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A proof of generic Green's conjecture in odd genus

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Abstract. In this note, we give a new proof of Voisin's theorem on canonical syzygies for generic curves of odd genus.

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Contents

1. Introduction

Our goal is to give a proof of Voisin's theorem in [Voi05] resembling [Kem21, Sections 1 and 2]. Green's conjecture on the syzygies of a canonical curve, see [Gre84], is one of the central conjectures on the syzygies of projective varieties. It was first proven for a generic curve by Voisin. This proof, using the geometry of $K3$ surfaces intimately, is a landmark result; however, it is somewhat long and complicated. Alternate proofs have been offered recently, including an approach using the tangent developable, see [AFP+19], which is very different to the approach we take here. Our approach here is rather formal and only uses the geometry of $K3$ surfaces for the first few steps to set up the problem, leading one to hope that it generalizes to new situations.

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2. The proof

The main idea is that while, in the notation below, the local complete intersection scheme $\mathcal{Z}$ is not finite over $\mathbf{P}$, $\widetilde{\mathcal{Z}}$ is finite over $\mathbf{P}_2$.

Let X denote a $K3$ surface with $\mathrm{Pic}(X)$ generated by a big and nef line bundle L' with $(L')^2 = 4k+2$ together with a smooth rational curve Δ with $(L' \cdot \Delta) = 0$ for $k \geq 4$; so we need a different $K3$ surface for each integer k.(1) Set $L := L'(-\Delta)$. Then L and L' are base-point-free, with $\mathrm{H}^1(X, L) = \mathrm{H}^1(X, L') = 0$; see [Kem21, Lemma 6]. We use the ***kernel bundle*** approach, see [EL12, Section 3], and so need to show $\mathrm{H}^1(\bigwedge^k M_L(L)) = 0$. We have the exact sequence

$$0 \longrightarrow \bigwedge^{k+1} M_L(L') \longrightarrow \bigwedge^{k+1} M_{L'}(L') \longrightarrow \bigwedge^{k} M_L(L) \longrightarrow 0,$$

where $M_L, M_{L'}$ are the kernel bundles in the notation of [AN10, Section 2.1]. Duality gives $\mathrm{H}^2(\bigwedge^{k+1} M_L(L')) \simeq \mathrm{H}^0(\bigwedge^k M_L(-\Delta)) \subseteq \bigwedge^k \mathrm{H}^0(L) \otimes \mathrm{H}^0(\mathcal{O}(-\Delta)) = 0$. So $\mathrm{H}^1(\bigwedge^k M_L(L)) = 0$ if the map $\mathrm{H}^1(\bigwedge^{k+1} M_L(L')) \to \mathrm{H}^1(\bigwedge^{k+1} M_{L'}(L'))$, induced from $M_L \subseteq M_{L'}$, is surjective.

Contract Δ via a map $\mu\colon X \to \hat{X}$, where $\hat{X}$ is a nodal $K3$ surface. We have a line bundle $\hat{L}$ on $\hat{X}$ with $\mu^*\hat{L} \simeq L'$ and the rank two bundle $\hat{E}$ on $\hat{X}$, with $h^0(\hat{X}, \hat{E}) = k+3$ and $\det \hat{E} = \hat{L}$, defined as the Lazarsfeld–Mukai bundle induced by a general g^1_{k+2} on $C \in |\hat{L}|$. Set E to be the vector bundle $\mu^*\hat{E}$ on X. Consider $E(-\Delta)$, a globally generated bundle with $h^0(X, E(-\Delta)) = k+1$; see [Kem21, Section 3]. Then $\mathrm{H}^0(X, E) \simeq \mathrm{H}^0(\hat{X}, \hat{E})$. Set $\mathbf{P} := \mathbf{P}(\mathrm{H}^0(X, E)) := \mathrm{Proj}(\mathrm{H}^0(X, E)^\vee) \simeq \mathbf{P}^{k+2}$. Let $p\colon X \times \mathbf{P} \to X$ and $q\colon X \times \mathbf{P} \to \mathbf{P}$ denote the projections. Let $\mathcal{Z} \subseteq X \times \mathbf{P}$ be the locus $\{(x, s) \mid s(x) = 0\} \subseteq X \times \mathbf{P}$, which is a projective bundle

(1)There is a typo in [Kem21, Section 3], where it is written $(L')^2 = 2k+2$.

over X, and thus irreducible. We have $\dim \mathcal{Z} = k+2$, and $q_{|\mathcal{Z}}$ is generically finite but has one-dimensional fibres over $\mathbf{P}(\mathrm{H}^0(X, E(-\Delta))$. Further, $\mathcal{Z}$ has codimension two in $X \times \mathbf{P}$, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\mathrm{id}} E \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^*L' \otimes I_{\mathcal{Z}} \longrightarrow 0,$$

which is a Koszul complex, where id is given by multiplication by id in

$$\mathrm{H}^0(E \boxtimes \mathcal{O}_{\mathbf{P}}(1)) \simeq \mathrm{H}^0(E) \otimes \mathrm{H}^0(E)^\vee.$$

Set $\mathbf{P}_2 := \mathbf{P}(\mathrm{H}^0(\hat{X}, \hat{E}))$. Then $\mu_* E = \mu_* \mu^* \hat{E} \simeq \hat{E}$, and we have an isomorphism $i\colon \mathbf{P} \to \mathbf{P}_2$. Let $\hat{\mathcal{Z}} \subseteq \hat{X} \times \mathbf{P}_2$ be the codimension two locus $\{(x,s) \mid s(x) = 0\} \subseteq \hat{X} \times \mathbf{P}(\mathrm{H}^0(\hat{X}, \hat{E}))$, which is an lci in a Cohen–Macaulay scheme and hence a Cohen–Macaulay scheme; see [Mat70, p. 112].

We have an exact sequence $0 \to \mathcal{O}_{\hat{X}} \boxtimes \mathcal{O}_{\mathbf{P}_2}(-2) \xrightarrow{\mathrm{id}} \hat{E} \boxtimes \mathcal{O}_{\mathbf{P}_2}(-1) \to \hat{p}^* \hat{L} \otimes I_{\hat{\mathcal{Z}}} \to 0$, where $\hat{p}\colon \hat{X} \times \mathbf{P}_2 \to \hat{X}$ and $\hat{q}\colon \hat{X} \times \mathbf{P}_2 \to \mathbf{P}_2$ denote the projections. We let

$$\tau := \mu \times i\colon X \times \mathbf{P} \longrightarrow \hat{X} \times \mathbf{P}_2.$$

Let v be the node of $\hat{X}$.

Lemma 2.1. *We have $\tau^* I_{\hat{\mathcal{Z}}} \simeq I_{\mathcal{Z}}$, $\tau_* I_{\mathcal{Z}} \simeq I_{\hat{\mathcal{Z}}}$ and $\tau_*(I_{\mathcal{Z}} \otimes p^* \mathcal{O}_X(-\Delta)) \simeq I_{\hat{\mathcal{Z}}} \otimes \hat{p}^* I_v$.*

Proof. We have the two exact sequences

$$0 \longrightarrow \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\mathrm{id}} E \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^*L' \otimes I_{\mathcal{Z}} \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathcal{O}_{\hat{X}} \boxtimes \mathcal{O}_{\mathbf{P}_2}(-2) \xrightarrow{\mathrm{id}} \hat{E} \boxtimes \mathcal{O}_{\mathbf{P}_2}(-1) \longrightarrow \hat{p}^* \hat{L} \otimes I_{\hat{\mathcal{Z}}} \longrightarrow 0.$$

Now $\mu_* \mathcal{O}_X \simeq \mathcal{O}_{\hat{X}}$, $R^i \mu_* \mathcal{O}_X \simeq \mathcal{O}_{\hat{X}} = 0$ for $i > 0$, $\mu^* \hat{E} \simeq E$ and $\mu_* E \simeq \hat{E}$. So the first two claims follow by applying τ^* or τ_* to the two exact sequences. For the third statement, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-\Delta) \boxtimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\mathrm{id}} E(-\Delta) \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^*L'(-\Delta) \otimes I_{\mathcal{Z}} \longrightarrow 0.$$

We may view τ as the blow-up of $\hat{X} \times \mathbf{P}_2$ in the singularity $v \times \mathbf{P}_2$. The claim then follows by applying τ_*. ☐

The following lemma is important for us.

Lemma 2.2. *The sheaves $q^* q_*(p^* L \otimes I_{\mathcal{Z}})$ and $q^* q_*(p^* L' \otimes I_{\mathcal{Z}})$ are locally free on $X \times \mathbf{P}$. Further, the natural morphisms $q^* q_*(p^* L \otimes I_{\mathcal{Z}}) \to p^* L \otimes I_{\mathcal{Z}}$ and $q^* q_*(p^* L' \otimes I_{\mathcal{Z}}) \to p^* L' \otimes I_{\mathcal{Z}}$ are surjective.*

Proof. We have

$$i_* q_* (p^* L' \otimes I_{\mathcal{Z}}) \simeq \hat{q}_* \tau_* (p^* L' \otimes I_{\mathcal{Z}}) \simeq \hat{q}_* \left(\hat{p}^* \hat{L} \otimes \tau_* I_{\mathcal{Z}}\right) \simeq \hat{q}_* \left(\hat{p}^* \hat{L} \otimes I_{\hat{\mathcal{Z}}}\right)$$

by Lemma 2.1. By miracle flatness $I_{\hat{\mathcal{Z}}}$ is flat over $\mathbf{P}_2$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\hat{X}} \xrightarrow{s} \hat{E} \longrightarrow \hat{L} \otimes I_{Z(s)} \longrightarrow 0,$$

and $\mathrm{H}^1(\mathcal{O}_{\hat{X}}) = 0$, $h^0(\hat{E}) = k+3$, so $h^0(\hat{X}, \hat{L} \otimes I_{Z(s)}) = k+2$ for any $s \neq 0$, so $q^* q_*(p^* L' \otimes I_{\mathcal{Z}})$ is locally free. Next,

$$i_* q_* (p^* (L'(-\Delta)) \otimes I_{\mathcal{Z}}) \simeq \hat{q}_* \tau_* (p^* (L'(-\Delta)) \otimes I_{\mathcal{Z}}) \simeq \hat{q}_* \left(\hat{p}^* \hat{L} \otimes I_{\hat{\mathcal{Z}}} \otimes \hat{p}^* I_v\right).$$

We have the exact sequence $0 \to I_v \xrightarrow{s} \hat{E} \otimes I_v \to \hat{L} \otimes I_{Z(s)} \otimes I_v \to 0$. Now $I_v \simeq \mu_* \mathcal{O}_X(-\Delta)$, $R^j \mu_* \mathcal{O}_X(-\Delta) = 0$ for $j > 0$ and $\hat{E} \otimes I_v \simeq \mu_*(E(-\Delta))$, so $h^0(I_v) = h^1(I_v) = h^1(\mathcal{O}_X(-\Delta)) = 0$ and $h^0(\hat{X}, \hat{L} \otimes I_{Z(s)} \otimes I_v) = h^0(X, E(-\Delta)) = k+1$, using [Kem21, Lemma 10] and the fact that $q^* q_*(p^* L \otimes I_{\mathcal{Z}})$ is locally free. Next, we show that

$$q^* q_*(p^* L' \otimes I_{\mathcal{Z}}) \longrightarrow p^* L' \otimes I_{\mathcal{Z}}$$

is surjective. We have a natural map $\mathrm{H}^0(E)\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)\to q^*q_*(p^*L'\otimes I_{\mathcal{Z}})$ and a surjection $E\boxtimes\mathcal{O}_{\mathbf{P}}(-1)\twoheadrightarrow p^*L'\otimes I_{\mathcal{Z}}$. The evaluation map $\mathrm{H}^0(E)\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)\to E\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)$ is surjective because E is globally generated. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(E)\otimes q^*\mathcal{O}_{\mathbf{P}}(-1) & \twoheadrightarrow & E\otimes q^*\mathcal{O}_{\mathbf{P}}(-1) \\ \downarrow & & \downarrow\!\!\downarrow \\ q^*q_*(p^*L'\otimes I_{\mathcal{Z}}) & \longrightarrow & p^*L'\otimes I_{\mathcal{Z}}, \end{array}$$

so the lower horizontal map must be surjective. We can show similarly that the map

$$q^*q_*(p^*L\otimes I_{\mathcal{Z}})\longrightarrow p^*L\otimes I_{\mathcal{Z}}$$

is surjective. Indeed, we have a natural map

$$\mathrm{H}^0(E(-\Delta))\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)\longrightarrow q^*q_*(p^*L\otimes I_{\mathcal{Z}})$$

and a natural surjection $E(-\Delta)\boxtimes\mathcal{O}_{\mathbf{P}}(-1)\hookrightarrow p^*L'(-\Delta)\otimes I_{\mathcal{Z}}=p^*L\otimes I_{\mathcal{Z}}$. The evaluation map from the group $\mathrm{H}^0(E(-\Delta))\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)$ to $E(-\Delta)\otimes q^*\mathcal{O}_{\mathbf{P}}(-1)$ is surjective because $E(-\Delta)$ is globally generated; see [Kem21, Lemma 10]. □

To proceed, we first need a lemma.

Lemma 2.3. *The sheaves $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})$ and $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)$ on $\mathbf{P}_2$ are line bundles.*

Proof. We have the exact sequence

$$0\longrightarrow\mathcal{O}_{\hat{X}}\xrightarrow{t}\hat{E}\longrightarrow\hat{L}\otimes I_{Z(t)}\longrightarrow 0$$

for $t\neq 0\in\mathrm{H}^0(\hat{E})$. Since $\mathrm{H}^1(\hat{E})=\mathrm{H}^2(\hat{E})=0$, this gives $h^1(\hat{X},\hat{L}\otimes I_{Z(t)})=h^2(\mathcal{O}_{\hat{X}})=1$, so $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})$ is a line bundle by Grauert's theorem. To study $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)$, we use Grauert's theorem plus the exact sequence

$$0\longrightarrow I_v\xrightarrow{s}\hat{E}\otimes I_v\longrightarrow\hat{L}\otimes I_{Z(s)}\otimes I_v\longrightarrow 0$$

of sheaves on $\hat{X}$, for $s\neq 0\in\mathrm{H}^0(\hat{E})$. Note that the finite map $\hat{\mathcal{Z}}\to\mathbf{P}_2$ is flat by miracle flatness. Now we claim $\mathrm{H}^i(\hat{X},\hat{E}\otimes I_v)=0$ for $i=1,2$. Indeed, $\mu_*\mathcal{O}_X(-\Delta)\simeq I_v$ and $R^i\mu_*\mathcal{O}_X(-\Delta)=0$ for $i>0$. Thus, $R^i\mu_*(E(-\Delta))\simeq\hat{E}\otimes R^i\mu_*\mathcal{O}_X(-\Delta)=0$ for $i>0$. So $h^i(\hat{X},\hat{E}\otimes I_v)=h^i(X,E(-\Delta))$ for $i>0$. But this vanishes, as $E(-\Delta)$ is a Lazarsfeld–Mukai bundle; see [Kem21, Lemma 10]. So $h^1(\hat{L}\otimes I_{Z(s)}\otimes I_v)=h^2(I_v)=h^2(\mathcal{O}_{\hat{X}})=1$, from the exact sequence

$$0\longrightarrow I_v\longrightarrow\mathcal{O}_{\hat{X}}\longrightarrow\mathcal{O}_v\longrightarrow 0.$$

So $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)$ is a line bundle. □

The next result is analogous to [Kem21, Lemma 1].

Proposition 2.4. *Define a coherent sheaf on $\mathbf{P}$ by $W:=\mathrm{Coker}\,(q_*(p^*L'\otimes I_{\mathcal{Z}})\to q_*p^*L')$. Then W is locally free of rank $k+1$. Further, define a coherent sheaf by $\widetilde{W}:=\mathrm{Coker}\,(q_*(p^*L\otimes I_{\mathcal{Z}})\to q_*p^*L)$. Then $\widetilde{W}$ is also locally free of rank $k+1$.*

Proof. Define $W':=\mathrm{Coker}(\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})\to\hat{q}_*\hat{p}^*\hat{L})$ and $\hat{W}:=\mathrm{Coker}(\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)\to\hat{q}_*(\hat{p}^*\hat{L}\otimes\hat{p}^*I_v))$ on $\mathbf{P}_2$. We will show they are locally free. For the first sheaf, we have the exact sequence

$$0\longrightarrow W'\longrightarrow\hat{q}_*\left(\hat{p}^*\hat{L}_{|\hat{\mathcal{Z}}}\right)\longrightarrow R^1\hat{q}_*\left(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\right)\longrightarrow 0.$$

By Lemma 2.3, $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})$ is a line bundle. The finite map $\hat{\mathcal{Z}}\to\mathbf{P}_2$ is flat by miracle flatness, so, by Grauert's theorem, $\hat{q}_*(\hat{p}^*\hat{L}_{|\hat{\mathcal{Z}}})$ is a vector bundle of rank $k+2$. So W' is locally free of rank $k+1$. Define $S:=\mathrm{Coker}(\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)\to\hat{q}_*\hat{p}^*\hat{L})$. We have an exact sequence $0\to\hat{q}_*(\hat{p}^*\hat{L}\otimes p^*I_v)\to\hat{q}_*\hat{p}^*\hat{L}\to\hat{q}_*(\hat{p}^*\hat{L}_{|p\times\mathbf{P}_2})\to 0$, and $\hat{q}_*(\hat{p}^*\hat{L}_{|p\times\mathbf{P}_2})$ is a line bundle on $\mathbf{P}_2$. Indeed, it suffices to show $R^1\hat{q}_*(\hat{p}^*\hat{L}\otimes\hat{p}^*I_v)=0$, and, by Grauert's theorem, for this it suffices to note $h^1(X,L) = h^1(X,L'(-\Delta)) = h^1(\hat{X},\hat{L}\otimes I_v) = 0$; see [Kem21, Lemma 6]. So we have an exact sequence

$$0 \longrightarrow \hat{W} \longrightarrow S \longrightarrow \hat{q}_*\left(\hat{p}^*\hat{L}_{|_{p\times \mathbf{P}_2}}\right) \longrightarrow 0,$$

where $\hat{q}_*(\hat{p}^*\hat{L}_{|_{p\times\mathbf{P}_2}})$ is a line bundle. To show $\hat{W}$ is locally free, it suffices to show S is locally free.

Let $T \subseteq X\times\mathbf{P}$ denote the closed subscheme $\mathcal{Z}\bigcup\Delta\times\mathbf{P}$ defined by the product ideal $I_{\mathcal{Z}}\,p^*I_\Delta$. Since $\Delta\times\mathbf{P}$ is a divisor, this ideal is isomorphic to $I_{\mathcal{Z}}\otimes p^*I_\Delta$. Set $\hat{T}\subseteq\hat{X}\times\mathbf{P}_2$ to be the closed subscheme defined by the product ideal $I_{\hat{\mathcal{Z}}}\,\hat{p}^*I_v$. Applying the left-exact functor τ_*, and using from Lemma 2.1 the fact that $\tau_*(I_{\mathcal{Z}}\otimes p^*I_\Delta)\simeq I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v$ and the fact that $\tau_*\mathcal{O}_{X\times\mathbf{P}}\simeq\mathcal{O}_{\hat{X}\times\mathbf{P}_2}$ (because, up to isomorphism, we can view τ as a blow-up in $v\times\mathbf{P}_2$), we get that the natural map $I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v\to\mathcal{O}_{\hat{X}\times\mathbf{P}_2}$ is injective, *i.e.* $I_{\hat{\mathcal{Z}}}\,\hat{p}^*I_v\simeq I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v$. So the zero locus of $I_{\hat{\mathcal{Z}}}\otimes I_{v\times\mathbf{P}_2}$ is $\hat{T}:=\hat{\mathcal{Z}}\bigcup\{v\times\mathbf{P}_2\}$. We have the exact sequence

$$0\longrightarrow S\longrightarrow\hat{q}_*\left(\hat{p}^*\hat{L}_{|_{\hat{T}}}\right)\longrightarrow R^1\hat{q}_*\left(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes I_{p\times\mathbf{P}_2}\right)\longrightarrow 0,$$

since $\mathrm{H}^1(\hat{X},\hat{L})=0$. Now $\hat{q}_*(\hat{p}^*\hat{L}_{|_{\hat{T}}})$ is a vector bundle of rank $k+3$ by Grauert's theorem. From Lemma 2.3, $R^1\hat{q}_*(\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes I_{v\times\mathbf{P}_2})$ is a line bundle. It follows that S is locally free of rank $k+2$ and thus

$$\hat{W}:=\operatorname{Coker}\left(\hat{q}_*\left(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v\right)\longrightarrow\hat{q}_*\left(\hat{p}^*\hat{L}\otimes\hat{p}^*I_v\right)\right)$$

is locally free. Now $i\colon\mathbf{P}\to\mathbf{P}_2$ is an isomorphism, so the sheaves $i^*W'\simeq\operatorname{Coker}(i^*\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})\to i^*\hat{q}_*(\hat{p}^*\hat{L}))$ and $i^*\hat{W}\simeq\operatorname{Coker}(i^*\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v)\to i^*\hat{q}_*(\hat{p}^*\hat{L}\otimes\hat{p}^*I_v))$ are locally free. But i is an isomorphism, so

$$i^*\hat{q}_*\left(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\otimes\hat{p}^*I_v\right)\simeq q_*(p^*L'\otimes I_Z\otimes p^*\mathcal{O}_X(-\Delta))\simeq q_*(p^*L\otimes I_Z).$$

Likewise $i^*\hat{q}_*(\hat{p}^*\hat{L}\otimes\hat{p}^*I_v)\simeq q_*(p^*L)$, $i^*\hat{q}_*(\hat{p}^*\hat{L})\simeq q_*(p^*L')$, $i^*\hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})\simeq q_*(p^*L'\otimes I_Z)$, $i^*\hat{W}\simeq\widetilde{W}$ and $i^*W'\simeq W$, which completes the proof. □

Let $\pi\colon B\to X\times\mathbf{P}$ be the blow-up along $\mathcal{Z}$, with exceptional divisor D, which is smooth since $\mathcal{Z}$ is smooth. Set $p':=p\circ\pi$, $q':=q\circ\pi$. The maps $q^*q_*(p^*L'\otimes I_{\mathcal{Z}})\to p^*L'\otimes I_{\mathcal{Z}}$ and $q^*q_*(p^*L\otimes I_{\mathcal{Z}})\to p^*L\otimes I_{\mathcal{Z}}$ are surjective by Lemma 2.2. Notice ${q'}_*(p'^*L\otimes I_D)\simeq q_*(p^*L\otimes I_{\mathcal{Z}})$, and likewise replacing L by L'. Noting I_D is a quotient of $\pi^*I_{\mathcal{Z}}$, we get surjective morphisms $q'^*{q'}_*(p'^*L\otimes I_D)\to p'^*L\otimes I_D$ and $q'^*{q'}_*(p'^*L'\otimes I_D)\to p'^*L'\otimes I_D$. Since $q'^*{q'}_*(p'^*L\otimes I_D)$ and $q'^*{q'}_*(p'^*L'\otimes I_D)$ are locally free and D is a divisor, we get ***vector bundles*** S_1 and S_2 defined by the exact sequences.

$$0\longrightarrow S_1\longrightarrow q'^*{q'}_*\left(p'^*L'\otimes I_D\right)\longrightarrow p'^*L'\otimes I_D\longrightarrow 0\quad\text{and}$$
$$0\longrightarrow S_2\longrightarrow q'^*{q'}_*\left(p'^*L\otimes I_D\right)\longrightarrow p'^*L\otimes I_D\longrightarrow 0.$$

We have surjections

$$q'^*q'_*p'^*L'\twoheadrightarrow p'^*L'\twoheadrightarrow p'^*L'_{|_D},\ q'^*q'_*p'^*L\twoheadrightarrow p'^*L\twoheadrightarrow p'^*L_{|_D}.$$

These maps induce surjective maps of sheaves

$$q'^*W\twoheadrightarrow p'^*L'_{|_D}\quad\text{and}\quad q'^*\widetilde{W}\twoheadrightarrow p'^*L_{|_D}.$$

Define vector bundles Γ_1,Γ_1, both of rank $k+1$, by the exact sequences

$$0\longrightarrow\Gamma_1\longrightarrow q'^*W\longrightarrow p'^*L'_{|_D}\longrightarrow 0\quad\text{and}\quad 0\longrightarrow\Gamma_2\longrightarrow q'^*\widetilde{W}\longrightarrow p'^*L_{|_D}\longrightarrow 0.$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_2 & \longrightarrow & \pi^*\mathcal{M}_L & \longrightarrow & \Gamma_2 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_1 & \longrightarrow & \pi^*\mathcal{M}_{L'} & \longrightarrow & \Gamma_1 & \longrightarrow & 0,
\end{array}$$

where the vertical maps are induced from the natural inclusion $L \hookrightarrow L'$ (induced from the effective divisor Δ) on X and where $\mathcal{M}_L := p^*M_L$, $\mathcal{M}_{L'} := p^*M_{L'}$.

We get a natural commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1\left(B,\pi^*\left(\bigwedge^{k+1}\mathcal{M}_L(p^*L')\right)\right) & \xrightarrow{f} & \mathrm{H}^1\left(B,\bigwedge^{k+1}\Gamma_2(p'^*L')\right) \\ \downarrow & & \downarrow g \\ \mathrm{H}^1\left(B,\pi^*\left(\bigwedge^{k+1}\mathcal{M}_{L'}(p^*L')\right)\right) & \xrightarrow{h} & \mathrm{H}^1\left(B,\bigwedge^{k+1}\Gamma_1(p'^*L')\right). \end{array}$$

Lemma 2.5. *The natural map $l\colon \widetilde{W} \to W$, induced by the map $L \hookrightarrow L'$, given by multiplication by a nonzero section $s_\Delta \in \mathrm{H}^0(\mathcal{O}_X(\Delta))$, is injective.*

Proof. Recall $W := \mathrm{Coker}(q_*(p^*L' \otimes I_{\mathcal{Z}}) \to q_*p^*L')$ and $\widetilde{W} := \mathrm{Coker}(q_*(p^*L \otimes I_{\mathcal{Z}}) \to q_*p^*L)$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{W} & \longrightarrow & q_*\left((p^*L)_{|\mathcal{Z}}\right) & \longrightarrow & R^1q_*(p^*L \otimes I_{\mathcal{Z}}) & \longrightarrow & 0 \\ & & \downarrow l & & \downarrow q_*(\cdot s_{p^*\Delta_{|\mathcal{Z}}}) & & \downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & q_*\left((p^*L')_{|\mathcal{Z}}\right) & \longrightarrow & R^1q_*(p^*L' \otimes I_{\mathcal{Z}}) & \longrightarrow & 0. \end{array}$$

To prove l is injective, it suffices to show $q_*(\cdot s_{p^*\Delta_{|\mathcal{Z}}})$ is injective. Since q_* is left-exact, for this it suffices to show multiplication by $s_{p^*\Delta_{|\mathcal{Z}}}$ yields an injective map $(p^*L)_{|\mathcal{Z}} \to (p^*L')_{|\mathcal{Z}}$, and for this it suffices to note $\mathcal{Z}$ is irreducible and is not contained in $\Delta \times \mathbf{P}$. □

We now prove the first of three statements needed for the proof.

Lemma 2.6. *The natural morphism $g\colon \mathrm{H}^1(B,\pi^*(\bigwedge^{k+1}\Gamma_2(p'^*L')) \to \mathrm{H}^1(B,\bigwedge^{k+1}\Gamma_1(p'^*L'))$ is an isomorphism. Further,* $\det\Gamma_i \simeq q'^*\mathcal{O}_{\mathbf{P}}(k+1)(-D)$ *for $i = 1,2$.*

Proof. The natural map $\widetilde{W} \to W$ is injective by Lemma 2.5. We now prove $\det\widetilde{W} \simeq \det W$. Since q_*p^*L and q_*p^*L' are trivial vector bundles, it is equivalent to prove $\det q_*(p^*L \otimes I_{\mathcal{Z}}) \simeq \det q_*(p^*L' \otimes I_{\mathcal{Z}})$, and then $\det\widetilde{W} \simeq \det W \simeq \det q_*(p^*L \otimes I_{\mathcal{Z}})^{-1}$. We have the exact sequence $0 \to \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\mathrm{id}} E \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \to p^*L' \otimes I_{\mathcal{Z}} \to 0$, and applying q_*, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow \mathrm{H}^0(E) \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow q_*(p^*L' \otimes I_{\mathcal{Z}}) \longrightarrow 0.$$

Since $h^0(E) = k+3$, this gives $\det q_*(p^*L' \otimes I_{\mathcal{Z}}) \simeq \mathcal{O}_{\mathbf{P}}(-k-1)$. Next, from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-\Delta) \boxtimes \mathcal{O}_{\mathbf{P}}(-2) \xrightarrow{\mathrm{id}} E(-\Delta) \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow p^*L'(-\Delta) \otimes I_{\mathcal{Z}} \longrightarrow 0$$

and $h^0(E(-\Delta)) = k+1$, see [Kem21, Lemma 10], we get

$$\det q_*(p^*L \otimes I_{\mathcal{Z}}) \simeq \det \mathrm{H}^0(E(-\Delta)) \otimes \mathcal{O}_{\mathbf{P}}(-1) \simeq \mathcal{O}_{\mathbf{P}}(-k-1),$$

as required. From the defining sequences for Γ_i, $i = 1,2$, we see $\det\Gamma_i \simeq \det q'^*W(-D) \simeq q'^*\mathcal{O}_{\mathbf{P}}(k+1)(-D)$. So, $\bigwedge^{k+1}\Gamma_2 \simeq q'^*\det\widetilde{W}(-D) \simeq q'^*\det W(-D) \simeq \bigwedge^{k+1}\Gamma_1$. The natural injective map

$$\bigwedge^{k+1}\Gamma_2 \hookrightarrow \bigwedge^{k+1} q'^*\widetilde{W} \hookrightarrow \bigwedge^{k+1} W,$$

which lands in $\bigwedge^{k+1}\Gamma_1$, is an injective map $\bigwedge^{k+1}\Gamma_2 \hookrightarrow \bigwedge^{k+1}\Gamma_1$ between isomorphic line bundles on a projective space and hence must be an isomorphism, as required. □

We now prove the second of the three needed statements.

Lemma 2.7. *The map $h\colon \mathrm{H}^1(B,\pi^*(\bigwedge^{k+1}\mathcal{M}_{L'}(p^*L')) \to \mathrm{H}^1(B,\bigwedge^{k+1}\Gamma_1(p'^*L'))$ is an isomorphism.*

Proof. We follow [Kem21, Section 2]. From Lemma 2.6, $\bigwedge^{k+1}\Gamma_1(p'^*L') \simeq q'^*\mathcal{O}_{\mathbf{P}}(k+1))\otimes p'^*L'\otimes I_D$. As in [Kem21, Corollary 2],(2) we compute

$$\mathrm{H}^1\left(B, \bigwedge^{k+1}\Gamma_1\left(p'^*L'\right)\right) \simeq \mathrm{H}^1\left(X\times\mathbf{P}, (L'\boxtimes\mathcal{O}_{\mathbf{P}}(k+1))\otimes I_{\mathcal{Z}}\right)$$

using the sequence

$$0\longrightarrow \mathcal{O}_X\boxtimes\mathcal{O}_{\mathbf{P}}(-2)\xrightarrow{\mathrm{id}} E\boxtimes\mathcal{O}_{\mathbf{P}}(-1)\longrightarrow p^*L'\otimes I_{\mathcal{Z}}\longrightarrow 0.$$

So $\mathrm{H}^1(B,\bigwedge^{k+1}\Gamma_1(p'^*L'))\simeq \mathrm{Sym}^{k-1}\mathrm{H}^0(X,E)^\vee$. Now

$$h^1\left(\pi^*\left(\bigwedge^{k+1}\mathcal{M}_{L'}(p^*L')\right)\right) = h^1\left(X\times\mathbf{P}, \bigwedge^{k+1}\mathcal{M}_{L'}(p^*L')\right) = h^1\left(\bigwedge^{k+1}M_{L'}(L')\right) = \dim \mathrm{K}_{k,2}(X,L').$$

Apply verbatim the method of [Kem21, Section 1] to prove Green's conjecture for the Gorenstein surface $\hat{X}$:

$$\mathrm{K}_{k-1,2}\left(\hat{X},\hat{L}\right)\simeq \mathrm{H}^1\left(\hat{X},\bigwedge^{k}M_{\hat{L}}\left(\hat{L}\right)\right)\simeq \mathrm{H}^1\left(X,\bigwedge^{k}M_{L'}(L')\right)\simeq \mathrm{K}_{k-1,2}(X,L')=0.$$

This vanishing implies $\dim\mathrm{K}_{k,2}(X,L') = \dim\mathrm{Sym}^{k-1}\mathrm{H}^0(X,E)^\vee$; see [Far17, Section 4.1]. So we have $h^1(\pi^*(\bigwedge^{k+1}\mathcal{M}_{L'}(p^*L')) = h^1(\bigwedge^{k+1}\Gamma_1(p'^*L'))$. To prove that h is an isomorphism, it suffices to prove it is surjective. From the sequence $0\to S_1\to\pi^*\mathcal{M}_{L'}\to\Gamma_1\to 0$ of vector bundles on B, we obtain the exact sequence

$$\cdots\longrightarrow S_1\otimes\bigwedge^{k}\pi^*\mathcal{M}_{L'}\left(p'^*L'\right)\longrightarrow\bigwedge^{k+1}\pi^*\mathcal{M}_{L'}\left(p'^*L'\right)\longrightarrow\bigwedge^{k+1}\Gamma_1\left(p'^*L'\right)\longrightarrow 0.$$

It suffices to show $\mathrm{H}^{1+i}(\mathrm{Sym}^i(S_1)\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_{L'}(p'^*L'))=0$ for $1\le i\le k+1$. As in [Kem21, Theorem 5], we have the exact sequence

$$0\longrightarrow S_1\longrightarrow q'^*q'_*\left(p'^*L'\otimes I_D\right)\longrightarrow p'^*L'\otimes I_D\longrightarrow 0$$

and define

$$\mathcal{G} := q'_*\left(p'^*L'\otimes I_D\right)\simeq q_*(p^*L'\otimes I_{\mathcal{Z}})\simeq i^*\hat{q}_*\left(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}}\right)\simeq i^*\hat{\mathcal{G}}$$

for $\hat{\mathcal{G}} := \hat{q}_*(\hat{p}^*\hat{L}\otimes I_{\hat{\mathcal{Z}}})$. By Grauert's theorem and the exact sequence $0\to\mathcal{O}_{\hat{X}}\xrightarrow{s}\hat{E}\to\hat{L}\otimes I_{Z(s)}\to 0$ for any $s\neq 0\in\mathrm{H}^0(\hat{E})$, we see $\hat{\mathcal{G}}$ and $\mathcal{G}$ are locally free of rank $k+2$. Now the defining sequence for S_1 gives the exact sequence

$$0\longrightarrow\mathrm{Sym}^i S_1\longrightarrow\mathrm{Sym}^i q'^*\mathcal{G}\longrightarrow\mathrm{Sym}^{i-1}q'^*\mathcal{G}\otimes p'^*L'\otimes I_D\longrightarrow 0$$

of bundles. It suffices that

$$\mathrm{H}^{1+i}\left(\mathrm{Sym}^i q^*\mathcal{G}\otimes\bigwedge^{k+1-i}\mathcal{M}_{L'}(p^*L')\right) = \mathrm{H}^i\left(\mathrm{Sym}^{i-1}q^*\mathcal{G}\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_{L'}\left(p^*L'^2\otimes I_{\mathcal{Z}}\right)\right)=0$$

for $1\le i\le k+1$, or

$$\mathrm{H}^{1+i}\left(\mathrm{Sym}^i \hat{q}^*\hat{\mathcal{G}}\otimes\bigwedge^{k+1-i}\mathcal{M}_{\hat{L}}\left(\hat{p}^*\hat{L}\right)\right) = \mathrm{H}^i\left(\mathrm{Sym}^{i-1}\hat{q}^*\hat{\mathcal{G}}\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_{\hat{L}}\left(\hat{p}^*\hat{L}^2\right)\otimes I_{\hat{\mathcal{Z}}}\right)=0$$

for $1\le i\le k+1$. For this, as in [Kem21, Section 3], we can use verbatim the proof of [Kem21, Theorem 5]. □

We can now prove the main theorem of this paper.

Theorem 2.8. *Let X denote a K3 surface of Picard rank 2, generated by a big and nef line bundle L' with $(L')^2 = 4k+2$, together with the class of a smooth rational curve Δ with $(L'\cdot\Delta)=0$ for $k\ge 4$, as above. Define the line bundle $L := L'(-\Delta)$. Then $\mathrm{K}_{k-1,2}(X,L)=0$.*

(2)Note that k has become $k+1$ as a smooth curve in $|L'|$ has genus $2k+2$

Proof. It remains to show $f\colon \mathrm{H}^1(B,\pi^*(\bigwedge^{k+1}\mathcal{M}_L(p^*L'))\to \mathrm{H}^1(B,\bigwedge^{k+1}\Gamma_2(p'^*L'))$ is surjective. From the exact sequence $0\to S_2\to\pi^*\mathcal{M}_L\to\Gamma_2\to 0$, we obtain the exact sequence

$$\cdots\longrightarrow S_2\otimes\bigwedge^{k}\pi^*\mathcal{M}_L\left(p'^*L'\right)\longrightarrow\bigwedge^{k+1}\pi^*\mathcal{M}_L\left(p'^*L'\right)\longrightarrow\bigwedge^{k+1}\Gamma_2\left(p'^*L'\right)\longrightarrow 0.$$

It suffices to show $\mathrm{H}^{1+i}(\mathrm{Sym}^i(S_2)\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_L(p'^*L'))=0$ for $1\le i\le k+1$.

Now, we have the exact sequence $0\to S_2\to q'^*q'_*(p'^*L\otimes I_D)\to p'^*L\otimes I_D\to 0$. Define $\mathcal{H}:=q'_*(p'^*L\otimes I_D)\simeq q_*(p^*L\otimes I_{\mathcal{Z}})$. From the exact sequence $0\to\mathcal{O}_X(-\Delta)\boxtimes\mathcal{O}_{\mathbf{P}}(-2)\xrightarrow{\mathrm{id}}E(-\Delta)\boxtimes\mathcal{O}_{\mathbf{P}}(-1)\to p^*L\otimes I_{\mathcal{Z}}\to 0$, by applying q_*, we get $\mathcal{H}\simeq\mathrm{H}^0(X,E(-\Delta))\otimes\mathcal{O}_{\mathbf{P}}(-1)$. We have the exact sequence $0\to\mathrm{Sym}^i S_2\to\mathrm{Sym}^i q'^*\mathcal{H}\to\mathrm{Sym}^{i-1}q'^*\mathcal{H}\otimes p'^*L'\otimes I_D\to 0$. It suffices to have

$$\mathrm{H}^{1+i}\left(\mathrm{Sym}^i q'^*\mathcal{H}\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_L(p^*L')\right)=\mathrm{H}^i\left(\mathrm{Sym}^{i-1}q'^*\mathcal{H}\otimes\bigwedge^{k+1-i}\pi^*\mathcal{M}_L\left(p^*L'^2\otimes I_{\mathcal{Z}}\right)\right)=0$$

for $1\le i\le k+1$. So, it suffices to have

$$\mathrm{H}^{1+i}\left(X\times\mathbf{P},\bigwedge^{k+1-i}M_L(L')\boxtimes\mathcal{O}_{\mathbf{P}}(-i)\right)=\mathrm{H}^i\left(X\times\mathbf{P},\left(\bigwedge^{k+1-i}M_L\left(L'^2\right)\boxtimes\mathcal{O}_{\mathbf{P}}(-i+1)\right)\otimes I_{\mathcal{Z}}\right)=0$$

for $1\le i\le k+1$, where $\mathbf{P}\simeq\mathbf{P}^{k+2}$. The first vanishing is immediate by the Künneth formula. For the second, by the above exact sequence, it is enough to have

$$\mathrm{H}^i\left(X\times\mathbf{P}^{k+2},\bigwedge^{k+1-i}M_L(L'\otimes E)\boxtimes\mathcal{O}_{\mathbf{P}}(-i)\right)=\mathrm{H}^{i+1}\left(X\times\mathbf{P}^{k+2},\bigwedge^{k+1-i}M_L(L')\boxtimes\mathcal{O}_{\mathbf{P}}(-i-1)\right)=0$$

for $1\le i\le k+1$, which is again immediate by the Künneth formula. □

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