algebraically closed field is $H$-birationally rigid, then it is also $G$-birationally rigid, answering a geometric version of Kollár's question in dimension 2 positively. On our way, we also investigate $G$-birational rigidity of two-dimensional quadrics and del Pezzo surfaces of degree 6.

**Keywords**. Del Pezzo surface, conic bundle, birational rigidity, Sarkisov program

**2020 Mathematics Subject Classification**. 14E07, 14E05, 14E30, 14J45, 14M22

Received by the Editors on July 25, 2023, and in revised form on August 22, 2024.
Accepted on September 17, 2024.

Egor Yasinsky
IMB, Université de Bordeaux, 351 Cours de la Libération, 33405 Talence Cedex, France
*e-mail:* egor.yasinsky@u-bordeaux.fr

## Contents

1. Birational rigidity . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
2. Sarkisov links on del Pezzo surfaces . . . . . . . . . . . . . . . . . . . . . . . 4
3. Del Pezzo surfaces of degree less than 6 and the projective plane . . . . . . . . . . . 5
4. Del Pezzo surfaces of degree 6 . . . . . . . . . . . . . . . . . . . . . . . . . 7
5. $G$-birational rigidity of quadric surfaces . . . . . . . . . . . . . . . . . . . . 12
6. The projective space of dimension 3 and the “mixed” arithmetico-geometric case . . . . 21
References . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

# 1. Birational rigidity

The present note is motivated by J. Kollár’s paper [Kol09] which studies the behaviour of birational rigidity of Fano varieties under extensions of algebraically closed fields. Let $\mathbf{k}$ be a field. Recall that a *Mori fibre space* is a projective morphism $\pi\colon X \to Y$ of algebraic varieties over $\mathbf{k}$ such that $X$ is $\mathbb{Q}$-factorial with terminal singularities, $\operatorname{rk}\operatorname{Pic}(X/Y) = 1$ and $-K_X$ is $\pi$-ample. When $Y = \operatorname{Spec}(\mathbf{k})$, the variety $X$ is simply a $\mathbb{Q}$-factorial terminal Fano variety of Picard rank 1. Roughly speaking, it is called *birationally rigid* if $X$ is not birational to the total space of any other Mori fibre space; see Definition 1.3 below for the details. In [Kol09] the following question was raised.

*Question* (Kollár). Let $X$ be a Fano variety over a field $\mathbf{k}$ such that $X$ is birationally rigid over the algebraic closure $\overline{\mathbf{k}}$. Is $X$ birationally rigid over $\mathbf{k}$?

From the modern point of view, one can naturally formulate the minimal model program in the *equivariant* setting; see [Pro21] for an overview. Let $X$ be an algebraic variety over a field $\mathbf{k}$ and $G$ be a group. Following Yu. Manin [Man66], one calls $X$ a *geometric $G$-variety* if $\mathbf{k} = \overline{\mathbf{k}}$ and there is an injective homomorphism $G \hookrightarrow \operatorname{Aut}(X)$. Another instance of this concept is an *arithmetic $G$-variety*, for which $\mathbf{k} \neq \overline{\mathbf{k}}$ and $G$ is the Galois group of the extension $\overline{\mathbf{k}}/\mathbf{k}$ acting on $X \times_{\operatorname{Spec}(\mathbf{k})} \operatorname{Spec}(\overline{\mathbf{k}})$ through the second factor; for simplicity, $\mathbf{k}$ will be always assumed perfect in this paper. In both cases, we refer to $X$ as a *$G$-variety* if no confusion arises.

*Remark* 1.1. Furthermore, one can consider the “mixed” case, when $\mathbf{k}$ is not assumed to be algebraically closed and $G$ acts by biregular automorphisms of $X$; *i.e.* one considers the action of $\operatorname{Gal}(\overline{\mathbf{k}}/\mathbf{k}) \times G$. For some results in this setting, see [DI09b, Tre16, Tre18, Tre19, RZ18, Yas22, CMYZ22, Avi20] and Section 6.2 below.

Now, let us give some precise definitions. We follow [CS16, Chapters 1–3].

**Definition 1.2**. A *$G$-Mori fibre space* is a $G$-equivariant surjective morphism $\pi\colon X \to Y$ of $G$-varieties such that $\pi$ has connected fibres, $X$ has terminal singularities, all $G$-invariant Weil divisors on $X$ are $\mathbb{Q}$-Cartier divisors, $\dim Y < \dim X$, $\operatorname{rk}\operatorname{Pic}(X)^G - \operatorname{rk}\operatorname{Pic}(Y)^G = 1$ and $-K_X$ is $\pi$-ample. If $Y$ is a point, we say that $X$ is a *$G$-Fano variety.*

**Definition 1.3** (see *e.g.* [CS16, Definition 3.1.1]). A $G$-Fano variety is called *$G$-birationally rigid* if the following two conditions are satisfied:

(1) There is no $G$-birational map $X \dashrightarrow X'$ such that $X'$ is a $G$-Mori fibre space over a positive-dimensional variety.
(2) If there is a $G$-birational map $\varphi\colon X \dashrightarrow X'$ such that $X'$ is a $G$-Fano variety, then $X \simeq X'$ and there is a $G$-birational self-map $\tau \in \mathrm{Bir}(X)$ such that $\varphi \circ \tau$ is a biregular $G$-morphism.

Assume, moreover, that the following holds:

(3) Every $G$-birational self-map $X \dashrightarrow X$ is actually $G$-biregular.

Then $X$ is called *$G$-birationally superrigid.*

One can then generalize the initial question as follows.

*Question* (Cheltsov–Kollár). Let $G$ be a group and $H \subseteq G$ be a subgroup. Assume that $X$ is an $H$-birationally rigid $H$-Fano variety. Is $X$ then $G$-birationally rigid?

Informally speaking, the non-triviality of this question lies *e.g.* in the fact that *a priori* $X$ may admit only $H$-birational maps to Fano varieties $X'$ with $\mathrm{rk}\,\mathrm{Pic}(X')^H > 1$, while enlarging the group to $G$ forces $\mathrm{Pic}(X')^G$ to be of rank 1. Kollár's original question addresses arithmetic $G$-varieties. The goal of this note is to answer its *geometric* counterpart in dimension 2 in the positive.

**Theorem**. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic zero and $G$ be a finite group. Let $S$ be a two-dimensional geometric $G$-Fano variety over* $\mathbf{k}$, *i.e. a smooth del Pezzo surface on which $G$ acts faithfully by automorphisms, so that* $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. *Assume that $H \subseteq G$ is a subgroup and $S$ is $H$-birationally rigid. Then $S$ is $G$-birationally rigid.*

At the moment, there are no known counter-examples to Cheltsov–Kollár's question, neither in the geometric nor in the arithmetic setting, although it is highly probable such a counter-example exists. On the other hand, the results [CS19] of I. Cheltsov and C. Shramov imply that the answer to the question is positive when $X = \mathbb{P}^3_{\mathbb{C}}$; see Section 6.1 below. By contrast, a natural generalization of Cheltsov–Kollár's question to the "mixed" setting of Remark 1.1 has a *negative* answer. This will be shown in Section 6.2.

Finally, we note that the analogue of Cheltsov–Kollár's question with rigidity replaced by *superrigidity* is easy to answer positively; *i.e.* one has the following.

**Proposition 1.4**. *Let $G$ be a finite group and $H \subseteq G$ be a subgroup. Suppose that $X$ is an $H$-Fano variety. If $X$ is $H$-birationally superrigid, then $X$ is $G$-birationally superrigid.*

*Proof.* Recall, see [CS16, Section 3.3], that $X$ is $G$-birationally superrigid if and only if for every $G$-invariant mobile linear system $\mathscr{M}_X$ on $X$, the pair $(X, \lambda\mathscr{M}_X)$ is canonical for $\lambda \in \mathbb{Q}_{>0}$ such that

$$\lambda\mathscr{M}_X \sim_{\mathbb{Q}} -K_X.$$

Since a $G$-invariant linear system $\mathscr{M}_X$ is also $H$-invariant, the result follows. □

*Remark* 1.5. The analogue of Proposition 1.4 holds when the notion of superrigidity is replaced with that of *solidity*. Recall, see [AO18], that a $G$-Fano variety $X$ is called *$G$-solid* if $X$ is not $G$-birational to a $G$-Mori fibre space with positive-dimensional base; *i.e.* only condition (1) of Definition 1.3 is required; see [Cor00, CS16, CS19, CS22]. Recently, $G$-solid rational surfaces over the field of complex numbers were classified in [Pin24].

**Notation**. We denote by $\mathrm{S}_n$ the symmetric group on $n$ letters and by $\mathrm{A}_n$ its alternating subgroup. Further, $\mathrm{D}_n$ denotes the dihedral group of order $2n$ with presentation

$$\mathrm{D}_n = \left\langle r, s \mid r^n = s^2 = (sr)^2 = \mathrm{id} \right\rangle,$$

and $\mathrm{C}_n$ denotes the cyclic group of order $n$. We denote by $\mathrm{V}_4$ the Klein four-group, isomorphic to $\mathrm{C}_2 \times \mathrm{C}_2$. Finally, $A_{\bullet}B$ denotes an extension of $B$ by $A$, not necessarily split.

### Acknowledgements

I would like to thank Ivan Cheltsov for bringing this problem to my attention. I am very grateful to Andrey Trepalin for useful remarks and for pointing out some gaps in the first version of this text. Finally, I thank the anonymous referees for their helpful comments.

## 2. Sarkisov links on del Pezzo surfaces

### 2.1. Geometric $G$-surfaces

In what follows, all surfaces are assumed smooth and projective and defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. Given a finite group $G$, a (geometric) *$G$-surface* is a triple $(S, G, \iota)$, where $S$ is a surface over $\mathbf{k}$ and $\iota\colon G \hookrightarrow \mathrm{Aut}(S)$ is a faithful $G$-action. A *$G$-morphism* of $G$-surfaces $(S_1, G, \iota_1) \to (S_2, G, \iota_2)$ is a morphism $f\colon S_1 \to S_2$ such that $f \circ \iota_1(G) = \iota_2(G) \circ f$. Similarly, one defines $G$-rational maps and $G$-birational maps. In what follows, we consider only rational $G$-surfaces equipped with the structure of a $G$-Mori fibre space, *i.e.* $G$-del Pezzo surfaces and $G$-conic bundles, whose $G$-invariant Picard ranks are 1 and 2, respectively.

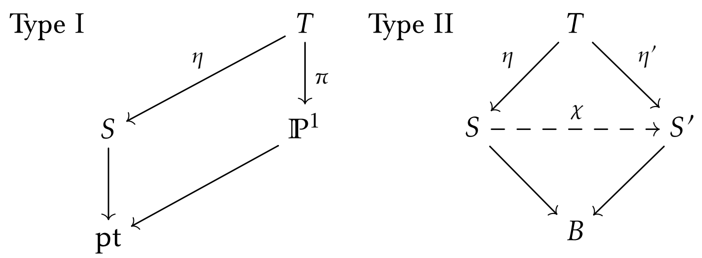
### 2.2. Sarkisov program

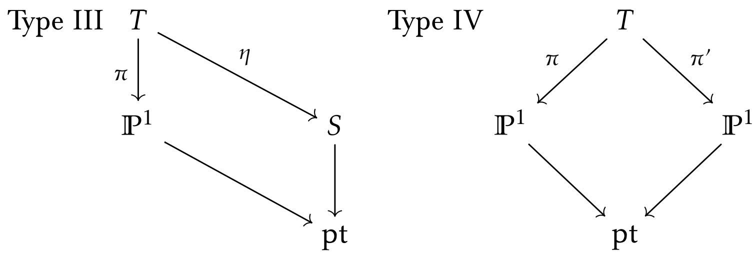
The proof of the main theorem will be based on the explicit geometry of del Pezzo surfaces (for which we refer to [Dol12, Chapter 8]) and, most importantly, the *Sarkisov program* in dimension 2. Any $G$-birational map between two $G$-surfaces can be decomposed into a sequence of birational $G$-morphisms and their inverses. A birational $G$-morphism $S \to T$ is a blow-up of a union of $G$-orbits on $T$. In this article, we often refer to a $G$-orbit of size $d \geqslant 1$ as a *$G$-point of degree $d$*. In particular, a $G$-point of degree 1 is simply a fixed point of $G$.

A $G$-birational map $\varphi$ between $G$-Mori fibre spaces $\pi\colon S \to B$ and $\pi'\colon S' \to B'$ is a diagram

$$\begin{array}{ccc} S & \overset{\varphi}{\dashrightarrow} & S' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & & B' \end{array}$$

which in general does not commute with the fibrations. Recall that in dimension 2, a rational $G$-Mori fibre space $\pi\colon S \to B$ is either a $G$-del Pezzo surface (if $B$ is a point) or a $G$-conic bundle (if $B = \mathbb{P}^1$). According to the equivariant version of the Sarkisov program, every $G$-birational map $\varphi\colon S \dashrightarrow S'$ of $G$-Mori fibre spaces is factorized into a composition of isomorphisms of $G$-Mori fibre spaces and *elementary Sarkisov $G$-links* of four types, depicted below.






For type I, $S$ is a $G$-del Pezzo surface and the Sarkisov link $\eta$ is the blow-up of a $G$-point on $S$, giving a $G$-conic bundle $\pi\colon T\to\mathbb{P}^1$; a link of type III is simply the inverse of I. For type II, the birational morphisms $\eta$ and $\eta'$ are the blow-ups of $G$-points on $S$ and $S'$, respectively. The induced Sarkisov link $\chi$ is either an elementary transformation of $G$-conic bundles (when $B\simeq\mathbb{P}^1$) or a $G$-birational map between $G$-del Pezzo surfaces (when $B=\mathrm{pt}$). Finally, a link of type IV is the choice of a conic bundle structure on a $G$-conic bundle $T$ which has exactly two such structures; note that in general such a link is not represented by a biregular automorphism of $T$, which exchanges $\pi$ and $\pi'$.

*Remark* 2.1. Recently, the Sarkisov program has been reformulated (and then successfully used to prove many structural results about Cremona groups) in terms of so-called *rank r fibrations*; see [LZ20, BLZ21]. For example, in the arithmetic case, one defines a rank $r$ fibration as a surface $S$ with a surjective morphism $\pi\colon S\to B$ with connected fibres, where $B$ is a point or a smooth curve, with relative Picard number equal to $r$ and $\pi$-ample anticanonical divisor $-K_S$. Of course, in rank 1 fibrations, we recognize the usual (arithmetic) $G$-del Pezzo surfaces and $G$-conic bundles. The key observation, based on the so-called *2-ray game*, is that rank 2 fibrations are in a one-to-one correspondence with Sarkisov links (and rank 3 fibrations correspond to the elementary relations between the links). As usual, there is a geometric counterpart of this theory; see *e.g.* [SZ21, Flo20, FZ24].

Sarkisov links between surfaces were classified by V. A. Iskovskikh in [Isk96, Theorem 2.6] in the arithmetic case (see also a recent exposition [LS21] by S. Lamy and J. Schneider) and restated in [DI09a, Section 7] in the geometric case. The following claim will be used systematically throughout the paper and is an immediate consequence of the Sarkisov program.

**Proposition 2.2**. *A del Pezzo surface $S$ is $G$-birationally rigid if and only if for every Sarkisov $G$-link $S\dashrightarrow S'$, the surfaces $S$ and $S'$ are $G$-isomorphic.*

## 3. Del Pezzo surfaces of degree less than 6 and the projective plane

In what follows, $S$ denotes a smooth del Pezzo surface over an algebraically closed field $\mathbf{k}$ of characteristic zero. Recall that $K_S^2\in\{1,2,\ldots,9\}$. Let $G\subseteq\mathrm{Aut}(S)$ be a finite group and $H\subsetneq G$ be a subgroup (we always stick to this notation in what follows). Assume that $\mathrm{Pic}(S)^H\simeq\mathbb{Z}$, so in particular $\mathrm{Pic}(S)^G\simeq\mathbb{Z}$. Note that the condition $\mathrm{Pic}(S)^H\simeq\mathbb{Z}$ immediately excludes two cases: first, when $K_S^2=7$, and second, when $K_S^2=8$ and $S$ is a blow-up of $\mathbb{P}^2$ in one point. Indeed, in both cases, there exists an $H$-invariant $(-1)$-curve on $S$ which can be $H$-equivariantly contracted, so $\mathrm{rk}\,\mathrm{Pic}(S)^H>1$.

**Proposition 3.1**. *Assume that $K_S^2\in\{1,2,3\}$ and $\mathrm{Pic}(S)^G\simeq\mathbb{Z}$. Then $S$ is $G$-birationally rigid.*

*Proof.* This is essentially the content of the so-called *Segre–Manin theorem* (which follows from the classification of Sarkisov links nowadays). If $S\dashrightarrow S'$ is a $G$-birational map to another $G$-Mori fibre space $S'$, then it decomposes into Sarkisov $G$-links of type II and isomorphisms, and for any such link $\chi\colon S\dashrightarrow S'$, there exists a commutative diagram

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{\eta} & & \searrow^{\eta'} & \\ S & & \overset{\chi}{\dashrightarrow} & & S' \\ & \searrow & & \swarrow & \\ & & \mathrm{pt}, & & \end{array} \tag{3.1}$$

where $\eta, \eta'$ are birational morphisms and $S', T$ are del Pezzo surfaces as well; see [DI09a, Propositions 7.12 and 7.13]. This immediately implies that $S$ is even $G$-superrigid when $K_S^2 = 1$. If $K_S^2 \in \{2, 3\}$, then up to automorphisms of $S$, any such link is a birational Bertini or Geiser involution; *i.e.* there exist a biregular involution $\sigma \in \mathrm{Aut}(T)$ and an automorphism $\delta \in \mathrm{Aut}(S)$ such that $\chi = \eta' \circ \sigma \circ \eta^{-1} \circ \delta$. Thus, in particular, we have $S' \simeq S$. Since $\sigma$ centralizes $G$, we conclude that $S'$ is $G$-isomorphic to $S$. ☐

*Remark* 3.2. In [dDM19], M. Mauri and L. das Dores classify completely those del Pezzo surfaces of degree 2 and 3 which are $G$-birationally superrigid.

**Proposition 3.3.** *Let $K_S^2 = 4$. Then $S$ is $G$-birationally rigid if and only if there are no $G$-fixed points on $S$. In particular, if $S$ is $H$-birationally rigid, then it is $G$-birationally rigid as well.*

*Proof.* By [DI09a, Propositions 7.12 and 7.13], every Sarkisov $G$-link $\chi$ starting from $S$ is either of type I or of type II. In the former case, $\chi$ is centred at a $G$-fixed point on $S$. So, if such a point exists, then its blow-up is a smooth cubic surface equipped with a structure of $G$-conic bundle, so $S$ is not $G$-birationally rigid. Suppose there are no $G$-fixed points on $S$. Then by *loc. cit.*, any Sarkisov $G$-link of type II starting from $S$ is either a birational Bertini involution (centred at a point of degree 3) or a birational Geiser involution (centred at a point of degree 2). In both cases, it leads to a $G$-isomorphic surface $S' \simeq S$, as in the proof of Proposition 3.1. Thus, $S$ is $G$-birationally rigid. ☐

Before going to the next case, let us recall the following useful statement.

**Lemma 3.4** (*cf.* [BB73, Lemma 2.4]). *Let $X$ be an irreducible algebraic variety and $G \subset \mathrm{Aut}(X)$ be a finite group. If $G$ fixes a point $p \in X$, then there is a faithful linear representation $G \hookrightarrow \mathrm{GL}(T_p X)$.*

*Remark* 3.5. Note that a cyclic group always has a fixed point on a rational variety over an algebraically closed field of characteristic zero. This follows from the holomorphic Lefschetz fixed-point formula.

We now proceed with del Pezzo surfaces of degree 5.

**Lemma 3.6** (*cf.* [DI09a, Theorem 6.4]). *Let $S$ be a del Pezzo surface of degree 5 and $G \subset \mathrm{Aut}(S)$ be a group such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Then $G$ is isomorphic to one of the following five groups:*

$$\mathrm{S}_5, \quad \mathrm{A}_5, \quad \mathrm{AGL}_1(\mathbb{F}_5), \quad \mathrm{D}_5, \quad \mathrm{C}_5.$$

Here, $\mathrm{AGL}_1(\mathbb{F}_5)$ denotes the general affine group of degree 1 over $\mathbb{F}_5$, defined by the presentation $\langle a, b \mid a^5 = b^4 = \mathrm{id}, bab^{-1} = a^3 \rangle$; it has the structure of a semidirect product $\mathrm{C}_5 \rtimes \mathrm{C}_4$ and is sometimes called the Frobenius group of order 20.

**Proposition 3.7** (*cf.* [Wol18] and [Che08, Example 6.3]). *Let $K_S^2 = 5$. If $S$ is $H$-birationally rigid, then it is $G$-birationally rigid.*

*Proof.* We use Lemma 3.6. By [DI09a, Propositions 7.12 and 7.13], every Sarkisov $G$-link starting from $S$ is of type II; *i.e.* it is a diagram (3.1) where $\eta$ blows up a $G$-point of degree $d$, and one of the following holds:

(1) $S \simeq S'$, $d = 4$, $\chi$ is a birational Bertini involution;

(2) $S \simeq S'$, $d = 3$, $\chi$ is a birational Geiser involution;
(3) $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $d = 2$;
(4) $S' \simeq \mathbb{P}^2$, $d = 1$.

Recall that $\mathrm{A}_5$ (and hence $\mathrm{S}_5$) has no faithful representations of degree 2. Therefore, neither $\mathrm{A}_5$ nor $\mathrm{S}_5$ can have an orbit of size 1 or 2 on $S$ by Lemma 3.4. Further, $\mathrm{S}_5$ and $\mathrm{A}_5$ have no subgroups of index 3 or 4; hence there are no birational Bertini or Geiser involutions for these groups. We conclude that $S$ is $G$-birationally superrigid for $G \in \{\mathrm{A}_5, \mathrm{S}_5\}$.

So, it remains to verify the statement for the following pairs $(G, H)$:

$$(\mathrm{C}_5 \rtimes \mathrm{C}_4, \mathrm{D}_5), \quad (\mathrm{C}_5 \rtimes \mathrm{C}_4, \mathrm{C}_5), \quad (\mathrm{D}_5, \mathrm{C}_5).$$

Note that $S$ is never $H$-birationally rigid for $H \in \{\mathrm{C}_5, \mathrm{D}_5\}$. Indeed, it is easy to show (using a holomorphic Lefschetz fixed-point formula, as was mentioned in Remark 3.5) that $\mathrm{C}_5$ has exactly *two* fixed points on $S$ and these points do not lie on $(-1)$-curves; see *e.g.* [Yas16, Lemma 4.16]. Therefore, there is a Sarkisov link (4) from above, which leads to $\mathbb{P}^2$. Furthermore, these two fixed points form an orbit under the action of the dihedral group $\mathrm{D}_5$, containing $\mathrm{C}_5$. So, there is a link (3) leading to $\mathbb{P}^1 \times \mathbb{P}^1$. □

Let us treat the del Pezzo surface of degree 9, *i.e.* the projective plane $S = \mathbb{P}^2$. We stick to the following classical (although perhaps outdated) terminology.

**Definition 3.8** (*cf.* [Bli17]). We call a subgroup $\iota\colon G \hookrightarrow \mathrm{GL}_n(\mathbf{k})$ *intransitive* if the representation $\iota$ is reducible, and *transitive* otherwise. Further, a transitive group $G$ is called *imprimitive* if there is a decomposition $\mathbf{k}^n = \bigoplus_{i=1}^m V_i$ into a direct sum of subspaces and $G$ transitively acts on the set $\{V_i\}$. A transitive group $G$ is called *primitive* if there is no such decomposition. Finally, we say that $G \subset \mathrm{PGL}_n(\mathbf{k})$ is (in)transitive or (im)primitive if its preimage in $\mathrm{GL}_n(\mathbf{k})$ is such a group.

The following is due to D. Sakovics.

**Theorem 3.9** (*cf.* [Sak19, Theorem 1.3]). *The projective plane $\mathbb{P}^2$ is $G$-birationally rigid if and only if $G$ is transitive and $G$ is not isomorphic to $\mathrm{A}_4$ or $\mathrm{S}_4$.*

**Corollary 3.10**. *If $\mathbb{P}^2$ is $H$-birationally rigid, then it is $G$-birationally rigid as well.*

*Proof.* Indeed, $G$ is transitive since $H$ is. Assume that $G \simeq \mathrm{A}_4$ or $G \simeq \mathrm{S}_4$. If $H \neq \mathrm{A}_4$, then $H$ must be one of the following groups: $\mathrm{C}_2$, $\mathrm{C}_3$, $\mathrm{C}_4$, $\mathrm{C}_2 \times \mathrm{C}_2$, $\mathrm{S}_3$, $\mathrm{D}_4$. But the irreducible representations of these groups are of degree 1 or 2; hence they fix a point on $\mathbb{P}^2$ and are not transitive, so we have a contradiction (in fact, there is an $H$-equivariant blow-up $\mathbb{F}_1 \to \mathbb{P}^2$). □

## 4. Del Pezzo surfaces of degree 6

Let $S$ be a del Pezzo surface of degree $K_S^2 = 6$. Recall that $S$ is a blow-up $\pi\colon S \to \mathbb{P}^2$ of three non-collinear points $p_1, p_2, p_3$, which we may assume to be $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, respectively. The set of $(-1)$-curves on $S$ consists of six curves: the exceptional divisors of blow-ups $e_i = \pi^{-1}(p_i)$ and the strict transforms of the lines $d_{ij}$ passing through $p_i$ and $p_j$. In the anticanonical embedding $S \hookrightarrow \mathbb{P}^6$, these exceptional curves form a "regular hexagon" $\Sigma$. This yields a homomomorphism to the symmetry group of this hexagon

$$\psi\colon \mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(\Sigma) \simeq \mathrm{D}_6 = \left\langle r, s \mid r^6 = s^2 = 1,\ srs = r^{-1} \right\rangle,$$

where $r$ is a rotation by $\pi/3$ and $s$ is a reflection, shown on Figure 1. The surface $S$ can be given as

$$\left\{([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 :\ x_0 y_0 = x_1 y_1 = x_2 y_2 \right\}. \tag{4.1}$$

The projection to the first factor $\mathbb{P}^2$ is the blow-down of three lines $\{x_1 = x_2 = 0\}$, $\{x_0 = x_2 = 0\}$ and $\{x_0 = x_1 = 0\}$ onto $p_1$, $p_2$ and $p_3$, respectively, while the projection to the second factor is the blow-down of $\{y_1 = y_2 = 0\}$, $\{y_0 = y_2 = 0\}$ and $\{y_0 = y_1 = 0\}$.

The kernel of $\psi$ is the maximal torus $T$ of $\mathrm{PGL}_3(\mathbf{k})$, isomorphic to $(\mathbf{k}^*)^3/\mathbf{k}^* \simeq (\mathbf{k}^*)^2$ and acting on $S$ by

$$(\lambda_0, \lambda_1, \lambda_2) \cdot ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([\lambda_0 x_0 : \lambda_1 x_1 : \lambda_2 x_2], [\lambda_0^{-1} y_0 : \lambda_1^{-1} y_1 : \lambda_2^{-1} y_2]) \tag{4.2}$$

The corresponding element of $T$ will be denoted by $[(\lambda_1, \lambda_2, \lambda_3)]$. The action of $T$ on $S \setminus \Sigma$ is faithful and transitive. The automorphism group of $\mathrm{Aut}(S)$ fits into the short exact sequence

$$1 \longrightarrow (\mathbf{k}^*)^2 \longrightarrow \mathrm{Aut}(S) \xrightarrow{\psi} \mathrm{D}_6 \longrightarrow 1$$

with $\psi(\mathrm{Aut}(S)) \simeq \mathrm{D}_6 \simeq \mathrm{S}_3 \times \mathrm{C}_2$. We denote by

$$\iota \colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto ([y_0 : y_1 : y_2], [x_0 : x_1 : x_2]) \tag{4.3}$$

the lift of the standard Cremona involution, whose image under $\psi$ generates $\langle r^3 \rangle \simeq \mathrm{C}_2$, the centre of $\psi(\mathrm{Aut}(S))$. Further, the symmetric group $\mathrm{S}_3$ naturally acts on the indices of the coordinates $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2])$. In what follows, we will denote by

$$\theta \colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto ([x_1 : x_2 : x_0], [y_1 : y_2 : y_0]) \tag{4.4}$$

the automorphism which is mapped to $r^2$, the rotation of order 3 of the hexagon $\Sigma$.

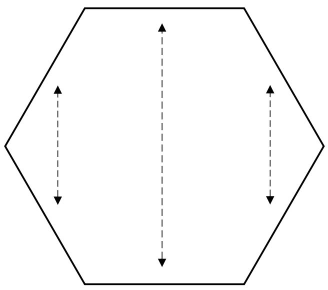


**Figure 1.** Action of $s$ on $\Sigma$

Note that

$$\rho = \iota \circ \theta \colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto ([y_1 : y_2 : y_0], [x_1 : x_2 : x_0]) \tag{4.5}$$

is an automorphism of order 6 such that $\psi(\rho)$ generates $\langle r \rangle$. Finally, the automorphism

$$\sigma \colon ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto ([y_0 : y_2 : y_1], [x_0 : x_2 : x_1]) \tag{4.6}$$

is mapped onto the reflection of $\Sigma$. The automorphisms $\sigma$ and $\rho$ generate a subgroup of $\mathrm{Aut}(S)$ which is mapped isomorphically onto $\mathrm{D}_6$ by $\psi$. In what follows, we sometimes call these actions $\theta$, $\iota$, $\rho$ and $\sigma$ "standard".

**Lemma 4.1.** *Let $S$ be a del Pezzo surface of degree 6 and $G \subset \mathrm{Aut}(S)$ be a finite group such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. If $G$ fixes a point on $S$, then $G \cap T = \mathrm{id}$.*

*Proof.* Assume that $G \cap T \neq \mathrm{id}$. Note that $T$ can be identified with a subgroup of $\mathrm{PGL}_3(\mathbf{k})$ which fixes three points $p_1$, $p_2$ and $p_3$. In particular, an element $t \in T$ fixing a point on $S \setminus \Sigma$ is necessarily trivial (of course, one can also deduce that from the explicit action of $T$ given above). Therefore, a fixed point $p \in S$ of $G$ lies on $\Sigma$. Note that it must be the intersection of two sides of $\Sigma$, as otherwise we have a $G$-invariant $(-1)$-curve, contradicting the minimality condition $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Similarly, if $p = \ell_1 \cap \ell_2$, where $\ell_1$, $\ell_2$ are some sides of $\Sigma$, then either $G$ preserves both $\ell_i$ and hence $\mathrm{rk}\,\mathrm{Pic}(S)^G > 1$, or $G$ switches $\ell_1$ and $\ell_2$. Denoting by $\ell_1'$ and $\ell_2'$ the other two sides which intersect $\ell_1$ and $\ell_2$, respectively, we easily see that $\ell_1'$ and $\ell_2'$ form a $G$-orbit of non-intersecting $(-1)$-curves, so $\mathrm{rk}\,\mathrm{Pic}(S)^G > 1$. □

The following elementary group-theoretic fact will be used several times below. We state it only for the reader's convenience.

**Lemma 4.2**. *The subgroups of the dihedral group* $\mathrm{D}_n = \langle r, s \mid r^n = s^2 = (sr)^2 = \mathrm{id}\rangle$ *are the following:*
Cyclic: $\langle r^d\rangle \simeq \mathrm{C}_{n/d}$ *and* $\langle r^k s\rangle$*, where* $d$ *divides* $n$ *and* $0 \leqslant k \leqslant n-1$*;*
Dihedral: $\langle r^d, r^k s\rangle \simeq \mathrm{D}_{n/d}$*, where* $d < n$ *divides* $n$ *and* $0 \leqslant k \leqslant d-1$.
*Moreover, all cyclic subgroups* $\langle r^d\rangle$ *are normal, one has* $\mathrm{D}_n/\langle r^d\rangle \simeq \mathrm{D}_d$*, and these are all normal subgroups when* $n$ *is odd. When* $n$ *is even, there are two more normal dihedral subgroups of index 2, namely* $\langle r^2, s\rangle$ *and* $\langle r^2, rs\rangle$.

**Lemma 4.3**. *Let* $S$ *be a del Pezzo surface of degree 6. If* $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$*, then* $G$ *is of the form*

$$N_{\bullet}\langle r\rangle \simeq N_{\bullet}\mathrm{C}_6, \quad N_{\bullet}\langle r^2, s\rangle \simeq N_{\bullet}\mathrm{S}_3 \quad \text{or} \quad N_{\bullet}\langle r, s\rangle \simeq N_{\bullet}\mathrm{D}_6,$$

*where* $N \simeq \mathrm{C}_n \times \mathrm{C}_m$ *is a subgroup of* $\mathrm{Ker}\,\psi \simeq (\mathbf{k}^*)^2$*. In particular, if* $G$ *fixes a point on* $S$*, then it is isomorphic to one of the following subgroups of* $\mathrm{Im}\,\psi = \mathrm{Aut}(\Sigma)$*:* $\mathrm{C}_6$, $\mathrm{S}_3$ *or* $\mathrm{D}_6$.

*Proof.* Let $G \subset \mathrm{Aut}(S)$ be a finite group. Assume that $\psi(G)$ is cyclic. In all the cases described in Lemma 4.2 except $\psi(G) = \langle r\rangle$, we clearly have a $G$-orbit of skew sides of the hexagon, which correspond to $(-1)$-curves. Hence $\mathrm{rk}\,\mathrm{Pic}(S)^G > 1$.

Let $\psi(G)$ be dihedral, *i.e.* $\langle r, s\rangle$, $\langle r^2, s\rangle$, $\langle r^2, rs\rangle$, $\langle r^3, s\rangle$, $\langle r^3, rs\rangle$ or $\langle r^3, r^2 s\rangle$. In the last three cases, one can always find a $G$-orbit of two disjoint $(-1)$-curves (a pair of opposite sides of the hexagon). Similarly, if $\psi(G) = \langle r^2, rs\rangle$, then we have a $G$-orbit consisting of three pairwise non-intersecting $(-1)$-curves. In the first two cases, one has $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Finally, the claim about fixed points follows from Lemma 4.1. □

*Remark* 4.4. Note that, although $\mathrm{D}_6 = \langle r, s\rangle$ contains two groups isomorphic to $\mathrm{S}_3$, only one of them gives $G$-invariant Picard number 1, namely $\langle r^2, s\rangle$, which we denote by $\mathrm{S}_3^{\mathrm{min}}$. The group $\langle r^2, rs\rangle$ will be denoted by $\mathrm{S}_3^{\mathrm{nmin}}$. Note that this is the quotient of $\mathrm{D}_6$ by its centre $\mathrm{Z}(\mathrm{D}_6) = \langle r^3\rangle \simeq \mathrm{C}_2$.

By [DI09a, Propositions 7.12 and 7.13], every Sarkisov link starting from $S$ is of type II and is represented by the diagram (3.1), where $\eta$ blows up a point of degree $d$ and one of the following holds:

(1) $S \simeq S'$, $d = 5$, $\chi$ is a birational Bertini involution;
(2) $S \simeq S'$, $d = 4$, $\chi$ is a birational Geiser involution;
(3) $d = 3$, $K_{S'}^2 = 6$;
(4) $d = 2$, $K_{S'}^2 = 6$;
(5) $d = 1$, $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Let us emphasize that in cases (3) and (4), the surfaces $S'$ *does not* have to be $G$-isomorphic[(1)] to $S$, as the following example shows. I am grateful to Andrey Trepalin for pointing this out.

*Example* 4.5. Let $\omega$ be a primitive 3rd root of unity, and consider the finite subgroup $G \subset \mathrm{Bir}(\mathbb{P}^2_{\mathbf{k}})$ generated by the following three elements:

$$\alpha\colon [x:y:z] \longmapsto [x:\omega y:\omega^2 z], \quad \beta\colon [x:y:z] \longmapsto [y:z:x], \quad \gamma\colon [x:y:z] \longmapsto [yz:xz:xy].$$

[(1)]Therefore, the user should be careful when using the statement of [DI09a, Proposition 7.13], whose notation is a bit misleading, in our opinion. For example, in the case of del Pezzo surfaces of degree 6 and links at points of degree 3 and 2, the authors write $S' \simeq S$, which might create the impression that this is an isomorphism of $G$-surfaces; compare with the case of del Pezzo surfaces of degree 8 and points of degree 4, where it is not written that $S' \simeq S$.

One has

$$G=\left\langle \alpha,\beta,\gamma \mid \alpha^3=\beta^3=\gamma^2=\mathrm{id},\ \alpha\beta=\beta\alpha,\ \beta\gamma=\gamma\beta,\ \gamma\alpha\gamma=\alpha^{-1}\right\rangle \simeq (\mathrm{C}_3\times\mathrm{C}_3)\rtimes\mathrm{C}_2,$$

where the copies of $\mathrm{C}_3$ are generated by $\alpha$ and $\beta$, and $\mathrm{C}_2$ is generated by the Cremona involution $\gamma$ and acts on $\mathrm{C}_3\times\mathrm{C}_3$ by coordinate exchange (*i.e.* we have the wreath product $G\simeq \mathrm{C}_3\wr\mathrm{C}_2$). The group $G$ is regularized on a del Pezzo surface $S$ of degree 6 given by Equation (4.1), which we identify with the blow-up of $\mathbb{P}^2$ in $p_1=[1:0:0]$, $p_2=[0:1:0]$ and $p_3=[0:0:1]$. The homomorphism $\psi\colon \mathrm{Aut}(S)\to \mathrm{D}_6$ induces a short exact sequence

$$1\longrightarrow \langle\alpha\rangle\longrightarrow G\longrightarrow\langle\beta,\gamma\rangle\longrightarrow 1, \tag{4.7}$$

where $\beta$ acts by permutation of coordinates in each triple $x_0,x_1,x_2$ and $y_0,y_1,y_2$, the involution $\gamma$ acts as in (4.3) and $\alpha$ acts as in (4.2). In particular, $\psi(G)\simeq \mathrm{C}_6$. Consider the lift of the three points $p_4=[1:1:1]$, $p_5=[1:\omega:\omega^2]$ and $p_6=[1:\omega^2:\omega]$ on $S$. They form a $G$-orbit in general position on $S$. A Sarkisov link (3.1) centred at these points leads to a del Pezzo surface $S'$ of degree 6. Let us denote by $L_{ij}$ the strict transforms on the cubic surface $T$ of the lines on $\mathbb{P}^2$ passing through $p_i$ and $p_j$. Similarly, $Q_{ijklr}$ will denote the strict transform of the conic passing through $p_i,p_j,p_k,p_l,p_r$. Then the morphism $\eta'$ blows down $Q_{12456}$, $Q_{23456}$ and $Q_{13456}$. The hexagon of $(-1)$-curves on $S'$ consists of the $\eta'$-images of $L_{45}$, $L_{46}$, $L_{56}$, $Q_{12346}$, $Q_{12345}$ and $Q_{12356}$. Hence the homomorphism $\psi'\colon \mathrm{Aut}(S')\to\mathrm{D}_6$ induces a short exact sequence

$$1\longrightarrow\langle\beta\rangle\longrightarrow G\longrightarrow\langle\alpha,\gamma\rangle\longrightarrow 1. \tag{4.8}$$

In particular, $\psi'(G)\simeq \mathrm{S}_3$. But this clearly implies that $S$ and $S'$ cannot be $G$-isomorphic.

So, one has to pay special attention to Sarkisov links centred at points of degrees 3 and 2. Proposition 4.8 below shows that the extra condition of being $H$-isomorphic eliminates the phenomena described in Example 4.5. To prove it, we will need some technical lemmas.

**Lemma 4.6**. *Let $S$ be a del Pezzo surface of degree 6 and $\tau$ be the toric automorphism (4.2) with $\lambda_0=1$, $\lambda_1=\omega$, $\lambda_2=\omega^2$, where $\omega$ is a primitive $3^{\mathrm{rd}}$ root of unity. Assume that a group $\Gamma\subset\mathrm{Aut}(S)$ fits into the short exact sequence*

$$1\longrightarrow\Gamma'\longrightarrow\Gamma\xrightarrow{\ \psi\ }\Gamma''\longrightarrow 1, \tag{4.9}$$

*where $\Gamma'=\langle\tau\rangle$ or $\Gamma'=\mathrm{id}$. Then one has the following:*

(1) *If $\Gamma''\simeq\mathrm{D}_6$, then $\Gamma$ is conjugate in $\mathrm{Aut}(S)$ to the subgroup $\Gamma_0\subset\mathrm{Aut}(S)$ generated by $\tau$, $\rho$ and $\sigma$ if $\Gamma'=\langle\tau\rangle$, and by $\rho$ and $\sigma$ if $\Gamma'=\mathrm{id}$.*
(2) *If $\Gamma''\simeq\mathrm{C}_6$, then $\Gamma$ is conjugate in $\mathrm{Aut}(S)$ to the subgroup $\Gamma_0\subset\mathrm{Aut}(S)$ generated by $\tau$ and $\rho$ if $\Gamma'=\langle\tau\rangle$, and by $\rho$ if $\Gamma'=\mathrm{id}$.*

*Proof.* We first prove the claim for $\Gamma'=\langle\tau\rangle$ and $\Gamma''\simeq\mathrm{D}_6$. Let $\overline{\rho}\in\Gamma$ and $\overline{\sigma}\in\Gamma$ be such that $\psi(\overline{\rho})$ and $\psi(\overline{\sigma})$ generate $\Gamma''$. We may thus assume that $\psi(\overline{\rho})=\psi(\rho)$ and $\psi(\overline{\sigma})=\psi(\sigma)$. Then $\Gamma$ is generated by $\tau$, $\overline{\rho}$ and $\overline{\sigma}$. The map $\overline{\rho}$ is given by

$$[(1,a,b)]\circ\rho=\overline{\rho}\colon ([x_0:x_1:x_2],[y_0:y_1:y_2])\longmapsto([y_1:ay_2:by_0],[x_1:a^{-1}x_2:b^{-1}x_0]) \tag{4.10}$$

for some $a,b\in\mathbf{k}^*$. The map

$$\beta\colon([x_0:x_1:x_2],[y_0:y_1:y_2])\longmapsto([x_0:ba^{-1}x_1:a^{-1}x_2],[y_0:ab^{-1}y_1,ay_2])$$

commutes with $\tau$ and satisfies $\beta\circ\overline{\rho}\circ\beta^{-1}=\rho$. Let $\widetilde{\rho}=\beta\circ\overline{\rho}\circ\beta^{-1}=\rho$ and $\widetilde{\sigma}=\beta\circ\overline{\sigma}\circ\beta^{-1}$. Then $\tau$, $\widetilde{\rho}$, $\widetilde{\sigma}$ generate the group $\beta\circ\Gamma\circ\beta^{-1}$. Since $\beta\in\ker\psi=T$, we have $\psi(\widetilde{\sigma})=\psi(\sigma)$ and thus

$$\widetilde{\sigma}=\mu\circ\sigma\colon([x_0:x_1:x_2],[y_0:y_1:y_2])\longmapsto([y_0:cy_2:dy_1],[x_0:c^{-1}x_2:d^{-1}x_1]) \tag{4.11}$$

for some $\mu = [(1,c,d)] \in T$. Therefore,

$$\widetilde{\sigma}^2\colon ([x_0:x_1:x_2],[y_0:y_1:y_2]) \longmapsto ([x_0:cd^{-1}x_1:c^{-1}dx_2],[y_0:c^{-1}dy_1:cd^{-1}y_2]) \tag{4.12}$$

is a power of $[(1,\omega,\omega^2)]$; hence $c = d\omega^i$ for $i \in \{0,1,2\}$. Since $(\sigma\circ\rho)^2 = \mathrm{id}$, we have that

$$(\widetilde{\sigma}\circ\widetilde{\rho})^2\colon ([x_0:x_1:x_2],[y_0:y_1:y_2]) \longmapsto ([cx_0:cx_1:d^2x_2],[c^{-1}y_0:c^{-1}y_1:d^{-2}y_2]) \tag{4.13}$$

is a power of $\tau = [(1,\omega,\omega^2)]$; hence $c = d^2$. Since $c = d\omega^i$, we conclude that $\mu = \tau^k$ for some $k \in \{0,1,2\}$, *i.e.* $\widetilde{\sigma} = \sigma\circ\tau^k$. Therefore, $\beta\circ\Gamma\circ\beta^{-1} = \langle\tau,\rho,\sigma\circ\tau^k\rangle = \Gamma_0$, as claimed. If $\Gamma'' \simeq \mathrm{C}_6$, then it is enough to conjugate the generator $\overline{\rho}$ to $\rho$.

If $\Gamma' = \mathrm{id}$ and $\Gamma'' \simeq \mathrm{D}_6$ or $\Gamma'' \simeq \mathrm{C}_6$, we again conjugate the generator $\overline{\rho}$ to $\rho$. Since $\widetilde{\sigma}$ and $\widetilde{\sigma}\circ\widetilde{\rho}$ are just involutions, (4.12) gives $c = d$, while (4.13) implies $c = d^2$. Hence $c = d = 1$ and we are done. We refer to [Pin24, Propositions 5.6, 5.7 and 5.8] for similar proofs in these cases. ◻

Let us fix some notation. Let $\chi\colon S_1 \dashrightarrow S_2$ be a Sarkisov $G$-link between del Pezzo surfaces of degree 6 and $H \subset G$ be a subgroup. We denote by $G_1 = \iota_1(G)$ and $H_1 = \iota_1(H)$ the embeddings of $G$ and $H$ into $\mathrm{Aut}(S_1)$, and by $G_2 = \iota_2(G)$ and $H_2 = \iota_2(H)$ the embeddings of $G$ and $H$ into $\mathrm{Aut}(S_2)$ induced by the map $\chi$. For each $i \in \{1,2\}$, we denote by $\psi_i\colon \mathrm{Aut}(S_i) \to \mathrm{Aut}(\Sigma_i) \simeq \mathrm{D}_6$ the homomorphism described above with $T_i = \ker\psi_i$; further, set $G_{i,T} = G_i \cap T_i$, $H_{i,T} = H_i \cap T_i$, $\widehat{G_i} = \psi_i(G_i)$ and $\widehat{H_i} = \psi_i(H_i)$. We have $H_{i,T} \subset G_{i,T}$ and $\widehat{H_i} \subset \widehat{G_i}$ for each $i \in \{1,2\}$.

We start with some restrictions on the "toric part" of our groups and then deal with the simplest case when this part of the larger group $G$ is trivial.

**Lemma 4.7**. *Let $S_1$ be a del Pezzo surface of degree 6 and $S_1 \dashrightarrow S_2$ be a Sarkisov $G$-link of type II, centred at a $G$-point of degree $d$, where $d \in \{2,3\}$. Let $H \subset G$ be a subgroup, and assume there is an $H$-link $S_1 \dashrightarrow S_2$ at the same point. Then the following hold:*

1. *The subgroups $H_{i,T}$ and $G_{i,T}$ are either trivial or of order $d$; i.e. for each $i$ we have the following possibilities:*
   (a) $H_{i,T} = G_{i,T} = \mathrm{id}$,
   (b) $H_{i,T} = \mathrm{id} \subset G_{i,T} \simeq \mathrm{C}_d$,
   (c) $H_{i,T} = G_{i,T} \simeq \mathrm{C}_d$.
2. *Assume that $S_2$ is $H$-isomorphic to $S_1$. If $G_{1,T} = \mathrm{id}$, then $S_2$ is $G$-isomorphic to $S_1$.*
3. *If $G_{1,T} \simeq \mathrm{C}_d$, then $G_{2,T} \simeq \mathrm{C}_d$.*

*Proof.* Since each $S_i$ is $H$-del Pezzo, by Lemma 4.3 the pair $(\widehat{H_i},\widehat{G_i})$ must be one of the following: $(\mathrm{C}_6,\mathrm{C}_6)$, $(\mathrm{S}_3,\mathrm{S}_3)$, $(\mathrm{D}_6,\mathrm{D}_6)$, $(\mathrm{C}_6,\mathrm{D}_6)$, $(\mathrm{S}_3,\mathrm{D}_6)$. Furthermore, since $S_i$ admits an $H_i$-orbit of degree $d$ (which is also a $G_i$-orbit), both $H_i$ and $G_i$ have index $d$ subgroups which fix a point on $S_i \setminus \Sigma_i$ and thus do not intersect $H_{i,T}$ and $G_{i,T}$, respectively. We observe that the order of $G_{i,T}$ is at most $d$ and deduce statement (1).

(2) Suppose that $G_{1,T} = \mathrm{id}$; then $H_1 \simeq \widehat{H_1}$, $G_1 \simeq \widehat{G_1}$, and the statement is tautological when $\widehat{H_1} = \widehat{G_1}$. If $\widehat{H_1} \neq \widehat{G_1}$, then $G_1 \simeq \widehat{G_1} = \psi_1(\mathrm{Aut}(S_1)) \simeq \mathrm{D}_6$. Since $G_1 \simeq G_2$, we must have $G_2 \simeq \widehat{G_2}$. Indeed, otherwise $G_{2,T} = G_2 \cap T_2 \neq \mathrm{id}$ and thus $G_{2,T} \simeq \mathrm{C}_d$. But then $\widehat{G_2} \simeq \mathrm{C}_2\times\mathrm{C}_2$ (if $d = 3$) or $\widehat{G_2} \simeq \mathrm{S}_3$ (if $d = 2$). In the former case, $S_2$ is not $G$-del Pezzo by Lemma 4.3, while in the latter case $G_{2,T} \simeq \mathrm{C}_2$ is necessarily the centre of $G_2 \simeq \mathrm{C}_2\times\mathrm{S}_3$; however, an involution from $T_2$ cannot commute with an automorphism $\tau'\circ\theta$, where $\tau' \in T_2$, mapped to a rotation of order 3. Now, having $G_1 \simeq \psi_1(\mathrm{Aut}(S_1))$ and $G_2 \simeq \psi_2(\mathrm{Aut}(S_2))$, both groups are conjugate to the "standard" one generated by $\rho$ and $\sigma$; see Lemma 4.6.

(3) Assume that $G_{1,T} \simeq \mathrm{C}_d$. If $G_{2,T} = \mathrm{id}$, then $G_1 \simeq G_2 \simeq \widehat{G_2} \in \{\mathrm{C}_6,\mathrm{S}_3^{\min},\mathrm{D}_6\}$. If $d = 3$, then $\widehat{G_1} \in \{\mathrm{C}_2,\mathrm{C}_2^2\}$ and thus $S_1$ is not $G$-del Pezzo. If $d = 2$, then $\widehat{G_1} \simeq \mathrm{C}_3$ or $G_{1,T} \simeq \mathrm{C}_2$ is the centre of $G_1 \simeq \mathrm{C}_2\times\mathrm{S}_3$. In the first case, $S_1$ is not $G$-del Pezzo, while the second case is impossible, as was noticed before. ◻

**Proposition 4.8**. *Let $S_1$ be a del Pezzo surface of degree 6 and $S_1 \dashrightarrow S_2$ be a Sarkisov $G$-link of type II, centred at a $G$-point of degree $d$, where $d \in \{2,3\}$. Let $H \subset G$ be a subgroup, and assume there is an $H$-link $S_1 \dashrightarrow S_2$ at the same point. If $S_2$ is $H$-isomorphic to $S_1$, then it is also $G$-isomorphic to $S_1$.*

*Proof.* By Lemma 4.7, we may assume $G_{1,T} \simeq \mathrm{C}_d$. Let $\alpha\colon S_1 \xrightarrow{\sim} S_2$ be an $H$-isomorphism from the statement; it maps $T_1 = S_1 \setminus \Sigma_1$ isomorphically onto $T_2 = S_2 \setminus \Sigma_2$ and induces isomorphisms $H_{1,T} \simeq H_{2,T}$ and $\widehat{H}_1 \simeq \widehat{H}_2$. Define $G_2' = \alpha \circ G_1 \circ \alpha^{-1} \subset \mathrm{Aut}(S_2)$. We claim that $G_2' = \gamma^{-1} \circ G_2 \circ \gamma$ for some $\gamma \in \mathrm{Aut}(S_2)$. This will finish the proof as the surfaces $S_1$ and $S_2$ are then $G$-isomorphic via the map $\gamma \circ \alpha$.

Set $\widehat{G_2'} = \psi_2(G_2')$ and $G_{2,T}' = G_2' \cap T_2$. Firstly, note that $\widehat{G_2} = \widehat{G_2'}$. Since $\widehat{G_2'} \simeq \widehat{G_1}$ and there is only one subgroup of $\mathrm{D}_6$ in each isomorphism class for which $S_2$ is $G$-del Pezzo, it is enough to show that $\widehat{G_2} \simeq \widehat{G_1}$. Both are subgroups of $\mathrm{D}_6$ of the same order, as $G_1 \simeq G_2$ and $G_{1,T} \simeq G_{2,T}$ by Lemma 4.7(3); hence it remains to see that one cannot have $\widehat{G_1} \simeq \mathrm{C}_6$ and $\widehat{G_2} \simeq \mathrm{S}_3$ (or vice versa); but this is implied by $\widehat{H_1} \simeq \widehat{H_2}$.

Secondly, we claim that $G_{2,T} = G_{2,T}'$. Recall that the group $\psi_2(\mathrm{Aut}(S_2)) \simeq \mathrm{S}_3 \times \mathrm{C}_2$ acts on $T_2$; namely, $\mathrm{S}_3$ acts on the torus $T_2 \simeq (\mathbf{k}^*)^3/\mathbf{k}^*$ by permuting the coordinates, and the action of $\mathrm{C}_2$ is the inversion; we denote this action of $\mathrm{D}_6$ on $T_2$ by $\varphi$. The groups $\widehat{G_2'}$ and $\widehat{G_2}$, being both $\mathrm{S}_3^{\min}$, $\mathrm{C}_6$ or $\mathrm{D}_6$, contain $g = r^2$. Let $\tau = (t, \mathrm{id}) \in T_2 \rtimes \rho_2(\mathrm{Aut}(S_2))$ be an element of order $d \in \{2,3\}$ which generates $G_{2,T}$ (or $G_{2,T}'$) and $(u,g)$ be an element of $G_2$ (respectively, of $G_2'$) which is mapped to $(\mathrm{id}, g)$ by $\psi_2$. Then $(u,g)^{-1} = (\varphi_{g^{-1}}(u^{-1}), g^{-1})$. Therefore, $(\varphi_{g^{-1}}(u^{-1}), g^{-1})(t,\mathrm{id})(u,g) = (\varphi_{g^{-1}}(u^{-1}tu), 1) = (\varphi_{g^{-1}}(t), 1)$ is a power of $\tau = (t, \mathrm{id})$. If $t = [(1,a,b)] \in T_2 \simeq (\mathbf{k}^*)^3/\mathbf{k}^*$, we must have $[(a,b,1)] = [(1,a,b)]$ or $[(a,b,1)] = [(1,a^2,b^2)]$, where $a$ and $b$ are primitive $d^{\text{th}}$ roots of unity. This implies that $d = 3$ and $\tau = ([(1,\omega,\omega^2)], \mathrm{id})$ or $\tau = ([(1,\omega^2,\omega)], \mathrm{id})$, which both generate the same subgroup of $T_2$.

We conclude by applying Lemma 4.6. Namely, the groups $G_2$ and $G_2'$ are both conjugate to $\langle \tau, \rho, \sigma \rangle$ or to $\langle \tau, \rho \rangle$ if their $\psi_2$-images are $\mathrm{D}_6$ or $\mathrm{C}_6$, respectively, or to the group $\langle \tau, H_2 \rangle$ if their $\psi_2$-images are $\mathrm{S}_3^{\min}$. □

**Proposition 4.9**. *If a del Pezzo surface of degree 6 is $H$-birationally rigid, then it is also $G$-birationally rigid.*

*Proof.* Possible Sarkisov $G$-links were described before Example 4.5; recall that $G$-birational Bertini or Geiser involutions always lead to a $G$-isomorphic surface. If $S$ admits a $G$-link to $\mathbb{P}^1 \times \mathbb{P}^1$ centred at a $G$-fixed point, then the same point is fixed by $H$ and hence there is an $H$-link to $\mathbb{P}^1 \times \mathbb{P}^1$, so we have a contradiction. Assume that there is a $G$-link $S \dashrightarrow S'$ at a $G$-orbit of cardinality 2 or 3. Then either this orbit contains an $H$-fixed point, or it gives rise to an $H$-link $S \dashrightarrow S'$. In the former case, we get an $H$-link to $\mathbb{P}^1 \times \mathbb{P}^1$, contradicting the $H$-birational rigidity of $S$, while in the second case, the $H$-birational rigidity of $S$ implies that $S'$ is $H$-isomorphic to $S$. Now Proposition 4.8 shows that $S'$ is also $G$-isomorphic to $S$. □

*Remark* 4.10. If there is a $G$-link $S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ centred at a $G$-fixed point, then $G \cap T = \mathrm{id}$ by Lemma 4.1 and hence $\psi$ maps $G$ isomorphically onto one of the following subgroups of $\psi(\mathrm{Aut}(S))$: $\mathrm{C}_6$, $\mathrm{S}_3^{\min}$ or $\mathrm{D}_6$. If their actions are in the "standard" form as in Lemma 4.6, the fixed point becomes $([1:1:1],[1:1:1])$. Making a Sarkisov link centred at this point, we arrive at $\mathbb{P}^1 \times \mathbb{P}^1$ acted on by $G$. Explicitly, one blows up the fixed point and then blows down the preimages of the three genus zero curves passing through this point; see [Isk08] for more details and related discussion.

## 5. $G$-birational rigidity of quadric surfaces

In this section, we investigate carefully the case of $S = \mathbb{P}^1 \times \mathbb{P}^1$. In Section 5.1, we study some Sarkisov links on $S$ and show how to complete the proof of our main theorem. In Section 5.2, we present a detailed analysis of finite group actions on $S$ and $G$-birational rigidity in each case, more in the spirit of [Sak19, Wol18].

### 5.1. Sarkisov $G$-links

By [DI09a, Propositions 7.12 and 7.13], every Sarkisov $G$-link starting from $S$ is either of type I and of the form

$$\begin{array}{ccc} & & T \\ & \overset{\eta}{\swarrow} & \downarrow \pi \\ S & & \mathbb{P}^1 \\ \downarrow & \swarrow & \\ \mathrm{pt}, & & \end{array} \tag{5.1}$$

where $\eta$ is a blow-up of a degree 2 point, $\pi\colon T \to \mathbb{P}^1$ is a $G$-conic bundle with two singular fibres (in fact, it is necessarily a del Pezzo surface of degree 6, see [Isk80, Theorem 5]) or is of type II and is represented by the diagram (3.1), where $\eta$ blows up a point of degree $d$, $\eta'$ blows up a point of degree $d'$, and one of the following holds:

(1) $S \simeq S'$, $d = d' = 7$, $\chi$ is a birational Bertini involution;
(2) $S \simeq S'$, $d = d' = 6$, $\chi$ is a birational Geiser involution;
(3) $S'$ is a del Pezzo surface of degree 5, $d = 5$, $d' = 2$;
(4) $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $d = d' = 4$;
(5) $S'$ is a del Pezzo surface of degree 6, $d = 3$, $d' = 1$;
(6) $S' \simeq \mathbb{P}^2$, $d = 1$, $d' = 2$.

In particular, if $G$ does not fix a point on $S$ and is not isomorphic to any of the groups

$$\mathrm{C}_5,\quad \mathrm{C}_6,\quad \mathrm{S}_3,\quad \mathrm{D}_5,\quad \mathrm{D}_6,\quad \mathrm{AGL}_1(\mathbb{F}_5),\quad \mathrm{A}_5,\quad \mathrm{S}_5, \tag{5.2}$$

then all $G$-links of type II from $S$ lead to a quadric surface $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (recall Lemmas 3.6, 4.1 and 4.3). *A priori*, the surface $S'$ does not have to be $G$-isomorphic to $S$ (we saw such phenomena in Section 4) unless we deal with $G$-birational Bertini and Geiser involutions. It turns out that $G$-links centred at a point of degree 4 also lead to a $G$-isomorphic del Pezzo surface $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. As was pointed out to me by Andrey Trepalin, this holds in the arithmetic, see [Tre23, Lemma 4.3], and even mixed settings; we limit ourselves to the geometric situation.

**Proposition 5.1.** *Let $\chi\colon S \dashrightarrow S'$ be a Sarkisov $G$-link of type II centred at a point of degree 4. Then $S'$ is $G$-isomorphic to $S$.*

*Proof.* Assume that $\chi = \eta' \circ \eta^{-1}\colon S \dashrightarrow S'$ is given by the diagram (3.1); then $T$ is a del Pezzo surface of degree 4. It is a blow-up of $\mathbb{P}^2_{\mathbf{k}}$ in five points $p_1, \ldots, p_5$ in general position. Denote by $E_1, \ldots, E_5$ the exceptional divisors of this blow-up, by $L_{ij}$ for $i, j \in \{1, \ldots, 5\}$ with $i < j$ the strict transforms of the lines through $p_i$ and $p_j$, and by $Q$ the strict transform of the conic through $p_1, \ldots, p_5$. These are sixteen $(-1)$-curves on $T$. Their intersection graph is the Clebsch strongly regular quintic graph on sixteen vertices, shown on Figure 2. Up to renumbering the points, we may assume that the curves $\Sigma = \{E_1, E_2, E_3, L_{45}\}$ are the exceptional divisors of the blow-up $\eta$, while the curves $\Sigma' = \{L_{12}, L_{13}, L_{23}, Q\}$ are the exceptional divisors of $\eta'$. Recall, see [Dol12, Corollary 8.2.40], that $\mathrm{Aut}(T)$ injects into the Weyl group $\mathrm{W}(D_5) \simeq \mathrm{C}_2^4 \rtimes \mathrm{S}_5$, the automorphism group of the Clebsch graph. The quartic del Pezzo surface $T$ is isomorphic to an intersection of two quadrics

$$\sum_{i=1}^{5} x_i^2 = \sum_{i=1}^{5} \lambda_i x_i^2 = 0$$

in $\mathbb{P}^4_{\mathbf{k}}$. The group $\mathrm{S}_5$ acts by permutation of coordinates (and naturally acts on the indices of $E_i$ and $L_{jk}$), while $\mathrm{C}_2^4$ acts as a diagonal subgroup of $\mathrm{PGL}_5(\mathbf{k})$. There are two types of involutions in this group, $\iota_{ij}$

and $\iota_{ijkl}$, which switch the signs of $x_i, x_j$ and $x_i, x_j, x_k, x_l$, respectively. Equivalently, $\iota_{ijkl}$ coincides with the automorphism $\jmath_t$, where $t \in \{1,2,3,4,5\} \setminus \{i,j,k,l\}$, switching the sign of $x_t$. As explained in [DD16, Section 7], the $\jmath_t$ are given by de Jonquières involutions of the plane model centred at $p_t$ and interchange $E_i$ with $L_{it}$ and $E_t$ with $Q$. To recover the action of $\jmath_t$ on the curves $L_{ij}$ for $i, j \neq t$, we notice that two disjoint curves $E_i$ and $E_j$ are both intersected by exactly two others, $L_{ij}$ and $Q$ (in other words, any two non-adjacent vertices of the Clebsch graph 2 have two common neighbours). Therefore, $\jmath_t(L_{ij})$ is the curve which intersects $L_{it}$ and $L_{jt}$, and it is different from $\jmath_t(Q) = E_t$; hence, it is $L_{sr}$, where $s, r \in \{1,2,3,4,5\} \setminus \{i,j,t\}$.

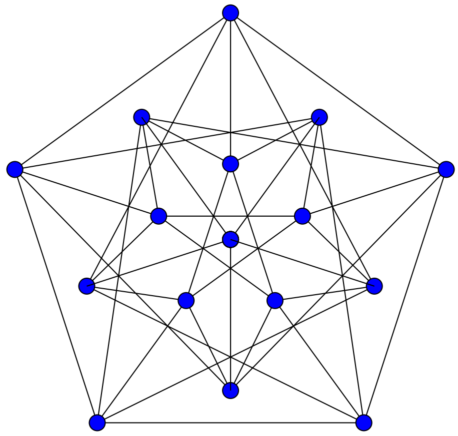


**Figure 2**. The Clebsch graph

Now, one easily checks that the involutions $\iota_{*5} = \jmath_* \circ \jmath_5$ and $\iota_{*4} = \jmath_* \circ \jmath_4$ do not preserve the set $\Sigma$. Hence, the subgroup of $\mathrm{W}(D_5)$ which preserves $\Sigma$ is generated by $(12), (123), \iota_{12}$ and $(45)$, and is isomorphic to $\mathrm{S}_4 \times \mathrm{C}_2$. The involution $\iota_{45}$ commutes with this group and maps $\Sigma$ onto $\Sigma'$. Since $\iota_{45}$ actually corresponds to an automorphism of $T$, we conclude that the blow-down $\eta'$ yields a $G$-isomorphic del Pezzo surface $S'$. □

**Corollary 5.2**. *The surface $\mathbb{P}^1 \times \mathbb{P}^1$ is $G$-birationally rigid (as a $G$-del Pezzo surface) if and only if the size of every $G$-orbit in general position is 4 or at least 6. (Here and everywhere below, by "general position" we mean that the blow-up of this orbit gives a del Pezzo surface.)*

*Proof.* The sufficiency follows from Proposition 5.1 and the fact that $G$-birational involutions of Bertini and Geiser yield a $G$-isomorphic surface. Conversely, if $\mathbb{P}^1 \times \mathbb{P}^1$ is $G$-birationally rigid, then it does not admit Sarkisov $G$-links centred at $G$-points of degree $d \in \{1,2,3,5\}$: by our assumption, the blow-up of every such orbit gives a del Pezzo surface $T$ with $\mathrm{Pic}(T)^G \simeq \mathbb{Z}^2$; hence the 2-ray game (see Remark 2.1) provides a Sarkisov $G$-link to a $G$-del Pezzo surface of degree $d' \neq 8$. □

*Remark* 5.3. At the same time, it is often possible to exclude the possibility of $G$-links $\chi\colon S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ centred at a point of degree 4. If such a link exists and is represented by the diagram (3.1), then $T$ is a del Pezzo surface of degree 4, so a natural obstruction to the existence of $\chi$ is the impossibility of an embedding $G \hookrightarrow \mathrm{Aut}(T)$. Luckily, all possible automorphism groups $\mathrm{Aut}(T)$ of smooth del Pezzo surfaces of degree 4 are classified, see [Hos96, Dol12], and they are the following (see the proof of Proposition 5.1 for the description of these semidirect products):

$$\mathrm{C}_2^4, \quad \mathrm{C}_2^4 \rtimes \mathrm{C}_2, \quad \mathrm{C}_2^4 \rtimes \mathrm{C}_4, \quad \mathrm{C}_2^4 \rtimes \mathrm{S}_3, \quad \mathrm{C}_2^4 \rtimes \mathrm{D}_5.$$

The following proposition finishes the proof of our main theorem. An alternative way will be sketched in Remark 5.15.

**Proposition 5.4**. *If the del Pezzo surface $\mathbb{P}^1 \times \mathbb{P}^1$ is $H$-birationally rigid, then it is also $G$-birationally rigid.*

*Proof.* Suppose that $S = \mathbb{P}^1 \times \mathbb{P}^1$ is not $G$-birationally rigid. By Corollary 5.2, there is a $G$-orbit $\Sigma$ of size $|\Sigma| \in \{1, 2, 3, 5\}$; moreover, $\Sigma$ is in general position on $S$. Write $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_r$, where each $\Sigma_i$ is an $H$-orbit and $|\Sigma_i| \leqslant |\Sigma_j|$ for $i \leqslant j$. Clearly, $r \geqslant 2$ since $r = 1$ implies that $\Sigma$ is an $H$-orbit and hence $S$ is not $H$-birationally rigid. If $r \geqslant 2$ and $|\Sigma| \in \{1, 2, 3\}$, then $H$ admits an orbit of size 1, and thus $S$ has an $H$-link to $\mathbb{P}^2$. The same reasoning applies to the case $|\Sigma| = 5$ and $r \geqslant 3$, and we conclude that $r = 2$, $|\Sigma_1| = 2$, $|\Sigma_2| = 3$. Since $\Sigma_1$ is in general position, Corollary 5.2 again gives a contradiction with $H$-birational rigidity. □

The main theorem is proven.

## 5.2. Finite groups acting on quadric surfaces

We now proceed with a deeper analysis of finite group actions on $S = \mathbb{P}^1 \times \mathbb{P}^1$. Recall that

$$\mathrm{Aut}(S) \simeq (\mathrm{PGL}_2(\mathbf{k}) \times \mathrm{PGL}_2(\mathbf{k})) \rtimes \mathrm{C}_2,$$

where the action of $\mathrm{C}_2$ is given by exchanging the factors. Finite subgroups of the direct product can be determined using the so-called *Goursat's lemma*. Recall that the *fibred product* of two groups $G_1$ and $G_2$ over a group $D$ is

$$G_1 \times_D G_2 = \{(g_1, g_2) \in G_1 \times G_2 \colon \alpha(g_1) = \beta(g_2)\},$$

where $\alpha\colon G_1 \to D$ and $\beta\colon G_2 \to D$ are some surjective homomorphisms. Although the notation does not reflect it, the data defining $G_1 \times_D G_2$ is not only the groups $G_1$, $G_2$ and $D$ but also the homomorphisms $\alpha$, $\beta$.

**Lemma 5.5** (Goursat's lemma, *cf.* [Gou89, p. 47]). *Let $A$ and $B$ be two groups. There is a bijective correspondence between subgroups $G \subset A \times B$ and 5-tuples $\{G_A, G_B, K_A, K_B, \varphi\}$, where $G_A$ is a subgroup of $A$, $K_A$ is a normal subgroup of $G_A$, $G_B$ is a subgroup of $B$, $K_B$ is a normal subgroup of $G_B$ and $\varphi\colon G_A/K_A \xrightarrow{\sim} G_B/K_B$ is an isomorphism. More precisely, the group corresponding to this 5-tuple is*

$$G = \{(a, b) \in G_A \times G_B \colon \varphi(aK_A) = bK_B\}.$$

*Conversely, let $G$ be a subgroup of $A \times B$. Denote by $p_A\colon A \times B \to A$ and $p_B\colon A \times B \to B$ the natural projections, and set $G_A = p_A(G)$ and $G_B = p_B(G)$. Further, let*

$$K_A = \ker p_B|_G = \{(a, \mathrm{id}) \in G,\ a \in A\},$$
$$K_B = \ker p_A|_G = \{(\mathrm{id}, b) \in G,\ b \in B\},$$

*whose images by $p_A$ and $p_B$ define normal subgroups of $G_A$ and $G_B$, respectively (denoted the same). Let $\pi_A\colon G_A \to G_A/K_A$ and $\pi_B\colon G_B \to G_B/K_B$ be the canonical projections. The map $\varphi\colon G_A/K_A \to G_B/K_B$, $\varphi(aK_A) = bK_B$, where $b \in B$ is any element such that $(a, b) \in G$, is an isomorphism. Furthermore, $G = G_A \times_D G_B$, where $D = G_A/K_A$, $\alpha = \pi_A$ and $\beta = \varphi^{-1} \circ \pi_B$.*

**Corollary 5.6.** *In the notation from Goursat's lemma, the subgroup $G \subset A \times B$ fits into the short exact sequence*

$$1 \longrightarrow K_A \times K_B \longrightarrow G \longrightarrow D \longrightarrow 1.$$

*Proof.* Indeed, the restriction of the homomorphism $\alpha \times \beta\colon G_A \times G_B \to D \times D$ to $G$ has kernel $K_A \times K_B$, and its image is isomorphic to $\Delta = \{(t, t) \in D \times D\} \simeq D$. □

Using Goursat's lemma, one can get the description of finite subgroups $G \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ for which $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)^G \simeq \mathbb{Z}$. Before doing that, let us recall the following classical result due to F. Klein.

**Proposition 5.7** (*cf.* [Kle19]). *If $\mathbf{k}$ is an algebraically closed field of characteristic zero, then every finite subgroup of $\mathrm{PGL}_2(\mathbf{k})$ is isomorphic to $\mathrm{C}_n$, $\mathrm{D}_n$ (where $n \geqslant 1$), $\mathrm{A}_4$, $\mathrm{S}_4$ or $\mathrm{A}_5$. Moreover, there is only one conjugacy class for each of these groups.*

Every group $G \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ fits into the short exact sequence

$$1 \longrightarrow G_\circ \longrightarrow G \longrightarrow \widehat{G} \longrightarrow 1,$$

where $G_\circ = G \cap (\mathrm{PGL}_2(\mathbf{k}) \times \mathrm{PGL}_2(\mathbf{k}))$ and $\widehat{G} \subseteq \mathrm{C}_2$.

**Proposition 5.8** (*cf.* [Tre18, Lemma 3.2]). *Let $G \subset \mathrm{Aut}(S)$ be a finite subgroup such that $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$. Then $G \simeq (F \times_D F)_\bullet \mathrm{C}_2$, where $F$ is cyclic, dihedral or one of the groups $\mathrm{A}_4$, $\mathrm{S}_4$, $\mathrm{A}_5$. Moreover, for every such group $G$, we have $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$.*

*Proof.* Since $G_\circ$ preserves the factors of $\mathbb{P}^1 \times \mathbb{P}^1$, the condition $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$ forces $\widehat{G} = \mathrm{C}_2$. If $G_1$ and $G_2$ are the images of $G_\circ$ under the projections of $\mathrm{PGL}_2(\mathbf{k}) \times \mathrm{PGL}_2(\mathbf{k})$ onto its factors, Goursat's lemma implies that $G_\circ = G_1 \times_D G_2$ for some $D$. Since $\widehat{G} \neq \mathrm{id}$, we must have $G_1 \simeq G_2$. Combining this with Proposition 5.7, we get the result. □

**Corollary 5.9**. *In the notation of Proposition 5.8, for every finite subgroup $G \subset \mathrm{Aut}(S)$ satisfying $\mathrm{Pic}(S)^G \simeq \mathbb{Z}$, one has the short exact sequence*

$$1 \longrightarrow K \times K \longrightarrow G_\circ \longrightarrow D \longrightarrow 1, \tag{5.3}$$

*where $K$ is a normal subgroup of $F$ and $F/K \simeq D$.*

*Proof.* We apply Corollary 5.6 and note that the action of $\mathrm{C}_2$ on the semidirect product $(\mathrm{PGL}_2(\mathbf{k}) \times \mathrm{PGL}_2(\mathbf{k})) \rtimes \mathrm{C}_2$ induces an isomorphism of the kernels $K_A \xrightarrow{\sim} K_B$. (In fact, for finite subgroups $A, K_1, K_2$ of $\mathrm{PGL}_2(\mathbf{k})$, an isomorphism $A/K_1 \simeq A/K_2$ always implies $K_1 \simeq K_2$ unless $A = \mathrm{D}_{2n}$; see Lemma 4.2.) □

**Lemma 5.10**. *Let $G \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ be a finite subgroup. Then, the following holds:*

1. *Assume there is a Sarkisov $G$-link $\chi \colon S \dashrightarrow T$ of type I as in (5.1). Then $G_\circ$ has a subgroup of the form $\mathrm{C}_n \times \mathrm{C}_m$ which is of index at most 2 in $G_\circ$.*
2. *Assume there exists a Sarkisov $G$-link $S \dashrightarrow \mathbb{P}^2$ of type II. Then $G_\circ$ is isomorphic to a direct product of at most two cyclic groups. In particular, the abelian group $G_\circ$ is generated by at most two elements.*

*Proof.* (1) Recall that the centre of $\chi$ is a $G$-point $\{p, q\}$ of degree 2. Therefore, the stabilizer $G' \subset G$ of $p$ is a normal subgroup of index 2 and $G'' = G_\circ \cap G'$ is of index at most 2 in $G_\circ$. Since $G''$ acts on $S$ fibrewise, the faithful representation $G'' \to \mathrm{GL}(T_pS)$ is reducible; *i.e.* it is a direct sum of two 1-dimensional representations. Hence $G'' \simeq \mathrm{C}_n \times \mathrm{C}_m$. In case (2), the group $G$ fixes a point on $S$, and the result follows similarly. □

We now proceed with determining for which groups $G \simeq (F \times_D F)_\bullet \mathrm{C}_2$ of Proposition 5.8 the surface $S = \mathbb{P}^1 \times \mathbb{P}^1$ is actually $G$-birationally rigid. In our analysis, we will often use Corollary 5.9 (including its notation) without explicitly mentioning it.

## 5.3. Case $F = \mathrm{C}_n$

Set $S = L_1 \times L_2$, where $L_1 \simeq L_2 \simeq \mathbb{P}^1$. Then $G_\circ \subseteq H_1 \times H_2$ and $H_1 \simeq H_2 \simeq \mathrm{C}_n$ are cyclic groups of order $n$, and $H_i$ faithfully acts on $L_i$. Recall that each $H_i$ fixes exactly two points on $L_i$. Hence, the group $G_\circ$ fixes four points on $S$. Let $p \in S$ be one of these fixed points. The group $G$ is a disjoint union of $G_\circ$ and $gG_\circ$, where $g \in G \setminus G_\circ$. Consider the set $\Omega = \{p, g(p)\}$. Since $g^2 \in G_\circ$, this set is $g$-invariant. Furthermore, since $g^{-1}G_\circ g = G_\circ$, the point $g(p)$ is fixed by $G_\circ$. Therefore, $\Omega$ is $G$-invariant. If $|\Omega| = 1$ (*i.e.* $g(p) = p$), then the stereographic projection from $p$ conjugates $G$ to a group acting on $\mathbb{P}^2$, so $S$ is not $G$-birationally rigid. Assume that $g(p) \neq p$. Then $g(p)$ and $p$ are in general position on $S$. Indeed, the coordinates $[x : y]$ on $L_1$ and $[z : t]$ on $L_2$ can be chosen so that the fixed points of $G_\circ$ are

$$([1:0],[1:0]),\ ([1:0],[0:1]),\ ([0:1],[1:0]),\ ([0:1],[0:1]).$$

The automorphism $g$ is of the form $([x:y],[z:t]) \mapsto (A[z:t], B[x:y])$, where $A, B \in \mathrm{PGL}_2(\mathbf{k})$. Assume that $g$ sends $([1:0],[1:0])$ onto $([1:0],[0:1])$, *i.e.* $A[1:0]=[1:0]$ and $B[1:0]=[0:1]$. Then, as discussed above, $g^2\colon ([x:y],[z:t]) \mapsto (AB[x:y], BA[z:t])$ fixes $([1:0],[1:0])$, which is impossible as $BA[1:0]=B[1:0]=[0:1]$. Similarly, $g$ cannot send $([1:0],[1:0])$ onto $([0:1],[1:0])$. We conclude that $p$ and $g(p)$ are not in a common fibre of either projection. Their blow-up gives a $G$-conic bundle $S' \to \mathbb{P}^1$ with two singular fibres; hence $S$ is not even $G$-solid.

*Remark* 5.11. As we already noticed in Remark 3.5, a cyclic group always has a fixed point on a quadric. Blowing this point up and contracting the strict transforms of the lines passing through it, we arrive at $\mathbb{P}^2$ (one often says that the group is *linearizable* in this case).

### 5.4. Case $F = \mathrm{D}_n$

Recall that $\mathrm{D}_2 \simeq \mathrm{V}_4 \simeq \mathrm{C}_2^2$, and set $\mathrm{D}_1 = \mathrm{C}_2$. Assume $n \geqslant 3$. Recall from Lemma 4.2 that proper normal subgroups of $\mathrm{D}_n$ are cyclic groups of order $n/d$ for each $d$ dividing $n$ (of index $2d$) and, if $n$ is even, dihedral of index 2. Therefore, we have the following two possibilities:

**5.4.1.** — The group $G_\circ$ fits into the short exact sequence

$$1 \longrightarrow \mathrm{D}_m \times \mathrm{D}_m \longrightarrow G_\circ \longrightarrow D \longrightarrow 1, \tag{5.4}$$

where either $D \simeq \mathrm{C}_2$ and $n = 2m$, $m \geqslant 2$, or $D = \mathrm{id}$ and $m = n \geqslant 3$.

**5.4.2.** — The group $G_\circ$ fits into the short exact sequence

$$1 \longrightarrow \mathrm{C}_m \times \mathrm{C}_m \longrightarrow G_\circ \longrightarrow \mathrm{D}_d \longrightarrow 1, \tag{5.5}$$

where $n = md$, $m \geqslant 1$. Note that this includes the extremal case $m = 1$, $G_\circ = \mathrm{D}_n$, $n \geqslant 3$.

**Proposition 5.12**. *In the notation from above, one has the following:*

1. *$S$ is $G$-birationally rigid in the case 5.4.1.*
2. *$S$ may fail to be $G$-birationally rigid in the case 5.4.2. If $\chi\colon S \dashrightarrow S'$ is a Sarkisov $G$-link to a different $G$-Mori fibre space, then either $S' = \mathbb{P}^2$, or $S'$ is a $G$-del Pezzo surface of degree 6 and $G \simeq \mathrm{D}_6$, or $S'$ is a $G$-del Pezzo surface of degree 5 and $G \simeq \mathrm{AGL}_1(\mathbb{F}_5)$, or $S'$ admits the structure of a $G$-conic bundle with two singular fibres.*

*Proof.* (1) First, there are no Sarkisov $G$-links of type I from $S$ or links to $\mathbb{P}^2$. Indeed, otherwise, $G_\circ$ has a subgroup $N_\circ \simeq \mathrm{C}_k \times \mathrm{C}_l$ of index at most two by Lemma 5.10. But then $N_\circ \cap (\mathrm{D}_m \times \mathrm{D}_m)$ must be an abelian group which can be generated by at most two elements and has index at most 2 in $\mathrm{D}_m \times \mathrm{D}_m$, which is not possible. Furthermore, $G_\circ$ obviously does not embed into the groups from (5.2). We are also able to show that there are no Sarkisov $G$-links of type II centred at a point of degree 4. Assume that it exists and is given by the diagram (3.1). Then $T$ is a del Pezzo surface of degree 4 in $\mathbb{P}^4$. Note that $|G| = 4n^2$ if $D \simeq \mathrm{C}_2$, and $|G| = 8n^2$ if $D = \mathrm{id}$. From Remark 5.3 we see that $G$ does not embed into $\mathrm{Aut}(T)$ unless $m = 2$; in particular, $G$ must contain a subgroup of the form $\Delta = \mathrm{C}_2^4$. It is easy to prove, see [Bea07, Lemma 3.1 and Proposition 3.11], that such a $\Delta$ is conjugate in $\mathrm{PGL}_5(\mathbf{k})$ to a subgroup of the diagonal torus; it acts on $T$ (which is an intersection of two quadrics in $\mathbb{P}^4$) by changing the signs of the ambient coordinates $x_k$ and consists of the projective transformations id, $\jmath_i$ for $i = 1, \ldots, 5$, $\jmath_i \circ \jmath_j$ for $1 \leqslant i < j \leqslant 5$, where $\jmath_k\colon x_k \mapsto -x_k$. The fixed-point locus of each $\jmath_k$ is an elliptic curve cut out on $T$ by the hyperplane $\{x_k = 0\}$, while the fixed-point loci of other non-trivial involutions in $\Delta$ consist of exactly four points. If $\mathrm{tr}(\delta)$ denotes the trace of the action of $\delta$ on $\mathrm{Pic}(T) \otimes \mathbb{C}$ and $\mathrm{Eu}(\cdot)$ denotes the topological Euler characteristic, then $\mathrm{Eu}(T^\delta) = \mathrm{tr}(\delta) + 2$ by the Lefschetz fixed-point formula (here $T^\delta$ denotes the fixed locus of $\delta$). Since $\mathrm{Eu}(T^{\jmath_k}) = 0$ and $\mathrm{Eu}(T^{\jmath_k \circ \jmath_l}) = 4$, we get $\mathrm{tr}(\jmath_k) = -2$ and $\mathrm{tr}(\jmath_k \circ \jmath_l) = 2$. Therefore, $\mathrm{rank}\,\mathrm{Pic}(T)^\Delta = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \mathrm{tr}(\delta) = 1$, which contradicts the existence of a Sarkisov link; see [DI09a, Section 6.1] or [Yas22, Section 2.1] for more details. We conclude that $S$ is $G$-birationally rigid.

(2) All possible Sarkisov $G$-links from $S$ were described at the beginning of the section; as we know, those centred at points of degrees 7, 6 and 4 lead to $G$-isomorphic del Pezzo surfaces. Suppose that $S'$ is a del Pezzo surface of degree 6. As $\chi^{-1}$ starts with blowing up a $G$-fixed point on $S'$, the group $G$ must be isomorphic to $\mathrm{C}_6$, $\mathrm{S}_3$ or $\mathrm{D}_6$, and hence $G_\circ$ is $\mathrm{C}_3$, $\mathrm{C}_6$ or $\mathrm{S}_3$. As we are in the setting of the exact sequence (5.5), we must have $G \simeq \mathrm{D}_6$. Next, if $S'$ is a del Pezzo surface of degree 5, then Lemma 3.6 and the exact sequence (5.5) show that we must have $m = 1$, $G_\circ \simeq \mathrm{D}_5$ and hence $G \simeq \mathrm{AGL}_1(\mathbb{F}_5)$. □

In fact, all possibilities described in Proposition 5.12(2) do occur. For $\mathrm{D}_6$-links to the del Pezzo surface of degree 6, this was already mentioned at the end of Section 4; see also [Isk08]. To construct $\mathrm{AGL}_1(\mathbb{F}_5)$-links to the del Pezzo surface of degree 5, recall that $\mathrm{AGL}_1(\mathbb{F}_5)$ has presentation $\langle \alpha, \beta \mid \alpha^5 = \beta^4 = \mathrm{id},\ \beta\alpha\beta^{-1} = \alpha^3 \rangle$. Let $\alpha \in \mathrm{Aut}(S') \simeq \mathrm{S}_5$ be an automorphism of order 5. It is easy to show, see [Yas16, Lemma 4.16], using the Lefschetz fixed-point formula, that $\alpha$ has exactly two fixed points, say $p$ and $q$, in general position on $S'$. Then $\alpha\beta \cdot p = \alpha^{-2}\beta\alpha \cdot p = \alpha^{-2}\beta \cdot p$, so $\beta \cdot p$ is a fixed point of $\alpha^3$ and hence of $\alpha$; *i.e.* $\beta \cdot p \in \{p, q\}$. The set $\{p, q\}$ is an orbit of $G$ in general position, and one can associate a link to it; see [Wol18, Theorem 1.1 and Lemma 4.2] for more details.

Next, let us provide an example of a link to a conic bundle and to $\mathbb{P}^2$.

*Example* 5.13. Assume that $m = 1$ and that $G_\circ = \mathrm{D}_n$ acts on $S = \mathbb{P}^1 \times \mathbb{P}^1$ "diagonally", *i.e.* by $([x:y], [z:t]) \mapsto (A[x:y], A[z:t])$, where $A \in \mathrm{PGL}_2(\mathbf{k})$ are elements of $\mathrm{D}_n$. Let us choose the coordinates so that the action of $\mathrm{D}_n = \langle r, s \colon r^n = s^2 = \mathrm{id}, srs = r^{-1} \rangle$ on each factor is given by $r\colon [x:y] \mapsto [x:\omega y]$, $s\colon [x:y] \mapsto [y:x]$, where $\omega$ denotes a primitive $n^{\mathrm{th}}$ root of unity. Consider two automorphisms of $S$

$$\tau_1\colon ([x:y],[z:t]) \longmapsto ([z:t],[x:y]),$$
$$\tau_2\colon ([x:y],[z:t]) \longmapsto ([t:z],[y:x]).$$

Then $\tau_1$ commutes with $G_\circ$, while $\tau_2$ defines a semidirect product $G_\circ \rtimes \langle \tau_2 \rangle$, where $\tau_2$ acts by the inversion of $r$ and preserves $s$. Note that $r$ has exactly two fixed points $[1:0]$ and $[0:1]$, which are permuted by $s$. Let

$$p_1 = ([1:0],[1:0]), \quad p_2 = ([1:0],[0:1]), \quad p_3 = ([0:1],[1:0]), \quad p_4 = ([0:1],[0:1]).$$

The sets $\Omega_1 = \{p_1, p_4\}$ and $\Omega_2 = \{p_2, p_3\}$ are invariant with respect to $G_\circ$, $\tau_1$ and $\tau_2$, and provide the orbits (in general position) for $G_1 = G_\circ \times \langle \tau_1 \rangle$ and $G_2 = G_\circ \rtimes \langle \tau_2 \rangle$. Blowing up these orbits gives a $G$-conic bundle $S' \to \mathbb{P}^1$ with two singular fibres (in fact, this is a del Pezzo surface of degree 6; see [Isk80, Theorem 5]).

Similarly, one can construct examples of links to $\mathbb{P}^2$. Note that if such a link exists, Lemma 3.4 implies that $G_\circ$ is an abelian group generated by at most two elements. In particular, $d \leqslant 2$ in (5.5). If $d = 1$, then $|G_\circ| = 2m^2$ and hence $G_\circ$ is an index 2 subgroup of $\mathrm{D}_m \times \mathrm{D}_m$, which is impossible. If $d = 2$, then $G_\circ$ is an index 4 subgroup therein and hence coincides with $\mathrm{C}_{2m} \times \mathrm{C}_{2m} \subset \mathrm{D}_{2m} \times \mathrm{D}_{2m}$. Making this group act on $\mathbb{P}^1 \times \mathbb{P}^1$ diagonally, as above, and taking a direct product with $\tau_1$, we get a linearizable action.

Finally, if $n = 2$, then $G_\circ = \mathrm{V}_4 \times_D \mathrm{V}_4 \subset \mathrm{V}_4 \times \mathrm{V}_4 \simeq \mathrm{C}_2^4$ and hence one has the following possibilities for $G_\circ$:

$$\mathrm{V}_4 \times \mathrm{V}_4 \ (D = \mathrm{id}), \quad (\mathrm{C}_2 \times \mathrm{C}_2) \times \mathrm{C}_2 \simeq \mathrm{C}_2^3 \ (D = \mathrm{C}_2), \quad \mathrm{V}_4 \ (D = \mathrm{V}_4).$$

When $G_\circ \simeq \mathrm{V}_4 \times \mathrm{V}_4 \simeq \mathrm{C}_2^4$, the same arguments as in the proof of Proposition 5.12(1) show that $S$ is $G$-birationally rigid. In the remaining two cases $D \simeq \mathrm{C}_2$ and $D \simeq \mathrm{V}_4$, one can construct examples similar to Example 5.13.

### 5.5. Case $F = \mathrm{A}_4$

If $K = \mathrm{A}_4$, then $G_\circ = \mathrm{A}_4 \times \mathrm{A}_4$. If $K = \mathrm{V}_4$, then we have a short exact sequence

$$1 \longrightarrow \mathrm{V}_4 \times \mathrm{V}_4 \longrightarrow G_\circ \longrightarrow \mathrm{C}_3 \longrightarrow 1, \tag{5.6}$$

while for $K = \mathrm{id}$ we simply get $G_\circ = \mathrm{A}_4$. Therefore, $G$ is of the form

$$(\mathrm{A}_4\times\mathrm{A}_4)_\bullet\mathrm{C}_2,\quad ((\mathrm{V}_4\times\mathrm{V}_4)_\bullet\mathrm{C}_3)_\bullet\mathrm{C}_2\quad\text{or}\quad \mathrm{A}_{4\bullet}\mathrm{C}_2. \tag{5.7}$$

Note that none of the groups $G_\circ$ admits a subgroup $\mathrm{C}_n\times\mathrm{C}_m$ of index at most 2; hence there are no Sarkisov $G$-links of type I on $S$ and no $G$-links leading to $\mathbb{P}^2$ by Lemma 5.10. Clearly, none of the extensions (5.7) is isomorphic to a group from the list (5.2). Since birational Geiser and Bertini involutions and links centred at points of degree 4 lead to a $G$-isomorphic surface, we get that $S$ is $G$-birationally rigid.

Although this is not necessary for our further purposes, let us proceed to explore the existence of $G$-links of type II at points of degree 4. If $G \simeq (\mathrm{A}_4\times\mathrm{A}_4)_\bullet\mathrm{C}_2$, then there are no such $G$-links as $|G|$ is divisible by 9 and hence $G$ does not embed into automorphism groups of del Pezzo surfaces of degree 4. If such a $G$-link existed for $G\simeq((\mathrm{C}_2^4)_\bullet\mathrm{C}_3)_\bullet\mathrm{C}_2$, then $G$ would embed into $\mathrm{Aut}(T)$, where $T$ is a quartic del Pezzo surface, and moreover $G = \mathrm{Aut}(T)\simeq \mathrm{C}_2^4\rtimes\mathrm{S}_3$. However, one has $\mathrm{Pic}(T)^G\simeq\mathbb{Z}$ in this case [DI09a, Theorem 6.9], which gives a contradiction.

Finally, if $K = \mathrm{id}$, the group $G_\circ = \mathrm{A}_4\times_{\mathrm{A}_4}\mathrm{A}_4\simeq\mathrm{A}_4$ acts on $S$ by

$$([x:y],[z:t])\longmapsto(g[x:y],\varphi(g)[z:t]),\quad g\in\mathrm{A}_4,$$

where $\varphi\in\mathrm{Aut}(\mathrm{A}_4)$ is a fixed automorphism. The extension $\mathrm{A}_{4\bullet}\mathrm{C}_2$ always splits, and one has $G\simeq\mathrm{A}_4\rtimes_\psi\langle\tau\rangle$, where $\tau\in G\setminus G_\circ$. The latter automorphism is of the form

$$\tau\colon([x:y],[z:t])\longmapsto(A[z:t],B[x:y]),$$

where $A,B\in\mathrm{PGL}_2(\mathbf{k})$. Since $\tau^2=\mathrm{id}$, we find that $B=A^{-1}$. Let us choose the coordinates on the first factor of $\mathbb{P}^1\times\mathbb{P}^1$ so that the derived subgroup $\mathrm{V}_4$ of $\mathrm{A}_4$ is generated by $[x:y]\mapsto[x:-y]$, $[x:y]\mapsto[y:x]$. A direct computation then shows that an element of order 3 is represented by one of the following matrices:

$$\begin{pmatrix} i & i\\ 1 & -1\end{pmatrix},\quad \begin{pmatrix} i & -i\\ 1 & 1\end{pmatrix},\quad \begin{pmatrix} -i & i\\ 1 & 1\end{pmatrix},\quad \begin{pmatrix} -i & -i\\ 1 & -1\end{pmatrix}. \tag{5.8}$$

Suppose that $\psi=\mathrm{id}$, so $G\simeq\mathrm{A}_4\times\langle\tau\rangle$. Below we give an example of $G$-actions which do not give rise to a Sarkisov link. Their systematic study will be provided elsewhere.

*Example* 5.14. Assume $\varphi=\mathrm{id}$, so the group $G_\circ$ acts on $\mathbb{P}^1\times\mathbb{P}^1$ "diagonally" by $([x:y],[z:t])\mapsto(g\cdot[x:y], g\cdot[z:t])$, where $g\in\mathrm{PGL}_2(\mathbf{k})$. Further, assume that $\tau$ is given by

$$([x:y],[z:t])\longmapsto([z:t],[x:y]). \tag{5.9}$$

Obviously, it commutes with $G_\circ$. If $\Omega = G\cdot p$ is an orbit of cardinality 4 on $S$, then the stabilizer of $p$ is a cyclic group $\langle\tau\rangle\times\langle\delta\rangle\simeq\mathrm{C}_6$, where $\delta\in G_\circ$ is an element of order 3. As the fixed locus of $\tau$ is the diagonal $\Delta\subset\mathbb{P}^1\times\mathbb{P}^1$, we have $p\in\Delta$. But $G_\circ$ preserves $\Delta$; hence we have $\Omega\subset\Delta$. So, the orbit $\Omega$ is not in general position on $S$, and hence there is no Sarkisov $G$-link starting from $\Omega$.

Now choose a non-identity automorphism $\varphi$; *e.g.* assume that $\varphi\in\mathrm{Aut}(\mathrm{A}_4)$ permutes the elements of $\mathrm{V}_4\subset\mathrm{A}_4$ such that $\mathrm{V}_4=\langle\alpha\rangle\times\langle\beta\rangle$ acts by

$$\alpha\colon([x:y],[z:t])\longmapsto([x:-y],[t:z]),\quad \beta\colon([x:y],[z:t])\longmapsto([y:x],[t:-z]).$$

Set $M=\left(\begin{smallmatrix} i & i\\ 1 & -1\end{smallmatrix}\right)$, and define an element of order 3 of $\mathrm{A}_4$ and $\tau$ as

$$\gamma\colon\left(\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}z\\t\end{bmatrix}\right)\longmapsto\left(M\begin{bmatrix}x\\y\end{bmatrix},M\begin{bmatrix}z\\t\end{bmatrix}\right),\quad \tau\colon\left(\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}z\\t\end{bmatrix}\right)\longmapsto\left(M\begin{bmatrix}z\\t\end{bmatrix},M^{-1}\begin{bmatrix}x\\y\end{bmatrix}\right),$$

respectively. One easily checks that $\alpha,\beta,\gamma$ generate the group $\langle\alpha,\beta,\gamma\colon\alpha^2=\beta^2=\gamma^3=\mathrm{id},\gamma\alpha\gamma^{-1}=\alpha\beta=\beta\alpha,\gamma\beta\gamma^{-1}=\alpha\rangle\simeq\mathrm{A}_4$ and $\tau$ commutes with this group. However, the automorphism

$$\left(\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}z\\t\end{bmatrix}\right)\longmapsto\left(\begin{bmatrix}x\\y\end{bmatrix},M\begin{bmatrix}z\\t\end{bmatrix}\right)$$

of $S$ conjugates the group $\mathrm{A}_4 \times \langle\tau\rangle$ to the one with "diagonal" action of $\mathrm{A}_4$ and $\tau$ acting as in (5.9).

### 5.6. Case $F = \mathrm{S}_4$

If $K = \mathrm{S}_4$, then $G_\circ = \mathrm{S}_4 \times \mathrm{S}_4$. If $K = \mathrm{id}$, then $G_\circ = \mathrm{S}_4$. In the remaining two cases $K = \mathrm{A}_4$ and $K = \mathrm{V}_4$, the group $G_\circ$ fits into the short exact sequence

$$1 \longrightarrow \mathrm{A}_4 \times \mathrm{A}_4 \longrightarrow G_\circ \longrightarrow \mathrm{C}_2 \longrightarrow 1 \tag{5.10}$$

or into the short exact sequence

$$1 \longrightarrow \mathrm{V}_4 \times \mathrm{V}_4 \longrightarrow G_\circ \longrightarrow \mathrm{S}_3 \longrightarrow 1. \tag{5.11}$$

By Lemma 5.10, there are no Sarkisov $G$-links to $\mathbb{P}^2$ and no Sarkisov $G$-links of type I starting from $S$. Furthermore, none of the extensions

$$(\mathrm{S}_4 \times \mathrm{S}_4)_\bullet\mathrm{C}_2, \quad ((\mathrm{A}_4 \times \mathrm{A}_4)_\bullet\mathrm{C}_2)_\bullet\mathrm{C}_2, \quad ((\mathrm{V}_4 \times \mathrm{V}_4)_\bullet\mathrm{S}_3)_\bullet\mathrm{C}_2, \quad \mathrm{S}_{4\bullet}\mathrm{C}_2 \tag{5.12}$$

of $\mathrm{C}_2$ by $G_\circ$ is isomorphic to a group from the list (5.2). Note that there are no Sarkisov $G$-links of type II starting at a point of degree 4. Indeed, otherwise, $G$ has a subgroup of index 4 which fixes a point on $S$, and hence $G_\circ$ has an abelian subgroup of the form $\mathrm{C}_n \times \mathrm{C}_m$ and of index 2 or 4, which is clearly not the case (alternatively, one can again argue using Remark 5.3). We conclude that $S$ is $G$-birationally rigid.

### 5.7. Case $F = \mathrm{A}_5$

Since $\mathrm{A}_5$ is simple, one has $K = \mathrm{id}$ or $K = \mathrm{A}_5$, so either $G \simeq (\mathrm{A}_5 \times_{\mathrm{A}_5} \mathrm{A}_5)_\bullet\mathrm{C}_2 \simeq \mathrm{A}_{5\bullet}\mathrm{C}_2$ or $G \simeq (\mathrm{A}_5 \times \mathrm{A}_5)_\bullet\mathrm{C}_2$. Clearly, none of these groups is isomorphic to a group from (5.2), unless in the first extension we get $G \simeq \mathrm{S}_5$. However, by Lemma 3.4, there exists(2) no Sarkisov $\mathrm{S}_5$-link $\chi\colon S \dashrightarrow S'$ to a del Pezzo surface $S'$ of degree 5, as the centre of $(\eta')^{-1}$ in (3.1) would be a point of degree 2. Further, none of the groups $G_\circ = \mathrm{A}_5$ or $G_\circ = \mathrm{A}_5 \times \mathrm{A}_5$ has a subgroup of the form $\mathrm{C}_n \times \mathrm{C}_m$ of index at most 2 in $G_\circ$, so there are no $G$-links of type I and no $G$-links to $\mathbb{P}^2$ by Lemma 5.10. By Remark 5.3, $S$ does not admit $G$-links centred at a point of degree 4. Similarly, $G$ does not embed into automorphism group of del Pezzo surfaces of degree 1 or 2; see [Dol12, Sections 8.7 and 8.8]. We conclude that $S$ is $G$-birationally superrigid.

*Remark* 5.15. The results of previous sections allow us to complete the proof of the main theorem in an alternative way. Indeed, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ may fail to be $G$-birationally rigid exactly in the following cases:

(a) $G = (\mathrm{C}_n \times_D \mathrm{C}_n)_\bullet\mathrm{C}_2$ is a group from Section 5.3 and $G_\circ = \mathrm{C}_n \times_D \mathrm{C}_n$.
(b) $G = (\mathrm{D}_n \times_D \mathrm{D}_n)_\bullet\mathrm{C}_2$ is a group from Section 5.4, where $G_\circ$ is as in the case 5.4.2.
(c) $G = (\mathrm{C}_2^3)_\bullet\mathrm{C}_2$ or $G = \mathrm{V}_{4\bullet}\mathrm{C}_2$ are groups from Section 5.4.

The groups $G_\circ$ from (a) are abelian and generated by at most two elements, hence cannot contain any of the groups $G_\circ$ of Sections 5.4–5.7 unless $G_\circ = \mathrm{V}_4$ is as in (c). However, in the latter case, $S$ is not $G$-birationally rigid. The groups $G_\circ$ from Sections 5.4–5.7 cannot embed into the groups $G_\circ$ from (c) by order reasons.

Suppose that $G_\circ$ is a group from (b). Then there exists a Sarkisov $G$-link $\chi\colon S \dashrightarrow S'$ leading to a different $G$-Mori fibre space $S' \to Z$, where $Z = \mathrm{pt}$ or $Z = \mathbb{P}^1$; all possibilities are given by Proposition 5.12. If $S' = \mathbb{P}^2$, then $G$ fixes a point on $S$, and hence the same is true for any subgroup $H \subset G$, showing that $H$ is linearizable as well (see Remark 5.11). If $S' \to \mathbb{P}^1$ is a $G$-conic bundle, then $G$ has an orbit of size 2 on $S$, and hence $H$ has an orbit of size 1 or 2; the result follows. If $S'$ is a $G$-del Pezzo surface of degree 6, then $G \simeq \mathrm{D}_6$ and $G_\circ \simeq \mathrm{D}_3$. Then $H_\circ \in \{\mathrm{id}, \mathrm{C}_2, \mathrm{C}_3\}$ and $S$ is not $H$-birationally rigid. Finally, if $S'$ is a del Pezzo surface of degree 5, then $G \simeq \mathrm{AGL}_1(\mathbb{F}_5)$ and $G_\circ \simeq \mathrm{D}_5$; hence, $H_\circ \in \{\mathrm{id}, \mathrm{C}_2, \mathrm{C}_5\}$ and once again $S$ fails to be $H$-birationally rigid.

(2) Alternatively, if such link existed, then the surface $T$ in the diagram (3.1) is a cubic surface with an action of $\mathrm{S}_5$. It is well known that $T$ must be the Clebsch diagonal cubic. One always has $\mathrm{Pic}(T)^{\mathrm{S}_5} \simeq \mathbb{Z}$; see [DI09a, Theorem 6.14].

# 6. The projective space of dimension 3 and the "mixed" arithmetico-geometric case

## 6.1. $G$-birational rigidity of $\mathbb{P}^3$

Inspired by Blichfeldt's classification [Bli17] of finite subgroups of $\mathrm{PGL}_4(\mathbb{C})$, I. Cheltsov and C. Shramov managed to describe all the cases in which $\mathbb{P}^3_{\mathbb{C}}$ is $G$-birationally rigid.

**Theorem 6.1** (*cf.* [CS19, Theorem 1.1]). *The 3-dimensional complex projective space $\mathbb{P}^3$ is $G$-birationally rigid if and only if $G$ is primitive and not isomorphic to $\mathrm{A}_5$ and $\mathrm{S}_5$.*

This immediately implies a positive answer to Cheltsov–Kollár's question for $X = \mathbb{P}^3$.

**Corollary 6.2.** *Let $G$ be a finite group and $H \subset G$ be its subgroup. If $\mathbb{P}^3$ is $H$-birationally rigid, then $\mathbb{P}^3$ is $G$-birationally rigid.*

*Proof.* Recall (see Definition 3.8) that finite subgroups of $\mathrm{Aut}(\mathbb{P}^3) \simeq \mathrm{PGL}_4(\mathbb{C})$ are either transitive (*i.e.* do not fix any point and do not leave any line invariant) or intransitive. Transitive groups are either imprimitive (*i.e.* leave a union of two skew lines invariant or have an orbit of length 4) or primitive.

Now, if $\mathbb{P}^3$ is $H$-birationally rigid, then $H$ is transitive, and hence $G$ is transitive as well. Clearly, if $G$ leaves a union of two skew lines invariant, then the same is true for $H$. If $G$ has an orbit of length 4 and $H$ has no orbit of length 4, then $H$ either fixes a point, or permutes two points in $\mathbb{P}^3$ and hence has an invariant line. Both cases are not possible as $H$ is transitive. So, $G$ is primitive. It remains to notice that $G$ is not isomorphic to $\mathrm{A}_5$ or $\mathrm{S}_5$ as all proper subgroups of $\mathrm{S}_5$, not isomorphic to $\mathrm{A}_5$, are not primitive; see *e.g.* [CS19, Appendix]. □

## 6.2. Cheltsov–Kollár's question in the arithmetico-geometric case

Assume that the base field $\mathbf{k}$ is not algebraically closed. A natural generalization of Cheltsov–Kollár's question to the "mixed" setting of Remark 1.1 would be: if $X$ is an $H$-birationally rigid $H$-Fano variety over $\overline{\mathbf{k}}$, then must $X$ be $G$-birationally rigid over $\mathbf{k}$? We claim that the answer to this question is *negative* already for $H = G$.

Let $p \equiv 1 \pmod 3$ be a prime number, fix a non-trivial homomorphism $\mathrm{C}_3 \to \mathrm{Aut}(\mathrm{C}_p)$, and consider the corresponding semidirect product $G = \mathrm{C}_p \rtimes \mathrm{C}_3$. By the result of C. Shramov [Shr21, Theorem 1.3(ii)], there exist a field $\mathbf{k}$ (of characteristic zero) and a non-trivial Severi–Brauer surface $S$ over $\mathbf{k}$ such that $G \hookrightarrow \mathrm{Aut}(S)$. Any Sarkisov $G$-link $\chi\colon S \dashrightarrow S'$ centred at a point of degree 3 leads to the *opposite* Severi–Brauer surface $S' = S^{\mathrm{op}}$; if $S$ corresponds to a central simple $\mathbf{k}$-algebra $A$, then by definition, $S^{\mathrm{op}}$ is the unique Severi–Brauer surface corresponding to $A^{\mathrm{op}}$, the inverse of $A$ in the Brauer group $\mathrm{Br}(\mathbf{k})$. Since $S^{\mathrm{op}}$ is never isomorphic to $S$, we conclude that $S$ is not $G$-birationally rigid. However, passing to the algebraic closure of $\mathbf{k}$, one has $S_{\overline{\mathbf{k}}} \simeq \mathbb{P}^2_{\overline{\mathbf{k}}}$, which is $G$-birationally rigid by Sakovics' Theorem 3.9. For more recent results on birational and biregular self-maps of Severi–Brauer surfaces, see [Shr20, Tre21, BSY22].

# References


[AO18] H. Ahmadinezhad and T. Okada, *Birationally rigid Pfaffian Fano 3-folds*, Algebr. Geom. **5** (2018), no. 2, 160–199, doi:10.14231/AG-2018-006.

[Avi20] A. A. Avilov, *Forms of the Segre cubic*, Math. Notes **107** (2020), no. 1, 3–9, doi:10.1134/S0001434620010010.

[Bea07] A. Beauville, *$p$-elementary subgroups of the Cremona group*, J. Algebra **314** (2007), no. 2, 553–564, doi:10.1016/j.jalgebra.2005.07.040.

[BB73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. Math. (2) **98** (1973), no. 3, 480–497, doi:10.2307/1970915.

[BLZ21] J. Blanc, S. Lamy and S. Zimmermann, *Quotients of higher dimensional Cremona groups*, Acta Math. **226** (2021), no. 2, 211–318, doi:10.4310/acta.2021.v226.n2.a1.

[BSY22] J. Blanc, J. Schneider and E. Yasinsky, *Birational maps of Severi-Brauer surfaces, with applications to Cremona groups of higher rank*, preprint arXiv:2211.17123 (2022).

[Bli17] H. F. Blichfeldt, *Finite Collineation Groups*, University of Chicago Press, 1917.

[Che08] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), no. 4, 1118–1144, doi:10.1007/s00039-008-0687-2.

[CMYZ22] I. Cheltsov, F. Mangolte, E. Yasinsky and S. Zimmermann, *Birational involutions of the real projective plane*, preprint arXiv:2208.00217 (2022).

[CS22] I. Cheltsov and A. Sarikyan, *Equivariant pliability of the projective space*, preprint arXiv:2202.09319 (2022).

[CS16] I. Cheltsov and C. Shramov, *Cremona groups and the icosahedron*, Monogr. Res. Notes Math., CRC Press, Boca Raton, FL, 2016, doi:10.1201/b18980.

[CS19] ———, *Finite collineation groups and birational rigidity*, Sel. Math., New Ser. **25** (2019), no. 5, Paper No. 71, doi:10.1007/s00029-019-0516-5.

[Cor00] A. Corti, *Singularities of linear systems and 3-fold birational geometry*, In: *Explicit birational geometry of 3-folds*, pp. 259–312, Cambridge Univ. Press, Cambridge, 2000, doi:10.1017/CBO9780511758942.007.

[Dol12] I. V. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge Univ. Press, Cambridge, 2012.

[DD16] I. Dolgachev and A. Duncan, *Fixed points of a finite subgroup of the plane Cremona group*, Algebr. Geom. **3** (2016), no. 4, 441–460, doi:10.14231/AG-2016-021.

[DI09a] I. V. Dolgachev and V. A. Iskovskikh, *Finite subgroups of the plane Cremona group*, In: *Algebra, arithmetic, and geometry. In honor of Yu. I. Manin. Vol. I*, pp. 443–548, Progr. Math. vol. 269, Birkhäuser, Boston, MA, 2009, doi:10.1007/978-0-8176-4745-2_11.

[DI09b] ———, *On elements of prime order in the plane Cremona group over a perfect field*, Int. Math. Res. Not. (2009), no. 18, 3467–3485, doi:10.1093/imrp/rnp061.

[dDM19] L. das Dores and M. Mauri, *$G$-birational superrigidity of Del Pezzo surfaces of degree 2 and 3*, Eur. J. Math. **5** (2019), no. 3, 798–827, doi:10.1007/s40879-018-0298-x.

[Flo20] E. Floris, *A note on the $G$-Sarkisov program*, Enseign. Math. (2) **66** (2020), no. 1-2, 83–92, doi:10.4171/LEM/66-1/2-5.

[FZ24] E. Floris and S. Zikas, *Umemura quadric fibrations and maximal subgroups of $Cr_n(\mathbb{C})$*, preprint arXiv:2402.05021 (2024).

[Gou89] E. Goursat, *Sur les substitutions orthogonales et les divisions régulières de l'espace*, Ann. Sci. Éc. Norm. Supér. (3) **6** (1889), 9–102, 1889, doi:10.24033/asens.317.

[Hos96] T. Hosoh, *Automorphism groups of quartic del Pezzo surfaces*, J. Algebra **185** (1996), no. 2, 374–389, doi:10.1006/jabr.1996.0331.

[Isk80] V. A. Iskovskikh, *Minimal models of rational surfaces over arbitrary fields*, Math. USSR, Izv. **14** (1980), no. 1, 17–39, doi:10.1070/IM1980v014n01ABEH001064.

[Isk96] ———, *Factorization of birational maps of rational surfaces from the viewpoint of Mori theory*, Russ. Math. Surv. **51** (1996), no. 4, 585–652, doi:10.1070/RM1996v051n04ABEH002962.

[Isk08] ———, *Two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group. II*, In: *Algebraic geometry in East Asia—Hanoi 2005*, pp. 251–267, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, doi:10.2969/aspm/05010251.

[Kle19] F. Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree* (translated by G. G. Morrice, with a new introduction and commentaries by P. Slodowy, translated by L. Yang, reprint of the English translation of the 1884 German original edition), CTM. Class. Top. Math., vol. 5, Higher Education Press, Beijing, 2019.

[Kol09] J. Kollár, *Birational rigidity of Fano varieties and field extensions*, Proc. Steklov Inst. Math. **264** (2009), no. 1, 96–101, doi:10.1134/S008154380901012X.

[LS21] S. Lamy and J. Schneider, *Generating the plane Cremona groups by involutions*, Algebr. Geom. **11** (2024), no. 1, 111-–162, doi:10.14231/ag-2024-004.

[LZ20] S. Lamy and S. Zimmermann, *Signature morphisms from the Cremona group over a non-closed field*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 10, 3133–3173, doi:10.4171/JEMS/983.

[Man66] Ju. I. Manin, *Rational surfaces over perfect fields*, Publ. Math. Inst. Hautes Étud. Sci. **30** (1966), 55–97 (Russian), doi:10.1007/BF02684356.

[Pin24] A. Pinardin, *$G$-solid rational surfaces*, Eur. J. Math. **10** (2024), no. 2, Paper No. 33, doi:10.1007/s40879-024-00747-z.

[Pro21] Y. G. Prokhorov, *Equivariant minimal model program*, Russ. Math. Surv. **76** (2021), no. 3, 461–542, doi:10.1070/RM9990.

[RZ18] M. F. Robayo and S. Zimmermann, *Infinite algebraic subgroups of the real Cremona group*, Osaka J. Math. **55** (2018), no. 4, 681–712.

[Sak19] D. Sakovics, *$G$-birational rigidity of the projective plane*, Eur. J. Math. **5** (2019), no. 3, 1090–1105, doi:10.1007/s40879-018-0261-x.

[SZ21] J. Schneider and S. Zimmermann, *Algebraic subgroups of the plane Cremona group over a perfect field*, Épijournal de Géom. Algébr. **5** (2021), Art. 14, doi:10.46298/epiga.2021.6715.

[Shr20] C. A. Shramov, *Birational automorphisms of Severi-Brauer surfaces*, Sb. Math. **211** (2020), no. 3, 466–480, doi:10.1070/SM9304.

[Shr21] ———, *Finite groups acting on Severi-Brauer surfaces*, Eur. J. Math. **7** (2021), no. 2, 591–612, doi:10.1007/s40879-020-00448-3.

[Tre16] A. Trepalin, *Quotients of conic bundles*, Transform. Groups **21** (2016), no. 1, 275–295, doi:10.1007/s00031-015-9342-9.

[Tre18] ———, *Quotients of del Pezzo surfaces of high degree*, Trans. Amer. Math. Soc. **370** (2018), no. 9, 6097–6124, doi:10.1090/tran/7130.

[Tre19] ———, *Quotients of del Pezzo surfaces*, Int. J. Math. **30** (2019), no. 12, 1950068, doi:10.1142/S0129167X1950068X.

[Tre21] ———, *Quotients of Severi-Brauer surfaces*, Dokl. Math. **104** (2021), no. 3, 390–393, doi:10.1134/S106456242106017X.

[Tre23] ———, *Birational classification of pointless del Pezzo surfaces of degree 8*, Eur. J. Math. **9** (2023), no. 1, Paper No. 2, doi:10.1007/s40879-023-00591-7.

[Wol18] J. Wolter, *Equivariant birational geometry of quintic del Pezzo surface*, Eur. J. Math. **4** (2018), no. 3, 1278–1292, doi:10.1007/s40879-018-0272-7.

[Yas16] E. Yasinsky, *Subgroups of odd order in the real plane Cremona group*, J. Algebra **461** (2016), 87–120, doi:10.1016/j.jalgebra.2016.04.019.

[Yas22] ———, *Automorphisms of real del Pezzo surfaces and the real plane Cremona group*, Ann. Inst. Fourier **72** (2022), no. 2, 831–899, doi:10.5802/aif.3460.