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Terminalizations of quotients of compact hyperkähler manifolds by induced symplectic automorphisms

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Abstract. Terminalizations of symplectic quotients are sources of new deformation types of irreducible symplectic varieties. We classify all terminalizations of quotients of Hilbert schemes of K3 surfaces or of generalized Kummer varieties, by finite groups of symplectic automorphisms induced from the underlying K3 or abelian surface. We determine their second Betti number and the fundamental group of their regular locus. In the Kummer case, we prove that the terminalizations have quotient singularities and determine the singularities of their universal quasi-étale cover. In particular, we obtain at least eight new deformation types of irreducible symplectic varieties of dimension 4. Finally, we compare our deformation types with those in papers by Fu–Menet and by Menet. The smooth terminalizations are only three and of $K3^{[n]}$ type, and surprisingly they all appeared in different places in the literature.

Keywords. Irreducible symplectic varieties, hyperkähler manifolds, symplectic automorphisms, terminalizations, singularities, Betti numbers

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Contents

1. Introduction

1.1. Irreducible symplectic varieties

Irreducible symplectic varieties play a key role in the classification of varieties with Kodaira dimension zero. In the last decades, fundamental results about their birational geometry, algebraic cycles and moduli theory have been proved; see for instance https://www.erc-hyperk.org/papers for a list of the latest advances in the field. The importance of irreducible symplectic varieties rests on the celebrated Beauville–Bogomolov decomposition, proved in increasing degree of generality in [Bea83, GKKP11, DG18, Dru18, Gue16, GGK19, HP19, Cam21, BGL22]: Any compact Kähler space with numerically trivial canonical class and klt singularities admits a quasi-étale cover[(1)] which can be written as the product of complex tori, strict Calabi–Yau varieties or irreducible symplectic varieties.

It is expected that the number of deformation types of irreducible symplectic varieties is finite in each dimension; see Remark 3.13. Therefore, it is natural to ask whether it is possible to even classify irreducible symplectic varieties, at least in low dimension. Despite active research in the field, irreducible symplectic varieties (especially smooth ones) are notoriously difficult to construct. At the moment, in the smooth case, there are in each dimension at most three known deformation types of irreducible symplectic manifolds, see [Bea83, O'G03, O'G99], namely those of

- Hilbert schemes $S^{[n]}$ of n points on a K3 surface S,
- generalized Kummer varieties $\mathrm{K}_n(A)$ associated to an abelian surface A,

[(1)]A quasi-étale cover is a finite morphism étale in codimension 1.

- two sporadic examples built by O'Grady in dimensions 6 and 10.

Dropping the smoothness assumption, we can generate more examples. For instance, there are, in [Men22] alone, at least 29 distinct deformation types of singular 4-dimensional irreducible symplectic orbifolds. The implicit hope is that while studying singular symplectic varieties, one may find some of them admitting a symplectic resolution, so ideally new smooth examples. Historically this is indeed how the O'Grady examples in dimensions 6 and 10 were discovered.

All known deformation types of irreducible symplectic varieties arise in the following ways:

- moduli spaces of semistable sheaves on K3 or abelian surfaces [PR23],
- compactifications of Lagrangian fibrations, see [MT07, ASF15, Mat16, SS22, BCG+24, LLX24],
- terminalizations $p\colon Y \to X/G$ of symplectic quotients of a symplectic variety X by a finite group G, see [Fuj83, FM21, Men22],

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow{\;p\;} & X/G. \end{array}$$

See also the survey [Per20].

The purpose of this paper is to study systematically terminalizations of quotients of known irreducible symplectic manifolds. In particular, we complete part of the classification program designed by Menet in [Men22, Section 1.3].

1.2. Criteria for an efficient classification of terminalizations

For a sensible and efficient description of the terminalizations above, some reductions and assumptions are in order. We first propose to restrict to the case of

projective $\mathbb{Q}$-factorial terminalizations Y of symplectic quotients X/G
with simply connected regular locus Y^{reg}.(2)

Although the combination of quotients and birational modifications of X/G is a source of many more irreducible symplectic varieties, they should be considered redundant as we explain in Section 4. Concretely, the reduction above requires that the candidate G-actions satisfy the following conditions; see Section 3.1, Section 3.2 and Proposition 8.1 for the equivalence.

Assumption 1.1. The following equivalent conditions hold:

(1) X/G has strictly canonical singularities.
(2) The singular locus of X/G has codimension 2.
(3) An element of G fixes a codimension 2 subvariety in X.(3)

Assumption 1.2. The finite group G acts on X in such a way that the automorphisms whose fixed locus in X has codimension two generate the entire group G.

In this paper, we study the case of X being a known irreducible symplectic manifold. In view of Assumption 1.1, we can rule out the case of manifolds of O'Grady type as explained in Remark 3.19. Without loss of generality, we can then restrict to the case of Hilbert schemes or generalized Kummer varieties.

Finite groups of symplectic automorphisms of them have been extensively studied in the literature; see Remark 3.19. However, the lattice-theoretic information of these classifications seems insufficient to prescribe

(2)Some authors call irreducible symplectic varieties with quotient singularities and simply connected regular locus *irreducible orbifolds*. We avoid this convention as it competes with the now well-established definition of irreducible symplectic varieties and it may cause confusion: An irreducible symplectic orbifold whose regular locus has nontrivial fundamental group would not be an irreducible orbifold!

(3)If X has $\mathbb{Q}$-factorial singularities, Assumption 1.1 is equivalent to the following condition:

(4) The $\mathbb{Q}$-factorial terminalization of X/G is a nontrivial morphism.

the geometry and the intersection theory of the fixed loci, and ultimately the geometry and singularities of Y. In order to maintain control over the geometry of the fixed loci, in this paper we assume the following.

Assumption 1.3. The finite group G acts on $S^{[n]}$ or $\mathrm{K}_n(A)$ via symplectic automorphisms induced by automorphisms of the underlying K3 or abelian surface S or A, respectively.

While Assumptions 1.1 and 1.2 are necessary to obtain an efficient classification (and should be required even in future works on the subject), Assumption 1.3 should be considered primarily as a technical requirement. Indeed, not all symplectic automorphisms with fixed loci of codimension 2 (so satisfying Assumption 1.1) are induced. Consider for instance the example of a non-induced automorphism of order 3 on a variety of $\mathrm{K3}^{[2]}$ type in [Nam01a, Example 17(iv)]; *cf.* also [Kaw09, Section 3].

There are certainly other classes of automorphisms whose fixed loci may be controlled effectively. For example, to also keep into account the Namikawa–Kawatani automorphism above, it would be interesting to also classify quotients of Fano varieties of lines on cubic fourfolds induced by automorphisms of the underlying cubic fourfold, or automorphisms of EPW sextics, or the more challenging automorphisms of moduli spaces of semistable sheaves induced by automorphisms of the underlying surface. We plan to tackle some of these other cases in the near future and include them in the classification program of [Men22, Section 1.3].

1.3. Classification results

We first show that the geometric Assumptions 1.1 and 1.3 impose group-theoretic constraints on G and on the dimension of X.

Theorem 1.4. *Let G be a finite group of induced symplectic automorphisms acting on $X \simeq S^{[m]}$ or $\mathrm{K}_n(A)$. Then X/G has strictly canonical singularities if and only if one of the following holds:*

- *$m = 2$ or $n = 2$, and G contains an involution.*
- *$n = 2$, and G contains a special type of automorphisms of order 3 as in Lemma 5.6(2).*
- *$n = 3$, and G contains a special type of involutions as in Lemma 5.6(1).*

In particular, X is isomorphic to $S^{[2]}$, $\mathrm{K}_2(A)$ or $\mathrm{K}_3(A)$.

Proof. This follows from Lemmas 5.4 and 5.6. □

Theorem 1.5 (*cf.* Corollary 5.9). *Away from the dissident locus (see Definition 3.6), a terminalization of X/G as in Theorem 1.4 is isomorphic to the blowup of the reduced singular locus.*

It is open whether Theorem 1.5 holds unconditionally without Assumption 1.3.

Theorem 1.6 (Second and third Betti numbers, *cf.* Proposition 6.1, Remark 10.5 and Proposition 7.1). *Let G be a finite group of induced symplectic automorphisms of $X = S^{[n]}$ or $\mathrm{K}_n(A)$. Let $q\colon X \to X/G$ be the quotient map, $p\colon Y \to X/G$ be a terminalization of X/G, and Σ be the singular locus of X/G. Denote by*

- *$F_g \subset X$ the (unique) component of the fixed locus of $g \in G$ of codimension 2, if any,*
- *L a lattice isomorphic to $H^2(X,\mathbb{Z})$,*
- *N_2 the number of components $q(F_g)$ in Σ with $\mathrm{ord}(g) = 2$,*
- *N_3 the number of components $q(F_g)$ in Σ with $\mathrm{ord}(g) = 3$.*

Then the following topological identities hold:

$$b_2(Y) = \mathrm{rk}\left(L^G\right) + N_2 + 2N_3 - \epsilon,$$
$$IH^3(Y,\mathbb{Q}) \simeq H^3(X,\mathbb{Q})^G,$$

where $IH^*(Y,\mathbb{Q})$ stands for the intersection cohomology of Y with rational coefficients, and ϵ equals 1 if $X = K_2(A)$ and $G_\circ \simeq BD_{12}$ (cf. Section 2), or 0 otherwise.

Theorem 1.7 (*cf.* Tables 4, 7 and 8). *For any action of a finite group of symplectic automorphisms of* $X \simeq S^{[2]}, \mathrm{K}_2(A)$ *or* $\mathrm{K}_3(A)$ *induced by the underlying K3 or abelian surface, the second Betti number and fundamental group of the regular locus of a projective terminalization* Y *of the quotient* X/G *are listed in Tables 4, 7 and 8.*

If $X \simeq S^{[2]}$, the topological invariants of Y depend only on the abstract isomorphism type of G, while in the Kummer case, they rely on the actual action of the group G and neither on the abstract isomorphism type nor on the induced action in cohomology; see Example 10.2. In any case, we find a group-theoretic description of these topological invariants depending on the embedding of G in the automorphism group of the underlying surface; see Proposition 6.4 and Corollary 8.4.

Theorem 1.8 (*cf.* Theorem 9.5). *Let* G *be a finite group of induced symplectic automorphisms acting on* $S^{[2]}$ *and* Y *a projective terminalization of* $S^{[2]}/G$ *with simply connected regular locus. There are at least five new deformation classes of such irreducible symplectic varieties* Y. *In particular, they are not deformation equivalent to any terminalization of quotients of Kummer fourfolds by groups of induced symplectic automorphisms, or a Fujiki fourfolds appearing in* [Men22, Theorem 1.11] (*cf. Definition 12.2*).

Theorem 1.9 (*cf.* Table 9). *Let* G *be a finite group of induced symplectic automorphisms acting on* $\mathrm{K}_2(A)$ *and* Y *a projective terminalization of* $\mathrm{K}_2(A)/G$ *with simply connected regular locus. The Betti numbers, Chern classes and singularities of* Y *are listed in Table 9.*

In particular, there exist at least three new deformation classes of irreducible symplectic orbifolds of dimension 4. *All other terminalizations are deformation equivalent to Fujiki varieties, with the exception of* $G = C_2$ (*called a Nikulin orbifold*) *and possibly* $G = BT_{24}$.

Theorem 1.10 (*cf.* Table 10 and Lemma 12.9). *Let* G *be a finite group of induced symplectic automorphisms acting on* $\mathrm{K}_3(A)$ *and* Y *a projective terminalization of* $\mathrm{K}_3(A)/G$ *with simply connected regular locus. The second Betti number and the singularities of* Y *are listed in Table 10. In particular,* Y *is deformation equivalent to one of the Fujiki sixfolds appearing in* [Men22, Section 6].

Corollary 1.11 (*cf.* Corollary 10.8). *Any projective terminalization of a quotient of* $\mathrm{K}_2(A)$ *or* $\mathrm{K}_3(A)$ *by a finite group of induced symplectic automorphisms has quotient singularities.*

If instead $X \simeq S^{[2]}$, the configurations of the singularities and topological invariants have already been studied in [Men22] for so-called *admissible* symplectic groups. We have been informed that Menet is working on non-admissible group actions.

It is natural to ask whether some of the previous terminalizations are smooth. We show that this happens only in three cases, and quite surprisingly they already appeared in the literature, scattered over three different places.

Theorem 1.12 (Smooth terminalizations). *Let* G *be a nontrivial finite group of induced symplectic automorphisms of* $X = S^{[n]}$ *or* $\mathrm{K}_n(A)$. *The quotient* X/G *admits a smooth terminalization if and only if*

(1) $X = S^{[2]}$ *and* $G \simeq C_2^4$, *see* [Fuj83, Proposition 14.5],

(2) $X = \mathrm{K}_2(A)$ *and* $G \simeq C_3^3$ *acting nontrivially on* $H^2(A, \mathbb{Z})$, *see* [Kaw09, Theorem 4.2],

(3) $X = \mathrm{K}_3(A)$ *and* $G \simeq C_2^5$, *see* [Flo24, Theorem 1.1].

All three terminalizations are birational to an irreducible symplectic manifold of $\mathrm{K3}^{[n]}$ *type.*

Proof. This follows by direct inspection of our tables; see Proposition 9.1, Table 9 and Proposition 10.7. See also Remark 10.3. □

1.4. Second Betti numbers

The study of terminalizations of quotients of symplectic manifolds goes back to the work of Fujiki. Nowadays, Fujiki varieties are terminalizations of certain quotients of squares of K3 surfaces; see Definition 12.2.

Their classification was initiated by Fujiki in [Fuj83] and recently completed in [Men22]. Terminalizations of cyclic quotients of $\mathrm{K}_2(A)$ and $S^{[2]}$ have also been studied by Fu and Menet in [FM21]. There, their interest was not to provide an exhaustive classification of all possible terminalizations arising, but rather to realize examples of irreducible symplectic fourfolds with different second Betti numbers.

In the following table, we compare the second Betti numbers of irreducible symplectic fourfolds constructed in [FM21], [Men22] and in the present paper.

b_2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
[FM21]	○		○	○	●	●		●	○			●		●							●
[Men22]		●	●	●	●	●		●	●			●		●							●
Present paper			●	●	●	●		●	●			●		●							●

Table 1. The first row lists all possible second Betti numbers of irreducible symplectic fourfolds. A circle in the table corresponds to known examples of such fourfolds: The column gives their second Betti number, and the row indicates a reference in the literature where the examples appear. Black circles correspond to examples with simply connected regular locus. White circles mean that the regular locus of all examples with a fixed Betti number (column) in a given reference (row) is not simply connected.

Observing the table, it is natural to ask the following:

(1) *Is there an irreducible symplectic variety Y with $b_2(Y) = 3$ and $\pi_1(Y^{\mathrm{reg}}) = 1$?* A nontrivial terminalization of a quotient of a symplectic variety will always have at least $b_2 \geq 4$. In fact, the example of [FM21] with $b_2 = 3$ is a quotient of a variety of $\mathrm{K3}^{[2]}$ type by an automorphism of order 11, but its regular locus is not simply connected.

(2) *Is there an irreducible symplectic variety Y with $b_2(Y) = 4$ and non-quotient singularities?* In [Men22], Menet exhibits some Fujiki orbifolds of dimension 4 with the smallest Betti number possible, namely $b_2 = 4$. This Betti number cannot be realized by terminalizing the quotient of $\mathrm{K}_2(A)$ or $S^{[2]}$ by induced symplectic automorphisms (see Tables 4 and 7), while examples in dimension 6 appear in Table 8. It would be interesting to find an example with ($\mathbb{Q}$-factorial terminal) non-quotient singularities since at the moment the global Torelli theorem is not known in this case; see [BL22, Theorem 1.1] and [Men20].

(3) *Are there examples of irreducible symplectic fourfolds, possibly with simply connected regular locus, with $b_2 = 9, 12, 13, 15$ or $16 < b_2 < 23$.*

(4) *Is there a conceptual explanation for the gap $16 < b_2 < 23$?* Note that the only examples with $b_2 = 23$ and $b_2 = 16$ are, respectively, $S^{[2]}$ and a Nikulin orbifold, *i.e.*, a terminalization of a quotient of $S^{[2]}$ by an involution. Further, a 4-dimensional irreducible symplectic orbifold with $b_2 = 23$ is necessarily smooth by [FM21, Theorem 1.3].

Terminalizations of sixfolds are less studied in the literature. In Table 2, we summarize the second Betti numbers of irreducible symplectic sixfolds constructed in this paper as terminalizations of quotients of $\mathrm{K}_3(A)$.

3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	○	○	○	○	●	●		●				●								●

Table 2. Second Betti numbers of terminalizations of quotients of $\mathrm{K}_3(A)$, in the same notation as in Table 1.

1.5. General results on terminalizations

In the perspective of producing examples of irreducible symplectic varieties, we prove new results about terminalizations of symplectic varieties of independent interest.

Proposition 1.13 (Terminalization of symplectic varieties, *cf.* Proposition 3.7). *Let $f\colon Y \to X$ be a proper birational morphism onto a symplectic variety X. Let X° be the complement of the dissident locus (see Definition 3.6) and $f^\circ\colon Y^\circ \to X^\circ$ be the unique symplectic resolution of X°.*

Then f is a terminalization of X if and only if f is a compactification of $f^\circ\colon Y^\circ \to X^\circ$ and we have $\operatorname{codim}(Y \setminus Y^\circ) \geq 2$.

Proposition 1.14 (Terminalization of symplectic varieties with only quotient singularities, *cf.* Corollary 3.11). *Let $f\colon Y \to X$ be a projective $\mathbb{Q}$-factorial terminalization of a complex symplectic variety X with only quotient singularities. Suppose that the divisor $E|_{f^{-1}(U)}$ is irreducible for any prime exceptional divisor $E \subset Y$ and for any connected open set $U \subseteq X$ in the Euclidean topology. Assume further that U is a connected Euclidean neighborhood of some $x \in X$, and let $T \to U$ be a projective terminalization of U. Then, up to shrinking U to a smaller neighborhood of x, $Y_U := f^{-1}(U)$ admits a locally trivial deformation to T. Furthermore, Y_U and T are locally analytically $\mathbb{Q}$-factorial over U (see Definition 3.8), and they have the same singularities (in the sense of Proposition 3.10).*

Remark. A deformation $\pi\colon \mathcal{X} \to S$ is called *locally trivial* if for any $x \in \mathcal{X}$, there exist analytic neighborhoods $\mathcal{U} \subset \mathcal{X}$ of x and $S_0 \subset S$ of $\pi(x)$ such that $\mathcal{U} \simeq S_0 \times U$, where $U := \pi^{-1}(\pi(x)) \cap \mathcal{U}$; see *e.g.* [Ser06, Section 1.2.1] or [BL22, Definition 4.1].

Proposition 1.15 (Fundamental group of terminalizations, *cf.* Proposition 8.1). *Let X be a simply connected smooth symplectic variety endowed with an action of a finite group G of symplectic automorphisms. Let $p\colon Y \to X/G$ be a terminalization of the quotient. The fundamental group of the regular locus of Y is*

$$\pi_1(Y^{\mathrm{reg}}) \simeq G/N,$$

where $N \triangleleft G$ is the normal subgroup generated by elements $\gamma \in G$ whose fixed locus in X has codimension 2. The universal quasi-étale cover of Y is a terminalization of the quotient X/N.

1.6. Outline

- In Section 3, we recall the definition of (irreducible) symplectic variety and describe properties of their terminalizations.
- In Section 4, we motivate the criteria of the classification and comment in particular on Assumptions 1.1 and 1.2.
- In Section 5.3, we specialize the previous results to the case of quotients of irreducible symplectic manifolds $S^{[n]}$ or $\mathrm{K}_n(A)$ by induced automorphism groups. For this purpose, in Section 5.2, we show that the codimension 2 fixed loci of induced symplectic automorphisms are subject to severe constraints.
- In Sections 6, 7 and 8, we provide group-theoretic formulas for the second and third Betti numbers of Y and the fundamental group of the regular locus of Y.
- We list the second Betti number and fundamental group of the regular locus of all terminalizations of induced symplectic quotients of $S^{[2]}$, $\mathrm{K}_2(A)$ and $\mathrm{K}_3(A)$, respectively in Table 4 (Section 9), Table 7 (Section 10.2) and Table 8 (Section 10.2).
- In Table 9 (Section 10.4), we list Betti numbers, Chern classes and singularities of the terminalizations of quotients $\mathrm{K}_2(A)/G$ with simply connected regular locus. The analysis of the singularities is carried on in Section 11.
- In Section 10.5 and in Table 10 (Section 10.4), we describe the singularities of the terminalizations of quotients $\mathrm{K}_3(A)/G$ with simply connected regular locus.

- In Section 12, we determine whether terminalizations with the same topological invariants are actually deformation equivalent.

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2. Notation

- Denote by $S^{(n)} := S^n/S_n$ the n-fold symmetric product of the surface S. A point in $S^{(n)}$ is an unordered n-tuple $[(x_1,\dots,x_n)]$ or the formal sum $x_1+\dots+x_n$ with $x_i \in S$.
- The Hilbert–Chow morphism

$$\epsilon\colon S^{[n]} \longrightarrow S^{(n)}, \quad \xi \longmapsto \sum_{x\in\xi} \mathrm{length}(\mathcal{O}_{\xi,x})\cdot x,$$

  sends any subscheme ξ of length n in the surface S to its weighted support. It is a symplectic resolution of the symmetric product $S^{(n)}$.
- The generalized Kummer variety $\mathrm{K}_n(A)$ is the fiber over 0 of the composition

$$A^{[n+1]} \xrightarrow{\epsilon} A^{(n+1)} \xrightarrow{s} A,$$

  where ϵ is the Hilbert–Chow morphism and s is the summation map. We denote by $A_0^{(n+1)}$ the fiber over 0 of s. The restriction $\epsilon\colon \mathrm{K}_n(A) \to A_0^{(n+1)}$ is a symplectic resolution.
- Let $A = \mathbb{C}^2/\Lambda$ be a complex torus with period lattice $\Lambda := H_1(A,\mathbb{Z})$. The group of symplectic automorphisms of A is

$$A \rtimes \mathrm{SL}(\Lambda),$$

  where A acts on itself by translation and $\mathrm{SL}(\Lambda) \subset \mathrm{SL}(2,\mathbb{C})$ is the group of linear automorphisms of the universal cover $\mathbb{C}^2$ with determinant 1 and preserving the period lattice Λ. The group of induced symplectic automorphisms of $\mathrm{K}_n(A)$ is

$$A[n+1] \rtimes \mathrm{SL}(\Lambda).$$

  Denote by τ_α the automorphism on $\mathrm{K}_n(A)$ induced by the translation $\alpha \in A[n+1]$.
- Given a group $G \subseteq A[n+1] \rtimes \mathrm{SL}(\Lambda)$, we write

$$G_{\mathrm{tr}} := G \cap A[n+1]$$

  for the normal subgroup of translations in G, and we set

$$G_\circ := G/G_{\mathrm{tr}} = \mathrm{Im}(\pi\colon G \hookrightarrow A[n+1] \rtimes \mathrm{SL}(\Lambda) \longrightarrow \mathrm{SL}(\Lambda)). \tag{2.1}$$

- We use the notation

C_n	cyclic group of order n,
S_n	symmetric group of degree n,
A_n	alternating group of degree n,
D_n	dihedral group of degree n,
Q_8	quaternion group,
BD_{12}	binary dihedral group of order 12,
BT_{24}	binary tetrahedral group of order 24.

- Let G be a group acting on a normal variety X, and let $q : X \to X/G$ be the quotient map. For any $x \in X$ and $g \in G$, we write

$$G_x := \{g \in G \mid g(x) = x\},$$
$$\mathrm{Fix}(g) := \{x \in X \mid g(x) = x\}.$$

  The isotropy (group) of a point z in X/G is the stabilizer of a point of the orbit $q^{-1}(z)$, up to conjugation.
- Assume that G acts on a smooth complex algebraic variety X of dimension n and fixes a point x. Then an analytic (or étale) neighborhood of $q(x)$ in X/G is isomorphic to the linear quotient T_xX/G of its tangent space. If G is cyclic of order k, then the action of G on $T_xX \simeq \mathbb{A}^n$ can be diagonalized and written as

$$(x_1, \dots, x_n) \longmapsto \left(\xi_k^{m_1} x_1, \dots, \xi_k^{m_n} x_n\right),$$

  where ξ_k is a primitive k^{th} root of unity and the integers m_i are called weights of the action. We usually abbreviate the quotient by this action as

$$\mathbb{A}^n / \tfrac{1}{k}(m_1, \dots, m_n).$$

  Definition 2.1. Let X be an algebraic variety of dimension $2n$. We denote by $a_k = a_k(X)$ the number of singularities of analytic type

$$\mathbb{A}^{2n} / \tfrac{1}{k}(1, -1, \dots, 1, -1).$$

3. Symplectic varieties and terminalizations

3.1. Symplectic varieties

We refer to [KM98] for the standard terminology in birational geometry. If X is a normal variety and $j \colon X^{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the regular locus, then $\Omega_X^{[p]} := j_* \Omega^p_{X^{\mathrm{reg}}}$ is the sheaf of reflexive p-forms on X.

Definition 3.1. Let X be a normal variety. A reflexive 2-form

$$\omega_X \in H^0\left(X, \Omega_X^{[2]}\right) = H^0\left(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}}\right)$$

is *symplectic* if its restriction to the regular locus of X, denoted by X^{reg}, is closed non-degenerate.

Definition 3.2. A normal variety X is *symplectic*, or equivalently has *symplectic singularities*, if

- it admits a symplectic form $\omega_X \in H^0(X, \Omega_X^{[2]})$,
- it has rational singularities.

By [KS21, Corollary 1.8], this means that a holomorphic symplectic form $\omega_{X^{\mathrm{reg}}}$ on X^{reg} extends to a (possibly degenerate) holomorphic 2-form $\omega_{\widetilde{X}}$ on a resolution $\widetilde{X} \to X$. We say that X admits a *symplectic resolution* if $\omega_{\widetilde{X}}$ is non-degenerate.

Proposition 3.3. *A symplectic variety has Gorenstein canonical singularities, and it is terminal if and only if the singular locus has codimension at least* 4.

Proof. See *e.g.* [Kol13, Claim 2.3.1] and [Nam01b, Corollary 1]. □

Definition 3.4. Let X be a variety with canonical singularities. A *terminalization* of X is a proper birational morphism $f\colon Y \to X$ such that Y has terminal singularities and $f^*K_X = K_Y$.

A terminalization of X exists by [BCHM10, Corollary 1.4.3], and it can be chosen projective, $\mathbb{Q}$-factorial (see Definition 3.8) and equivariant with respect to a group G acting on X. Further, it is also unique up to isomorphism in codimension 1; see [KM98, Corollary 3.54].

3.2. Terminalization of symplectic varieties

Proposition 3.5. *Let $f\colon Y \to X$ be a crepant birational modification of a symplectic variety X, e.g. a terminalization of X. Then f is semismall; i.e.,* $\dim(Y \times_X Y) \leq \dim X$.

Proof. See *e.g.* [Kal06, Lemma 2.11], [Los22, Proposition 2.14] and [Tig25, Proposition 2.15]. □

Definition 3.6. Let X be a symplectic variety. Let X° be the largest open set of points x in X such that either X is smooth at x, or the formal completion $\widehat{X}_x$ admits a decomposition $\widehat{M}_x \times \widehat{W}_x$, where M_x is a smooth scheme and W is a canonical surface singularity. The complement $X \setminus X^\circ$ is called the *dissident locus.*

In other words, X° is the union of the strata of dimension $\dim X$ and $\dim X - 2$ of the stratification of X constructed in [Kal06, Theorem 2.3]. Note that X° admits a unique symplectic resolution $f^\circ\colon Y^\circ \to X^\circ$ obtained by repeatedly blowing up the singular locus of X° (or of its blowup), as in the surface case.

Proposition 3.7. *Let $f\colon Y \to X$ be a proper birational modification of a symplectic variety X. Then f is a terminalization of X if and only if f is a normal compactification of $f^\circ\colon Y^\circ \to X^\circ$ and* $\operatorname{codim}(Y \setminus Y^\circ) \geq 2$.

Proof. Suppose that f is a terminalization of X. By the uniqueness of minimal surface resolution, f must restrict to f° over X°. Now, if f extracts a divisor $E \subseteq Y \setminus Y^\circ$, then

$$\begin{aligned}\dim(E \times_X E) &= 2\dim E - \dim f(E) \geq 2\dim X - 2 - \dim(X \setminus X^\circ)\\ &\geq \dim X - 2 + 4 > \dim X,\end{aligned}$$

which contradicts Proposition 3.5. Hence, $\operatorname{codim}(Y \setminus Y^\circ) \geq 2$.

Conversely, a compactification of $f^\circ\colon Y^\circ \to X^\circ$ such that $\operatorname{codim}(Y \setminus Y^\circ) \geq 2$ is isomorphic in codimension 1 to a terminalization of X, so it is terminal. □

Definition 3.8. A normal algebraic or analytic variety X is $\mathbb{Q}$*-factorial* if for every Weil divisor D on X, there is an $m \in \mathbb{N}$ such that mD is Cartier. A normal complex analytic variety X is *locally analytically* $\mathbb{Q}$*-factorial* if every open set $U \subseteq X$ in the Euclidean topology is $\mathbb{Q}$-factorial.

Let $f\colon Y \to X$ be a proper morphism of normal complex varieties; then Y is *locally analytically* $\mathbb{Q}$*-factorial over* X if $Y_U := f^{-1}(U)$ is $\mathbb{Q}$-factorial for any open set $U \subseteq X$ in the Euclidean topology.

Lemma 3.9. *Let $f\colon Y \to X$ be a proper birational morphism of normal complex algebraic or analytic varieties with exceptional divisor $E = \sum_i E_i$. Suppose that*

(†) *the divisors $E_i|_{f^{-1}(U)}$ are irreducible for any connected open set $U \subseteq X$ in the Euclidean topology.*

If X is locally analytically $\mathbb{Q}$-factorial and Y is $\mathbb{Q}$-factorial, then Y is locally analytically $\mathbb{Q}$-factorial.

Proof. By assumption (†), any prime exceptional divisor of $f|_{Y_U}\colon Y_U := f^{-1}(U) \to U$ is $E_i|_{Y_U}$ for some i. Then by the localization formula, we have

$$\bigoplus_i \mathbb{Q} E_i|_{Y_U} \longrightarrow \mathrm{Cl}(Y_U)_{\mathbb{Q}} \longrightarrow \mathrm{Cl}(Y_U \setminus E)_{\mathbb{Q}} \longrightarrow 0.$$

Since X is locally analytically $\mathbb{Q}$-factorial, then $\mathrm{Cl}(Y_U \setminus E)_{\mathbb{Q}} \simeq \mathrm{Cl}(U \setminus f(E))_{\mathbb{Q}} \simeq \mathrm{Cl}(U)_{\mathbb{Q}} \simeq \mathrm{Pic}(U)_{\mathbb{Q}}$. Since Y is $\mathbb{Q}$-factorial, a multiple of $E_i|_{Y_U}$ is Cartier. We conclude that $\mathrm{Cl}(Y_U)_{\mathbb{Q}}$ is generated by Cartier divisors; *i.e.*, Y is locally analytically $\mathbb{Q}$-factorial over X. □

Proposition 3.10. *Let $f: Y \to X$ be a projective $\mathbb{Q}$-factorial terminalization of a complex symplectic variety X with exceptional divisor $E = \sum_i E_i$. Suppose that*

(1) *X is locally analytically $\mathbb{Q}$-factorial,*
(2) *the formal completion $\hat{X}_x$ of X at any singular point $x \in X$ admits a $\mathbb{G}_m$-action with only positive weights on the maximal ideal of $\mathcal{O}_{\hat{X}_x}$ and on the local symplectic form,*
(3) *the divisors $E_i|_{f^{-1}(U)}$ are irreducible for any connected open set $U \subseteq X$ in the Euclidean topology.*

Assume further that U is a connected Euclidean neighborhood of some $x \in X$, and let $T \to U$ be a projective terminalization of U which is locally analytically $\mathbb{Q}$-factorial over U. Then, up to shrinking U to a smaller neighborhood of x if necessary, $Y_U := f^{-1}(U)$ admits a locally trivial deformation to T. In particular, Y_U and T have the same singularities; i.e., for any $t \in T$, there exist a $y \in Y_U$ and a formal isomorphism $\hat{T}_t \simeq \hat{Y}_{U,y}$.

Proof. We closely follow [Nam08]. For any $x \in X$, there exists a pointed affine symplectic scheme (Z, z), (i) with a good $\mathbb{G}_m$-action fixing z, (ii) algebraizing $\hat{X}_x$, *i.e.*, $\hat{X}_x \simeq \hat{Z}_z$, and (iii) only depending on $\hat{X}_x$ and the weights of the $\mathbb{G}_m$-action; see [Nam08, Lemma A.2 and the proof of Lemma 22]. The local $\mathbb{G}_m$-action on $\hat{X}_x$ lifts to $\hat{Y}_x := \hat{X}_x \times_X Y$ and linearizes a $f|_{\hat{Y}_x}$-ample line bundle; see [Nam08, Steps 1 and 2 of Proposition A.7, Lemma A.8].

Now, by [Nam08, Proposition A.5], there exists a $\mathbb{G}_m$-equivariant projective morphism $g: W = W(Y_U) \to Z$ such that $\hat{Y}_x \simeq \hat{W}_z := \hat{Z}_z \times_Z W$. By Artin's approximation [Art69, Corollary 2.4], there exists an analytic open neighborhood $z \in V \subset Z$ such that, up to shrinking U, the following diagram commutes:

$$\begin{array}{ccccccc} Y & \hookleftarrow & Y_U & \xrightarrow{\simeq} & W_V & \hookrightarrow & W \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle g_V} & & \downarrow{\scriptstyle g} \\ X & \hookleftarrow & U & \xrightarrow{\simeq} & V & \hookrightarrow & Z. \end{array}$$

Since Y_U is a $\mathbb{Q}$-factorial terminalization of U by Lemma 3.9, and since the $\mathbb{G}_m$-action retracts W into W_V, we conclude that W_V and W are terminal and $\mathbb{Q}$-factorial too. Applying the same construction to $T \to U$, we obtain two projective $\mathbb{Q}$-factorial terminalizations of Z

$$W(Y_U) \xrightarrow{g} Z \xleftarrow{g'} W(T).$$

The result then follows from [Nam08, Corollary 25]. □

Corollary 3.11. *Let $f: Y \to X$ be a projective $\mathbb{Q}$-factorial terminalization of a complex symplectic variety X with only quotient singularities. Suppose that the divisor $E|_{f^{-1}(U)}$ is irreducible for any prime exceptional divisor $E \subset Y$ and for any connected open set $U \subseteq X$ in the Euclidean topology. Assume further that U is a connected Euclidean neighborhood of some $x \in X$, and let $T \to U$ be a projective terminalization of U. Then, up to shrinking U to a smaller neighborhood of x, $Y_U := f^{-1}(U)$ admits a locally trivial deformation to T. Furthermore, Y_U and T are locally analytically $\mathbb{Q}$-factorial over U, and they have the same singularities in the sense of Proposition 3.10.*

Proof. If X has quotient singularities, then assumptions (1) and (2) in Proposition 3.10 hold. The local analytic $\mathbb{Q}$-factoriality of Y and T follows from Lemma 3.9. □

3.3. Irreducible symplectic varieties

Definition 3.12. A symplectic compact Kähler[(4)] variety (X, ω_X) is an *irreducible* (*holomorphic*) *symplectic variety* (IHS variety for short) if for any finite quasi-étale cover $g: X' \to X$, the exterior algebra of reflexive forms $H^0(X', \Omega_{X'}^{[\bullet]})$ on X' is generated by the reflexive pullback $g^*\omega_X$.

[(4)] We refer to [Gra62, Section 3.3, p. 346] or [BL22, Section 2.3] for the notion of (possibly singular) compact Kähler space.

Remark 3.13 (Finiteness results for IHS varieties). It is expected that the number of deformation types of irreducible symplectic varieties is finite in each dimension. For instance, in any given dimension, there are only finitely many diffeomorphism types of irreducible symplectic manifolds with isomorphic Beauville–Bogomolov lattice (H^2, q); see [Huy03, Theorem 4.3] and the refinement [Kam18, Theorem 4.4]. Topological bounds are known in low dimension: The second Betti number of a 4-dimensional irreducible symplectic orbifold is at most 23 by [Gua01, FM21], and in the smooth fourfold case is either at most 8 or 23 by [Gua01], and conjecturally only $5, 6, 7$ or 23 according to [BS22, Corollary 1.3] (*cf.* [DHMV24, Theorem 9.3]); see also the recent survey [BD22]. It is expected that the same bound of at most 23 holds for smooth sixfolds too; see [Kur16, Saw22] for partial results. A conjectural bound for the second Betti numbers of irreducible symplectic manifolds in arbitrary dimension is proposed in [KL20].

Proposition 3.14. *Birational projective $\mathbb{Q}$-factorial terminal IHS varieties are deformation equivalent. In particular, a projective $\mathbb{Q}$-factorial terminalization of an IHS variety is unique up to deformation.*

Proof. See [BL22, Corollary 6.17]. ☐

Definition 3.15. An automorphism $\varphi\colon X \to X$ of a symplectic variety (X, ω_X) is *symplectic* if $\varphi^*\omega_X = \omega_X$.

Lemma 3.16. *Let G be a symplectic group acting on a symplectic variety X. Any irreducible components of the fixed locus of G has even dimension.*

Proof. If X is smooth, the fixed locus $\mathrm{Fix}(G)$ is smooth and symplectic as its tangent bundle is the G-fixed part of the tangent bundle of X, thus symplectic and even-dimensional:

$$T_{\mathrm{Fix}(G)} = \left((T_X)|_{\mathrm{Fix}(G)}\right)^G.$$

In general, we stratify X into smooth G-invariant locally closed symplectic subsets as in [Kal06, Theorem 2.3] and reduce to the argument in the smooth case. ☐

Proposition 3.17. *The sets of symplectic varieties and of IHS varieties are both closed under projective terminalizations or finite quotients by symplectic groups.*

Proof. The terminalization of a symplectic variety is symplectic by construction. Let $q\colon X \to X/G$ be a symplectic quotient of a symplectic variety. Any G-invariant symplectic form descends to X/G, and X/G has rational singularities by [KM98, Proposition 5.13]. Hence, X/G is symplectic.

Now suppose that X is an IHS variety. The statement for projective terminalization is proved in [Sch20, Proposition 12].[5] The statement for symplectic quotients $q\colon (X, \omega_X) \to X/G$ is proved in [Mat15, Lemma 2.2] when X is smooth. The argument in the singular case is essentially identical. We give a self-contained proof for completeness. Let $g\colon X' \to X/G$ be a quasi-étale cover of X/G and Z be the normalization of an irreducible component of $X \times_{X/G} X'$ that maps surjectively onto X and X'. All maps in the commutative square

$$\begin{array}{ccc} Z & \xrightarrow{q'} & X' \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[q]{} & X/G \end{array}$$

are quasi-étale. Since X is an IHS variety, the algebra $H^0(Z, \Omega_Z^{[\bullet]})$ is generated by the pullback $\omega_Z := \tilde{g}^*(\omega_X)$. Since q' is quasi-étale and $\tilde{g}^*(\omega_X)$ descends to a symplectic form $\omega_{X'}$ on X', the inequalities

$$\dim\langle\omega_Z\rangle = \dim H^0\left(Z, \Omega_Z^{[\bullet]}\right) \geq \dim H^0\left(X', \Omega_{X'}^{[\bullet]}\right) \geq \dim\langle\omega_{X'}\rangle$$

are identities, and so $H^0(X', \Omega_{X'}^{[\bullet]})$ is generated by $\omega_{X'}$. Hence, X/G is an IHS variety. ☐

(5) The notion of primitive symplectic in *loc. cit.* stands for IHS varieties.

Corollary 3.18. *Let G be a finite symplectic group acting on a symplectic variety X or an IHS variety. A projective terminalization of the quotient X/G is symplectic or an IHS variety, respectively.*

Remark 3.19. Finite groups of symplectic automorphisms of known irreducible symplectic manifolds have been extensively studied in the literature. Particularly relevant for the present paper are the classifications of finite symplectic groups acting on $\mathrm{K3}^{[2]}$ by Höhn and Mason in [HM19] (see also the preliminary results in [Mon13]), and on $\mathrm{K}_2(A)$ by Mongardi, Tari and Wandel in [MTW18, Section 5]. See also [Mon16, KMO23].

In view of Assumption 1.1, O'Grady examples instead are less interesting for our classification purposes. Indeed, the only symplectic automorphism of an irreducible symplectic manifold of O'Grady 10 type is the identity by [GGOV24]. On the other hand, all symplectic automorphisms of an irreducible symplectic manifold of O'Grady 6 type act trivially on the second cohomology group by [GOV23], and their fixed loci have codimension at least 4 by [MW17, Section 6], so they do not satisfy Assumption 1.1.

4. Remarks on the criteria of the classification

Given an irreducible symplectic variety X, it is possible to obtain new such varieties by taking quotients, birational contractions or terminalizations. When one classifies birational modifications of symplectic quotients, in order to avoid redundancy, it is convenient to restrict to projective $\mathbb{Q}$-factorial terminalizations Y of symplectic quotients X/G by a finite group G, with simply connected regular locus Y^{reg}. If X has $\mathbb{Q}$-factorial singularities, this amounts to Assumptions 1.1 and 1.2.

Symplectic quotients X/G of irreducible symplectic varieties X are irreducible symplectic varieties themselves; see Proposition 3.17. However, they are less interesting from a classification viewpoint as the geometry of X/G can be essentially recovered from the G-equivariant geometry of X. For instance, we have the following:

- The rational cohomology of X/G is isomorphic to the G-invariant part of the rational cohomology of X:
$$H^*(X/G,\mathbb{Q}) = H^*(X,\mathbb{Q})^G.$$
Note, however, that the relation between the integral cohomology $H^*(X/G,\mathbb{Z})$ and $H^*(X,\mathbb{Z})$ is more subtle, and even for a single involution, their connection is not trivial; see for instance [KM18].
- The fundamental group of the regular locus of X/G is an extension of G by $\pi_1(X^{\mathrm{reg}})$:
$$1 \longrightarrow \pi_1(X^{\mathrm{reg}}) \longrightarrow \pi_1((X/G)^{\mathrm{reg}}) \longrightarrow G \longrightarrow 1.$$
In particular, if X^{reg} is simply connected, then $\pi_1((X/G)^{\mathrm{reg}}) \simeq G$
- The deformations of X/G are the deformations of X preserving the group action.

More conceptually, the building blocks of the Beauville–Bogomolov decomposition are defined only up to quasi-étale cover. Nonetheless, for the symplectic factors of the decomposition, one can actually choose a distinguished representative in the class of all quasi-étale covers of a fixed symplectic factor, namely its universal quasi-étale cover. In other words, the irreducible symplectic factors Y in the Beauville–Bogomolov decomposition can always be chosen so that the regular locus Y^{reg} is algebraically simply connected, *i.e.*, the algebraic fundamental group $\hat{\pi}_1(Y^{\mathrm{reg}})$ of the regular locus is trivial. This is indeed possible since the algebraic fundamental group of an irreducible symplectic variety is known to be finite by [GGK19, Corollary 13.2].[6] The conclusion is that we should only classify irreducible symplectic varieties Y with $\hat{\pi}_1(Y^{\mathrm{reg}}) = 1$ (conjecturally, $\pi_1(Y^{\mathrm{reg}}) = 1$) as all other irreducible symplectic varieties are quasi-étale quotients of them.

[6] Actually, the same is expected to hold for the topological fundamental group: This is implicit in [GGK19, Section 13], explicitly conjectured in [Wan22, Conjecture 3] and proved by Engel, Filipazzi, Greer, Mauri and Svaldi in [EFG+25] under the assumption that Y admits a Lagrangian fibration.

Birational transformations of X/G are also potential new sources of irreducible symplectic varieties. However, to preserve the non-degeneracy of the symplectic form, one is only allowed to extract divisors with discrepancy zero. Any such birational modification is dominated by a $\mathbb{Q}$-factorial terminalization Y, see [BCHM10, Corollary 1.4.3], and moreover it can be recovered by the Mori theory of Y itself. Therefore, it is superfluous to study symplectic birational modifications of X/G other than its $\mathbb{Q}$-factorial terminalizations. Actually, it has always been clear in the literature that $\mathbb{Q}$-factorial and terminal irreducible symplectic varieties form a particularly agreeable class of varieties for their well-behaved birational and deformation theory:

- Birational projective $\mathbb{Q}$-factorial terminal irreducible symplectic varieties are deformation equivalent. In particular, any two projective $\mathbb{Q}$-factorial terminalizations of the same irreducible symplectic variety are deformation equivalent. See [BL22, Corollary 6.17].
- Deformations of projective $\mathbb{Q}$-factorial terminal irreducible symplectic varieties are equisingular. In particular, the Betti numbers and the fundamental group of the regular locus are deformation invariants. See [Nam06].
- The global Torelli theorem holds for $\mathbb{Q}$-factorial Kähler terminal irreducible symplectic varieties with $b_2 \geq 5$. See [BL22, Theorem 1.1].(7)

Also note that if Assumption 1.1 holds, then the cohomology class of each exceptional divisor of a terminalization $Y \to X/G$ in $H^2(Y, \mathbb{Z})$ remains of type $(1,1)$ only along a divisor in the Kuranishi family of Y. This implies that the general deformation of Y cannot come from the quotient-terminalization construction and should be considered as an honestly new deformation type of irreducible symplectic variety.

Finally, observe that the projectivity condition can always be achieved by [BCHM10, Corollary 1.4.3]. The projectivity of terminalizations obtained by gluing local terminalization is discussed in Sections 3.2 and 11.1.

5. Induced symplectic automorphisms and terminalizations

In this section, we show that terminalizations of quotients of $S^{[n]}$ or $K_n(A)$ by induced symplectic automorphism groups can be obtained via a single explicit blowup away from the dissident locus; see Corollary 5.9.

5.1. Induced automorphism

Let X be either a Hilbert scheme $S^{[n]}$ of a K3 surface S, or a generalized Kummer variety $\mathrm{K}_n(A)$ associated to an abelian surface A.

Definition 5.1. An automorphism $\phi\colon S \to S$ of a K3 surface S induces an automorphism of $S^{[n]}$. We call such an automorphism of $S^{[n]}$ *induced.*

Definition 5.2. An automorphism $\phi\colon A \to A$ of the abelian surface A (not necessarily fixing the origin) induces an automorphism $\phi^{(n+1)}\colon A^{(n+1)} \to A^{(n+1)}$ of its symmetric product $A^{(n+1)}$. If $\phi^{(n+1)}$ preserves $A_0^{(n+1)}$, then it lifts to an automorphism of $\mathrm{K}_n(A)$. We call such an automorphism of $\mathrm{K}_n(A)$ *induced.*

Note that an induced automorphism on $S^{[n]}$ or $\mathrm{K}_n(A)$ is symplectic if and only if the underlying automorphism of S or A is symplectic.

(7)It is expected that the assumption on the Betti number can be removed, and this is known if the irreducible symplectic variety has only quotient singularities; see [Men20].

5.2. Codimension 2 fixed loci of induced authomorphisms

Let G be a finite group of induced symplectic automorphisms of $X = \mathrm{K}_n(A)$ or $S^{[n]}$. In this section, we describe the connected components of codimension 2 fixed by automorphisms $g \in G$. The importance of these loci lies in the fact that their images in X/G are the centers of blowups giving a terminalization $Y \to X/G$. Their geometry is severely constrained: We show that they occur only if the orders of g and n are either 2 or 3. Further, these fixed components are all of K3 or K3$^{[2]}$ type; see also [KMO23, Theorems 1.0.2 and 1.0.4].

Denote the Hilbert–Chow morphism by $\epsilon\colon S^{[n]} \to S^{(n)}$ or $\epsilon\colon \mathrm{K}_n(A) \to A_0^{(n+1)}$ as in Section 2.

Lemma 5.3. *Let G be a finite group of symplectic automorphisms of X. Let $F \subset X$ be a subvariety of codimension 2 fixed by an element of the group. Then the restriction $\epsilon|_F$ is generically finite.*

Proof. If $F \not\subseteq \mathrm{Exc}(\epsilon)$, then $\epsilon_{|F}$ is birational, so generically finite. Then suppose $F \subseteq \mathrm{Exc}(\epsilon)$. If $F = \epsilon^{-1}(\epsilon(F))$, then F is uniruled, which is impossible as F is the fixed locus of a symplectic automorphism and hence F is symplectic so not uniruled. It follows that $F \subsetneq \epsilon^{-1}(\epsilon(F))$ and $\dim(\epsilon^{-1}(z) \cap F) < \dim(\epsilon^{-1}(z))$ for a general $z \in \epsilon(F)$. By [Kal06, Lemma 2.11] or Proposition 3.5, the morphism ϵ is semismall; *i.e.*, $\dim(X) = \dim\left(X \times_{\epsilon(X)} X\right)$. We get

$$\begin{aligned}
2n &= \dim\left(X \times_{\epsilon(X)} X\right)\\
&\geq \dim\left(\epsilon^{-1}(\epsilon(F)) \times_{\epsilon(F)} \epsilon^{-1}(\epsilon(F))\right) \geq \dim(\epsilon(F)) + 2\dim\left(\epsilon^{-1}(z)\right)\\
&> \dim(\epsilon(F)) + 2\dim\left(\epsilon^{-1}(z) \cap F\right)\\
&= \dim(F) + \dim\left(\epsilon^{-1}(z) \cap F\right) = 2n - 2 + \dim\left(\epsilon^{-1}(z) \cap F\right),
\end{aligned}$$

and so $\dim\left(\epsilon^{-1}(z) \cap F\right) \leq 1$. If $\dim\left(\epsilon^{-1}(z) \cap F\right) = 0$, then $\epsilon_{|F}$ is generically finite. Otherwise, $\dim(\epsilon(F))$ is odd, which is impossible since $\epsilon(F)$ is symplectic as fixed locus of a symplectic automorphism of $\epsilon(X)$. □

Lemma 5.4 (Order of automorphisms with large fixed locus). *Let F be a subvariety of codimension 2 fixed by an induced symplectic automorphism $g\colon X \to X$. Then*

- $\mathrm{ord}(g) = 2$ *and* $X = S^{[2]}$, $\mathrm{K}_2(A)$ *or* $\mathrm{K}_3(A)$, *or*
- $\mathrm{ord}(g) = 3$ *and* $X = \mathrm{K}_2(A)$.

Proof. Let G be the cyclic group $\langle g \rangle$ acting indifferently on the surface $M := S$ or A, on the singular symplectic variety $X_{\mathrm{sing}} := S^{(n)}$ or $A_0^{(n)}$, or on the symplectic resolution $X := S^{[n]}$ or $\mathrm{K}_{n-1}(A)$. Stratify the surface M according to length of G-orbits; *i.e.*,

$$M = \bigsqcup_{i=1}^{\mathrm{ord}(g)} M_i \quad \text{with} \quad M_i := \{m \in M : |Gm| = i\}.$$

The locus $M_{\mathrm{ord}(g)}$ is open and dense, while M_i with $i < \mathrm{ord}(g)$ consists of at most finitely many points as g is symplectic. A g-fixed point z of X_{sing} is a union of orbits for the action of G on M; *i.e.*, $z = \{Gm_1, \ldots, Gm_r\}$ for some $m_j \in M$ such that $\sum_{j=1}^r |Gm_j| = n$.

For any partition $\underline{n} = \sum_{i=1}^{\mathrm{ord}(g)} i \cdot n_i$ of n, define $F_{\underline{n}} \subseteq X_{\mathrm{sing}}$ as the union of the g-fixed subvarieties whose general points $z = \{Gm_1, \ldots, Gm_r\}$ satisfy $|\{j \in [r] : |Gm_j| = i\}| = n_i$. The finite morphism

$$\prod_{i=1}^{\mathrm{ord}(g)} M_i^{n_i} \longrightarrow M^{(n)}, \quad (m_i) \longmapsto \{Gm_i\},$$

contains $F_{\underline{n}}$ in its image, and so

$$\dim F_{\underline{n}} \leq \sum_{i=1}^{\operatorname{ord}(g)} n_i \dim M_i = n_{\operatorname{ord}(g)} \dim M_{\operatorname{ord}(g)} = 2n_{\operatorname{ord}(g)}$$

since $\dim M_i = 0$ for $i \neq \operatorname{ord}(g)$ and $\dim M_{\operatorname{ord}(g)} = \dim M = 2$. In particular, we obtain

$$\operatorname{codim}_{X_{\text{sing}}}\left(F_{\underline{n}}\right) \geq 2n - 2n_{\operatorname{ord}(g)} - 2\epsilon,$$

where $\epsilon = 0$ or 1 if $M = S$ or A, respectively. Finally, the equation $\operatorname{codim}_{X_{\text{sing}}}(F_{\underline{n}}) = 2$ admits a solution only in the cases listed in Lemma 5.4. By Lemma 5.3, the solutions for X_{sing} are solutions for the symplectic resolution X too. ☐

Remark 5.5. Note that on $X = S^{[2]}$, any induced involution g fixes a locus of codimension 2, namely the strict transform of $\{[(x, g(x))] | x \in S\}$ in $S^{(2)}$. Thus, for $X = S^{[2]}$ the condition on g in Lemma 5.4 is actually necessary and sufficient. See also [KMO22, Theorems 1.1 and 1.2].

In the Kummer case, the automorphisms fixing a locus of codimension 2 (see Lemma 5.4 above) admit a particularly explicit description.

Lemma 5.6 (Induced involutions and automorphisms of order 3 on $\mathrm{K}_n(A)$).

(1) *An induced symplectic involution g of $\mathrm{K}_n(A)$, with $n = 2$ or 3, fixes a subvariety F of codimension 2 if and only if*

$$g = \tau_\alpha(-\mathrm{id}) \in A \rtimes \mathrm{SL}(\Lambda)$$

with $\alpha \in A[3]$ if $n = 2$, or $\alpha \in A[2]$ if $n = 3$.

(2) *An induced symplectic automorphism g of order 3 of $\mathrm{K}_2(A)$ fixes a subvariety F of codimension 2 if and only*

$$g = \tau_\alpha \circ T \in A \rtimes \mathrm{SL}(\Lambda)$$

with $T^3 = 1$, $T \neq \mathrm{id}$ and $T(\alpha) = \alpha$ (equivalently, T is a linear automorphism of order 3 commuting with τ_α).

Proof. Since $-\mathrm{id}$ is the only involution of $\mathrm{SL}(2, \mathbb{C})$, any induced involution g of $\mathrm{K}_2(A)$ is of the form

$$\tau_\alpha(-\mathrm{id}) \in A[3] \rtimes (-\mathrm{id}),$$

and it fixes the strict transform of the surface $\{[(x, g(x), -x - g(x))] \mid x \in A\} \subset A_0^{(3)}$; see also [KM18, Theorem 7.5].

If $n = 3$, induced involutions of $\mathrm{K}_3(A)$ are of the form either

$$\tau_\alpha(-\mathrm{id}) \in A[4] \rtimes (-\mathrm{id}) \quad \text{or} \quad \tau_\alpha \in A[2],$$

but the only involutions g that fix a fourfold in $A_0^{(4)}$, namely $\{[(x, g(x), y, g(y))] \mid x, y \in A\}$, are those of the form $\tau_\alpha(-\mathrm{id}) \in A[2] \rtimes (-\mathrm{id})$.

An order 3 automorphism g fixes a surface F in $\mathrm{K}_2(A)$ if and only if it fixes the surface

$$\epsilon(F) = \left\{\left[\left(x, g(x), g^2(x)\right)\right] \mid x \in A\right\}$$

in $A_0^{(3)}$ by Lemma 5.3. This occurs if and only if g satisfies $1 + g + g^2 = 0$, *i.e.*, if and only if $T \in \mathrm{SL}(\Lambda)$ has minimal polynomial $1 + t + t^2$, *i.e.*, $T^3 = \mathrm{id}$, $T \neq \mathrm{id}$ and $T(\alpha) = \alpha$. ☐

Remark 5.7. Lemmas 5.4 and 5.6 imply that an induced symplectic automorphism of $S^{[n]}$ or $\mathrm{K}_n(A)$ fixes at most one subvariety of codimension 2. When it exists, such a subvariety F is a crepant resolution of $\epsilon(F)$ and is isomorphic to a K3 surface or a Hilbert square of a K3 surface.

Table 3. Codimension 2 subvarieties F fixed by an induced symplectic automorphism g of X. We denote by S_2 and S_3 the minimal resolutions of A/g.

$\mathrm{ord}(g)$	X	F	$\epsilon(F)$	g
2	$S^{[2]}$	S	$[(x, g(x))]$	any involution
2	$\mathrm{K}_2(A)$	S_2	$[(x, -x+\alpha, -\alpha)]$	$\tau_\alpha(-\mathrm{id})$ with $\alpha \in A[3]$
3	$\mathrm{K}_2(A)$	S_3	$[(x, T(x)+\alpha, T^2(x)-\alpha)]$	$\tau_\alpha \circ T$ with $T^3 = 1$, $T \neq \mathrm{id}$, $T(\alpha) = \alpha$
2	$\mathrm{K}_3(A)$	$S_2^{[2]}$	$[(x, -x+\alpha, y, -y+\alpha)]$	$\tau_\alpha(-\mathrm{id})$ with $\alpha \in A[2]$

5.3. Terminalizations via explicit blowups

Notation 5.8. Let G be a finite group of induced symplectic automorphisms of $X = S^{[n]}$ or $\mathrm{K}_n(A)$. Let $q\colon X \to X/G$ be the quotient map, $p\colon Y \to X/G$ be a terminalization of X/G, and Σ be the singular locus of X/G. Denote by $F_g \subset X$ the (unique by Remark 5.7) component of the fixed locus of $g \in G$ of codimension 2, if any:

$$\begin{array}{ccc} & & X \supset F_g \\ & & \downarrow q \\ Y & \xrightarrow{p} & X/G \supset \Sigma := \mathrm{Sing}(X/G) \supseteq q(F_g). \end{array}$$

Corollary 5.9 is a refinement of Proposition 3.7 in our special context. It asserts that in order to obtain a terminalization of X/G away from the dissident locus, it suffices to blow up once the irreducible components of the singular locus of codimension 2—no need to repeat the process—in the same way as a single blowup suffices to resolve the surface singularities of type A_1 and A_2.

Corollary 5.9. *We use the notation of Definition 3.6. Away from the dissident locus, the terminalization Y is isomorphic to the blowup of the reduced singular locus of X/G; i.e.,*

$$Y^\circ \simeq \mathrm{Bl}_{\Sigma \cap (X/G)^\circ}(X/G)^\circ.$$

Proof. By Proposition 3.3, the quotient X/G is not terminal if and only if the fixed locus of some element of G has a component of codimension 2. This occurs only if $\mathrm{ord}(g) = 2$ or 3, and in the precise cases detailed in Lemma 5.4. Geometrically, this implies that a normal slice to a general point in $q(F_g)$ is a canonical surface singularity of type A_1 or A_2, which can be resolved with a single blowup. We conclude by applying Proposition 3.7. ☐

Remark 5.10. Up to a small $\mathbb{Q}$-factorial modification, see [Kol13, Corollary 1.37], which is an isomorphism away from the dissident locus of X/G, we can suppose that Y is $\mathbb{Q}$-factorial too.

6. Second Betti number of a terminalization

Proposition 6.1. *We use Notation 5.8. Let*
- *L be a lattice isomorphic to $H^2(X, \mathbb{Z})$,*
- *N_2 be the number of components $q(F_g)$ in Σ with $\mathrm{ord}(g) = 2$,*
- *N_3 be the number of components $q(F_g)$ in Σ with $\mathrm{ord}(g) = 3$.*

Then the identity

$$b_2(Y) = \mathrm{rk}\left(L^G\right) + N_2 + 2N_3$$

holds except in the case where $X = \mathrm{K}_2(A)$ and $G_\circ \simeq BD_{12}$, treated in Remark 10.5.

Remark 6.2. Let $X := S^{[n]}$ or $\mathrm{K}_2(A)$ with $M := S$ or A, respectively. Recall that

$$H^2(X,\mathbb{Z}) \simeq H^2(M,\mathbb{Z}) \oplus \mathbb{Z}e,$$

where $2e$ is the class of the ϵ-exceptional divisor. Since the group G of induced automorphisms preserves the ϵ-exceptional divisor, we obtain

$$H^2(X/G,\mathbb{Q}) \simeq H^2(X,\mathbb{Q})^G \simeq H^2(M,\mathbb{Q})^G \oplus \mathbb{Q}e.$$

We conclude that

$$\mathrm{rk}\left(L^G\right) = \mathrm{rk}\left(H^2(M)^G\right) + 1.$$

Proof of Proposition 6.1. The blowup formula (or the decomposition theorem) gives

$$H^2(Y,\mathbb{Q}) = IH^2(Y,\mathbb{Q}) \simeq H^2(X/G,\mathbb{Q}) \oplus \bigoplus_i H^0(E_i,\mathbb{Q}),$$

where the sum runs over all the p-exceptional divisors E_i. Then, it suffices to compute the p-exceptional divisors. As shown in Section 5.3, the terminalization $p\colon Y \to X/G$ extracts an exceptional prime divisor for each $q(F_g)$ with transversal A_1 singularities, and at most two exceptional prime divisors for each $q(F_g)$ with transversal A_2 singularities.

The latter case occurs only if g has order 3 and $X = \mathrm{K}_2(A)$. Suppose that this is indeed the case, and denote simply by F the g-fixed surface. Consider the blowup of $\mathrm{K}_2(A)/G$ along $q(F)$,

$$p_{q(F)}\colon \mathrm{Bl}_{q(F)}(\mathrm{K}_2(A)/G) \longrightarrow \mathrm{K}_2(A)/G.$$

A neighborhood U of a general point in $q(F)$ is locally analytically isomorphic to the product $\mathbb{A}^2 \times (\mathbb{A}^2/C_3)$. In particular, the restriction of $p_{q(F)}$ over U extracts two exceptional prime divisors. Globally, these may be contained in two distinct $p_{q(F)}$-exceptional prime divisors of $\mathrm{Bl}_{q(F)}(\mathrm{K}_2(A)/G)$, or be two branches of the same non-normal $p_{q(F)}$-exceptional divisor. The latter case occurs only when $G_\circ \simeq BD_{12}$, as explained in Lemma 6.3. We conclude that if $G_\circ \not\simeq BD_{12}$, the terminalization $p\colon Y \to X/G$ extracts exactly two exceptional prime divisors for each $q(F_g)$ with transversal A_2 singularities, whence the statement. □

Lemma 6.3. *Suppose that a finite group G of induced symplectic automorphisms of $\mathrm{K}_2(A)$ contains an element g of order 3 fixing a surface F. Then the blowup $p'\colon Y' \to \mathrm{K}_2(A)/G$ of $\mathrm{K}_2(A)/G$ along $q(F)$ extracts two exceptional prime divisors, unless g is contained in a subgroup of G which is isomorphic to the binary dihedral group BD_{12} and splits the quotient $G \to G_\circ \simeq BD_{12} \subset \mathrm{SL}(\Lambda)$ (cf. Section 2). In this case, the exceptional divisor of p' is irreducible.*

Proof. The G-orbit of F, denoted by $G\cdot F$, consists of r irreducible components

$$F := F_g, F_{g_1}, \ldots, F_{g_j} = F_{h_j^{-1} g h_j} = h_j^{-1}(F_g), \ldots, F_{g_{r-1}},$$

where $g_j := h_j^{-1} g h_j$ for some $h_j \in G$. Then consider the blowup of $\mathrm{K}_2(A)$ along $G\cdot F$,

$$p_1\colon X_1 := \mathrm{Bl}_{G\cdot F}\mathrm{K}_2(A) \longrightarrow \mathrm{K}_2(A),$$

with exceptional divisors $\widetilde{E}_0 := p_1^{-1}(F), \widetilde{E}_1 \ldots, \widetilde{E}_{r-1}$.

Denote by ξ_3 a primitive third root of unity, and let $Z \subset \mathrm{K}_2(A)$ be the locus of points in $\mathrm{K}_2(A)$ whose stabilizer is neither trivial nor conjugate to $\langle g\rangle$. The normal bundle of F_{g_j} in $\mathrm{K}_2(A)$ splits into the sum of two $\langle g_j\rangle$-equivariant line bundles:

$$N_{F_{g_j}/\mathrm{K}_2(A)} \simeq L_{\xi_3} \oplus L_{\bar{\xi}_3},$$

where g_j acts on L_{ξ_3} or $L_{\bar{\xi}_3}$ by scaling by ξ_3 and $\bar{\xi}_3$, respectively. Therefore, away from Z, $\widetilde{E}_j$ is $\langle g_j\rangle$-equivariantly isomorphic to $\mathbb{P}\left(N_{F_{g_j}/\mathrm{K}_2(A)}\right) \simeq \mathbb{P}\left(L_{\xi_3} \oplus L_{\bar{\xi}_3}\right)$ with two $\langle g_j\rangle$-fixed sections of $\widetilde{E}_j \to F_{g_j}$, denoted by

$$s_{\xi_3,j} := \mathrm{Im}\left(\mathbb{P}\left(L_{\xi_3}\right) \hookrightarrow \mathbb{P}\left(L_{\xi_3} \oplus L_{\bar{\xi}_3}\right)\right) \quad \text{and} \quad s_{\bar{\xi}_3,j} := \mathrm{Im}\left(\mathbb{P}\left(L_{\bar{\xi}_3}\right) \hookrightarrow \mathbb{P}\left(L_{\xi_3} \oplus L_{\bar{\xi}_3}\right)\right).$$

Let $p_2\colon X_2 \to X_1$ be the simultaneous blowup of the closure of the sections $s_{\xi_3,j}$ and $s_{\bar{\xi}_3,j}$ for $j = 0,\dots,r-1$, with exceptional divisors $\widetilde{E}_{\xi_3,j}$ and $\widetilde{E}_{\bar{\xi}_3,j}$. Since the centers of the blowups p_1 and p_2 are G-invariant, X_2 inherits an action by G. Further, away from $q(Z)$, the quotient X_2/G is isomorphic to the blowup $p_3\colon Y_1 \to Y'$ along the double locus of the p'-exceptional locus E; it suffices to verify it in the local model $\mathbb{A}^2 \times (\mathbb{A}^2/C_3)$. Therefore, away from Z, there exists a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}_2(A) & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & X_2 \\ \downarrow q & & & & \downarrow q_1 \\ \mathrm{K}_2(A)/G & \xleftarrow{p'} & Y' & \xleftarrow{p_3} & Y_1 \simeq X_2/G, \end{array} \tag{6.1}$$

where the horizontal arrows are blowups and the vertical arrows are G-quotient maps.

The p'-exceptional divisor E is the image under $p_3 \circ q_1$ of the distinct divisors $\widetilde{E}_{\xi_3,j}$ and $\widetilde{E}_{\bar{\xi}_3,j}$. Suppose that E is irreducible. Then there exists a $\iota \in G$ such that

$$\iota\left(\widetilde{E}_{\xi_3,0}\right) = \widetilde{E}_{\bar{\xi}_3,0}.$$

Note that the subgroup $H \subseteq G$ generated by g and ι has the following presentation:

$$H = \left\langle g, \iota \mid \iota g \iota^{-1} = g^2, \iota^{2k} = 1 \right\rangle.$$

Indeed, the following facts hold:

- The automorphism ι preserves F_g, so it exchanges $\widetilde{E}_{\xi_3,0}$ and $\widetilde{E}_{\bar{\xi}_3,0}$, and thus it has even order.
- The automorphism ι preserves the locus $\epsilon(F) = \{[(x, g(x), g^2(x))] \mid x \in A\}$, so either $\iota g = g\iota$ or $\iota g = g^2\iota$.
- For any $(x,v) \in L_{\xi_3} \subset N_{F/\mathrm{K}_2(A)}$ with $x \in F_g$ and $v \in L_{\xi_3,x}$, we have

$$g \cdot (x,v) = (g \cdot x, dg_x(v)) = (x, \xi_3 v).$$

  Hence, we obtain

$$\begin{aligned} g \cdot (\iota \cdot x, d\iota_x(v)) &= g\iota \cdot (x,v) = \iota g^m \cdot (x,v) \\ &= \iota \cdot (x, \xi_3^m v) = (\iota \cdot x, d\iota_x(\xi_3^m v)) = (\iota \cdot x, \xi_3^m d\iota_x(v)), \end{aligned}$$

  i.e., $\iota(L_{\xi_3}) = L_{\xi_3^m}$. Since ι must exchange L_{ξ_3} and $L_{\bar{\xi}_3}$, we must have $m = 2$, *i.e.*, $\iota g = g^2\iota$.

Further constraints on the subgroup H are imposed by the fact that G is a group of symplectic automorphisms coming from an abelian surface A. Indeed, since $G \subset A \rtimes \mathrm{SL}(\Lambda)$, the order of ι must be 2, 4 or 6.

- If $\mathrm{ord}(\iota) = 2$, then $H \simeq S_3$ and H projects isomorphically onto $H_\circ \subset \mathrm{SL}(\Lambda)$, which gives a contradiction since no subgroup of $\mathrm{SL}(\Lambda)$ is isomorphic to S_3.
- If $\mathrm{ord}(\iota) = 4$, then $H \simeq BD_{12}$ and H projects isomorphically onto $G_\circ \subset \mathrm{SL}(\Lambda)$ since BD_{12} is a maximal finite subgroup of $\mathrm{SL}(\Lambda)$; see Section 10.1.
- If $\mathrm{ord}(\iota) = 6$, we replace ι with ι^3 and obtain the same contradiction as in the case $\mathrm{ord}(\iota) = 2$.

We conclude that E is irreducible if and only if g is contained in the subgroup $H \simeq BD_{12} \simeq G_\circ$. ☐

We provide an alternative group-theoretic characterization of N_2 and N_3.

Proposition 6.4. *In the notation of Proposition 6.1, we have the following:*

1. N_2 *is the number of conjugacy classes of involutions of* G *if* $X = S^{[2]}$ *or* $\mathrm{K}_2(A)$.
2. N_2 *is the number of conjugacy classes of involutions satisfying Lemma 5.6(1) if* $X = \mathrm{K}_3(A)$.
3. N_3 *is the number of conjugacy classes of subgroups of* G *of order* 3 *satisfying Lemma 5.6(2) if* $X = \mathrm{K}_2(A)$.
4. $N_2 = N_3 = 0$ *in all other cases.*

Proof. Recall that the pointwise stabilizer of F_g is the group generated by g; in symbols,

$$\mathrm{Stab}(F_g) := \{g \in G \mid \forall \underline{x} \in F_g,\ g(\underline{x}) = \underline{x}\} = \langle g \rangle.$$

It is a standard and general fact that $q(F_g) = q(F_{g'})$ if and only if $g' \in h^{-1}\operatorname{Stab}(F_g)h$ for some $h \in G$, hence $q(F_g) = q(F_{g'})$ if and only if $g' \in h^{-1}\langle g\rangle h$ for some $h \in G$. Together with Section 5.2, this gives the group-theoretic characterization of N_2 and N_3 of the statement. □

Lemma 6.5. *If $X = \mathrm{K}_2(A)$, then*

$$N_2 = \begin{cases} 1 & \text{if } |G| \text{ is even,} \\ 0 & \text{if } |G| \text{ is odd.} \end{cases}$$

Proof. If $|G|$ is odd, then we have $N_2 = 0$. Otherwise, any two involutions in G, namely $t_1 := \tau_\alpha(-\mathrm{id})$ and $t_2 := \tau_{\alpha'}(-\mathrm{id})$, are conjugate to each other as $(t_2t_1)^{-1}t_1(t_2t_1) = t_2$; hence $N_2 = 1$. □

7. Third Betti number of a terminalization

Proposition 7.1. *We use Notation 5.8. The third intersection cohomology group of Y is the G-invariant part of the third cohomology group of X; i.e.,*

$$IH^3(Y,\mathbb{Q}) \simeq H^3(X,\mathbb{Q})^G.$$

Proof. Let $\Sigma_2 = \bigcup F_g$ be the components of the singular locus of X/G of pure codimension 2. The decomposition theorem for the semismall terminalization $p\colon Y \to X/G$ gives

$$Rp_*\mathcal{IC}_Y = \mathcal{IC}_{X/G} \oplus \mathcal{IC}_{\Sigma_2}\left(R^2p_*\mathbb{Q}_Y\right)[-2] \oplus \mathcal{S}, \tag{7.1}$$

where $\mathcal{S}$ is a summand of the decomposition theorem supported in codimension at least 4. Over a dense open set of Σ_2, the constructible sheaf $R^2p_*\mathbb{Q}_Y$ is a trivial local system (of rank 1 or 2, more precisely $\mathrm{ord}(g)-1$). The normalization $\nu\colon q(F_g)^\nu \to q(F_g)$ and X/G have quotient singularities, so their intersection complexes are trivial local systems:

$$\mathcal{IC}_{X/G} = \mathbb{Q}_{X/G}, \quad \mathcal{IC}_{q(F_g)} = \nu_*\mathcal{IC}_{q(F_g)^\nu} = \nu_*\mathbb{Q}_{q(F_g)^\nu}.$$

Therefore, we can rewrite (7.1) as

$$Rp_*\mathcal{IC}_Y = \mathbb{Q}_{X/G} \oplus \bigoplus_{q(F_g)\subseteq\Sigma_2} \nu_*\mathbb{Q}_{q(F_g)^\nu}^{\mathrm{ord}(g)-1}[-2] \oplus \mathcal{S}.$$

Taking H^3, we obtain that

$$IH^3(Y,\mathbb{Q}) = H^3(X,\mathbb{Q})^G \oplus \bigoplus_{q(F_g)\subseteq\Sigma_2} H^1\left(q\left(F_g\right)^\nu,\mathbb{Q}\right)^{\mathrm{ord}(g)-1}.$$

Let $\operatorname{Stab}(\{F_g\}) := \{g \in G \mid g(F_g) = F_g\}$ be the setwise stabilizer of F_g. Then the Galois quotient

$$F_g \to F_g/\operatorname{Stab}(\{F_g\}) = q(F_g)^\nu$$

induces the inclusion

$$H^1\left(q\left(F_g\right)^\nu,\mathbb{Q}\right) = H^1\left(F_g,\mathbb{Q}\right)^{\operatorname{Stab}(\{F_g\})} \subseteq H^1\left(F_g,\mathbb{Q}\right).$$

Since in our cases F_g is simply connected, we conclude that $IH^3(Y,\mathbb{Q}) = H^3(X,\mathbb{Q})^G$. □

Proposition 7.2. *We use Notation 5.8. Suppose further that $X = \mathrm{K}_n(A)$ and $G_\circ \neq 1$. Then*

$$H^3(Y,\mathbb{Q}) = IH^3(Y,\mathbb{Q}) = 0.$$

Proof. There exists a G-equivariant isomorphism

$$H^1(A,\mathbb{Z}) \oplus H^3(A,\mathbb{Z}) \simeq H^3(\mathrm{K}_n(A),\mathbb{Z})/\mathrm{Tors};$$

see [KM18, Corollary 6.3] or [O'G21, Theorem 2.7], or the classical version with rational coefficients in [GS93, Theorem 7]. A nontrivial symplectic linear automorphism $g \in G_\circ$ acting on $T_0 A \simeq H^{0,1}(A)$ does not fix any vector, so by Proposition 7.1,

$$0 = H^1(A,\mathbb{Q})^G \oplus H^3(A,\mathbb{Q})^G \simeq H^3(\mathrm{K}_n(A),\mathbb{Q})^G \simeq IH^3(Y,\mathbb{Q}).$$

Finally, note that $H^3(Y,\mathbb{Q}) \simeq IH^3(Y,\mathbb{Q})$ since Y has quotient singularities by Corollary 1.11. □

8. Fundamental group of the regular locus of a terminalization

Proposition 8.1. *Let X be a simply connected smooth complex symplectic variety endowed with an action of a finite group G of symplectic automorphisms. Let $p\colon Y \to X/G$ be a terminalization of the quotient. The fundamental group of the regular locus of Y is*

$$\pi_1(Y^{\mathrm{reg}}) \simeq G/N,$$

where $N \triangleleft G$ is the normal subgroup generated by the elements $\gamma \in G$ whose fixed locus in X has codimension 2. The universal quasi-étale cover of Y is a terminalization of the quotient X/N.

Proposition 8.1 is a refinement of [Men22, Proposition 2.13].

Remark 8.2. The fundamental group of the regular locus of a terminalization of X/G is actually independent of the choice of the given terminalization since all terminalizations of X/G are isomorphic in codimension 1. In general, however, the fundamental group of the regular locus of a variety is not a birational invariant. For instance, the fundamental group of the regular locus of the singular Kummer surface $A_0^{(2)}$ is infinite, but its minimal resolution is simply connected.

Remark 8.3. The subgroup N generated by elements in G whose fixed locus in X admits a component of codimension 2 is a normal subgroup of G. Indeed, the property of an element of having a component of the fixed locus of a certain codimension is invariant up to conjugation: If g fixes a locus F of codimension m, then hgh^{-1} fixes the locus $h(F) \simeq F$ of the same codimension. It follows that any element conjugate to a generator of N is in N; hence N is normal.

Proof of Proposition 8.1. The quotient map $q\colon X \to X/G$ is étale over the regular locus of X/G. Therefore, we have a short exact sequence

$$1 \longrightarrow \pi_1\left(q^{-1}((X/G)^{\mathrm{reg}})\right) \longrightarrow \pi_1((X/G)^{\mathrm{reg}}) \longrightarrow G \longrightarrow 1.$$

Since X is simply connected and q is étale in codimension 1, we have $\pi_1((X/G)^{\mathrm{reg}}) \simeq G$. As $(X/G)^{\mathrm{reg}}$ can be identified with a Zariski dense open subset of Y^{reg}, we have a surjective map

$$G \simeq \pi_1((X/G)^{\mathrm{reg}}) \twoheadrightarrow \pi_1(Y^{\mathrm{reg}}).$$

Let F be a codimension 2 subvariety of X fixed by an element of G. An analytic neighborhood U of a general point of $q(F)$ in X/G is isomorphic to an analytic open set of $\mathbb{A}^{\dim X-2} \times W$, where W is the canonical surface singularity $\mathbb{A}^2/\mathrm{Stab}(F)$. The restriction of a terminalization $p\colon Y \to X/G$ to U is isomorphic to an analytic simply connected open subset $\widetilde{U}$ of $\mathbb{A}^{\dim X-2} \times \widetilde{W}$, where $\widetilde{W}$ is the unique (simply connected) minimal resolution of W. By inclusions, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Stab}(F) = \pi_1\left(\mathbb{A}^2 \setminus \{0\}/\mathrm{Stab}(F)\right) = \pi_1(U \cap (X/G)^{\mathrm{reg}}) & \longrightarrow & \pi_1\left(\widetilde{U}\right) = 1 \\ \downarrow & & \downarrow \\ \pi_1((X/G)^{\mathrm{reg}}) & \twoheadrightarrow & \pi_1(Y^{\mathrm{reg}}). \end{array}$$

Therefore, there exists a surjective map

$$G/N \twoheadrightarrow \pi_1(Y^{\mathrm{reg}}).$$

We prove that the previous surjection is invertible. Let $p_N\colon Y_N \to X/N$ be a G/N-equivariant terminalization of X/N. We obtain the following commutative square:

$$\begin{array}{ccc} X/N & \xleftarrow{p_N} & Y_N \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X/G & \xleftarrow{p} & Y_N/(G/N), \end{array}$$

where the horizontal arrows are birational morphisms and the vertical arrows G/N-quotient maps. Let $(X/G)^\circ$ be the complement of the dissident locus; see Definition 3.6. By the definition of N, q_1 is étale over $(X/G)^\circ$, so $p^{-1}((X/G)^\circ)$ is a symplectic resolution of $(X/G)^\circ$ built via the same sequence of blowups which gives Y_N over $(X/N)^\circ$. We conclude that $Y_N/(G/N)$ is a terminalization of X/G by Proposition 3.7, and by Remark 8.2, there exists a surjective morphism

$$\pi_1(Y^{\mathrm{reg}}) \twoheadrightarrow G/N. \qquad \square$$

Corollary 8.4. *We use Notation 5.8. The fundamental group of the regular locus of Y is*

$$\pi_1(Y^{\mathrm{reg}}) \simeq G/N,$$

where N is the normal subgroup generated by all elements of

- *order 2 if $X = S^{[2]}$,*
- *order 2 and 3 satisfying Lemma 5.6(1) and (2) if $X = \mathrm{K}_2(A)$,*
- *order 2 satisfying Lemma 5.6(1) if $X = \mathrm{K}_3(A)$.*

In all other cases, $\pi_1(Y^{\mathrm{reg}}) \simeq G$.

9. Terminalizations of quotients of Hilbert schemes on K3 surfaces

Symplectic actions of finite groups on $S^{[2]}$ have been classified in [HM19, Table 12]. Here we restrict to the groups G of even order whose action comes from an action on the underlying K3 surface S (which are marked with the label *Type K3* in the fourth column of [HM19, Table 12]). Since any involution gives rise to a surface with transversal A_1 singularities in $S^{[2]}/G$ (see Remark 5.5), the previous conditions grant that the quotient $S^{[2]}/G$ is not terminal, as required by our criteria of classification (*cf.* Section 4).

In Table 4, for any such group G, we list

- the group ID as in GroupNames,
- an alias of G as abstract group,
- $\mathrm{rk} := \mathrm{rk}\left(H^2(S^{[2]})^G\right) = \mathrm{rk}\left(H^2(S)^G\right) + 1$, as computed in [HM19, Table 12, fifth column],
- the number N_2 of codimension 2 components of the singular locus of $S^{[2]}/G$, as computed in Proposition 6.4(1),
- $b_2(Y) = \mathrm{rk} + N_2$, see Proposition 6.1 and Lemma 5.4 for the fact that $N_3 = 0$,
- $\pi_1(Y^{\mathrm{reg}}) \simeq G/N$, where N is the subgroup generated by involutions, see Corollary 8.4.

We highlight in gray the quotients whose terminalization has simply connected regular locus.

Table 4. Terminalizations of $S^{[2]}/G$

ID	G	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\mathrm{reg}})$
2,1	C_2	15	1	16	{1}
4,1	C_4	9	1	10	C_2
4,2	C_2^2	11	3	14	{1}
6,1	S_3	9	1	10	{1}
6,2	C_6	7	1	8	C_3

ID	G	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\mathrm{reg}})$
8,1	C_8	5	1	6	C_4
8,2	$C_2 \times C_4$	7	3	10	C_2
8,3	D_4	8	3	11	{1}
8,4	Q_8	6	1	7	C_2^2
8,5	C_2^3	9	7	16	{1}
10,1	D_5	7	1	8	{1}
12,1	BD_{12}	5	1	6	S_3
12,3	A_4	7	1	8	C_3
12,4	D_6	7	3	10	{1}
12,5	$C_2 \times C_6$	5	3	8	C_3
16,2	C_4^2	5	3	8	C_2^2
16,3	$C_2^2 \rtimes C_4$	6	5	11	C_2
16,6	$M_4(2)$	4	2	6	C_4
16,8	$Q_8 \rtimes C_2$	5	2	7	C_2
16,9	Q_{16}	4	1	5	D_4
16,11	$C_2 \times D_4$	7	7	14	{1}
16,12	$C_2 \times Q_8$	5	3	8	C_2^2
16,13	$C_4 \circ D_4$	6	4	10	{1}
16,14	C_2^4	8	15	23	{1}
18,3	$C_3 \times S_3$	5	1	6	C_3
18,4	$C_3 \rtimes S_3$	7	1	8	{1}
20,3	$C_5 \rtimes C_4$	5	1	6	C_2
24,3	$Q_8 \rtimes C_3$	4	1	5	A_4
24,8	$C_3 \rtimes D_4$	5	3	8	{1}
24,12	S_4	6	2	8	{1}
24,13	$C_2 \times A_4$	5	3	8	C_3
32,6	$C_2^3 \rtimes C_4$	5	5	10	C_2
32,7	$C_4.D_4$	4	4	8	C_2
32,11	$C_4 \wr C_2$	4	3	7	C_2
32,27	$C_2^2 \wr C_2$	6	10	16	{1}
32,31	$C_4 \cdot_4 D_4$	5	5	10	C_2
32,44	$C_8.C_2^2$	4	3	7	C_2
32,49	$D_4 \circ D_4$	6	10	16	{1}
36,9	$C_3^2 \rtimes C_4$	5	1	6	C_2
36,10	S_3^2	5	3	8	{1}
36,11	$C_3 \times A_4$	5	1	6	C_3^2
48,3	$C_4^2 \rtimes C_3$	5	1	6	A_4
48,29	$Q_8 \rtimes S_3$	4	2	6	{1}
48,30	$A_4 \rtimes C_4$	4	3	7	S_3
48,48	$C_2 \times S_4$	5	5	10	{1}
48,49	$C_2^2 \times A_4$	4	7	11	C_3
48,50	$C_2^2 \rtimes A_4$	6	5	11	C_3
60,5	A_5	5	1	6	{1}
64,32	$C_2 \wr C_4$	4	6	10	C_2
64,35	$C_4^2 \rtimes_3 C_4$	4	4	8	C_2^2

ID	G	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\text{reg}})$
64,136	$D_4 \cdot_9 D_4$	4	6	10	C_2
64,138	$C_2 \wr C_2^2$	5	9	14	$\{1\}$
64,242	$C_2^4 \rtimes C_2^2$	5	9	14	$\{1\}$
72,40	$S_3 \wr C_2$	4	3	7	$\{1\}$
72,41	$C_3^2 \rtimes Q_8$	4	1	5	C_2^2
72,43	$C_3 \rtimes S_4$	5	2	7	$\{1\}$
80,49	$C_2^4 \rtimes C_5$	4	3	7	C_5
96,64	$C_4^2 \rtimes S_3$	4	2	6	$\{1\}$
96,70	$C_2^4 \rtimes C_6$	4	4	8	C_3
96,195	$A_4 \rtimes D_4$	4	6	10	$\{1\}$
96,204	$C_2^3 \rtimes A_4$	4	4	8	C_3
96,227	$C_2^2 \rtimes S_4$	5	5	10	$\{1\}$
120,34	S_5	4	2	6	$\{1\}$
128,931	$C_4^2 \rtimes_5 D_4$	4	7	11	$\{1\}$
144,184	A_4^2	4	3	7	C_3^2
160,234	$C_2^4 \rtimes D_5$	4	4	8	$\{1\}$
168,42	$\mathrm{GL}_3(\mathbb{F}_2)$	4	1	5	$\{1\}$
192,955	$C_2^4 \rtimes D_6$	4	6	10	$\{1\}$
192,1023	$C_4^2 \rtimes A_4$	5	3	8	C_3
192,1493	$C_2^3 \rtimes S_4$	4	6	10	$\{1\}$
288,1026	$A_4 \rtimes S_4$	4	4	8	$\{1\}$
360,118	A_6	4	1	5	$\{1\}$
384,18135	F_{384}	4	4	8	$\{1\}$
960,11357	M_{20}	4	2	6	$\{1\}$

Proposition 9.1. *All terminalizations in Table 4 are singular with the exception of* $G \simeq C_2^4$.

Proof. If the terminalization Y is smooth, then in particular the quotient $S^{(2)}/G$ does not admit an isolated singularity $[(x,y)]$ with $x \neq y$. In fact, such points lie in the locus where the birational morphism $Y \to S^{(2)}/G$ is an isomorphism.

Equivalently, for any $g \in G$, there exists no point $(x,y) \in S^2$ such that $(g(x), g(y)) = (x,y)$ and $y \notin Gx$. Otherwise, such a g-fixed point would not be of the form $(x, \iota(x))$ for any involution $\iota \in G$, so it would not lie on a codimension 2 component of the fixed locus of some element $g \in G$, and it certainly would give rise to a singularity of Y.

Equivalently, if Y is smooth, then the following statement holds true.

Assumption 9.2. For any $g \in G$, the fixed locus $\mathrm{Fix}(g) \subset S$ lies in a fiber of $\pi : S \to S/G$.

We deduce the following Lemmas 9.3 and 9.4.

Lemma 9.3. *Under Assumption 9.2,* $\mathrm{ord}(g) \cdot |\mathrm{Fix}(g)|$ *divides* $|G|$ *for any* $g \in G$.

Proof of Lemma 9.3. Given $g \in G$ and $x \in \mathrm{Fix}(g) \subset S$, any point of the orbit Gx is fixed by a conjugate of g since $\mathrm{Fix}(hgh^{-1}) = h\,\mathrm{Fix}(g)$ with $h \in G$. Moreover, $\mathrm{Fix}(g) \subseteq Gx$ by Assumption 9.2, so the orbit Gx is the disjoint union of the fixed loci of conjugates of g, and $|\mathrm{Fix}(g)|$ divides $|Gx|$. We conclude that

$$\mathrm{ord}(g) \cdot |\mathrm{Fix}(g)| \text{ divides } |G_x| \cdot |Gx| = |G|. \qquad \square$$

Lemma 9.4. *Under Assumption 9.2, there exists a bijective correspondence*

$$\{\textit{conjugacy classes of involutions of } G\} \longleftrightarrow \{\textit{singular points in } S/G \textit{ with even isotropy}\}$$
$$[\iota] \longmapsto \pi(\mathrm{Fix}(\iota)).$$

Proof of Lemma 9.4. The correspondence $\iota \mapsto \pi(\mathrm{Fix}(\iota))$ is well defined since $\mathrm{Fix}(\iota)$ lies in the same π-fiber by Assumption 9.2. It is also independent of the representative of $[\iota]$ since $\mathrm{Fix}(g\iota g^{-1}) = g\,\mathrm{Fix}(\iota)$. The inverse map sends a singular point q to the conjugacy class of the unique involution of G_x for any $x \in \pi^{-1}(q)$. The uniqueness of such an involution follows from the faithfullness of the action of G_x as finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ on the tangent space T_xS (recall that there is a unique involution in $\mathrm{SL}(2,\mathbb{C})$). □

By [Gua01], if Y is a smooth IHS fourfold, then either $3 \le b_2(Y) \le 8$ or $b_2(Y) = 23$, and according to Table 4, the latter occurs only if $G \simeq C_2^4$. All terminalizations Y in Table 4 with $b_2(Y) \le 8$ and $\pi_1(Y^{\mathrm{reg}}) = \{1\}$ fail to satisfy the necessary conditions for smoothness detailed in Lemmas 9.3 and 9.4. In order to apply these lemmas, we use the classification of the singularities of S/G obtained in [Xia96] and the computation of the cardinality of $\mathrm{Fix}(g_n) \subset S$ for a symplectic automorphism g_n on S of order n, contained for instance in [Nik80, Section 5]:

n	2	3	4	5	6	7	8
$\lvert\mathrm{Fix}(g_n)\rvert$	8	6	4	4	2	3	2

More precisely, we are able to exclude all the cases, as
- if $|G| = 160$ or 288, Lemma 9.4 fails,
- if $|G| = 48, 96, 384$ or 960, Lemma 9.3 fails since $3 \mid |G|$ but $3 \cdot 6 \nmid |G|$,
- for all other groups G, Lemma 9.3 fails since $2 \mid |G|$ but $2 \cdot 8 \nmid |G|$. □

Theorem 9.5. *Let G be a finite group of induced symplectic automorphisms acting on $S^{[2]}$, and let Y be a projective terminalization of $S^{[2]}/G$ with simply connected regular locus. There are at least five new deformation classes of such irreducible symplectic varieties Y. In particular, they are not deformation equivalent to any terminalization of quotients of Kummer fourfolds by groups of induced symplectic automorphisms, or a Fujiki fourfolds appearing in* [Men22, Theorem 1.11].

ID	G	$b_2(Y)$
10,1	D_5	8
60,5	A_5	6
120,34	S_5	6
168,42	$\mathrm{GL}_3(\mathbb{F}_2)$	5
360,118	A_6	5

Proof. If the projective terminalizations $Y_1 \to S^{[2]}/G_1$ and $Y_2 \to S^{[2]}/G_2$ are deformation equivalent, then $b_2(Y_1) = b_2(Y_2)$ and $\sqrt{|G_1|/|G_2|}$ is a rational number; see [Men22, Proposition 3.21, Proof of Proposition 1.13]. We then conclude by direct inspections of Tables 4 and 9 and [Men22, Section 5].

Note that the terminalization $Y \to S^{[2]}/D_5$ has $b_2(Y) = 8$, but it cannot be deformation equivalent to any of the new terminalizations with $b_2 = 8$ in Table 9. Indeed, the subgroup $C_5 \triangleleft D_5$ fixes two points z_1, z_2 on S lying in different D_5-orbits (*cf.* [Xia96, Theorem 3]), and so the point $(z_1, z_2) \in S^2$ corresponds to an isolated singularity of Y with isotropy C_5, but this singularity never appears in Table 9. □

Remark 9.6. The terminalizations of $S^{[2]}/G$ in Table 4 are Fujiki varieties $S(G)^{[2]}_{\theta}$ with trivial involution $\theta = \mathrm{id}$; see Definition 12.2. More information on their singularities is available in [Men22], provided that G is an *admissible* group of induced symplectic automorphisms; see [Men22, Definition 1.10].

10. Terminalizations of quotients of generalized Kummer manifolds

In this section, we compute the second Betti number and the fundamental groups of the regular locus of terminalizations of quotients of $\mathrm{K}_2(A)$ and $\mathrm{K}_3(A)$ by finite groups of induced symplectic automorphisms; see Tables 7 and 8, respectively.

10.1. Symplectic automorphisms of an abelian surfaces

Let G be a finite group of symplectic automorphisms of an abelian surface A. In the notation of Section 2, the group $G \subseteq A[n+1] \rtimes \mathrm{SL}(\Lambda)$ fits in the short exact sequence

$$1 \longrightarrow G_{\mathrm{tr}} \longrightarrow G \longrightarrow G_\circ \longrightarrow 1.$$

By the classification of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ together with [Fuj88, Lemma 3.3], $G_\circ$ is isomorphic to $\{1\}$, C_m for $m \in \{2,3,4,6\}$, Q_8, BD_{12} or BT_{24}. Moreover, by [Fuj88, Remarks 3.6 and 3.12], $(A, G_\circ)$ is deformation equivalent to one of the following:

$$\begin{array}{llll}
(A, \langle -\mathrm{id}\rangle \simeq C_2), & & & \\
(E^2, \langle g_m\rangle \simeq C_m) & \text{for } E = \mathbb{C}/\langle 1, \xi_m\rangle, & g_m = \begin{pmatrix} \xi_m & 0 \\ 0 & \xi_m^{-1}\end{pmatrix}, & \xi_m = e^{\frac{2\pi i}{m}}, \\
(E^2, \langle h, k\rangle \simeq Q_8) & \text{for } E = \mathbb{C}/\langle 1, i\rangle, & h = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, & k = g_4 = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}, \\
(\mathbb{H}/\Gamma, \langle i, j\rangle \simeq Q_8) & \text{for } \mathbb{H} = \mathbb{R}[1,i,j,k], & \Gamma = \mathbb{Z}[1,i,j,t], & t = \frac{1+i+j+k}{2}, \\
(E^2, \langle h, l\rangle \simeq BD_{12}) & \text{for } E = \mathbb{C}/\langle 1, \xi_6\rangle, & h = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, & l = g_6 = \begin{pmatrix} \xi_6 & 0 \\ 0 & \xi_6^{-1}\end{pmatrix}, \\
(\mathbb{H}/\Gamma, \Gamma^\times \simeq BT_{24}) & \text{for } \mathbb{H} = \mathbb{R}[1,i,j,k], & \Gamma = \mathbb{Z}[1,i,j,t], & t = \frac{1+i+j+k}{2}, \\
 & & \Gamma^\times = \langle r, t\rangle, & r = \frac{1+i+j-k}{2};
\end{array}$$

see also [Fuj88, Proposition 3.7, Lemma 2.6] and the surveys [Pie22, Section 2.2] and [KMO23, Appendix 2]. Therefore, without loss of generality, we can assume that $G_\circ$ acts on A as above, and we identify actions of G up to conjugation in $A[n+1] \rtimes \mathrm{SL}(\Lambda)$. In fact, the topological invariants we are interested in, that is, $b_2(Y)$ and $\pi_1(Y^{\mathrm{reg}})$, are independent on the deformation type of the pair (A, G) and invariant under conjugation in $A[n+1] \rtimes \mathrm{SL}(\Lambda)$.

Lemma 10.1. *Let G be a finite group of symplectic automorphisms of an abelian surface A. Then,*

$$\mathrm{rk}(H^2(A)^G) = \begin{cases} 6 & \text{if } G_\circ \simeq C_2, \\ 4 & \text{if } G_\circ \simeq C_3, C_4, C_6, \\ 3 & \text{if } G_\circ \simeq Q_8, BD_{12}, BT_{24}. \end{cases}$$

Proof. Note that the group

$$A \rtimes (-\mathrm{id}) = \ker\{A \rtimes \mathrm{SL}(\Lambda) \longrightarrow \mathrm{SL}(\Lambda) \longrightarrow \mathrm{PSL}(\Lambda)\}$$

acts trivially on $H^2(A)$, so $H^2(A)^G = H^2(A)^{G_\circ}$ and if $-\mathrm{id} \in G_\circ$, we have $H^2(A)^G = H^2(A)^{G_\circ/\langle -\mathrm{id}\rangle}$. The claim then follows from [Fuj88, Section 6]. ☐

10.2. Second Betti numbers and fundamental groups of terminalizations

Let G be a finite group of induced symplectic automorphisms of $\mathrm{K}_2(A)$ or $\mathrm{K}_3(A)$. In Tables 7 and 8, we list

- the group ID of G as in GroupNames, when available (otherwise, we write NA),

- an alias of G as abstract group (we express G as a (split or non-split) extension of $G_\circ$ by $G_{\rm tr}$ and, when available, we adopt the enumeration of extensions in GroupNames; otherwise, we add the subscript * for unnumbered extensions),
- $\mathrm{rk} := \mathrm{rk}\left(H^2(\mathrm{K}_2(A))^G\right) = \mathrm{rk}(H^2(A)^G) + 1$, where the ranks are computed in Lemma 10.1,
- the number N_i of components of codimension 2 of the singular locus of $\mathrm{K}_n(A)/G$ with transversal A_{i-1} singularities, see Proposition 6.4,
- $b_2(Y)$, see Proposition 6.1 and Remark 10.5,
- $\pi_1(Y^{\rm reg})$, see Corollary 8.4.

The explicit values of N_i, $b_2(Y)$ and the groups $\pi_1(Y^{\rm reg})$ can be obtained using GAP.(8) We highlight in gray the quotients whose terminalization has simply connected regular locus.

Example 10.2. Let ξ_3 be a primitive third root of unity, and let E be an elliptic curve with complex multiplication $\xi_3 \curvearrowright E\colon x \mapsto \xi_3 \cdot x$. Consider the symplectic automorphism $g_3(x_1,x_2) = (\xi_3 x_1, \xi_3^{-1} x_2)$. Choose $a = (a_1,a_2)$ and $b = (b_1,b_2)$ in $E^2[3]$ such that $g_3(a) = a$ and $g_3(b) \neq b$. Denote by $\tau_a, \tau_b\colon E^2 \to E^2$ the translations $\tau_a(x_1,x_2) = (x_1 + a_1, x_2 + a_2)$ and $\tau_b(x_1,x_2) = (x_1 + b_1, x_2 + b_2)$. Now, both $\tau_a g_3$ and $\tau_b g_3$ induce the same action on $H^*(\mathrm{K}_2(E^2),\mathbb{Z})$. However, the quotient $\mathrm{K}_2(E^2)/\langle \tau_a g_3\rangle$ has strictly canonical singularities, while $\mathrm{K}_2(E^2)/\langle \tau_b g_3\rangle$ is terminal. The actions correspond to the two distinct rows for the cyclic group C_3 in Table 7.

Remark 10.3 (Group actions *vs.* abstract realization). In Example 10.2, we pointed out that different actions of the same abstract group may lead to terminalizations with different deformation type. This explains why in the tables below, the same abstract group may appear multiple times. *A priori*, the table should include all possible actions, namely all possible subgroups of $A[n+1] \rtimes G_\circ$. A GAP code can easily provide all of them and their relevant invariants, but to avoid redundancy,

we identify groups which are conjugate in $A[n+1] \rtimes G_\circ$
or give the same string of invariants $[G, G_\circ, \mathrm{rk}, N_i, b_2, \pi_1]$.

Note that the latter condition leaves open the possibility that there may be terminalizations with different deformation type but same string of invariants; see for instance Remark 10.4. In this regard, we check the following facts:

- Using a GAP code, we observe that the actions whose quotients have a smooth terminalization, *i.e.*, C_3^3 in Table 7 and C_2^5 in Table 8, are unique up to conjugation in $A[n+1] \rtimes G_\circ$.
- By elementary algebraic considerations, the actions whose quotients admit a terminalization with simply connected regular locus (the most relevant according to Section 4) are all affine; *i.e.*, the group G is conjugate to the semidirect product $G_{\rm tr} \rtimes G_\circ$ by an element in $A[n+1]$; see Lemma 10.6.

Remark 10.4 (Quaternion group). The moduli space of pairs $(A, G_\circ)$, where A is an abelian surface and $G_\circ$ is a symplectic group of linear automorphisms, is connected except for $G_\circ = Q_8$, in which case it has two connected components represented by the pairs (E^2, Q_8) and $(\mathbb{H}/\Gamma, Q_8)$ defined in Section 10.1; see [Fuj88, Remark 3.12]. The action of Q_8 on E^2 is maximal; *i.e.*, it is not contained in any other finite subgroup of $\mathrm{Aut}(E^2)$ (fixing the origin), while the action on $\mathbb{H}/\Gamma$ is the restriction of the action of BT_{24} in Section 10.1. The induced actions on $\mathrm{K}_3(A)$ of their overgroups give rise to terminalizations with different b_2 and π_1; we distinguish the two cases in Table 9.

Accidentally, the terminalizations of $\mathrm{K}_2(E^2)/G$ and $\mathrm{K}_2(\mathbb{H}/\Gamma)/G$, with $G_\circ = Q_8$, have the same b_2 and π_1 (because the groups $E^2[3] \rtimes Q_8$ and $\mathbb{H}/\Gamma[3] \rtimes Q_8$ turn out to be abstractly isomorphic to the group of ID 648,730). Therefore, we write them only once in Table 7. Mind, however, that the terminalizations are not deformation equivalent. For instance, the terminalizations of the quotients $\mathrm{K}_2(E^2)/Q_8$ and $\mathrm{K}_2(\mathbb{H}/\Gamma)/Q_8$

(8)A GAP code containing all the calculations is available from the authors upon request.

have different singularities:

$$\begin{aligned} \mathrm{Sing}\left(\mathrm{K}_2\left(E^2, Q_8\right)\right): &\quad 4Q_8 + 6C_4 + 29C_2, \\ \mathrm{Sing}(\mathrm{K}_2(\mathbb{H}/\Gamma, Q_8)): &\quad 2Q_8 + 9C_4 + 28C_2, \end{aligned}$$

where $m \cdot G_x$ means that the terminalization has m isolated singularities with isotropy G_x. These singularities are computed in the same way as in Section 11.5; we omit the details.

Remark 10.5 (Binary dihedral group). If $G \simeq C_3^{2k} \rtimes BD_{12}$ with $k = 0, 1$ or 2, then there exists a unique conjugacy class of subgroups of order 3 satisfying Lemma 5.6(2) and further contained in a subgroup of G isomorphic to BD_{12}, splitting the projection $G \simeq C_3^{2k} \rtimes BD_{12} \to BD_{12}$. So if $G_\circ \simeq BD_{12}$, by Lemma 6.3, the formula in Proposition 6.1 acquires a correction term as follows:

$$b_2(Y) = \mathrm{rk}\left(L^G\right) + N_2 + 2N_3 - 1.$$

Table 7. Terminalizations of $\mathrm{K}_2(A)/G$

ID	G	$G_\circ$	rk	N_2	N_3	$b_2(Y)$	$\pi_1(Y^{\mathrm{reg}})$
2,1	C_2			1	0	8	$\{1\}$
6,1	$C_3 \rtimes C_2$			1	0	8	$\{1\}$
18,4	$C_3^2 \rtimes_2 C_2$	C_2	7	1	0	8	$\{1\}$
54,14	$C_3^3 \rtimes C_2$			1	0	8	$\{1\}$
162,54	$C_3^4 \rtimes C_2$			1	0	8	$\{1\}$
3,1	C_3			0	0	5	C_3
3,1	C_3			0	1	7	$\{1\}$
9,2	C_3^2			0	0	5	C_3^2
9,2	C_3^2			0	3	11	$\{1\}$
27,3	$C_3^2 \rtimes C_3$			0	0	5	$C_3^2 \rtimes C_3$
27,3	$C_3^2 \rtimes C_3$	C_3	5	0	1	7	C_3
27,5	C_3^3			0	0	5	C_3^3
27,5	C_3^3			0	9	23	$\{1\}$
81,12	$C_3^3 \rtimes_2 C_3$			0	0	5	$C_3^3 \rtimes_2 C_3$
81,12	$C_3^3 \rtimes_2 C_3$			0	3	11	C_3
243,37	$C_3^4 \rtimes_1 C_3$			0	1	7	C_3^2
4,1	C_4			1	0	6	C_2
36,9	$C_3^2 \rtimes C_4$	C_4	5	1	0	6	C_2
324,164	$C_3^4 \rtimes_4 C_4$			1	0	6	C_2
6,2	C_6			1	1	8	$\{1\}$
18,3	$C_3 \rtimes C_6$			1	2	10	$\{1\}$
54,13	$C_3^2 \rtimes_4 C_6$	C_6	5	1	5	16	$\{1\}$
54,5	$C_3^2 \rtimes C_6$			1	1	8	$\{1\}$
162,40	$C_3^3 \rtimes_4 C_6$			1	2	10	$\{1\}$
486,146	$C_3^4 \rtimes_4 C_6$			1	1	8	$\{1\}$
8,4	Q_8			1	0	5	C_2^2
72,41	$C_3^2 \rtimes Q_8$	Q_8	4	1	0	5	C_2^2
648,730	$C_3^4 \rtimes Q_8$			1	0	5	C_2^2
12,1	BD_{12}			1	1	6	C_2
108,37	$C_3^2 \rtimes_3 BD_{12}$			1	3	10	C_2

ID	G	$G_\circ$	rk	N_2	N_3	$b_2(Y)$	$\pi_1(Y^{\text{reg}})$
972,NA	$C_3^4 \rtimes_* BD_{12}$			1	1	6	C_2
24,3	BT_{24}			1	1	7	$\{1\}$
216,153	$C_3^2 \rtimes BT_{24}$	BT_{24}	4	1	1	7	$\{1\}$
1944,NA	$C_3^4 \rtimes_* BT_{24}$			1	1	7	$\{1\}$

Table 8. Terminalizations of $\mathrm{K}_3(A)/G$

ID	G	$G_\circ$	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\text{reg}})$
2,1	C_2			0	7	C_2
2,1	C_2			1	8	$\{1\}$
4,2	C_2^2			0	7	C_2^2
4,2	C_2^2			2	9	$\{1\}$
8,5	C_2^3			0	7	C_2^3
8,3	$C_4 \rtimes C_2$			1	8	C_2
8,3	$C_4 \rtimes C_2$			0	7	$C_4 \rtimes C_2$
8,5	C_2^3			4	11	$\{1\}$
16,11	$C_2^3 \rtimes C_2$			0	7	$C_2^3 \rtimes C_2$
16,11	$C_2^3 \rtimes C_2$			2	9	C_2
16,14	C_2^4			0	7	C_2^4
16,14	C_2^4			8	15	$\{1\}$
32,46	$(C_2^2 \times C_4) \rtimes_5 C_2$	C_2	7	0	7	$(C_2^2 \times C_4) \rtimes_5 C_2$
32,34	$C_4^2 \rtimes_6 C_2$			0	7	$C_4^2 \rtimes_6 C_2$
32,51	C_2^5			0	7	C_2^5
32,51	C_2^5			16	23	$\{1\}$
32,46	$(C_2^2 \times C_4) \rtimes_5 C_2$			4	11	C_2
32,34	$C_4^2 \rtimes_6 C_2$			1	8	C_2^2
64,211	$(C_2 \times C_4^2) \rtimes_{11} C_2$			0	7	$(C_2 \times C_4^2) \rtimes_{11} C_2$
64,261	$(C_2^3 \times C_4) \rtimes_7 C_2$			0	7	$(C_2^3 \times C_4) \rtimes_7 C_2$
64,211	$(C_2 \times C_4^2) \rtimes_{11} C_2$			2	9	C_2^2
64,261	$(C_2^3 \times C_4) \rtimes_7 C_2$			8	15	C_2
128,2172	$(C_2^2 \times C_4^2) \rtimes_{23} C_2$			0	7	$(C_2^2 \times C_4^2) \rtimes_{23} C_2$
128,2172	$(C_2^2 \times C_4^2) \rtimes_{23} C_2$			4	11	C_2^2
128,1599	$C_4^3 \rtimes_{15} C_2$			0	7	$C_4^3 \rtimes_{15} C_2$
128,1599	$C_4^3 \rtimes_{15} C_2$	C_2	7	1	8	C_2^3
256,29630	$(C_2 \times C_4^3) \rtimes_* C_2$			2	9	C_2^3
256,29630	$(C_2 \times C_4^3) \rtimes_* C_2$			0	7	$(C_2 \times C_4^3) \rtimes_* C_2$
512,NA	$C_4^4 \rtimes C_2$			1	8	C_2^4
3,1	C_3					C_3
12,3	$C_2^2 \rtimes C_3$					$C_2^2 \rtimes C_3$
48,3	$C_4^2 \rtimes C_3$	C_3	5	0	5	$C_4^2 \rtimes C_3$
48,50	$C_2^4 \rtimes_2 C_3$					$C_2^4 \rtimes_2 C_3$
192,1020	$(C_2^2 \times C_4^2) \rtimes_3 C_3$					$(C_2^2 \times C_4^2) \rtimes_3 C_3$
768,1083578	$C_4^4 \rtimes C_3$					$C_4^4 \rtimes C_3$
4,1	C_4			0	5	C_4
4,1	C_4			1	6	C_2

ID	G	$G_\circ$	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\mathrm{reg}})$
8,2	$C_2\times C_4$			0	5	$C_2\times C_4$
8,2	$C_2\times C_4$			2	7	C_2
16,3	$C_2^2\rtimes C_4$			3	8	C_2
16,10	$C_2^2\times C_4$			4	9	C_2
16,3	$C_2^2\rtimes C_4$			0	5	$C_2^2\rtimes C_4$
16,10	$C_2^2\times C_4$			0	5	$C_2^2\times C_4$
32,22	$C_2^3\rtimes_2 C_4$			0	5	$C_2^3\rtimes_2 C_4$
32,6	$(C_2\times C_4)\rtimes C_4$			2	7	C_2^2
32,6	$(C_2\times C_4)\rtimes C_4$			1	6	C_4
32,6	$(C_2\times C_4)\rtimes C_4$			2	7	C_4
32,6	$(C_2\times C_4)\rtimes C_4$			0	5	$(C_2\times C_4)\rtimes C_4$
32,22	$C_2^3\rtimes_2 C_4$			6	11	C_2
64,34	$C_4^2\rtimes C_4$			0	5	$C_4^2\rtimes C_4$
64,34	$C_4^2\rtimes C_4$			1	6	$C_2^2\rtimes C_2$
64,90	$(C_2^2\times C_4)\rtimes_3 C_4$	C_4	5	0	5	$(C_2^2\times C_4)\rtimes_3 C_4$
64,90	$(C_2^2\times C_4)\rtimes_3 C_4$			2	7	C_4
64,90	$(C_2^2\times C_4)\rtimes_3 C_4$			4	9	$C_2\times C_2$
64,60	$C_2^4\rtimes_3 C_4$			0	5	$C_2^4\rtimes_3 C_4$
64,60	$C_2^4\rtimes_3 C_4$			10	15	C_2
128,856	$(C_2\times C_4^2)\rtimes_9 C_4$			0	5	$(C_2\times C_4^2)\rtimes_9 C_4$
128,513	$(C_2^3\times C_4)\rtimes_6 C_4$			6	11	C_2^2
128,513	$(C_2^3\times C_4)\rtimes_6 C_4$			4	9	C_4
128,513	$(C_2^3\times C_4)\rtimes_6 C_4$			0	5	$(C_2^3\times C_4)\rtimes_6 C_4$
128,856	$(C_2\times C_4^2)\rtimes_9 C_4$			2	7	$C_2^2\rtimes C_2$
256,5681	$(C_2^2\times C_4^2)\rtimes_* C_4$			3	8	$C_2^2\rtimes C_2$
256,5681	$(C_2^2\times C_4^2)\rtimes_* C_4$			0	5	$(C_2^2\times C_4^2)\rtimes_* C_4$
256,1534	$(C_2^2\times C_4^2)\rtimes_* C_4$			2	7	$C_2\times C_4$
256,1534	$(C_2^2\times C_4^2)\rtimes_* C_4$			4	9	C_2^3
512,NA	$(C_2\times C_4^3)\rtimes_* C_4$			2	7	$C_2^3\rtimes C_2$
512,NA	$(C_2\times C_4^3)\rtimes_* C_4$			1	6	$C_2^2\rtimes C_4$
1024,NA	$C_4^4\rtimes_* C_4$			1	6	$C_2^4\rtimes_1 C_2$
6,2	C_6			1	6	C_3
24,13	$C_2^2\rtimes C_6$			2	7	C_3
96,72	$C_4^2\rtimes_2 C_6$	C_6	5	1	6	$C_2^2\rtimes C_3$
96,229	$C_2^4\rtimes_4 C_6$			6	11	C_3
384,18223	$(C_2^2\times C_4^2)\rtimes_* C_6$			2	7	$C_2^2\rtimes C_3$
1536,NA	$C_4^4\rtimes_* C_6$			1	6	$C_2^4\rtimes_2 C_3$
8,4	Q_8			0	4	Q_8
8,4	Q_8			1	5	C_2^2
16,4	$C_2.Q_8$			0	4	$C_2.Q_8$
16,12	$C_2\times Q_8$	Q_8	4	0	4	$C_2\times Q_8$
16,4	$C_2.Q_8$			2	6	C_2
16,12	$C_2\times Q_8$			2	6	C_2^2
32,29	$C_2^2\rtimes Q_8$			0	4	$C_2^2\rtimes Q_8$
32,29	$C_2^2\rtimes Q_8$			4	7	C_2^2
64,224	$C_2^3\rtimes_2 Q_8$			5	9	C_2^2

ID	G	$G_\circ$	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\mathrm{reg}})$
64,23	$C_2^3{\cdot_2}Q_8$			5	9	C_2^2
64,23	$C_2^3{\cdot_2}Q_8$			0	4	$C_2^3{\cdot_2}Q_8$
64,224	$C_2^3 \rtimes_2 Q_8$			0	4	$C_2^3 \rtimes_2 Q_8$
128,764	$(C_2^2 \times C_4) \rtimes Q_8$	Q_8	4	1	5	$C_2{\cdot_2}C_2^2$
128,764	$(C_2^2 \times C_4) \rtimes Q_8$			4	8	C_2^3
128,761	$C_2^4 \rtimes_2 Q_8$			0	4	$C_2^4 \rtimes_2 Q_8$
128,761	$C_2^4 \rtimes_2 Q_8$			7	11	C_2^2
256,298	$(C_2^3 \times C_4){\cdot_*}Q_8$			3	7	$C_2 \times C_4$
256,25861	$(C_2^3 \times C_4) \rtimes_* Q_8$			5	9	C_2^3
256,298	$(C_2^3 \times C_4){\cdot_*}Q_8$			4	8	$C_2{\cdot_1}C_2^2$
256,25861	$(C_2^3 \times C_4) \rtimes_* Q_8$			2	6	$C_2{\cdot_2}C_2^2$
512,NA	$(C_2^2 \times C_4^2) \rtimes_* Q_8$			3	7	$C_2^2 \rtimes C_2^2$
512,NA	$(C_2^2 \times C_4^2) \rtimes_* Q_8$			2	6	$C_2^2.C_2^2$
1024,NA	$(C_2 \times C_4^3) \rtimes_* Q_8$			2	6	$C_2^3 \rtimes_2 C_2^2$
1024,NA	$(C_2 \times C_4^3){\cdot_*}Q_8$			1	5	$C_2^3{\cdot_1}C_2^2$
2048,NA	$C_4^4 \rtimes_* Q_8$			1	5	$C_2^4 \rtimes_1 C_2^2$
8,4	Q_8			0	4	Q_8
8,4	Q_8			1	5	C_2^2
16,12	$C_2 \times Q_8$			2	6	C_2^2
16,12	$C_2 \times Q_8$			0	4	$C_2 \times Q_8$
32,2	$C_2^2{\cdot_2}Q_8$			0	4	$C_2^2{\cdot_2}Q_8$
32,47	$C_2^2 \times Q_8$			0	4	$C_2^2 \times Q_8$
32,2	$C_2^2{\cdot_2}Q_8$			4	8	C_2^2
32,47	$C_2^2 \times Q_8$			4	8	C_2^2
64,74	$C_2^3 \rtimes_1 Q_8$			0	4	$C_2^3 \rtimes Q_8$
64,74	$C_2^3 \rtimes_1 Q_8$			5	9	C_2^2
128,36	$C_2^4{\cdot_4}Q_8$			0	4	$C_2^4{\cdot_4}Q_8$
128,1572	$C_2^4 \rtimes_6 Q_8$	$Q_8 \subset BT_{24}$	4	7	11	C_2^2
128,36	$C_2^4{\cdot_4}Q_8$			7	11	C_2^2
128,1572	$C_2^4 \rtimes_6 Q_8$			0	4	$C_2^4 \rtimes_6 Q_8$
256,3378	$(C_2^3 \times C_4) \rtimes_* Q_8$			0	4	$(C_2^3 \times C_4) \rtimes_* Q_8$
256,3378	$(C_2^3 \times C_4) \rtimes_* Q_8$			2	6	Q_8
256,3378	$(C_2^3 \times C_4) \rtimes_* Q_8$			5	9	C_2^3
512,NA	$(C_2^2 \times C_4^2){\cdot_*}Q_8$			1	5	C_4^2
512,NA	$(C_2^2 \times C_4^2) \rtimes_* Q_8$			1	5	$C_2 \times Q_8$
512,NA	$(C_2^2 \times C_4^2){\cdot_*}Q_8$			2	6	$C_2^2 \rtimes C_4$
512,NA	$(C_2^2 \times C_4^2) \rtimes_* Q_8$			4	8	C_2^4
1024,NA	$(C_2 \times C_4^3) \rtimes_* Q_8$			1	5	$C_4^2 \rtimes_5 C_2$
1024,NA	$(C_2 \times C_4^3) \rtimes_* Q_8$			2	6	$C_2^4 \rtimes_1 C_2$
2048,NA	$C_4^4 \rtimes_* Q_8$			1	5	$C_2^4 \rtimes_1 C_2^2$
12,1	BD_{12}			1	5	S_3
48,30	$C_2^2 \rtimes BD_{12}$			2	6	S_3
192,1495	$C_2^4 \rtimes_4 BD_{12}$	BD_{12}		5	9	S_3
192,185	$C_4^2 \rtimes BD_{12}$			1	5	$C_2^2 \rtimes S_3$
768,1088649	$(C_2^2 \times C_4^2) \rtimes BD_{12}$			2	6	$C_2^2 \rtimes S_3$

4

ID	G	$G_\circ$	rk	N_2	$b_2(Y)$	$\pi_1(Y^{\text{reg}})$
3072,NA	$C_4^4 \rtimes BD_{12}$			1	5	$C_2^4 \rtimes_3 S_3$
24,3	BT_{24}	BT_{24}	4	1	5	A_4
96,3	$C_2^2.BT_{24}$			2	6	A_4
96,203	$C_2^2 \rtimes BT_{24}$			2	6	A_4
384,4	$C_2^4 ._* BT_{24}$			3	7	A_4
384,5868	$C_2^4 \rtimes_* BT_{24}$			3	7	A_4
1536,NA	$(C_2^2 \times C_4^2) \rtimes_* BT_{24}$			2	6	$C_2^2 \rtimes A_4$
1536,NA	$(C_2^2 \times C_4^2)._* BT_{24}$			1	5	$C_2^2.A_4$
6144,NA	$C_4^4 \rtimes_* BT_{24}$			1	5	$C_4^2 \rtimes A_4$

10.3. Technical digression: Affine actions

In this technical section, we show that all actions giving a terminalization with simply connected regular locus are affine. The possible groups arising are listed in Tables 7 and 8 and satisfy one of the assumptions (1)–(5) in Lemma 10.6.

Lemma 10.6 (Affine groups). *Let G be a finite group of induced symplectic automorphisms of* $\mathrm{K}_n(A)$. *Then G is conjugate by an element of $A[n+1]$ to the affine subgroup $G_{\text{tr}} \rtimes G_\circ$ if*

(1) *$n = 2$ and $G_\circ \simeq C_2$,*
(2) *$n = 2$ and $G_\circ \simeq C_6$,*
(3) *$n = 2$ and $G_\circ \simeq BT_{24}$,*
(4) *$n = 2$, $G_\circ \simeq C_3$ and $N_3 \neq 0$,*
(5) *$n = 3$, $G_\circ \simeq C_2$ and $N_2 \neq 0$.*

Proof. We follow the notation of Section 10.1. It suffices to prove that, up to conjugation in $A[n+1]$, the quotient $G \to G_\circ$ splits. If $\{g_i\}$ are generators of $G_\circ \subset \mathrm{SL}(\Lambda)$, and $\tau_{\alpha_i} g_i$ is a lift of g_i in $G \subset A[n+1] \rtimes \mathrm{SL}(\Lambda)$, we show that there exists an $\alpha \in A[n+1]$ such that the $\tau_\alpha(\tau_{\alpha_i} g_i)\tau_{-\alpha} \subset \mathrm{SL}(\Lambda)$ generate $G_\circ$; *i.e.*, $G_\circ \subset \tau_\alpha G \tau_{-\alpha}$ splits the quotient $\tau_\alpha G \tau_{-\alpha} \to G_\circ$.

(1) Let $\tau_\alpha(-\mathrm{id}) \in G$ be a lift of $-\mathrm{id} \in C_2$. Conjugating by τ_α, we write

$$\tau_\alpha(\tau_\alpha(-\mathrm{id}))\tau_{-\alpha} = -\mathrm{id} \in \tau_\alpha G \tau_{-\alpha}.$$

(2) Let $\tau_\alpha g_6 \in G$ be a lift of $g_6 \in C_6$. Observe that $(\mathrm{id} - g_6)$ is an automorphism of $A[3]$, so we can pick a β such that $\beta - g_6(\beta) = -\alpha$, which gives

$$\tau_\beta(\tau_\alpha g_6)\tau_{-\beta} = g_6 \in \tau_\beta G \tau_{-\beta}.$$

(3) Let $\tau_\alpha t$ be a lift of t to G. Observe that $(\mathrm{id} - t)$ is an automorphism of $A[3]$, so we can pick a γ such that $\gamma - t(\gamma) = -\alpha$, which implies

$$\tau_\gamma(\tau_\alpha t)\tau_{-\gamma} = t \in \tau_\gamma G \tau_{-\gamma}.$$

Let $\tau_\beta r$ be a lift of r to $\tau_\gamma G \tau_{-\gamma}$. As $t^3 = -\mathrm{id} \in \tau_\gamma G \tau_{-\gamma}$ and $(\tau_\beta r)^3 = \tau_{\beta + r(\beta) + r^2(\beta)}(-\mathrm{id}) \in \tau_\gamma G \tau_{-\gamma}$, we write

$$\tau_{\beta + r(\beta) + r^2(\beta)}(\tau_\beta r)\tau_{-\beta - r(\beta) - r^2(\beta)} = r \in \tau_\gamma G \tau_{-\gamma}.$$

(4) As $N_3 \neq 0$, by Lemma 5.6(2), there exists an

$$\alpha \in \Pi_{g_3} := \{\alpha \in A[3] \,|\, g_3(\alpha) = \alpha\} = \mathrm{Im}(\mathrm{id} - g_3)$$

such that $\tau_\alpha g_3 \in G$. Pick a β such that $\beta - g_3(\beta) = -\alpha$. Then

$$\tau_\beta(\tau_\alpha g_3)\tau_{-\beta} = g_3 \in \tau_\beta G \tau_{-\beta}.$$

(5) As $N_2 \neq 0$, by Lemma 5.6(1), there exists an $\alpha \in A[2]$ such that $\tau_\alpha(-\mathrm{id}) \in G$. Pick a β such that $2\beta = \alpha$. Then

$$\tau_\beta(\tau_\alpha(-\mathrm{id}))\tau_{-\beta} = -\mathrm{id} \in \tau_\beta G \tau_{-\beta}. \qquad \square$$

10.4. Terminalizations with simply connected regular locus

Tables 9 and 10 are refinements of Tables 7 and 8 for terminalizations Y of quotients with simply connected regular locus.

- We list the group ID, the alias of G, the integers N_i and $b_2(Y)$ as in Tables 7 and 8.
- We list the numbers a_k of isolated singularities in Y of analytic type $\mathbb{A}^{2n}/\frac{1}{k}(1,-1,\dots,1,-1)$; see Definition 2.1, the computations in Section 11 for $n=2$, and in Proposition 10.7 for $n=3$.
- If $n=3$, we list the number s_2 of surfaces of Y with general transversal singularities of type $\frac{1}{2}(1,1,-1,-1)$; see Proposition 10.7.
- If $n=2$, we list the topological Euler characteristic χ, the fourth Betti number b_4 and the Chern numbers c_4 and c_2^2 of Y, which are functions of $b_2(Y)$ and a_k as follows:(9)

$$b_4(Y) = 10b_2(Y) - b_3(Y) + 46 - a_2 - 2a_3 - 3a_4 \qquad \text{by [FM21, Proposition 3.6],}$$

$$\chi(Y) = 12b_2(Y) - 3b_3(Y) + 48 - a_2 - 2a_3 - 3a_4 \qquad \text{by [FM21, Proposition 3.6],}$$

$$c_4(Y) = \chi(Y) - \frac{a_2}{2} - \frac{2a_3}{3} - \frac{3a_4}{4} \qquad \text{by [Bla96, Theorem 2.14],}$$

$$c_2^2(Y) = \frac{1}{3}c_4(Y) + 720 - 240\left(\frac{a_2}{2^5} + \frac{2a_3}{27} + \frac{9a_4}{2^6}\right) \qquad \text{by [FM21,].}$$

  Note that we can apply the previous identities since our terminalizations have quotient singularities; see Corollary 1.11.
- If Y is deformation equivalent to a known IHS variety, we write the latter in the last column; this analysis follows from Proposition 12.3 for $n=2$ and Theorem 1.12 for $n=3$. The notation $\mathrm{K}_n(A,G)$ stands for a projective terminalization of $\mathrm{K}_n(A)/G$, while $S(G)^{[n]}_\theta$ is the Fujiki variety; see Notation 12.1 and Definition 12.2. The question mark in the correspondence of $G = BT_{24}$ indicates that it shares the singularities and topological invariants of $S(S_3^2 \rtimes C_2)^{[2]}_{\mathrm{id}}$, but we could not decide whether the two are deformation equivalent; see also Remark 12.4. Note that $\mathrm{K}_2(A,C_2)$ is studied in [KM18] and also appeared in [FM21]. In all other cases, we declare the deformation type to be *new*.

Table 9. Terminalizations of $\mathrm{K}_2(A)/G$ with simply connected regular locus

ID	G	N_2	N_3	b_2	a_2	a_3	a_4	b_4	χ	c_4	c_2^2	
2,1	C_2		0	8	36	0	0	90	108	90	480	[KM18]
6,1	$C_3 \rtimes C_2$		0	8	36	13	0	64	82	166/3	712/3	new
18,4	$C_3^2 \rtimes_2 C_2$	1	0	8	36	16	0	58	76	142/3	544/3	new
54,14	$C_3^3 \rtimes C_2$		0	8	36	13	0	64	82	166/3	712/3	$\mathrm{K}_2(A,S_3)$
162,54	$C_3^4 \rtimes C_2$		0	8	36	0	0	90	108	90	480	$\mathrm{K}_2(A,C_2)$
3,1	C_3		1	7	0	12	0	92	108	100	540	$S(C_3^2)^{[2]}_{-\mathrm{id}}$
9,2	C_3^2	0	3	11	0	15	0	126	150	140	500	$S(C_3)^{[2]}_{-\mathrm{id}}$
27,5	C_3^3		9	23	0	0	0	276	324	324	828	$S^{[2]}$

(9)Recall that in our case, $b_3(Y) = 0$ by Proposition 7.2.

6,2	C_6		1	8	28	12	0	74	92	70	320	$S(C_3 \rtimes S_3)^{[2]}_{\mathrm{id}}$
18,3	$C_3 \rtimes C_6$		2	10	28	12	0	94	116	94	328	$S(S_3)^{[2]}_{\mathrm{id}}$
54,13	$C_3^2 \rtimes_4 C_6$	1	5	16	28	0	0	178	212	198	576	$S(C_2)^{[2]}_{\mathrm{id}}$
54,5	$C_3^2 \rtimes C_6$		1	8	28	20	0	58	76	146/3	512/3	$S(C_3 \rtimes S_3)^{[2]}_{(-\mathrm{id},\mathrm{id})}$
162,40	$C_3^3 \rtimes_4 C_6$		2	10	28	12	0	94	116	94	328	$S(S_3)^{[2]}_{\mathrm{id}}$
486,146	$C_3^4 \rtimes_4 C_6$		1	8	28	12	0	74	92	70	320	$S(C_3 \rtimes S_3)^{[2]}_{\mathrm{id}}$
24,3	BT_{24}		1	7	20	12	3	63	79	235/4	275	$S(S_3^2 \rtimes C_2)^{[2]}_{\mathrm{id}}$?
216,153	$C_3^2 \rtimes BT_{24}$	1	1	7	20	16	3	55	71	577/12	601/3	new
1944,NA	$C_3^4 \rtimes BT_{24}$		1	7	20	12	3	63	79	235/4	275	$\mathrm{K}_2(A, BT_{24})$

Table 10. Terminalizations of $\mathrm{K}_3(A)/G$ with simply connected regular locus

ID	G	N_2	b_2	a_2	s_2	
2, 1	$\langle -\mathrm{id}\rangle$	1	8	140	0	$S(C_2^4)^{[3]}$
4, 2	$C_2 \times \langle -\mathrm{id}\rangle$	2	9	112	7	$S(C_2^3)^{[3]}$
8, 5	$C_2^2 \times \langle -\mathrm{id}\rangle$	4	11	64	18	$S(C_2^2)^{[3]}$
16, 14	$C_2^3 \times \langle -\mathrm{id}\rangle$	8	15	0	28	$S(C_2)^{[3]}$
32, 51	$C_2^4 \times \langle -\mathrm{id}\rangle$	16	23	0	0	$S^{[3]}$

10.5. Singularities of terminalizations of quotients of $\mathrm{K}_3(A)$

We determine the singularities of the terminalizations of $\mathrm{K}_3(A)/G$ with simply connected regular locus.

Proposition 10.7. *Let $X = \mathrm{K}_3(A)$ and $G = C_2^i \times \langle -\mathrm{id}\rangle$ for $0 \le i \le 4$. Then the singular locus of Y consists only of a_2 isolated points of type $\mathbb{A}^6/\frac{1}{2}(1,1,1,1,1,1)$ and s_2 surfaces with general transversal singularities $\mathbb{A}^4/\frac{1}{2}(1,1,1,1)$, where*

$$a_2 = 4\Big(42 - 7\cdot 2^i + \frac{1}{3}(2^i - 1)(2^i - 2)\Big),$$
$$s_2 = (2^i - 1)(8 - 2^{i-1}).$$

Proof. The fixed loci of an automorphism in G are computed for instance in [Flo24, Lemma 2.10, Proposition 2.12].

(1) Any nontrivial translation $\tau_\alpha \in A[2]$ fixes eight K3 surfaces $V_{\alpha,\theta}$ in $\mathrm{K}_3(A)$, where

$$\epsilon(V_{\alpha,\theta}) = \{[(x, x+\alpha, -x+\theta, -x+\alpha+\theta)] \,|\, x \in A\}$$

with $\theta \in A[2]$ and $V_{\alpha,\theta} = V_{\alpha,\theta+\alpha}$.

(2) Any involution $\tau_\alpha(-\mathrm{id}) \in A[2] \times \langle -\mathrm{id}\rangle$ fixes the fourfold W_α of $\mathrm{K3}^{[2]}$-type, where

$$\epsilon(W_\alpha) = \{[(x, -x+\alpha, y, -y+\alpha)] \,|\, x, y \in A\},$$

and 140 isolated fixed points of the form

$$[(\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_1 - \varepsilon_2 - \varepsilon_3)] \quad \text{with } 2\varepsilon_i = \alpha \text{ and the } \varepsilon_i \text{ pairwise disjoint.}$$

Observe that these fixed loci satisfy the following intersection rules:

- Two fourfolds W_α and W_β intersect along the surface $V_{\alpha+\beta,\alpha} = V_{\alpha+\beta,\beta}$.
- Three fourfolds W_α, W_β and W_γ intersect in four points of the form

$$[(\varepsilon, \varepsilon+\alpha+\beta, \varepsilon+\alpha+\gamma, \varepsilon+\beta+\gamma)] \quad \text{with } 2\varepsilon = \alpha+\beta+\gamma.$$

Thus, $W_\alpha \cap W_\beta \cap W_\gamma$ consists of 4 of the 140 isolated points fixed by $\tau_{\alpha+\beta+\gamma}(-\mathrm{id})$.

Let z be an isolated point of $\mathrm{Fix}(\tau_\alpha(-\mathrm{id}))$. Then one of the following cases holds:

(i) $G_z = \langle \tau_\alpha(-\mathrm{id})\rangle$, and z corresponds to a singular point of Y.

(ii) $G_z = \langle \tau_\alpha(-\mathrm{id}), \tau_\beta(-\mathrm{id})\rangle \simeq C_2^2$, and

$$z \in W_\beta \cap V_{\alpha+\beta,\theta} \left(= W_\beta \cap W_\theta \cap W_{\alpha+\beta+\theta}\right)$$

for some $\theta \neq \alpha, \beta$ in $A[2]$, equivalently $\tau_\theta(-\mathrm{id}) \notin G$. Indeed, z is an isolated fixed point only for the involution $\tau_\alpha(-\mathrm{id})$ as can be seen by writing $2\varepsilon_i = \alpha$, so z must lie in W_β. Locally at $q(z)$, the terminalization $q\colon Y \to X/G$ is isomorphic to

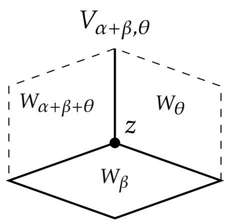
$$\mathrm{Tot}(T^*_{\mathbb{P}^1}) \times \mathbb{A}^4/\tfrac{1}{2}(1,1,1,1) \longrightarrow \mathbb{A}^2/\tfrac{1}{2}(1,1) \times \mathbb{A}^4/\tfrac{1}{2}(1,1,1,1);$$

thus it contains only a singular surface with general transversal singularities of type $\frac{1}{2}(1,1,1,1)$.

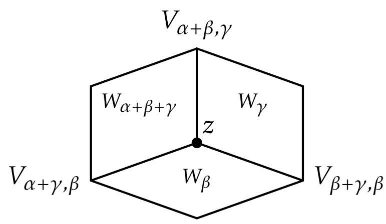
(iii) $G_z = \langle \tau_\alpha(-\mathrm{id}), \tau_\beta(-\mathrm{id}), \tau_\gamma(-\mathrm{id})\rangle \simeq C_2^3$, and

$$z \in W_\beta \cap W_\gamma \cap W_{\alpha+\beta+\gamma}.$$

Locally at $q(z)$, the quotient X/G is isomorphic to the triple product of a canonical surface singularity of type A_1, which admits a symplectic resolution.



(ii) $G_z \simeq C_2^2$



(iii) $G_z \simeq C_2^3$

As a result, Y has only (quotient) singularities of the types appearing in the statement of Proposition 10.7, with the invariants a_2 and s_2 as follows:

$$\begin{aligned}
s_2 &= \#\text{ surfaces in}\,\mathrm{Sing}(X/G)\\
&= (\#\text{ surfaces in } X \text{ fixed by a translation}) - (\#\text{ such surfaces lying on a fixed fourfold})\\
&= 8\cdot(\#\text{ nontrivial translations in } G) - (\#\text{ intersection of two fixed fourfolds})\\
&= 8(2^i-1) - \binom{2^i}{2},\\
a_2 &= \#\text{ isolated singular points of } X/G\\
&= \Big(\#\text{ isolated points in}\,\mathrm{Fix}(\tau_\alpha(-\mathrm{id})) \text{ for some } \tau_\alpha(-\mathrm{id}) \in G\\
&\qquad \text{and not lying on } W_\beta \text{ for any } \beta \in G_{\mathrm{tr}}\Big)/(\#\text{orbits of such points})\\
&= 2^i \cdot \Big((\#\text{ points fixed by } \tau_\alpha(-\mathrm{id}) \text{ and not lying on } W_\alpha)\\
&\qquad - (\#\text{ such points lying on } W_\beta \text{ for some } \beta \in G_{\mathrm{tr}})\Big)/2^i.
\end{aligned}$$

The points $z \in \mathrm{Fix}(\tau_\alpha(-\mathrm{id}))$ lying on $W_\beta \setminus W_\alpha$ for $\beta \neq \alpha$ are in particular fixed by $\tau_{\alpha+\beta}$; hence they lie on one of the seven fixed surfaces $V_{\alpha+\beta,\theta} = W_\theta \cap W_{\alpha+\beta+\theta}$ for $\theta \neq \alpha, \beta$ in $A[2]$. Thus, for each choice of

$\beta \in G_{\mathrm{tr}} \setminus \{\alpha\}$, there are $4 \cdot 7$ such points $z \in W_\beta \cap W_\theta \cap W_{\alpha+\beta+\theta}$. However, note that when θ is in G_{tr}, we count the same point z three times. Indeed, if $\theta \in G_{\mathrm{tr}}$, then z is a point of type (iii), and it lies on three fixed fourfolds W_β, W_θ and $W_{\alpha+\beta+\theta}$. So,

$$\begin{aligned}&(\text{\# isolated points fixed by } \tau_\alpha(-\mathrm{id}) \text{ and lying on } W_\beta \text{ for some } \beta \in G_{\mathrm{tr}})\\ &\quad = 4 \cdot 7 \cdot (\text{\# translations in } G_{\mathrm{tr}} \setminus \{\alpha\}) - 4 \cdot 2 \cdot (\text{\# choices of } \{\beta, \theta\} \text{ in } G_{\mathrm{tr}} \setminus \{\alpha\})/3\\ &\quad = 4 \cdot 7 \cdot (2^i - 1) - 4 \cdot 2 \cdot \binom{2^i - 1}{2}\frac{1}{3},\end{aligned}$$

and we conclude

$$a_2 = 140 - 4 \cdot 7 \cdot (2^i - 1) + 4 \cdot 2 \cdot \binom{2^i - 1}{2}\frac{1}{3}. \qquad \square$$

Corollary 10.8 (Quotient singularities). *Any projective terminalization of a quotient of* $\mathrm{K}_2(A)$ *or* $\mathrm{K}_3(A)$ *by a finite group of induced symplectic automorphisms has quotient singularities.*

Proof. By direct inspection of the singularities of $K_n(A)/G$, where G is one of the groups in Table 9 (more precisely, because of the analysis of the local model of terminalizations in Lemma 11.1 and the projectivity of their gluing explained in Section 11.1) and Proposition 10.7, we see that the projective terminalizations with simply connected regular locus have quotient singularities. The result in general follows since any other projective terminalization is deformation equivalent to a quotient of a terminalization with simply connected regular locus by Proposition 8.1. $\square$

11. Singularities of quotients of generalized Kummer fourfolds

In this section, we analyze the singularities of the quotients $\mathrm{K}_2(A)/G$ (see Table 9) and describe local models for their terminalizations (see Section 11.2). A result of Namikawa grants that the singularity type of global terminalizations agrees with that of the local models; see Section 11.1. One of the difficulties, compared to similar previous investigations, is that our groups G are not necessarily cyclic and may contain several translations. This implies that the intersections and combinatorics of fixed loci of all elements $g \in G$ make the analysis technically more challenging. To navigate this complexity, we display the configuration of the singularities of $\mathrm{K}_2(A)/G$ in some schematic pictures in Section 11.5: The diagrams clarify the relative position and isotropy of each stratum of the singular locus.

We pursue the desired analysis of the singularities of $\mathrm{K}_2(A)/G$ and their terminalizations, as follows:

- We describe the G-fixed locus on $\mathrm{K}_2(A)$; see Table 11. This can be done in terms of the G-fixed locus of A_0^3, except where the Hilbert–Chow morphism $\epsilon\colon \mathrm{K}_2(A) \to A_0^3$ is not an isomorphism, especially on the punctual Hilbert scheme $\epsilon^{-1}(0) \simeq \mathbb{P}(1,1,3)$; see Section 11.3.
- In Section 11.5, we provide an algorithm that extracts the configuration of the singularities of $\mathrm{K}_2(A)/G$ from the intersection theory and combinatorics of the G-fixed loci. We run this explicitly for the new deformation types that appear in Table 9 and represent the singularities in a diagram (see Figure 3 and Section 11.5.3).
- We provide local models for the singularities of $\mathrm{K}_2(A)/G$, describe the singularities of a local terminalization and show that they can be glued to a projective terminalization of $\mathrm{K}_2(A)/G$ by results of Namikawa; see Sections 3.2 and 11.1.

11.1. Projective terminalizations

Gluing together local analytic models of terminalizations may lead to a non-projective global terminalization, as in [Fuj83, Proposition 13.3]. One may wonder whether the local singularities of a global projective terminalization differ from that of an arbitrary local model. This is not our case.

The only local terminalization that is not obtained by blowing up the reduced singular locus, and which may potentially affect the projectivity of the global terminalization, corresponds to a singularity with isotropy BT_{24}, namely singularity (9) in Lemma 11.1. Two local symplectic resolutions of such a singularity are described in [LS12]. They are blowups of local analytic Weil divisors followed by the blowup of the singular locus of the previous blowup. In particular, the exceptional locus is irreducible. We do not know whether the same sequence of birational transformations can be carried out globally on $\mathrm{K}_2(A)/G$, namely if the effective Weil divisor extends (at least its class in the class group does). Nevertheless, by Corollary 3.11, any projective terminalization of X/G should be locally isomorphic to one of the two symplectic resolutions obtained by Lehn and Sorger. In fact, Bellamy showed that these are the only symplectic resolutions of such a quotient singularity; see [Bel16, Section 4.3]. We conclude that, in our case, a projective terminalization of $\mathrm{K}_2(A)/G$ can indeed be obtained by gluing local models of terminalization, which are listed in Lemma 11.1.

11.2. Local models of some symplectic singularities and their terminalizations

Lemma 11.1. *Let G be a finite group with a faithful complex symplectic representation V of dimension 4.*

(1) *If $G \simeq C_k$ for $k = 2, 3, 4$ or 6 and V/G has an isolated (terminal) singularity, then*
$$V/G \simeq \mathbb{A}^4/\tfrac{1}{k}(1, 1, -1, -1).$$
(2) *If $G \simeq C_4$ and $\mathrm{Sing}(V/G)$ is an irreducible surface generically of transversal A_1-singularities, then*
$$V/G \simeq \mathbb{A}^4/\tfrac{1}{4}(1, -1, 2, 2),$$
and a terminalization of V/G has two singularities of type $\frac{1}{2}(1, 1, 1, 1)$; i.e., $a_2 = 2$.

(3) *If $G \simeq C_6$ and $\mathrm{Sing}(V/G)$ is an irreducible surface generically of transversal A_1-singularities, then*
$$V/G \simeq \mathbb{A}^4/\tfrac{1}{6}(1, -1, 3, 3),$$
and a terminalization of V/G has two singularities of type $\frac{1}{3}(1, 1, -1, -1)$; i.e., $a_3 = 2$.

(4) *If $G \simeq C_6$ and $\mathrm{Sing}(V/G)$ is an irreducible surface generically of transversal A_2-singularities, then*
$$V/G \simeq \mathbb{A}^4/\tfrac{1}{6}(1, -1, 2, 2),$$
and a terminalization of V/G has three singularities of type $\frac{1}{2}(1, 1, 1, 1)$; i.e., $a_2 = 3$.

(5) *If $G \simeq C_6$ and $\mathrm{Sing}(V/G)$ consists of two surfaces generically of transversal A_1- and A_2-singularities, respectively, then*
$$V/G \simeq \mathbb{A}^2/C_2 \times \mathbb{A}^2/C_3.$$
(6) *If $G = C_3 \times C_3$, then*
$$V/G \simeq \mathbb{A}^2/C_3 \times \mathbb{A}^2/C_3.$$
(7) *If $G = S_3$ and V/G has singularities in codimension 2, then*
$$V/G \simeq \mathfrak{h} \oplus \mathfrak{h}^*/S_3,$$
where S_3 acts by permutation on the hyperplane $\mathfrak{h} = \{x \in \mathbb{A}^3 \mid \sum_i x_i = 0\}$.

(8) *If $G = C_3 \times S_3 = C_3^2 \rtimes C_2$ and $\mathrm{Sing}(V/G)$ consists of two surfaces generically of transversal A_1- and A_2-singularities, respectively, then*
$$V/G \simeq (\mathfrak{h} \otimes \chi) \oplus (\mathfrak{h} \otimes \chi)^*/C_3 \times S_3,$$
where $\mathfrak{h}$ is the irreducible 2-dimensional representation lifted from S_3 and χ is a nontrivial character of order 3.

(9) *If $G = BT_{24}$ and $\mathrm{Sing}(V/G)$ is an irreducible surface generically of transversal A_2-singularities, then*
$$V/G \simeq \rho \oplus \rho^*/BT_{24},$$
where ρ is the (unique up to dual) irreducible 2-dimensional representation of BT_{24} generated by complex reflections of order 3.

The quotients V/G as in (5), (6), (7), (8) *and* (9) *all admit a smooth terminalization.*

Proof. The symplectic form ω_V induces a G-equivariant isomorphism $V \simeq V^*$, and W is an irreducible subrepresentation of V if and only if its dual W^* is so too. Therefore, V decomposes in irreducible representations in one the following ways:

- $\chi_1 \oplus \chi_1^* \oplus \chi_2 \oplus \chi_2^*$ if and only if G is abelian,
- $\chi_1 \oplus \chi_1^* \oplus \rho$ with $\rho \simeq \rho^*$ symplectic,
- $\rho \oplus \rho^*$,
- V,

where χ_i and ρ are irreducible G-representations of dimension 1 and 2, respectively.

First consider the abelian cases: (1)–(6). The computation of the weights of the action is elementary. We comment on the singularities of a terminalization. In cases (2) and (3), a terminalization is obtained in the following way. Let $p_V : \mathrm{Bl}_F(V) \to V$ be the blowup of the plane $F \subset V$ with nontrivial stabilizer. The action of $G = C_{2k}$, with $k = 2$ or 3, lifts to $\mathrm{Bl}_F(V)$ and in particular on $p_V^{-1}(0) \simeq \mathbb{P}^1$ via $[x : y] \mapsto [\xi_{2k}x : \xi_{2k}^{-1}y]$. We obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{Bl}_F(V) & \xrightarrow{/C_2} & \mathrm{Bl}_F(V)/C_2 & \xrightarrow{/C_k} & \mathrm{Bl}_F(V)/C_{2k} \\ {\scriptstyle p_V}\downarrow & & & & \downarrow \\ V & & \longrightarrow & & V/C_{2k}. \end{array}$$

Since $C_2 = \langle \xi_{2k}^k \rangle$ fixes only the p_V-exceptional divisor, the quotient $\mathrm{Bl}_F(V)/C_2$ is smooth, and the residual C_k-action fixes the points $[0 : 1]$ and $[1 : 0]$ in $p_V^{-1}(0)/C_2 \simeq \mathbb{P}^1$. Hence, the terminalization $\mathrm{Bl}_F(V)/C_k \to V/C_k$ has exactly two singular points of type $\frac{1}{k}(1,-1,1,-1)$. A similar argument gives a proof of (4) by chasing fixed points as above in a local version of diagram (6.1) in Lemma 6.3. Finally, note that $\mathrm{Bl}_0(\mathbb{A}^2/C_2) \times \mathrm{Bl}_0(\mathbb{A}^2/C_3)$ and $(\mathrm{Bl}_0(\mathbb{A}^2/C_3))^2$ give symplectic resolutions in cases (5) and (6), respectively.

We are left with the non-abelian cases: (7)–(9).

- The only irreducible 2-dimensional representation $\mathfrak{h}$ of S_3 is not symplectic; it is generated by complex reflections. So we must have $V \simeq \mathfrak{h} \oplus \mathfrak{h}^*$.
- The irreducible 2-dimensional representations of $C_3 \times S_3$ are $\mathfrak{h}$, $\mathfrak{h} \otimes \chi$, $(\mathfrak{h} \otimes \chi)^*$. The representation V cannot be $\mathfrak{h} \oplus \mathfrak{h}^*$; otherwise, the $(C_3 \times S_3)$-action would factor through S_3. We must therefore have $V \simeq (\mathfrak{h} \otimes \chi) \oplus (\mathfrak{h} \otimes \chi)^*$.
- BT_{24} has seven irreducible representations: three characters $1, \chi, \chi^*$ lifted from $BT_{24} \twoheadrightarrow BT_{24}/Q_8 = C_3$; three 2-dimensional representations ρ_{symp}, $\rho = \rho_{\mathrm{symp}} \times \chi$, ρ^*; and a 3-dimensional representation. The reducible faithful 4-dimensional representations of BT_{24} are

$$1 \oplus 1 \oplus \rho_{\mathrm{symp}}, \quad \chi \oplus \chi^* \oplus \rho_{\mathrm{symp}}, \quad \rho_{\mathrm{symp}} \oplus \rho_{\mathrm{symp}}, \quad \rho \oplus \rho^*.$$

 Only the last representation admits a plane with generic stabilizer exactly C_3. So $V \simeq \rho \oplus \rho^*$.

The quotient V/G admits a smooth terminalization in cases (7), (8) and (9); see [Bel09, Corollary 1.2] or [LS12, Theorem 1]. □

11.3. Fixed points of the punctual Hilbert scheme

Let g be a symplectic automorphism of the complex torus A. The g-fixed points lying in the locus where the Hilbert–Chow morphism $\epsilon : \mathrm{K}_2(A) \to A_0^{(3)}$ is an isomorphism are fixed points in $A_0^{(3)}$, and they can be described as triples of points in A partitioned by g-orbits. The g-fixed points z in the ϵ-exceptional locus deserve additional analysis.

The positive-dimensional fibers $\epsilon^{-1}(\epsilon(z))$, with their reduced structure, are isomorphic to

(1) $\mathbb{P}(T_\alpha^* A) \simeq \mathbb{P}^1$ if $\epsilon(z) = [(\alpha, \alpha, \beta)]$ and $g(\alpha) = \alpha$ and $g(\beta) = \beta$, or
(2) $\mathcal{H}_3 \simeq \mathbb{P}(1,1,3)$ if $\epsilon(z) = [(\alpha, \alpha, \alpha)]$ and $g(\alpha) = \alpha$.

In the former case, g acts on $T^*_\alpha A$ with weights $(1,-1)$, which gives the following lemma.

Lemma 11.2. *The automorphism g acting on the rational curve $\epsilon^{-1}(\alpha,\alpha,\beta)\simeq \mathbb{P}(T^*_\alpha A)$ fixes either the whole curve if $\operatorname{ord}(g)=2$, or two points corresponding to the eigenlines of g if $\operatorname{ord}(g)>2$.*

In the latter case, the fiber $\epsilon^{-1}(\epsilon(z))$ is the so-called punctual Hilbert scheme $\mathcal{H}_3$ of three points on a plane, isomorphic to the weighted projective space $\mathbb{P}(1,1,3)$; see [Bri77, Section IV.2, p. 76] or [Gor18, Section 3]. It parametrizes ideals of colength 3 supported on a single point, say $0\in T_0A\simeq \mathbb{A}^2_{x,y}$, namely

- the square $\mathfrak{m}^2$ of the maximal ideal $\mathfrak{m}=(x,y)$,
- the curvilinear ideals I of colength 3, *i.e.*, ideals containing the ideal of a smooth curve passing through the origin. In symbols, $I=(f,\mathfrak{m}^3)$, where $f\in\mathfrak{m}$ and $df\neq 0$.

Note that if $\frac{\partial f}{\partial x}\neq 0$ and $\frac{\partial f}{\partial y}\neq 0$, we can write

$$I=(x+c_0y+c_1y^2,\mathfrak{m}^3)=\left(\frac{1}{c_0}x+y+\frac{c_1}{c_0^3}x^2,\mathfrak{m}^3\right)$$

using the equivalences $x^2+c_0xy\equiv 0$ and $xy+c_0y^2\equiv 0$ modulo I. This gives the transition functions of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(3)=\mathbb{P}(1,1,3)\setminus[0:0:1]=\mathcal{H}_3\setminus\mathfrak{m}^2$. In particular, the zero-section of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(3)$, isomorphic to $\mathbb{P}(T^*_0A)\simeq\mathbb{P}^1_{[\lambda:\,\mu]}$, represents the curvilinear ideals cosupported on the lines through the origin; *i.e.*, $I([\lambda:\mu])=(\lambda x+\mu y,\mathfrak{m}^3)$.

Lemma 11.3. *Let V be a 2-dimensional symplectic representation of the finite group G. Denote by $\mathbb{C}(k)$ the $\mathbb{C}^*$-character given by $t\cdot v=t^kv$, and let $W(k):=W\otimes\mathbb{C}(k)$ for any vector space W. Then the Hilbert scheme $\mathcal{H}_3$ of three points on V is G-equivariantly isomorphic to*

$$V^*(1)\oplus(\det V^*)^{\otimes 2}(3)/\!\!/\mathbb{C}^*\simeq\mathbb{P}(1,1,3).$$

Proof. Consider the curvilinear ideal $I=(x+c_0y+c_1y^2,\mathfrak{m}^3)$ and the matrix representation of the action of an element $g\in G$ on V,

$$g=M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

The ideal $g^*I=(f\circ g^{-1},\mathfrak{m}^3)$ is generated by

$$\left(\frac{dx-by}{\det M}+c_0\frac{-cx+ay}{\det M}y+c_1\frac{(-cx+ay)^2}{\det M^2},\mathfrak{m}^3\right)=\left(x+\frac{-b+c_0a}{d-c_0c}+c_1\frac{ad-bc}{(d-c_0c)^3}y^2,\mathfrak{m}^3\right).$$

In the quasi-homogeneous coordinates $[x_1:x_2:x_3]$ of $\mathbb{P}(1,1,3)$, we write

$$[1:c_0:c_1]\longmapsto\left[\frac{d-c_0c}{\det M}:\frac{-b+c_0a}{\det M}:\frac{c_1}{\det M^2}\right],$$

or equivalently $[\underline{x}:x_3]\mapsto[(M^{-1})^t\underline{x}:\det((M^{-2})^t)x_3]$, where $\underline{x}=(x_1,x_2)$. ☐

Note that G-fixed points of $\mathcal{H}_3\simeq\mathbb{P}(1,1,3)$ are G-invariant subspaces of $V^*\oplus(\det V^*)^{\otimes 2}$. We obtain the following elementary corollary.

Corollary 11.4. *We use the notation of Lemma 11.3. A symplectic automorphism $g\in G$ fixes*

- *$\mathbb{P}(V^*)$ and $\mathfrak{m}^2$ if $\operatorname{ord}(g)=2$,*
- *two lines through $\mathfrak{m}^2$ corresponding to the eigenlines of g if $\operatorname{ord}(g)=3$,*
- *two points in $\mathbb{P}(V^*)$ corresponding to the eigenlines of g and $\mathfrak{m}^2$ if $\operatorname{ord}(g)>3$.*

Let G be a group of symplectic automorphisms of the abelian variety A (fixing the origin). To determine the points of $\mathcal{H}_3$ with nontrivial stabilizers, we proceed as follows:

(1) Note that the stabilizer of $\mathfrak{m}^2$ is G.

(2) Determine the stabilizers for the action of G on $\mathbb{P}(T_0^*A)$, *i.e.*, the eigenspaces of all elements $g \in \mathrm{SL}(T_0^*A)$.
(3) Note that if the automorphism $g \in G$ of order 3 fixes the point $z \in \mathbb{P}(T_0^*A)$, then the line through z and $\mathfrak{m}^2$ has generic stabilizer C_3.

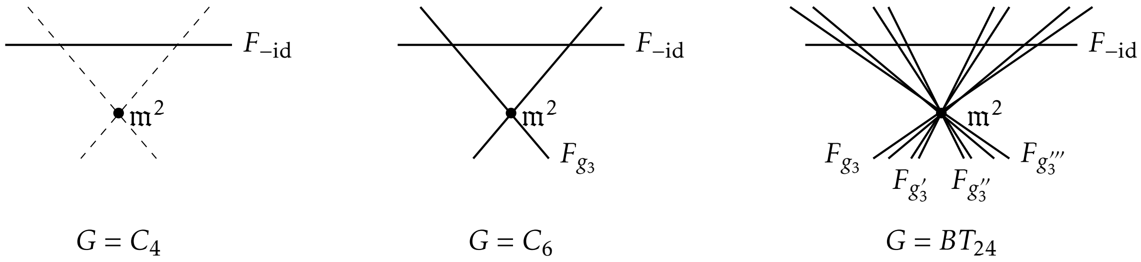


Figure 1. The picture represents the loci with nontrivial stabilizer in the punctual Hilbert scheme $\mathcal{H}_3$ with respect to the action of the group G. We draw $\mathcal{H}_3$ as a cone with vertex $\mathfrak{m}^2$, the horizontal section is $\mathbb{P}(V^*)$, and the segments from $\mathbb{P}(V^*)$ to $\mathfrak{m}^2$ are lines parametrizing ideals $I = (f, \mathfrak{m}^3)$ with fixed df.

11.4. Fixed points of $\mathrm{K}_2(A)$

Lemma 11.5. *Let G be a finite group with a faithful symplectic linear action on A. In Table 11, we provide the number of surfaces and isolated points in* $\mathrm{Fix}(G)$ *in* $\mathrm{K}_2(A)$*, and their inclusion in surfaces F_g fixed by an automorphism $g \in G$.*

Table 11. Fixed loci of some linear actions on $\mathrm{K}_2(A)$

G	G-fixed surface	G-fixed points	Relative position of fixed loci
C_2	1	36	36 pts $\notin F_{-\mathrm{id}}$
C_3	1	12	12 pts $\notin F_{g_3}$
C_4	0	16	8 pts $\in F_{-\mathrm{id}}$ 8 pts $\notin F_{-\mathrm{id}}$
C_6	0	12	2 pts $= F_{-\mathrm{id}} \cap F_{g_3}$ 4 pts $\in F_{-\mathrm{id}} \setminus F_{g_3}$ 6 pts $\in F_{g_3} \setminus F_{-\mathrm{id}}$
BT_{24}	0	2	2 pts $= \bigcap_{\mathrm{ord}(g)=3} F_g$

Proof. Cases C_2 and C_3 are classical; see for instance [Tar15, Section 1.2.1] and [FM21, Section 5.5]. We focus on the remaining cases. For any G-fixed point z in $\mathrm{K}_2(A)$, the image $\epsilon(z) = [(x, y, -x-y)]$ in $A_0^{(3)}$ is G-fixed too, and $\{x, y, -x-y\}$ is a union of orbits for the action of G on A, equivalently a union of fibers of the quotient $A \to A/G$.

(3) If $G \simeq C_4 = \langle g_4 \rangle$, the singularities of A/G are $4A_3 + 6A_1$; see [Fuj88, Lemma 3.19] and also [Pie22, Proposition 2.7]. We denote the point/orbit in A over the singularities $4A_3$ by $0, q_1, q_2$ and q_3, and the orbits over $6A_1$ are $\{x, g_4(x)\}$ for some $x \in A[2] \setminus \{0, q_1, q_2, q_3\}$. If z is a G-fixed point in $\mathrm{K}_2(A)$, then one of the following holds:
- $\epsilon(z) = [(0,0,0)]$, and G fixes two points in $\epsilon^{-1}(0,0,0)$ lying in $F_{-\mathrm{id}}$ and the point $\mathfrak{m}^2 \notin F_{-\mathrm{id}}$; see Corollary 11.4.

- $\epsilon(z) = [(q_i, q_i, 0)] \in \epsilon(F_{-\mathrm{id}})$, and G fixes two points in $\epsilon^{-1}(q_i, q_i, 0)$ lying on $F_{-\mathrm{id}}$; see Lemma 11.2.
- $\epsilon(z) = [(q_1, q_2, q_3)] \notin \epsilon(F_{-\mathrm{id}})$.
- $\epsilon(z) = [(x, g_4(x), -x - g_4(x))] \notin \epsilon(F_{-\mathrm{id}})$ for some $x \in A[2] \setminus \{0, q_1, q_2, q_3\}$.

(4) If $G \simeq \langle g_3, -\mathrm{id}\rangle \simeq C_6$, the singularities of A/G are $A_5 + 4A_2 + 5A_1$. The point/orbit over A_5 is 0, the orbits over $4A_2$ are $\{x, -x\}$ for some $x \neq 0$ with $g_3(x) = x$, and the orbits over $5A_1$ are $\{x, g_3(x), g_3^2(x)\}$ for some $x \in A[2] \setminus \{0\}$. If z is a G-fixed point in $\mathrm{K}_2(A)$, then one of the following holds:
- $\epsilon(z) = [(0,0,0)]$, and G fixes two points in $\epsilon^{-1}(0,0,0)$ lying in $F_{-\mathrm{id}} \cap F_{g_3}$ and the point $\mathfrak{m}^2 \in F_{g_3} \setminus F_{-\mathrm{id}}$; see Corollary 11.4.
- $\epsilon(z) = [(x, 0, -x)] \in F_{-\mathrm{id}} \setminus F_{g_3}$ for some $x \neq 0$ with $g_3(x) = x$.
- $\epsilon(z) = [(x, g_3(x), {g_3}^2(x))] \in F_{g_3} \setminus F_{-\mathrm{id}}$ for some $x \in A[2] \setminus \{0\}$.

(5) If $G \simeq BT_{24}$, the singularities of A/G are $E_6 + D_4 + 4A_2 + A_1$. The point/orbit over E_6 is 0, the orbit over D_4 is $\{q_1, q_2, q_3\}$, and all other orbits of G have cardinality greater than 3. If z is a G-fixed point in $\mathrm{K}_2(A)$, then one of the following holds:
- $\epsilon(z) = [(0,0,0)]$, and G fixes $\mathfrak{m}^2 = \epsilon^{-1}(0,0,0) \cap \bigcap_{g \in G:\mathrm{ord}(g)=3} F_g$; see Corollary 11.4.
- $\epsilon(z) = [(q_1, q_2, q_3)] \in \bigcap_{g \in G:\mathrm{ord}(g)=3} F_g$. □

Lemma 11.6. *Let G be a finite group of induced symplectic automorphisms of* $\mathrm{K}_2(A)$.

(1) *If $G \simeq \langle \tau_\alpha \rangle \simeq C_3$, then G fixes 27 points.*

(2) *If $G \simeq \langle \tau_\alpha, -\mathrm{id}\rangle \simeq S_3$, then G fixes the unique intersection point of all surfaces fixed by an involution of G.*

(3) *If $G \simeq \langle \tau_\alpha, g_3\rangle \simeq C_3^2$, then G fixes the three intersection points between a pair of surfaces fixed by an element of G.*

(4) *If $G \simeq \langle g_3, \tau_\alpha, -\mathrm{id}\rangle \simeq C_3 \times S_3$, then G fixes the unique intersection point of all surfaces fixed by an element of $G \setminus \langle g_3\rangle$.*

Proof. Let z be a point whose stabilizer G_z contains τ_α with $\alpha \neq 0$. Then z is of the form $[(x, x+\alpha, x-\alpha)]$ with $x \in A[3]$, and there are $27 = |A[3]|/3$ such points z. In particular, we have the following:
- If $G_z = \langle \tau_\alpha, -\mathrm{id}\rangle$, then $x \in \langle \alpha \rangle$, so
$$p = [(0, \alpha, -\alpha)] \in F_{-\mathrm{id}} \cap F_{\tau_\alpha(-\mathrm{id})} \cap F_{\tau_{-\alpha}(-\mathrm{id})}.$$
- If $G_z = \langle \tau_\alpha, g_3\rangle$, then one of the following holds:
  - $g_3(x) = x$, *i.e.*, $z \in F_{\tau_\alpha g_3} \cap F_{\tau_{-\alpha} g_3} \setminus F_{g_3}$.
  - $g_3(x) = x - \alpha$, *i.e.*, $z \in F_{g_3} \cap F_{\tau_{-\alpha} g_3} \setminus F_{\tau_\alpha g_3}$.
  - $g_3(x) = x + \alpha$, *i.e.*, $z \in F_{g_3} \cap F_{\tau_\alpha g_3} \setminus F_{\tau_{-\alpha} g_3}$.

  Note that any pair of fixed surfaces intersects in three points of the form $[(x, x+\alpha, x-\alpha)]$.
- If $G_z = \langle g_3, \tau_\alpha, -\mathrm{id}\rangle$, then combining the two cases above, we obtain
$$z = [(0, \alpha, -\alpha)] \in F_{-\mathrm{id}} \cap F_{\tau_\alpha(-\mathrm{id})} \cap F_{\tau_{-\alpha}(-\mathrm{id})} \cap F_{\tau_\alpha g_3} \cap F_{\tau_{-\alpha} g_3} \setminus F_{g_3}.$$
□

11.5. Singularities of symplectic quotients

The singular locus of $\mathrm{K}_2(A)/G$ is stratified in

(1) locally closed surfaces with isotropy C_2 or C_3,

(2) points in the closure of the surfaces in (1) with isotropy strictly greater than C_2 or C_3,

(3) remaining isolated singular points.

The points of type (2) are images under the quotient map $q\colon K_2(A) \to K_2(A)/G$ of

(2.1) the intersection of surfaces $F_g \cap F_h$ fixed by some $g, h \in G$,

(2.2) fixed points in F_g for the residual action of $N_G(\langle g\rangle)/\mathrm{ncl}(g)$, where $N_G(\langle g\rangle)$ is the normalizer of the cyclic subgroup $\langle g\rangle$ generated by g, and $\mathrm{ncl}(g)$ is the normal subgroup generated by g in $N_G(\langle g\rangle)$.

In order to determine the singularities of X/G effectively, we use the following algorithm:

(1) List the possible stabilizers of points of X for the action of G.
(2) Determine all the points of type 1, (2.1) and (2.2).
(3) Note that the number of remaining isolated fixed points with isotropy m is

$$\sum_{g\in G,\ \mathrm{ord}(g)=m} \Big((\#\text{ isolated } g\text{-fixed points}) - (\#\ g\text{-fixed points of type (2.1) and (2.2)})\Big)\Big/(|G|/\mathrm{ord}(g))$$

One may run the algorithm for all groups in Table 9, but for brevity we make the following expository choice. For the terminalizations which are deformation equivalent to a Fujiki variety (see Proposition 12.3), the singularities have been already computed in [Men22, Theorem 1.11], and we refer the reader to *loc. cit.* Here we study in detail the singularities of the new deformation types of IHS fourfolds in Table 9, namely $G_\circ = C_2$ (see Section 11.5.1) and $G = C_3^2 \rtimes BT_{24}$ (see Section 11.5.2). For the only remaining case $G = BT_{24}$, for which we do not know yet if it is deformation equivalent to other Fujiki varieties (see Remark 12.4), we provide the diagram of the singularities of $\mathrm{K}_2(A)/G$ and leave the details to the reader; see Section 11.5.3.

11.5.1. Groups with $G_0 = C_2$.— Suppose $G_{\mathrm{tr}} \simeq C_3^{\oplus i}$ for some $i = 0,\dots,4$. Since any point $z \in \mathrm{K}_2(A)$ cannot be fixed by more than one translation up to multiples (*i.e.*, $G_z \cap G_{\mathrm{tr}} = \{1\}$ or $\langle \tau_\alpha\rangle$), the possible nontrivial stabilizers of points in $\mathrm{K}_2(A)$ for the action of G are

$$\begin{aligned} \langle \tau_\beta\rangle &\simeq C_3, & \beta &\in G_{\mathrm{tr}} \setminus \{0\},\\ \langle \tau_\alpha(-\mathrm{id})\rangle &\simeq C_2, & \alpha &\in G_{\mathrm{tr}},\\ \langle \tau_\alpha(-\mathrm{id}), \tau_\beta\rangle &\simeq S_3, & \alpha &\in G_{\mathrm{tr}},\ \beta \in G_{\mathrm{tr}} \setminus \{0\}. \end{aligned}$$

The singular points of Y correspond to the isolated singularities of X/G. Indeed, as $N_2 = 1$ and $N_3 = 0$, the singular locus contains a unique irreducible component of codimension 2, namely $q(F_{-\mathrm{id}})$, with points of isotropy C_2 or S_3. By Lemma 11.1(7), the terminalization $Y \to X/G$ is a symplectic resolution in a neighborhood of $q(F_{-\mathrm{id}})$.

The singularities of X/G away from $q(F_{-\mathrm{id}})$ are images of isolated points in X fixed by elements $g \in G$. Given the list of possible stabilizers, the isolated points of the fixed locus of an involution do not lie on any surface fixed by any other involution. We obtain that

$$\begin{aligned} a_2 &= \#\text{ isolated singular points of } X/G \text{ with isotropy } C_2\\ &= (\#\text{ isolated points in } X \text{ fixed by an involution in } G)/(\#\text{orbits of such points})\\ &= (\#\text{ involutions in } G)\cdot(\#\text{ isolated points fixed by} -\mathrm{id})/(|G|/2)\\ &= 3^i \cdot 36/(2\cdot 3^i/2) = 36. \end{aligned}$$

On the contrary, if $g = \tau_\beta$ is a translation, an isolated fixed point may lie on a surface $F_{\tau_\alpha(-\mathrm{id})}$. In that case, the point is the unique intersection of the three surfaces

$$F_{\tau_\alpha(-\mathrm{id})} \cap F_{\tau_{\alpha+\beta}(-\mathrm{id})} \cap F_{\tau_{\alpha-\beta}(-\mathrm{id})} = [(-\alpha+\beta, -\alpha-\beta, -\alpha)].$$

Since there are exactly $\frac{1}{3}\binom{3^i}{2}$ such points, we obtain that

$$\begin{aligned} a_3 &= \#\text{ isolated singular points of } X/G \text{ with isotropy } C_3 \\ &= \Big((\#\text{ isolated points in } X \text{ fixed by a translation}) \\ &\qquad -(\#\text{ isolated such points lying on a fixed surface})\Big)\Big/(\#\text{orbits of such points}) \\ &= \Big((\#\text{ subgroups } \langle\tau_\beta\rangle\subset G)\cdot(\#\text{ isolated points in } X \text{ fixed by } \tau_\beta) \\ &\qquad -(\#\text{ isolated such points lying on a fixed surface})\Big)\Big/(|G|/3) \\ &= \left(\frac{3^i-1}{2}\cdot 27-\frac{1}{3}\binom{3^i}{2}\right)\Bigg/\frac{2\cdot 3^i}{3}=\frac{(3^i-1)(3^{4-i}-1)}{4}\in\{0,13,16\}. \end{aligned}$$

11.5.2. Group $C_3^2 \rtimes BT_{24}$ (ID 216,153).— We first determine the possible nontrivial stabilizers of points of $\mathrm{K}_2(A)$ under the action of $G = C_3^2 \rtimes BT_{24}$; see Figure 2.

We note in particular that there are no stabilizers isomorphic to S_3 and Q_8. To this end, observe that all subgroups of G isomorphic to S_3 and Q_8 are conjugate, so it suffices to show that the groups $S_3 \simeq \langle\tau_\alpha, -\mathrm{id}\rangle$ and $Q_8 = \langle i, j, k\rangle \subset BT_{24}$ are not stabilizers of any point in $\mathrm{K}_2(A)$.

- In the former case, any point $[(0,\alpha,-\alpha)] \in \mathrm{Fix}(\langle\tau_\alpha,-\mathrm{id}\rangle)$ is also fixed by $\tau_\alpha g_\alpha$, where $g_\alpha \in BT_{24}$ is the unique automorphism of order 3 fixing the line $\langle\alpha\rangle \in G_{\mathrm{tr}}$.
- In the latter case, any point $z \in \mathrm{K}_2(A)$ fixed by Q_8 is fixed by BT_{24} too. Indeed, $\epsilon(z) \in A_0^{(3)}$ is either $[(0,0,0)]$ or $[(q_1,q_2,q_3)]$, as in the proof of Lemma 11.5 for BT_{24}, and they are both fixed by BT_{24}. Further, the only Q_8-fixed point of the punctual Hilbert scheme $(0,0,0)$ is $\mathfrak{m}^2$, which is fixed by BT_{24} too.

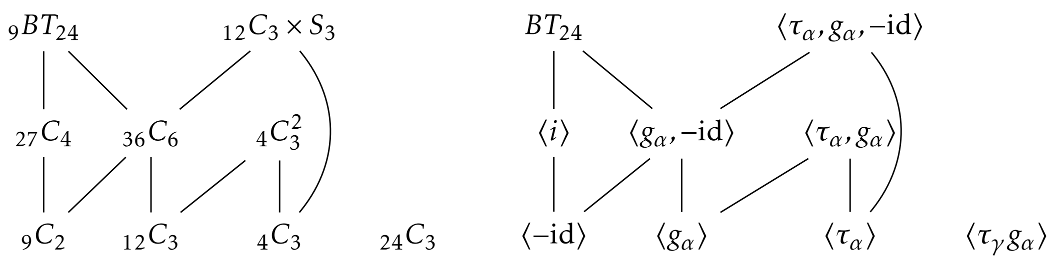


Figure 2. On the left, the poset of nontrivial stabilizers of points of $\mathrm{K}_2(A)$ under the action of $G = C_3^2 \rtimes BT_{24}$, up to conjugation. The left subscript denotes the number of conjugate subgroups. On the right, we provide a representative for each conjugacy class. Note that $\alpha,\gamma \in G_{\mathrm{tr}}$ with $g_\alpha(\gamma) \neq \gamma$.

As $N_2 = 1$ and $N_3 = 1$ (*cf.* Table 7), the only surfaces in the singular locus of X/G are $q(F_{-\mathrm{id}})$ and $q(F_{g_\alpha})$. The residual groups acting on the K3 surfaces $F_{-\mathrm{id}}$ and F_{g_α} are $A_4 = BT_{24}/{-\mathrm{id}}$ and $S_3 = \langle\tau_\alpha, -\mathrm{id}\rangle$, respectively. The singularities of the quotients $F_{-\mathrm{id}}/A_4$ and F_{g_α}/S_3 are

$$\begin{aligned} F_{-\mathrm{id}}/A_4 &: \quad 6A_2 + 4A_1, \\ F_{g_\alpha}/S_3 &: \quad 3A_2 + 8A_1; \end{aligned}$$

see [Xia96, Theorem 3, #17, #6]. This suffices to describe the singular points of $\mathrm{K}_2(A)/G$ lying on $q(F_{-\mathrm{id}}) \cup q(F_{g_3})$ (*cf.* Figure 3):

- The six points of type A_2 in $F_{-\mathrm{id}}/A_4$ correspond to
  - three points in $q(F_{-\mathrm{id}})$ with isotropy C_6,
  - two points in the intersection of $q(F_{-\mathrm{id}})$ and $q(F_{g_\alpha})$ with isotropy C_6,
  - The point $[(\alpha,-\alpha,0)] \subset q(F_{-\mathrm{id}}) \cap q(F_{g_\alpha})$, with isotropy $C_3 \times S_3$.

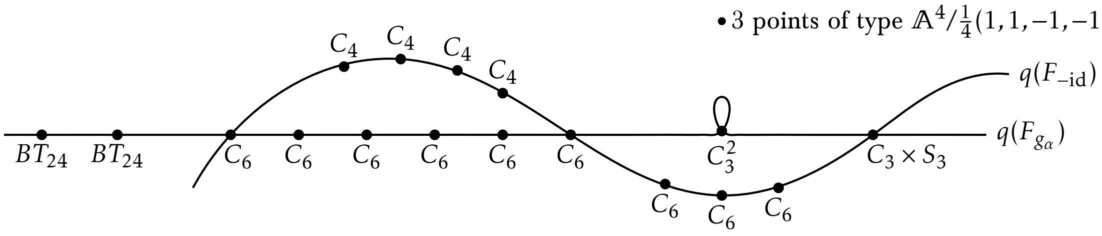


Figure 3. Singularities of X/G for $G = C_3^2 \rtimes BT_{24}$ (ID 216,153).

- The four points of type A_1 in $F_{-\mathrm{id}}/A_4$ correspond to four points in $q(F_{-\mathrm{id}})$ with isotropy C_4.
- The three points of type A_2 in F_{g_α}/S_3 corresponds to
  - the point $[(\alpha,-\alpha,0)] \subset q(F_{-\mathrm{id}}) \cap q(F_{g_\alpha})$, with isotropy $C_3 \times S_3$,
  - the point $[(x+\alpha, x-\alpha, x)] \in q(F_{g_\alpha})$, with $x \in \Pi_{g_\alpha} \setminus G_{\mathrm{tr}}$ and isotropy C_3^2.

  Note that two points of type A_2 in F_{g_α}/S_3 are identified by the normalization map $F_{g_\alpha}/S_3 \to q(F_{g_\alpha})$.
- The eight points of type A_1 in F_{g_α}/S_3 correspond to
  - four points in $q(F_{g_\alpha})$ with isotropy C_6,
  - two points in the intersection of $q(F_{-\mathrm{id}})$ and $q(F_{g_\alpha})$, with isotropy C_6,
  - two points in $q(F_{g_\alpha})$ with isotropy BT_{24}.

We are left to determine the isolated singular points in $\mathrm{K}_2(A)/G$ according to their stabilizer type. By Lemma 11.5, an isolated fixed point in $\mathrm{K}_2(A)$ can only have stabilizer isomorphic to C_4, C_3 or C_2.

Isotropy C_4. As all subgroups in G isomorphic to C_4 are conjugate, it suffices to consider the fixed points of the linear automorphism $g_4 \in BT_{24}$. By Lemma 11.5 for C_4, the automorphism g_4 fixes eight points not lying on $F_{-\mathrm{id}}$, but two of them are fixed by the whole BT_{24}, and they lie on F_{g_α}; see Lemma 11.5 for BT_{24}. Thus, the number of isolated singular points with isotropy C_4 in X/G is

$$(8-2)\cdot(\#\text{subgroups conjugate to } \langle g_4\rangle)/(|G|/|C_4|) = 6\cdot 27/(216/4) = 3. \tag{11.1}$$

Isotropy C_2. As all subgroups in G isomorphic to C_2 are conjugate, it suffices to consider the points fixed by $-\mathrm{id}$. By Lemma 11.5 for C_2, the involution $-\mathrm{id}$ fixes 36 points not lying on $F_{-\mathrm{id}}$, but

- 2 of them are fixed by the whole BT_{24} and lie on F_{g_α},
- 18 of them have stabilizer $\langle i\rangle, \langle j\rangle$ or $\langle k\rangle \simeq C_4$,
- 16 of them lie on a fixed surface F_g for some $g \in BT_{24}$ of order 3.

So there are no isolated points with isotropy C_2 in X/G.

Isotropy ${}_{12}C_3$. Consider the subgroups in G conjugate to $\langle g_\alpha\rangle$. By Lemma 11.5 for C_3, the automorphism g_α fixes 12 points not lying on F_{g_α}, but

- 1 lies on $F_{-\mathrm{id}} \cap F_{\tau_\alpha g_\alpha} \cap F_{\tau_{-\alpha} g_\alpha}$,
- 2 lie on $F_{\tau_\alpha g_\alpha} \cap F_{\tau_{-\alpha} g_\alpha}$,
- 9 lie on a single surface $F_{\tau_\beta(-\mathrm{id})}$ with $\beta \in \{0, \pm\alpha\}$.

So there are no isolated points with isotropy C_3 conjugate to $\langle g_\alpha\rangle$ in X/G.

Isotropy ${}_4C_3$. Consider the subgroups in G conjugate to $\langle\tau_\alpha\rangle$. By Lemma 11.6(1), the translation τ_α fixes 27 isolated points of the form $z = [(x, x+\alpha, x-\alpha)]$ for $x \in A[3]$. If $x \in G_{\mathrm{tr}}$, then z lies on $F_{\tau_{-x}(-\mathrm{id})}$. Thus, the number of isolated points with isotropy C_3 conjugate to $\langle\tau_\alpha\rangle$ in X/G is

$$(27-9)\cdot(\#\text{subgroups conjugate to } \langle\tau_\alpha\rangle)/(|G|/|C_3|) = (27-9)\cdot 4/(216/3) = 1. \tag{11.2}$$

Isotropy ${}_{24}C_3$. Consider the subgroups of G conjugate to $\tau_\gamma g_\alpha$ with $\gamma \in G_{\mathrm{tr}} \setminus \Pi_{g_\alpha}$. By Lemma 11.7, the number of isolated points with isotropy conjugate to $\langle\tau_\gamma g_\alpha\rangle$ is

$$27\cdot(\#\text{subgroups conjugate to } \langle\tau_\gamma g_\alpha\rangle)/(|G|/|C_3|) = 27\cdot 24/(216/3) = 9. \tag{11.3}$$

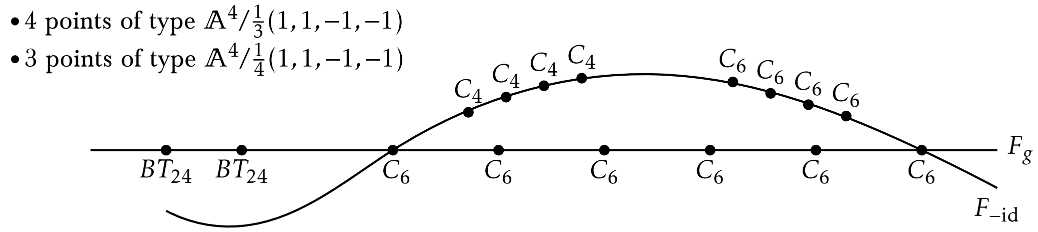
Finally, combining Lemma 11.1 and (11.1)–(11.3), we conclude that

$$a_2 = 3\cdot 4 + 4\cdot 2 = 20, \quad a_3 = 10 + 2\cdot 3 = 16, \quad a_4 = 3.$$

Lemma 11.7. *There are precisely* 27 *points in* $\mathrm{K}_2(A)$ *fixed by* $\tau_\gamma g_\alpha$ *with* $g_\alpha(\gamma) \neq \gamma$.

Proof. Let $z \in \mathrm{K}_2(A)$ with $G_z = \langle \tau_\gamma g_\alpha \rangle$, and write $\epsilon(z) = [(x, y, -x-y)] \in A_0^{(3)}$, where x and y are fixed by $\tau_\gamma g_\alpha$. Note that $x, y \in A[9] \cap (g_\alpha - \mathrm{id})^{-1}(\gamma)$, which consists of nine elements. If $x \neq y, -2y, 4y$, then z does not lie on the exceptional locus of ϵ, and we have $(9\cdot 6)/3! = 9$ such points z. Otherwise, $\epsilon(z) = [(x, x, -2x)]$ and $\tau_\gamma g_\alpha$ fixes two points in $\epsilon^{-1}(x, x, -2x)$ by Lemma 11.2. In total, we obtain 18 more points fixed by $\tau_\gamma g_\alpha$ lying on the exceptional locus. ☐

11.5.3. Group $G = BT_{24}$ (ID 24,3).—



12. Birational orbifolds

In this section, we show some exceptional birational maps between terminalizations of different quotients of IHS varieties; see Proposition 12.3. While the determination of Betti numbers, fundamental groups and singularities are essentially algorithmic, determining whether projective terminalizations of two different quotients X_1/G_1 and X_2/G_2 are deformation equivalent represents a subtle task. An obvious necessary condition is that their deformation invariants coincide, namely the corresponding rows in Table 9 or in [Men22, Table in Theorem 1.11] are identical. When this is the case (with the single open exception of Remark 12.4), we find an explicit birational map between X_1/G_1 and X_2/G_2, so that their terminalizations are deformation equivalent; see Propositions 3.14 and 12.3. The idea is to write G_1 as an extension of G_2 by a normal subgroup N_1 and then to show that X_2 is birational to X_1/N_1, equivariantly with respect to the given G_2-action on X_2 and the residual G_2-action on X_1/N_1. As a result, we obtain

$$X_1/G_1 = (X_1/N_1)/G_2 \sim_{\mathrm{bir.}} X_2/G_2.$$

Even when $X_1 = X_2 = \mathrm{K}_2(A)$, we can still run the argument: It suffices to find an isogeny $f\colon A \to A$ such that $N_1 = \ker(f)$. The ultimate goal is to merge the classification of irreducible symplectic varieties in this paper with [Men22, Theorem 1.11], avoiding redundancy.

Notation 12.1. Let $a\colon G \times \mathrm{K}_n(A) \to \mathrm{K}_n(A)$ be the action of a finite group of symplectic automorphisms G on $\mathrm{K}_n(A)$. A projective terminalization of the quotient $\mathrm{K}_n(A)/G$ is denoted by $\mathrm{K}_n(A, a)$. In the following, we always assume that the action a is induced by a symplectic action on the underlying abelian surface A, and we simply write $\mathrm{K}_n(A, G)$ when the action a of G is clear.

Definition 12.2. Let G be a finite group of symplectic automorphisms of a K3 surface S. Let $\theta\colon G \to G$ be an involution (which may also be the identity). The group G acts on S^n by

$$g(x_1, x_2, x_3, \ldots, x_n) = (g(x_1), \theta(g)(x_2), x_3, \ldots, x_n),$$

and the symmetric group S_n permutes the factors of S^n. A *Fujiki variety*, denoted by $S(G)_\theta^{[n]}$, is a terminalization of the quotient $S^n/\langle G, S_n\rangle$. In particular, we have

$$S(G)_\theta^{[n]} \sim_{\mathrm{bir.}} S^n/\langle G, S_n\rangle.$$

Proposition 12.3. *The following couples or triples of symplectic orbifolds with simply connected regular locus are deformation equivalent:*

(1) $\mathrm{K}_2(A, C_2) \sim \mathrm{K}_2(A, C_3^4 \rtimes C_2)$,
(2) $\mathrm{K}_2(A, S_3) \sim \mathrm{K}_2(A, C_3^3 \rtimes C_2)$,
(3) $\mathrm{K}_2(A, C_3) \sim S(C_3^2)^{[2]}_{-\mathrm{id}}$,
(4) $\mathrm{K}_2(A, C_3^2) \sim S(C_3)^{[2]}_{-\mathrm{id}}$,
(5) $\mathrm{K}_2(A, C_6) \sim \mathrm{K}_2(A, C_3^4 \rtimes C_6) \sim S(C_3 \rtimes S_3)^{[2]}_{\mathrm{id}}$,
(6) $\mathrm{K}_2(A, C_3 \rtimes C_6) \sim \mathrm{K}_2(A, C_3^3 \rtimes_4 C_6) \sim S(S_3)^{[2]}_{\mathrm{id}}$,
(7) $\mathrm{K}_2(A, C_3^2 \rtimes_4 C_6) \sim S(C_2)^{[2]}_{\mathrm{id}}$
(8) $\mathrm{K}_2(A, C_3^2 \rtimes C_6) \sim S(C_3 \rtimes S_3)^{[2]}_{(-\mathrm{id},\mathrm{id})}$,
(9) $\mathrm{K}_2(A, BT_{24}) \sim \mathrm{K}_2(A, C_3^4 \rtimes BT_{24})$,
(10) $\mathrm{K}_3(A, C_2^i \times C_2) \sim S(C_2^{4-i})^{[3]}_{\mathrm{id}}$ *for* $0 \le i \le 4$.

In all cases above, the group G acts on A as the affine group $G_{\mathrm{tr}} \rtimes G_\circ$ (see Lemma 10.6 and Equation (2.1)). For suitable choices of surfaces A and S and actions, the orbifolds in each row are actually birational.

Remark 12.4. The IHS orbifolds $\mathrm{K}_2(A, BT_{24})$ and $S(S_3^2 \rtimes C_2)^{[2]}_{\mathrm{id}}$ share the same Betti numbers and singularities. They could be a pair of deformation equivalent orbifolds, but the lemmas in this section are not sufficient to decide it.

Proof of Proposition 12.3. The proposition follows from Equations (12.1) and (12.5) below, Proposition 3.14, and

- Lemma 12.6 for (1), (2), (5), (6) and (9),
- Lemma 12.7 for (5), (6) and (7),
- Lemma 12.8 for (3), (4) and (8),
- Lemma 12.9 for (10).

In order to apply Lemmas 12.7 and 12.8 in cases (3)–(8), we first deform the pair (A, G) to (E^2, G), where E has complex multiplication. This is possible since the moduli space of such pairs (A, G) is connected by [Fuj88, Proposition 3.7]. ☐

Definition 12.5. Let $f : X \to Y$ be a morphism of algebraic varieties. An automorphism $h : X \to X$ descends along f to an automorphism $\bar{h} : Y \to Y$ if the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\bar{h}} & Y. \end{array}$$

Vice versa, we say that $\bar{h}$ lifts to h along f.

12.1. Birational orbifolds in dimension 4

Let G be a finite group of induced symplectic automorphisms of $\mathrm{K}_2(A)$. By construction, we have the following birational map:

$$\mathrm{K}_2(A, G) \sim_{\mathrm{bir.}} A_0^{(3)}/G \simeq A^2/(S_3 \times G), \tag{12.1}$$

where G acts diagonally on A^2 and the action of S_3 on A^2 is given by

$$\sigma(x, y) = (y, -x - y), \quad \tau(x, y) = (y, x). \tag{12.2}$$

Lemma 12.6. *Let* $H \simeq C_3^k \subseteq A[3]$ *for* $0 \le k \le 4$, *acting by translation on* A. *Assume that* $G_\circ$ *contains* $-\mathrm{id}$. *Then the following quotients are isomorphic:*

$$\frac{(A/H)^2}{S_3 \times (A[3]/H) \rtimes G_\circ} \simeq \frac{A^2}{S_3 \times H \rtimes G_\circ},$$

where the linear symplectic group $G_\circ$ *and the translation group* H (*respectively,* $A[3]/H$) *act diagonally on* A^2 (*respectively,* $(A/H)^2$).

Proof. Consider the isogeny $f_0 \colon A^2 \to A^2$ given by $f_0(x,y) = (x+2y, x-y)$, whose kernel is the diagonal copy of $A[3]$ in $A^2[3]$. The automorphisms σ, τ in (12.2) and $g \in G_\circ$ descend along f_0 to

$$\bar\sigma(x,y) = (-x-y, x), \quad \bar\tau(x,y) = (x+y, -y), \quad \bar g = g.$$

Note that the group $\langle \bar\sigma, \bar\tau\rangle = \langle \bar\sigma^2, \bar\sigma^2\bar\tau\rangle = \langle \sigma, -\tau\rangle$ acts via the standard action of S_3 up to a sign. Hence, the action of $S_3 \times A[3] \rtimes G_\circ = S_3 \times \ker(f_0) \rtimes G_\circ$ descends along f_0 to the action of $S_3 \times G_\circ$ since

$$\left\langle \bar\sigma, \bar\tau, \overline{A[3]}, \overline{-\mathrm{id}}, \bar g\right\rangle = \left\langle \bar\sigma^2, \bar\sigma^2\bar\tau, -\mathrm{id}, g\right\rangle = \langle \sigma, \tau, -\mathrm{id}, g\rangle.$$

Further, for any $(\alpha,\beta) \in A^2[3]$, the translation $\tau_{(\alpha,\beta)}$ descends along f_0 to $\bar\tau_{(\alpha,\beta)} = \tau_{(\alpha-\beta,\alpha-\beta)}$. In particular, the anti-diagonal $H^- := \{(\alpha,-\alpha)\} \subset H^2 \subset A^2[3]$ descends along f_0 to the diagonal $H := \{(\alpha,\alpha)\} \subset H^2 \subset A^2[3]$. We conclude that

$$\frac{(A/H)^2}{S_3 \times (A[3]/H) \rtimes G_\circ} \simeq \frac{A^2}{S_3 \times (H^- \times A[3]) \rtimes G_\circ} \simeq \frac{A^2}{S_3 \times H \rtimes G_\circ}. \qquad \square$$

Let ξ_3 be a primitive third root of unity, and let E be an elliptic curve with complex multiplication $\xi_3 \curvearrowright E \colon x \mapsto \xi_3 \cdot x$. Denote by $g_3 \colon E^2 \to E^2$ the diagonal automorphism $g_3(x_1,x_2) = (\xi_3 x_1, \xi_3^{-1} x_2)$.

Lemma 12.7. *Let* G' *be a finite symplectic group acting diagonally on* E^2, *and set* $G := \langle \Pi_{g_3}, g_3, G'\rangle$. *The group* G' *acts on the K3 surface* $S \sim_{\mathrm{bir.}} E^2/\langle g_3\rangle$, *and the following orbifolds are birational:*

$$\mathrm{K}_2(E^2, G) \sim_{\mathrm{bir.}} S(G')^{[2]}_{\mathrm{id}}$$

Proof. We follow closely [Kaw09, Proof of Theorem 4.2]. As in (12.1), there exists a birational map

$$\mathrm{K}_2(E^2, G) \sim_{\mathrm{bir.}} E^4/(S_3 \times G).$$

Consider the isogeny $f_1 \colon E^4 \to E^4$ given by

$$f_1(x_1,x_2,x_3,x_4) = \left(\xi_3^2 x_1 - x_3, \xi_3^2 x_2 - x_4, -\xi_3 x_1 + x_3, -\xi_3 x_2 + x_4\right),$$

whose kernel is the diagonal copy of Π_{g_3} in $E^4[3]$. The automorphisms σ, τ and g_3 descend along f_1 to $\bar\sigma$, $\bar\tau$ and $\bar g_3$ such that

$$\begin{aligned} \bar g_3\bar\sigma(x_4,x_1,x_2,x_3) &= \left(\xi_3 x_4, \xi_3^2 x_1, x_2, x_3\right), \\ \bar g_3\bar\sigma^2(x_4,x_1,x_2,x_3) &= \left(x_4, x_1, \xi_3 x_2, \xi_3^2 x_3\right), \\ \bar\tau\bar\sigma(x_4,x_1,x_2,x_3) &= (x_2,x_3,x_4,x_1). \end{aligned} \tag{12.3}$$

In particular, we obtain

$$E^4/\langle \bar\sigma, \bar\tau, \bar g_3\rangle = E^4/\langle \bar g_3\bar\sigma, \bar g_3\bar\sigma^2, \bar\tau\bar\sigma\rangle \simeq \left((E^2/g_3) \times (E^2/g_3)\right)/\bar\tau\bar\sigma \sim_{\mathrm{bir.}} S^{[2]}. \tag{12.4}$$

An element $g' \in G'$ is of the form

$$g'(x_1,x_2,x_3,x_4) = (cx_1 + a, dx_2 + b, cx_3 + a, dx_2 + b)$$

for some $c, d \in \mathbb{C}$ and $a, b \in E[3]$, and it descends along f_1 to

$$\bar g'(x_4,x_1,x_2,x_3) = \left(dx_4 + \bar b, cx_1 + \bar a, dx_2 + \bar b, cx_1 + \bar a\right),$$

with $\bar{a} = \xi_3^2 a - a$ and $\bar{b} = \xi_3^2 b - b$. Since $\xi_3 \bar{a} = \bar{a}$ and $\xi_3 \bar{b} = \bar{b}$, the morphism $\bar{g}'$ commutes with all the automorphisms $\bar{\sigma}$, $\bar{\tau}$ and $\bar{g}_3$.

We conclude that

$$\mathrm{K}_2(E^2, G) \sim_{\mathrm{bir.}} E^4/(S_3 \times G) \simeq E^4/(\langle \bar{\sigma}, \bar{\tau}, \bar{g}_3 \rangle \times G') \sim_{\mathrm{bir.}} S^{[2]}/G' \sim_{\mathrm{bir.}} S(G')^{[2]}_{\mathrm{id}}. \qquad \square$$

Lemma 12.8. *The following orbifolds are birational:*

- $\mathrm{K}_2(E^2, C_3) \sim_{\mathrm{bir.}} S(C_3^2)^{[2]}_{-\mathrm{id}}$ *with* $C_3 = \langle g_3 \rangle$ *and* $S \sim_{\mathrm{bir.}} E^2/\langle g_3 \rangle$,
- $\mathrm{K}_2(E^2, C_3^2) \sim_{\mathrm{bir.}} S_\alpha(C_3)^{[2]}_{-\mathrm{id}}$ *with* $C_3^2 = \langle g_3, \tau_\alpha \rangle$ *and* $S_\alpha \sim_{\mathrm{bir.}} E^2/\langle g_3, \tau_\alpha \rangle$,
- $\mathrm{K}_2(E^2, C_3^2 \rtimes C_6) \sim_{\mathrm{bir.}} S_\alpha(C_3 \rtimes S_3)^{[2]}_{\theta}$ *with* $C_3^2 \rtimes C_6 = \langle g_3, -\mathrm{id}, \tau_\alpha, \tau_\beta \rangle$, $g_3(\beta) \neq \beta$ *and* $\theta = (-\mathrm{id}, \mathrm{id})$ *acting on* $C_3 \rtimes S_3$.

Proof. Consider the isogeny $f_2 \colon E^4 \to E^4$ given by

$$f_2(x_1, x_2, x_3, x_4) = \left(x_1 + x_3, x_2 + x_4, \xi_3 x_1 + \xi_3^2 x_3, \xi_3 x_2 + \xi_3^2 x_4\right),$$

whose kernel is the anti-diagonal copy of Π_{g_3} in $E^4[3]$; *i.e.*,

$$\ker(f_2) = \{(a, b, -a, -b) \in E^4 \,|\, \xi_3(a) = a, \xi_3(b) = b\} \subseteq E^4[3].$$

The automorphisms σ, τ in (12.2) and g_3 lift along f_2 to the automorphisms $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{g}_3$ such that

$$\begin{aligned} \tilde{g}_3 \tilde{\sigma}(x_4, x_1, x_2, x_3) &= \left(\xi_3 x_4, \xi_3^2 x_1, x_2, x_3\right), \\ \tilde{g}_3 \tilde{\sigma}^2(x_4, x_1, x_2, x_3) &= \left(x_4, x_1, \xi_3 x_2, \xi_3^2 x_3\right), \\ \tilde{\tau} \tilde{\sigma}^2(x_4, x_1, x_2, x_3) &= (x_2, x_3, x_4, x_1). \end{aligned}$$

Thus, the group $\langle \tilde{\sigma}, \tilde{g}_3 \rangle \simeq C_3^2$ acts on $E^2_{x_4, x_1} \times E^2_{x_2, x_3}$ as $\langle g_3 \rangle \times \langle g_3 \rangle$, while the group $\langle \ker(f_2), \tilde{\tau}\tilde{\sigma}^2 \rangle \simeq C_3^2 \rtimes C_2$ acts on $E^4/\langle \tilde{\sigma}, \tilde{g}_3 \rangle \sim_{\mathrm{bir.}} S^2$ as the group $\langle C_3^2, C_2 \rangle$ in Definition 12.2 with $\theta = -\mathrm{id}$.

From the short exact sequence

$$1 \longrightarrow C_3^2 = \langle \tilde{\sigma}, \tilde{g}_3 \rangle \longrightarrow (C_3^3 \rtimes C_2) \times C_3 = \langle \ker(f_2), \tilde{\sigma}, \tilde{\tau}, \tilde{g}_3 \rangle \longrightarrow C_3^2 \rtimes C_2 = \left\langle \ker(f_2), \tilde{\tau}\tilde{\sigma}^2 \right\rangle \longrightarrow 1,$$

we obtain that

$$\begin{aligned} \mathrm{K}_2(E^2, C_3) \sim_{\mathrm{bir.}} E^4/(S_3 \times C_3) &= E^4/\langle \sigma, \tau, g_3 \rangle \simeq E^4/\langle \ker(f_2), \tilde{\sigma}, \tilde{\tau}, \tilde{g}_3 \rangle = E^4/((C_3^3 \rtimes C_2) \times C_3) \\ &\simeq \left((E^2/g_3) \times (E^2/g_3)\right)/(C_3^2 \rtimes C_2) \sim_{\mathrm{bir.}} S^2/(C_3^2 \rtimes C_2) \sim_{\mathrm{bir.}} S(C_3^2)^{[2]}_{-\mathrm{id}}. \end{aligned}$$

A 3-torsion point α in the diagonal $E^2[3] \subset E^4[3]$ lifts along f_2 to its opposite $-\alpha$, up to a translation in $\ker(f_2)$. If $\alpha = (a, b)$ is a nonzero translation in Π_{g_3}, then $\langle \ker(f_2), \tilde{\tau}_\alpha \rangle$ is generated by three translations

$$\tau_1 \coloneqq (a, 0, 0, b), \quad \tau_2 \coloneqq (0, b, a, 0), \quad \tau_3 \in \ker(f_2) \setminus \langle (a, -b, -a, b) \rangle.$$

From the short exact sequence

$$1 \longrightarrow C_3^4 = \langle \tilde{\sigma}, \tilde{g}_3, \tau_1, \tau_2 \rangle \longrightarrow \langle \ker(f_2), \tilde{\sigma}, \tilde{\tau}, \tilde{g}_3, \tilde{\tau}_\alpha \rangle = \left(C_3^3 \rtimes C_2\right) \times C_3^2 \longrightarrow S_3 = \left\langle \tau_3, \tilde{\tau}\tilde{\sigma}^2 \right\rangle \longrightarrow 1,$$

we obtain that

$$\begin{aligned} \mathrm{K}_2(E^2, C_3^2) \sim_{\mathrm{bir.}} E^4/\left(S_3 \times C_3^2\right) &\simeq E^4/\left(\left(C_3^3 \rtimes C_2\right) \times C_3^2\right) \\ &\simeq \left(E^2/\langle g_3, \tau_{(b,a)} \rangle\right)^2/S_3 \sim_{\mathrm{bir.}} S_\alpha^2/S_3 \sim_{\mathrm{bir.}} S(C_3)^{[2]}_{-\mathrm{id}}. \end{aligned}$$

Quotienting further by $S_3 = \langle \tau_\beta, -\mathrm{id} \rangle$ with $g_3(\beta) = \beta + \alpha$, we also obtain

$$\mathrm{K}_2\left(E^2, C_3^2 \rtimes C_6\right) \sim_{\mathrm{bir.}} S_\alpha \left(C_3 \rtimes S_3\right)^{[2]}_{\theta}. \qquad \square$$

12.2. Birational orbifolds in dimension 6

The following Lemma 12.9 was communicated to the authors by Menet. By construction, we have a birational map

$$\mathrm{K}_3(A,G) \sim_{\text{bir.}} A_0^{(4)}/G \simeq A^3/(S_4 \times G), \tag{12.5}$$

where G acts diagonally on A^3 and the action of S_4 on A^2 is given by

$$\sigma_{12}(x,y,z) = (y,x,z), \quad \sigma_{13}(x,y,z) = (z,y,x), \quad \sigma_{14}(x,y,z) = (-x-y-z,y,z).$$

Lemma 12.9. *Let $H \simeq C_2^k \subseteq A[2]$ for $0 \le k \le 4$, and set $G := A[2]/H \times \langle -\mathrm{id}\rangle$. The group H acts by translation on A, and it induces an action on the corresponding Kummer surface $T \sim_{\text{bir.}} A/\langle -\mathrm{id}\rangle$. Then the following orbifolds are birational:*

$$\mathrm{K}_3(A/H,G) \sim_{\text{bir.}} T(H)^{[3]}_{\mathrm{id}}.$$

Proof. Consider the isogeny $f_3\colon A^3 \to A^3$ given by $f_3(x,y,z) = (x+y,x+z,y+z)$, whose kernel is the diagonal copy of $A[2]$ in $A^3[2]$. The automorphisms σ_{12}, σ_{13}, σ_{14} and $-\mathrm{id}$ descend along f_3 to, respectively, the permutations (23), $(13) \in S_3$ of the factors of A^3, and

$$\bar\sigma_{14}(x,y,z) = (-y,-x,z) = (-\mathrm{id},-\mathrm{id},\mathrm{id})(12)(x,y,z), \qquad \overline{-\mathrm{id}} = (-\mathrm{id},-\mathrm{id},-\mathrm{id}).$$

Hence, the action of $S_4 \times A[2] \times \langle -\mathrm{id}\rangle = S_4 \times \ker(f_3) \times \langle -\mathrm{id}\rangle$ descends along f_3 to the action of $S_3 \times \langle -\mathrm{id}\rangle^3$, and

$$\frac{A^3}{S_4 \times A[2] \times \langle -\mathrm{id}\rangle} \simeq \frac{A^3}{S_3 \times \langle -\mathrm{id}\rangle^3} \sim_{\text{bir.}} T^{[3]}.$$

Further, for any $(\alpha,\beta,\gamma) \in A^3[2]$, the translation $\tau_{(\alpha,\beta,\gamma)}$ of A^3 descends along f_3 to

$$\bar\tau_{(\alpha,\beta,\gamma)} = (\tau_\alpha,\tau_\alpha,\mathrm{id})(\tau_\beta,\mathrm{id},\tau_\beta)(\mathrm{id},\tau_\gamma,\tau_\gamma).$$

In particular, the action of $H^3 \subseteq A^3[2]$ descends along f_3 to the action of $H^3 \subset \langle H, S_3\rangle$ as in Definition 12.2 with $\theta = \mathrm{id}$. We conclude that

$$\frac{(A/H)^3}{S_4 \times A[2]/H \times \langle -\mathrm{id}\rangle} \simeq \left(\frac{A^3}{S_4 \times A[2] \times \langle -\mathrm{id}\rangle}\right)\Big/ H^3 \simeq \frac{A^3}{\langle H,S_3\rangle \times \langle -\mathrm{id}\rangle^3} \sim_{\text{bir.}} T(H)^{[3]}_{\mathrm{id}}. \qquad \square$$

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