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Quotient singularities by permutation actions are canonical

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Abstract. The quotient variety associated to a permutation representation of a finite group has only canonical singularities in arbitrary characteristic. Moreover, the log pair associated to such a representation is Kawamata log terminal except in characteristic two, and log canonical in arbitrary characteristic.

Keywords. Quotient singularities, canonical singularities, permutation representations, positive characteristic, motivic integration

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Contents

1. Introduction

In birational geometry, there are four important classes of singularities. In order from the smallest class are terminal singularities, canonical singularities, log terminal singularities and log canonical singularities. It is well known that quotient singularities are log terminal in characteristic zero. Moreover, we have a representation-theoretic criterion for quotient singularities being even canonical or terminal due to Reid, Shepherd-Barron and Tai; see [Rei87, Section (4.11)]. These are not generally true in positive characteristic. It is interesting to ask the following.

Problem 1.1. Does there exist a representation-theoretic criterion for quotient singularities belonging to these classes of singularities?

Let V be an n-dimensional affine space $\mathbb{A}_k^n$ over a field k of characteristic $p \geq 0$. Suppose that a finite group G acts on V linearly and effectively, and let X denote the quotient variety V/G and π denote the quotient map $V \to X$. There exists a unique effective $\mathbb{Q}$-divisor B on X such that $\pi^*(K_X + B) = K_V$. The aim of this paper is to prove the following two theorems, giving a partial answer to the above problem.

Theorem 1.2. *Suppose that G acts on V by permutations of coordinates. Then, the quotient variety X has only canonical singularities.*

Theorem 1.3. *Suppose that G acts on V by permutations of coordinates. Then, the log pair (X, B) is log canonical. Moreover, it is Kawamata log terminal if $p \neq 2$.*

Note that if $p = 0$ and if G has no pseudo-reflections, then Theorem 1.2 is a direct consequence of the Reid–Shepherd-Barron–Tai criterion. Hochster and Huneke [HH92, Section 6, p. 77] observed that in positive characteristic, the invariant ring associated to a permutation representation is always F-pure, and hence log canonical from a result of Hara and Watanabe [HW02]. Since "canonical" implies "log canonical," Theorem 1.2 strengthens the last fact.

Remark 1.4. Hashimoto and Singh [HS23, Remark 4.2] noted that Hochster and Huneke's observation also holds for more general monomial representations. This may suggest that the above theorems could be extended to monomial representations by replacing "canonical" in Theorem 1.2 with "log terminal." Note that a similar approach to Theorem 1.2 via the implication "strongly F-regular" ⇒ "log terminal" does not work. Firstly, "canonical" is stronger than "log terminal." Secondly, a quotient variety X/G is never strongly F-regular, provided that p divides the order of G and G contains no pseudo-reflections (see [Bro05, Corollary 2] and [Yas12, Corollary 3.3]).

There are also a few cases in positive characteristic where the theorems are known. Fan [Fan23] constructed a crepant resolution of the quotient variety $\mathbb{A}_k^4/A_4$ associated to the standard permutation action of the alternating group A_4 on $\mathbb{A}_k^4$ in characteristic 2. In particular, this quotient variety has only canonical singularities. If the symmetric group S_n acts on $\mathbb{A}_k^{2n} = (\mathbb{A}_k^2)^n$ by permutations of the n components, then the quotient variety $\mathbb{A}_k^{2n}/S_n$ is nothing but the n^{th} symmetric product of $\mathbb{A}_k^2$ and admits a crepant resolution, the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{A}_k^2)$ of n points on $\mathbb{A}_k^2$; see [Bea83, BK05, KT01]. In particular, $\mathbb{A}_k^{2n}/S_n$ has only canonical singularities. As a consequence, if a finite group G acts on $\mathbb{A}_k^{2n}$ via a homomorphism $G \to S_n$, then $\mathbb{A}_k^{2n}/G$ has only canonical singularities.

The main ingredient of the proofs of the theorems is the wild McKay correspondence proved in [Yas24]. It is formulated as the following equality in a certain version of the Grothendieck ring of varieties:

$$\mathrm{M}_{\mathrm{st}}(X,B) = \int_{\Delta_G} \mathbb{L}^{n-\mathbf{v}}. \tag{1.1}$$

The left-hand side is the stringy motive of the log pair (X,B). It is defined as the volume of the arc space of X with respect to the so-called Gorenstein motivic measure with respect to the log canonical divisor $K_X + B$. When the pair admits a log resolution, the stringy motive is explicitly determined in terms of the resolution data. In the right-hand side of the above equality, the domain Δ_G of the integral is a kind of moduli space of G-torsors over the punctured formal disk $\operatorname{Spec} k((t))$. Roughly speaking, it is the union of countably many k-varieties, and if k is algebraically closed, then its k-points correspond to G-torsors over the punctured formal disk $\operatorname{Spec} k((t))$. The symbol $\mathbb{L}$ means the class of the affine line in the version of the Grothendieck ring of varieties that we consider and $\mathbf{v}$ is a function $\Delta_G \to \frac{1}{\#G}\mathbb{Z}_{\geq 0}$ associated to the representation $G \curvearrowright V$. For example, when the representation is a permutation representation, which is of our main interest in this paper, the function $\mathbf{v}$ is determined by discriminant exponents of étale $k((t))$-algebras thanks to a result in [WY15], where $k((t))$ denotes the field of Laurent power series (see Section 3.2 for more details). There exists a decomposition of Δ_G into countably many constructible subsets C_j, $j \in J$, such that for each j, the restriction $\mathbf{v}|_{C_j}$ of $\mathbf{v}$ is constant. The integral of the right-hand side of (1.1) is then defined as the countable sum $\sum_{j\in J}\{C_j\}\mathbb{L}^{n-\mathbf{v}(C_j)}$. If the action $G \curvearrowright V$ does not have a pseudo-reflection, then the following variant also holds: If X_{sing} denotes the singular locus of X and $o \in \Delta_G$ denotes the point corresponding to the trivial G-torsor, then we have

$$\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}} = \{X_{\mathrm{sing}}\} + \int_{\Delta_G \setminus \{o\}} \mathbb{L}^{n-\mathbf{v}}.$$

Here $\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}$ is the stringy motive of X along X_{sing} and is defined as the volume of the space of arcs passing through X_{sing}. Since this invariant contains information about the minimal discrepancy of X, evaluating the integral on the right-hand side of the last equality can provide information about singularities of X.

We conclude this introduction by mentioning known results related to Problem 1.1 which have not been mentioned above. When $p > 0$ and G is a cyclic p-group, there are representation-theoretic criterions as desired in the problem above; see [Yas14, Yas19, TY21, Tan22]. Chen, Du and Gao [CDG20] showed that in characteristic 3 a quotient variety by the cyclic group of order 6 containing pseudo-reflections admits a crepant resolution and hence has only canonical singularities. Yamamoto [Yam21a] constructed crepant resolutions of quotient varieties $\mathbb{A}_k^3/G$ in characteristic 3 for some class of groups G without pseudo-reflections, including the case $G = S_3$, which shows again that the quotient variety in question has only canonical singularities. He also showed that if $G = (\mathbb{Z}/3\mathbb{Z})^2$ acts on $\mathbb{A}_k^3$ without pseudo-reflections in characteristic 3, the quotient variety $\mathbb{A}_k^3/G$ is not log canonical, see [Yam21b], while for any proper subgroup $H \subsetneq G$, the quotient variety $\mathbb{A}_k^3/H$ has only canonical singularities. This shows that the problem cannot be reduced to the case of cyclic groups as in the case of characteristic zero.

The outline of the paper is as follows. In Section 2, we collect known results which are necessary to prove our main results. In particular, we define basic classes of singularities, then recall basic facts on moduli spaces of G-torsors over $\operatorname{Spec} k((t))$ as well as the wild McKay correspondence. In Section 3, we give key dimension estimates on loci in moduli spaces. In Section 4, we prove the main theorems.

Throughout the paper, we work over a base field k of characteristic $p \geq 0$. By a variety, we mean a separated integral scheme of finite type over k.

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2. Preliminaries

2.1. Singularities in the minimal model program

Let (X,B) be a log pair. Namely, X is a normal variety over k and B is a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. For a proper birational morphism $f\colon Y \to X$ with Y normal and a prime divisor E on Y, the discrepancy of (X,B) at E, denoted by $\operatorname{discrep}(E;X,B)$, is defined to be the multiplicity of $K_Y - f^*(K_X+B)$.

Definition 2.1. We say that (X,B) is *Kawamata log terminal* (resp. *log canonical*) if $\operatorname{discrep}(E;X,B) > -1$ (resp. $\operatorname{discrep}(E;X,B) \geq -1$) for every proper birational morphism $f\colon Y \to X$ and every prime divisor E on Y.

We identify a $\mathbb{Q}$-Gorenstein normal variety X with the log pair $(X,0)$.

Definition 2.2. We say that a $\mathbb{Q}$-Gorenstein normal variety X has only *terminal singularities* (resp. *canonical singularities*) if $\operatorname{discrep}(E;X) := \operatorname{discrep}(E;X,0) > 0$ (resp. $\operatorname{discrep}(E;X) \geq 0$) for every proper birational morphism $f\colon Y \to X$ with Y a normal variety and every prime divisor E on Y.

2.2. Stringy motives

The Grothendieck ring of varieties, denoted by $K_0(\mathbf{Var}_k)$, is defined as the quotient of the free abelian group generated by the isomorphism classes of k-varieties by the so-called scissor relation. It has a natural structure of commutative ring. We denote the class of a k-variety Z in this ring or in its variants by $\{Z\}$. The class $\{\mathbb{A}^1_k\}$ of the affine line $\mathbb{A}^1_k$ plays a special role and is denoted by $\mathbb{L}$. We need the version of the Grothendieck ring of varieties that was denoted by $\widehat{\mathcal{M}'_{k,r}}$ in [Yas24]. Here r denotes a positive integer which is factorial enough so that all the $\mathbb{Q}$-Cartier divisors that we will consider are r-Cartier and finite groups that we will consider have orders dividing r. To obtain this ring, we modify $K_0(\mathbf{Var}_k)$ by adjoining the fractional power $\mathbb{L}^{1/r}$ of $\mathbb{L}$ and its inverse, taking a quotient by imposing an extra relation among elements, and taking the completion with respect to some filtration. In particular, the ring $\widehat{\mathcal{M}'_{k,r}}$ contains fractional powers $\mathbb{L}^{n/r}$, $n \in \mathbb{Z}$, of $\mathbb{L}$. Since the ring is also complete with respect to a certain topology, we can consider some type of infinite sums in it and discuss their convergence and divergence.

Let (X,B) be a log pair. We can define a motivic measure on the arc space $\mathrm{J}_\infty X$ of X in terms of the log canonical divisor K_X+B, which is often called the Gorenstein (or $\mathbb{Q}$-Gorenstein) measure. The *stringy motive* $\mathrm{M}_{\mathrm{st}}(X,B)$ is defined to be the volume of the entire arc space $\mathrm{J}_\infty X$ with respect to this measure. It is expressed as a countable sum $\sum_i \{Z_i\}\mathbb{L}^{m_i}$, where the Z_i are k-varieties and the m_i are elements of $\frac{1}{r}\mathbb{Z}$. This sum converges if and only if for each $s \in \mathbb{R}$, there are at most finitely many indices i such that $\dim Z_i + m_i \geq s$. If this is the case, $\mathrm{M}_{\mathrm{st}}(X,B)$ is an element of the ring $\widehat{\mathcal{M}'_{k,r}}$; otherwise, we formally put it to be ∞. In either case, we define the *dimension* of $\mathrm{M}_{\mathrm{st}}(X,B)$ as

$$\dim \mathrm{M}_{\mathrm{st}}(X,B) := \sup_i (\dim Z_i + m_i) \in \frac{1}{r}\mathbb{Z} \cup \{\infty\}.$$

When $B = 0$, we write $\mathrm{M}_{\mathrm{st}}(X, 0)$ simply as $\mathrm{M}_{\mathrm{st}}(X)$. The *stringy motive of X along the singular locus X_{sing}*, denoted by $\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}$, is defined to be the volume of the space of those arcs on X that pass through X_{sing}. Its dimension is similarly defined.

When X is smooth, we have $\mathrm{M}_{\mathrm{st}}(X) = \{X\}$. In the general case, $\mathrm{M}_{\mathrm{st}}(X, B)$ can be regarded as a modification of $\{X\}$ obtained by incorporating singularities of the pair (X, B). Suppose that there exists a log resolution $f : Y \to X$ of the pair and write

$$K_Y - f^*(K_X + B) = \sum_{i \in I} a_i E_i,$$

where the E_i are prime divisors on Y and the a_i are rational numbers. Then, we can express $\mathrm{M}_{\mathrm{st}}(X, B)$ by the following formula:

$$\mathrm{M}_{\mathrm{st}}(X, B) = \begin{cases} \sum_{I' \subset I} \{E_{I'}^{\circ}\} \prod_{i \in I'} \frac{\mathbb{L}-1}{\mathbb{L}^{a_i+1}-1} & (\forall i,\, a_i > -1), \\ \infty & (\text{otherwise}). \end{cases}$$

In particular, assuming the existence of a log resolution, we have that $\mathrm{M}_{\mathrm{st}}(X, B) \neq \infty$ if and only if (X, B) is Kawamata log terminal. Note however that stringy motives are defined whether the pair admits a log resolution or not. When $B = 0$, we have the following equality:

$$\inf_{(f,E);\, f(E) \subset X} \operatorname{discrep}(E; X) = d - 1 - \dim \mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}} \in \frac{1}{r}\mathbb{Z} \cup \{-\infty\}.$$

Here the pair (f, E) runs over the pairs of a proper birational morphism $f : Y \to X$ with Y a normal variety and a prime divisor E on Y such that $f(E) \subset X_{\mathrm{sing}}$. In particular, X has only canonical singularities if and only if

$$\dim \mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}} \leq d - 1.$$

For more details about the relation between stringy motives and discrepancies, we refer the reader to [Yas19, Section 2], [Yas24, Section 16] and [Yas21, Section 6.6].

2.3. P-Moduli spaces of formal torsors

In [TY23], the authors developed the theory of P-schemes and P-moduli spaces in order to construct moduli spaces of torsors over the punctured formal disk $\operatorname{Spec} k((t))$. A *P-morphism* $f : Y \to X$ of schemes over k is a collection of compatible maps $f(L): Y(L) \to X(L)$ for algebraically closed fields L/k such that there exist a surjective morphism $Z \to Y$ locally of finite presentation (a sur covering) and a morphism $Z \to X$ such that for each algebraically closed field L/K, the following diagram of the induced maps is commutative:

$$\begin{array}{ccc} Z(L) & & \\ \downarrow & \searrow & \\ Y(L) & \longrightarrow & X(L). \end{array}$$

The *category of P-schemes over k* has schemes over k as its objects and P-morphisms as its morphisms. An important property of this category is the following: When X and Y are locally of finite type and separated over k, a P-morphism $f : Y \to X$ is an isomorphism in the category of P-schemes if and only if for every algebraically closed field L/k, $f(L)$ is bijective; see [TY23, Lemma 4.32]. Since an ordinary morphism of k-schemes induces a P-morphism in the obvious way, there exists a natural functor from the category of schemes over k to the one of P-schemes over k:

$$\mathrm{P} \colon \{\text{scheme over } k\} \longrightarrow \{\text{P-scheme over } k\}$$
$$X \longmapsto X^{\mathrm{P}}.$$

Fixing a finite group G, let us consider the functor

$$\begin{aligned} F_G\colon \{\text{affine scheme over } k\} &\longrightarrow \{\text{Set}\} \\ \operatorname{Spec} R &\longmapsto \{G\text{-torsor over } \operatorname{Spec} R((t))\}/\cong. \end{aligned}$$

Here $R((t))$ denotes the ring of Laurent power series with coefficients in R. This functor has a strong P-moduli space which is of the form $(\coprod_{i\in I} W_i)^{\mathrm{P}}$, where I is a countable set and the W_i are k-varieties. Roughly speaking, this means that for each G-torsor over $\operatorname{Spec} R((t))$, we have the induced P-morphism $\operatorname{Spec} R \to \coprod_{i\in I} W_i$, and for each algebraically closed field L/k, the induced map

$$\{G\text{-torsor over } \operatorname{Spec} L((t))\}/\cong \longrightarrow \left(\coprod_{i\in I} W_i\right)(L)$$

is bijective. See [TY23, Definition 4.16] for the precise meaning. We denote this P-moduli space by Δ_G. The P-scheme $(\coprod_{i\in I} W_i)^{\mathrm{P}}$ is unique up to unique isomorphism; if $\coprod_{j\in J} V_j$ is another scheme satisfying the same property, then there exist a scheme Z and morphisms $Z \to \coprod_{i\in I} W_i$ and $Z \to \coprod_{j\in J} V_j$ which are of finite type and geometrically bijective; see [TY23, Corollary 4.33]. In what follows, we choose one scheme-model $\coprod_{i\in I} W_i$ of the P-scheme Δ_G and identify Δ_G with it.

2.4. The wild McKay correspondence

Let $V = \mathbb{A}^n_k$ and let G be a finite group which acts on $\mathbb{A}^n_k$ linearly and effectively. Let $X = V/G$ be the associated quotient variety. There exists a unique effective $\mathbb{Q}$-divisor on B which has support along the branch divisor of the quotient map $V \to X$ and satisfies $\pi^*(K_{V/G} + B) = K_V$. Note that $B = 0$ if and only if $X \to V$ is étale in codimension one, which holds if and only if G contains no pseudo-reflection. (An element $g \in G$ is called a *pseudo-reflection* if the fixed-point locus V^G has codimension one in V.)

Let $\Delta_G = \coprod_{i\in I} W_i$ be as above. We call a subset $S \subset \Delta_G$ *constructible* if $S = \bigcup_{j\in J} S_j$ for a finite subset $J \subset I$ and constructible subsets $S_j \subset W_j$, $j \in J$. To the given linear action of G on V, we can associate a function $\mathbf{v}\colon \Delta_G \to \frac{1}{\#G}\mathbb{Z}_{\geq 0}$, for example, following [WY15, Definition 3.3] or [TY23, Definition 9.1]. This function has the following properties:

(1) If o denotes the point of Δ_G corresponding to the trivial G-torsor, then $\mathbf{v}(o) = 0$. The function $\mathbf{v}$ takes positive values on $\Delta_G \setminus \{o\}$.
(2) There exist countably many constructible subsets C_j, $j \in J$, of $\Delta_G \setminus \{o\}$ such that $\Delta_G \setminus \{o\} = \bigsqcup_{j\in J} C_j$ and $\mathbf{v}|_{C_j}$ is constant for each j.

With this notation, we can define the following motivic integrals:

$$\int_{\Delta_G} \mathbb{L}^{-\mathbf{v}} := 1 + \sum_{j\in J} \{C_j\}\mathbb{L}^{-\mathbf{v}(C_j)},$$
$$\int_{\Delta_G\setminus\{o\}} \mathbb{L}^{-\mathbf{v}} := \sum_{j\in J} \{C_j\}\mathbb{L}^{-\mathbf{v}(C_j)}.$$

If the infinite sum on the right-hand side converges in the sense explained in Section 2.2, then the integrals $\int_{\Delta_G} \mathbb{L}^{-\mathbf{v}}$ and $\int_{\Delta_G\setminus\{o\}} \mathbb{L}^{-\mathbf{v}}$ are defined as elements of the ring $\widehat{\mathcal{M}'_{k,r}}$ in [Yas24]. Otherwise, we put

$$\int_{\Delta_G} \mathbb{L}^{-\mathbf{v}} = \int_{\Delta_G\setminus\{o\}} \mathbb{L}^{-\mathbf{v}} = \infty.$$

Whether this condition holds or not, the dimensions of these integrals are well defined.

Theorem 2.3 (*cf.* [Yas24, Corollary 1.4 and the proof of Corollary 16.4]).

(1) *We have*

$$\mathrm{M}_{\mathrm{st}}(X,B)=\int_{\Delta_G}\mathbb{L}^{n-\mathbf{v}}.$$

(2) *If G has no pseudo-reflection, then*

$$\mathrm{M}_{\mathrm{st}}(X)_{X_{\mathrm{sing}}}=\{X_{\mathrm{sing}}\}+\int_{\Delta_G\setminus\{o\}}\mathbb{L}^{n-\mathbf{v}}.$$

Proposition 2.4 (*cf.* [Yas24, Corollary 1.4]). *Suppose that G contains no pseudo-reflection. Fix a decomposition $\Delta_G\setminus\{o\}=\bigsqcup_{j\in J}C_j$ as above. If $\dim C_j-\mathbf{v}(C_j)\le -1$ for every $j\in J$, then X has only canonical singularities.*

Proof. From [Yas19, Theorem 1.2], X has only canonical singularities if and only if

$$n-1-\max\left\{\dim X_{\mathrm{sing}},\max_j\left(n+\dim C_j-\mathbf{v}\left(C_j\right)\right)\right\}\ge 0.$$

This inequality is equivalent to that for every j, we have $\dim C_j-\mathbf{v}(C_j)\le -1$. □

Proposition 2.5. *We keep the notation as above.*

(1) *If*

$$\lim_{j\in J}\dim C_j-\mathbf{v}(C_j)=-\infty,$$

then the pair (X,B) is Kawamata log terminal.

(2) *If $\sup\{\dim C_j-\mathbf{v}(C_j)\mid j\in J\}<+\infty$, then the pair (X,B) is log canonical.*

Proof. Note that when G has no pseudo-reflections, assertion (1) is the same as [Yas24, Corollary 1.4(2)]. Let Y be a normal variety, and let $Y\to X$ be a proper birational morphism. Let B' be the $\mathbb{Q}$-divisor on Y such that (Y,B') is crepant over (X,B). From [Yas24, Theorem 16.2], $\mathrm{M}_{\mathrm{st}}(Y,B')=\mathrm{M}_{\mathrm{st}}(X,B)$. If (X,B) is not Kawamata log terminal (resp. log canonical), then for some proper birational morphism $Y\to X$, the divisor B' has multiplicity at most -1 at some prime divisor E. Then the standard computation of stringy invariants (in a neighborhood of a general point of E) shows that $\mathrm{M}_{\mathrm{st}}(Y,B')$ does not converge. Translating these conditions to ones on $\dim C_j-\mathbf{v}(C_j)$ shows assertion (1). If (X,B) was not log canonical, a similar reasoning shows that the dimensions of terms in an infinite sum defining $\mathrm{M}_{\mathrm{st}}(Y,B')=\mathrm{M}_{\mathrm{st}}(X,B)=\int_{\Delta_G}\mathbb{L}^{n-\mathbf{v}}$ are not bounded above, which shows assertion (2). □

3. Permutation actions and dimension estimates

From now on, suppose that G is a subgroup of S_n and acts on $V=\mathbb{A}^n_k$ by the induced permutation action. Let $X=V/G$ be the quotient variety with quotient map $\pi\colon V\to X$, and let B be the effective $\mathbb{Q}$-divisor on X such that $\pi^*(K_X+B)=K_V$.

3.1. The Gorenstein index of X

Lemma 3.1. *The quotient variety X is 2-Gorenstein; that is, $2K_X$ is Cartier. Moreover, if $p=2$, then X is 1-Gorenstein, that is, K_X is Cartier.*

Proof. We first consider the case $p\neq 2$. Let $x_1,\dots,x_n$ be coordinates of the affine space V. Let R be the reduced divisor on V defined by

$$\prod_{\substack{i>j\\(i,j)\in G}}(x_i-x_j)=0. \tag{3.1}$$

Here (i,j) denotes a transposition. We easily see that the ramification divisor of $\pi\colon V \to X$ has support R. The ramification index of π along every irreducible component of R is 2. In particular, π is tamely ramified at general points of R. It follows that $(\pi^*\omega_X)^{**} \cong \omega_V(-R)$ (for example, see [Kol13, Section 2.41]). The sheaf $\omega_V(-R)$ is generated by the global section

$$\eta := \prod_{\substack{i>j \\ (i,j)\in G}} (x_i - x_j) \cdot dx_1 \wedge \cdots \wedge dx_n. \tag{3.2}$$

For $\sigma \in G$, we have

$$\begin{aligned} \sigma(\eta) &= \prod_{\substack{i>j \\ (i,j)\in G}} (x_{\sigma(i)} - x_{\sigma(j)}) \cdot dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(n)} \\ &= \pm \prod_{\substack{i>j \\ (i,j)\in G}} (x_{\sigma(i)} - x_{\sigma(j)}) \cdot dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Since $(\sigma(i),\sigma(j)) = \sigma \circ (i,j) \circ \sigma^{-1}$, the transposition $(\sigma(i),\sigma(j))$ runs over all transpositions in G when (i,j) does so. Therefore, $\sigma(\eta) = \pm\eta$. It follows that $\eta^{\otimes 2} \in (\omega_V(-R))^{\otimes 2}$ is invariant under the G-action and gives a global generator of $\omega_X^{[2]} = (\omega_X^{\otimes 2})^{**}$. This shows X is 2-Gorenstein.

We next consider the case $p = 2$. The ramification divisor of the quotient map $\pi\colon V \to X$ is again the reduced divisor R defined by the same equation as in the case $p \neq 2$. To understand the ramification of π along R, we first consider the case $G = \{1_G, (1,2)\}$. The coordinate ring of X is then

$$k[x_1 + x_2, x_1x_2, x_3, \dots, x_n] \subset k[x_1, x_2, x_3, \dots, x_n].$$

In particular, $X \cong \mathbb{A}_k^n$. Pulling back the generator

$$d(x_1 + x_2) \wedge d(x_1x_2) \wedge dx_3 \wedge \cdots \wedge dx_n$$

of ω_X by π, we get the n-form

$$\begin{aligned} &(dx_1 + dx_2) \wedge (x_2dx_1 + x_1dx_2) \wedge dx_3 \wedge \cdots \wedge dx_n \\ &= (x_1 + x_2)dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_n. \end{aligned}$$

This is a generator of $\omega_V(-R)$, where R is defined by the function $x_1 + x_2$ in the current situation. Thus, we get $\pi^*\omega_X = \omega_V(-R)$. Let us return to the case of a general group G and take a general $\bar{k}$-point $b = (b_1, \dots, b_n)$ of R with $b_i = b_j$ for some (i,j). Here $\bar{k}$ denotes an algebraic closure of k. The stabilizer group of this point is generated by the transposition (i,j), which interchanges $x_i - b_i$ and $x_j - b_j$ for the local coordinates $x_1 - b_1, \dots, x_n - b_n$. This shows that the morphism

$$\operatorname{Spec} \widehat{\mathcal{O}_{V_{\bar{k}},b}} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}_{X_{\bar{k}},\bar{b}}}$$

between the formal neighborhoods of b and of its image $\bar{b}$ on X is isomorphic to the one in the case where $G = \{1_G, (1,2)\}$ and b is the origin. Thus, the ramification of $\pi\colon V \to X$ looks like the above special case, and we still have $(\pi^*\omega_X)^{**} = \omega_V(-R)$ in the general case. The n-form η given by Formula (3.2) is again a global generator of $\omega_V(-R)$. Since we are working in characteristic 2, the same computation as in the case $p \neq 2$ shows

$$\sigma(\eta) = \pm\eta = \eta.$$

Thus, η is G-invariant and gives a global generator of ω_X, which shows that X is 1-Gorenstein. □

Lemma 3.2. *If $p \neq 2$, then the multiplicity of B at every irreducible component of its support is $1/2$. If $p = 2$, the multiplicity of B at every irreducible component of its support is 1.*

Proof. If $p \neq 2$, then $\pi\colon V \to X$ is tamely ramified along general points of the ramification divisor R with ramification index 2. As is well known, the multiplicity of the branch divisor at a prime divisor is then given by 1/2. Suppose that $p = 2$. As in the proof of Lemma 3.1, we may reduce to the case $G = \{1_G, (1,2)\}$. Let $B' = \pi(R) = \mathrm{Supp}(B)$. The prime divisor B' is defined by the ideal $x_1 + x_2$ of the coordinate ring

$$k[x_1 + x_2, x_1 x_2, x_3, \ldots, x_n].$$

Thus, $\pi^* B' = R$. In the proof of Lemma 3.1, we showed $\pi^*\omega_X = \omega_V(-R)$. It follows that $\pi^*(\omega_X(B')) = \omega_V$ and hence $B = B'$, which shows the lemma in the case $p = 2$. □

Corollary 3.3. *The $\mathbb{Q}$-divisor B is 2-Cartier. Moreover, it is Cartier if $p = 2$.*

Proof. We first consider the case $p \neq 2$. Consider the log pair $(V/S_n, D)$ associated to the standard permutation action of S_n. As is well known, the support of D is defined by the discriminant polynomial, and the coefficient of D is 1/2 from Lemma 3.2. Since V/S_n is isomorphic to $\mathbb{A}^n_k$, so, in particular, smooth, $K_{V/S_n} + D$ is 2-Cartier. If $h\colon X \to V/S_n$ denotes the natural map, then

$$K_X + B = h^*\left(K_{V/S_n} + D\right).$$

Since K_X and $h^*(K_{V/S_n} + D)$ are both 2-Cartier (see Lemma (3.1)), B is also 2-Cartier.

We next consider the case $p = 2$ and again consider the log pair $(V/S_n, D)$ associated to the standard action of S_n on V. In this case, D has coefficient 1 from Lemma 3.2. Hence $K_{V/S_n} + D$ is 1-Cartier, and so is

$$K_X + B = h^*\left(K_{V/S_n} + D\right).$$

Since K_X and $K_{V/S_n} + D$ are both Cartier (see Lemma 3.1), B is also Cartier. □

3.2. Dimensions of loci in moduli spaces

The inclusion map $G \hookrightarrow S_n$ induces a map $\Delta_G \to \Delta_{S_n}$ which sends a G-torsor P to the contracted product $P \wedge^G S_n$ (for example, see [CF15, Proposition 2.2.2.12] or [Ems17, Section 4.4, Item 5]). (Strictly speaking, we have a canonical P-morphism $\Delta_G \to \Delta_{S_n}$. To find a scheme-morphism inducing this P-morphism, we need to replace the chosen scheme-model $\coprod_{j \in J} W_j$ of Δ_G with another model $\coprod_{i \in I} V_i$ given with a universal bijection $\coprod_{i \in I} V_i \to \coprod_{j \in J} W_j$.) The map $\Delta_G \to \Delta_{S_n}$ is quasi-finite, and the cardinality of each fiber is at most $\#S_n/\#G$. Let Δ_n denote the P-moduli space of degree n finite étale covers of $\operatorname{Spec} k((t))$. We have an isomorphism $\Delta_{S_n} \to \Delta_n$, sending an S_n-torsor $A \to \operatorname{Spec} k((t))$ to $A/S_{n-1} \to \operatorname{Spec} k((t))$, where S_{n-1} is identified with the stabilizer of $1 \in \{1, 2, \ldots, n\}$ (see [TZ20, Proposition 2.7] and [TY23, Definition 8.2]). We denote by ψ_G the composite map

$$\psi_G \colon \Delta_G \longrightarrow \Delta_{S_n} \longrightarrow \Delta_n.$$

Since this is quasi-finite, for every constructible subset $C \subset \Delta_G$, we have $\dim C = \dim \psi_G(C)$.

Let $\mathbf{d}\colon \Delta_n \to \mathbb{Z}_{\geq 0}$ be the discriminant exponent function. From [WY15, Lemma 2.6 and Theorem 4.8], the $\mathbf{v}$-function on Δ_G (associated to the permutation action $G \curvearrowright \mathbb{A}^n_k$ defined as above) factors as follows:

$$\mathbf{v}\colon \Delta_G \xrightarrow{\psi_G} \Delta_n \xrightarrow{\mathbf{d}/2} \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

From [TY23, Theorem 9.8] (see also [Yas24, Lemma 14.3]), the function $\mathbf{v}$ is locally constructible. Namely, there exists a decomposition of Δ_G into countably many constructible subsets C_i, $i \in I$, such that $\mathbf{v}|_{C_i}$ is constant for every i. Since $\mathbf{d}/2$ is identical to the $\mathbf{v}$-function associated to the standard permutation action of S_n on $\mathbb{A}^n_k$, the function $\mathbf{d}/2$ on Δ_n, as well as $\mathbf{d}$, is locally constructible. This shows that for each $d \in \mathbb{Z}_{\geq 0}$,

$$\Delta_{n,d} := \mathbf{d}^{-1}(d)$$

is a locally constructible subset of Δ_n. In fact, this set is also quasi-compact and constructible; see [Yas25, Corollary 4.12].

Let $\Delta_n^\circ \subset \Delta_n$ denote the subspace of geometrically connected covers (see [TY23, Definition 8.6]), which is a locally constructible subset of Δ_n; see [TY23, Lemma 8.7]. Let $\nu = (\nu_1, \nu_2, \dots, \nu_l)$ be a partition of n by positive integers; that is, the ν_i are positive integers satisfying $n = \sum_{i=1}^{l} \nu_i$. We have the map

$$\eta_\nu \colon \prod_{i=1}^{l} \Delta_{\nu_i}^\circ \longrightarrow \Delta_n, \quad (A_i)_{1\le i\le l} \longmapsto \coprod_{i=1}^{l} A_i.$$

For $d \ge 0$, let

$$\Delta_{n,d}^\circ := \Delta_n^\circ \cap \Delta_{n,d}.$$

If $\delta = (\delta_1, \dots, \delta_l)$ is a partition of d by non-negative integers, then η_ν restricts to

$$\eta_{\nu,\delta} \colon \prod_{i}^{l} \Delta_{\nu_i,\delta_i}^\circ \longrightarrow \Delta_{n,d}.$$

This map is quasi-finite; in particular, the source of the map has the same dimension as its image. Moreover, $\Delta_{n,d}$ is covered by the images of the maps $\eta_{\nu,\delta}$ as ν and δ run over partitions as above. Thus, we can estimate the dimension of $\Delta_{n,d}$ in terms of those of the $\Delta_{\nu_i,\delta_i}^\circ$.

The following result from [Yas25] is the key in the proofs of our main results.

Theorem 3.4 (*cf.* [Yas25]). *We have*

$$\dim \Delta_{n,d}^\circ = \begin{cases} 0 & (p \nmid n,\ d = n-1), \\ -\infty & (p \nmid n,\ d \ne n-1), \\ \lceil (d-n+1)/p \rceil & (p \mid n,\ p \nmid (d-n+1),\ d-n+1 \ge 0), \\ -\infty & (p \mid n,\ p \nmid (d-n+1),\ d-n+1 < 0), \\ -\infty & (p \mid n,\ p \mid (d-n+1)). \end{cases}$$

Here we follow the convention that $\dim \emptyset = -\infty$ *and that if* $p = 0$, *then* $p \nmid n$.

Remark 3.5. Theorem 3.4 can be regarded as a motivic version of a formula by Krasner [Kra66, Kra62]. See [Yas25] for more details.

Corollary 3.6. *Let d be a positive integer, and let n be an integer with $n \ge 2$.*

(1) *If $n \ge 4$, then*

$$\dim \Delta_{n,d}^\circ - \frac{d}{2} \le -1.$$

(2) *If $n = 3$, then*

$$\dim \Delta_{n,d}^\circ - \frac{d}{2} \le \begin{cases} -1 & (p \ne 3), \\ -1/2 & (p = 3). \end{cases}$$

(3) *If $n = 2$, then*

$$\dim \Delta_{n,d}^\circ - \frac{d}{2} = \begin{cases} -1/2 & (p \ne 2,\ d = 1), \\ -\infty & (p \ne 2,\ d > 1), \\ 0 & (p = 2,\ d \text{ even}), \\ -\infty & (p = 2,\ d \text{ odd}). \end{cases}$$

(4) *We always have*

$$\dim \Delta_{n,d}^\circ - \frac{d}{2} \le 0.$$

Proof. If $\Delta^{\circ}_{n,d}$ is empty, then it has dimension $-\infty$ and the desired inequalities are obvious. In what follows, we only consider the situation where $\Delta^{\circ}_{n,d}$ is not empty.

(1) If $p \nmid n$ and $d = n-1$, then

$$\dim \Delta^{\circ}_{n,d} - \frac{d}{2} = 0 - \frac{n-1}{2} \le -\frac{3}{2} \le -1.$$

If $p \mid n$, $p \nmid (d-n+1)$ and $d-n+1 \ge 0$, then

$$\dim \Delta^{\circ}_{n,d} - \frac{d}{2} = \left\lceil \frac{d-n+1}{p} \right\rceil - \frac{d}{2}.$$

Let $h(d)$ denote the right-hand side. If $d-n+1$ and $(d+1)-n+1$ are both coprime to p, then

$$h(d+1) = h(d) - \frac{1}{2} < h(d).$$

If $(d+1)-n+1$ is divisible by p, then

$$h(d+2) = h(d) + 1 - 1 = h(d).$$

Thus, $h(d)$ is a weakly decreasing function, as d runs over integers satisfying $p \nmid (d-n+1)$ and $d-n+1 \ge 0$, and attains the maximum when $d-n+1 = 1$, equivalently when $d = n$. Thus, the maximum of $h(d)$ is

$$\left\lceil \frac{d-n+1}{p} \right\rceil - \frac{d}{2} = 1 - \frac{n}{2} \le -1.$$

Here the last inequality follows from the assumption $n \ge 4$.

(2) If $p \ne 3$ and $d = 2$, then

$$\dim \Delta^{\circ}_{n,d} - \frac{d}{2} = -1.$$

Suppose $p = 3$. If $3 \nmid (d-2)$ and $d-2 \ge 0$, then

$$\begin{aligned}\dim \Delta^{\circ}_{3,d} - \frac{d}{2} &\le \left\lceil \frac{d-2}{3} \right\rceil - \frac{d}{2} \\ &\le \frac{d-2}{3} + \frac{2}{3} - \frac{d}{2} \\ &\le \frac{d}{3} - \frac{d}{2} \\ &< 0.\end{aligned}$$

Since $\dim \Delta^{\circ}_{3,d} - d/2$ is a half integer, it is at most $-1/2$.

(3) The only nontrivial case is when $p = 2$ and d is even. In this case,

$$\dim \Delta^{\circ}_{n,d} = \left\lceil \frac{d-1}{2} \right\rceil = \frac{d}{2}.$$

(4) This is a direct consequence of the previous three assertions. □

3.3. Better estimations derived from the non-existence of transposition

Although Corollary 3.6 is good enough for most cases, it is not enough to show Theorem 1.2 in a few exceptional cases. Assuming that G contains no pseudo-reflections, we obtain better estimations in characteristics 2 and 3.

In what follows, we denote by K the field $k((t))$ of Laurent power series. Let $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be its absolute Galois group. Let $\wp\colon K \to K$ be the Artin–Schreier map $f \mapsto f^p - f$.

Lemma 3.7. *Suppose $p = 2$. Let $L_1, \dots, L_m$ be quadratic field extensions of K such that the corresponding elements $[L_1], \dots, [L_m] \in K/\wp K$ (via Artin–Schreier theory) are linearly independent over $\mathbb{F}_2$. Let $\varphi\colon \Gamma_K \to S_n$ be the map corresponding to the étale K-algebra*

$$L = L_1 \times \cdots \times L_m \times K^{n-2m}.$$

Then, $\mathrm{Im}(\varphi)$ *contains a transposition.*

Proof. Let $\alpha_1, \dots, \alpha_m\colon \Gamma_K \to \mathbb{F}_2$ be the maps corresponding to $L_1, \dots, L_m$, respectively. After a suitable rearrangement of elements of $\{1, \dots, n\}$, $\mathrm{Im}(\varphi)$ is contained in

$$\mathbb{F}_2^m = \langle (1,2), (3,4), \dots, (2m-1, 2m) \rangle \subset S_n.$$

The induced morphism $\Gamma_K \to \mathbb{F}_2^m$ is given by

$$\Gamma_K \ni \gamma \longmapsto (\alpha_1(\gamma), \dots, \alpha_m(\gamma)) \in \mathbb{F}_2^m,$$

and its image is an $\mathbb{F}_2$-linear subspace from Artin-Schreier theory. Since $\alpha_1, \dots, \alpha_m$ are linearly independent over $\mathbb{F}_2$, there is no non-zero linear map $\mathbb{F}_2^m \to \mathbb{F}_2$ such that the composition $\Gamma_K \to \mathbb{F}_2^m \to \mathbb{F}_2$ is the trivial map onto $0 \in \mathbb{F}_2$. This is equivalent to that $\Gamma_K \to \mathbb{F}_2^m$ is surjective. In particular, the image of $\Gamma_K \to G \subset S_n$ contains the transposition $(1,2) \in \langle (1,2), (3,4), \dots, (2m-1, 2m) \rangle = \mathbb{F}_2^m$. □

Lemma 3.8. *Suppose $p = 3$. Let L' be a separable cubic field extension of K. Let $\varphi\colon \Gamma_K \to S_n$ be the map corresponding to the étale K-algebra $L = L' \times K^{n-3}$. If L'/K is not Galois, then* $\mathrm{Im}(\varphi)$ *contains a transposition.*

Proof. Let us regard S_3 as the subgroup of S_n which permutes 1, 2 and 3 in $\{1, 2, \dots, n\}$. Then, after a suitable rearrangement of elements of $\{1, \dots, n\}$, we get $\mathrm{Im}(\varphi) \subset S_3$. If L'/K is not Galois, then the Galois closure has Galois group isomorphic to S_3. This means that $\mathrm{Im}(\varphi)$ is S_3. In particular, it contains the transposition $(1,2)$. □

Lemma 3.9. *Suppose $p = 3$. Let $\Delta^\circ_{3,d,\mathrm{Gal}} \subset \Delta^\circ_{3,d}$ be the locus of Galois extensions. Then*

$$\dim \Delta^\circ_{3,d,\mathrm{Gal}} - \frac{d}{2} \le -1.$$

Proof. For $d > 0$, let $\Delta_{\mathbb{Z}/3\mathbb{Z},d}$ denote the locus of $\mathbb{Z}/3\mathbb{Z}$-torsors with discriminant exponent d. The locus $\Delta^\circ_{3,d,\mathrm{Gal}}$ is the image of the forgetting map $\Delta_{\mathbb{Z}/3\mathbb{Z},d} \to \Delta^\circ_3$. If L/K is a cubic Galois extension with ramification jump j ($j > 0$, $p \nmid j$), then $d = 2(j+1)$; see [Ser79, Section V.3, Lemma 3]. Thus, the locus $\Delta_{\mathbb{Z}/3\mathbb{Z},d}$ is also the locus with ramification jump $j = d/2 - 1$. From [Yas14, Proposition 2.11], we have

$$\Delta_{\mathbb{Z}/3\mathbb{Z},d} = \mathbb{G}_m \times \mathbb{A}_k^{j - \lfloor j/3 \rfloor - 1}.$$

In particular, $\dim \Delta_{\mathbb{Z}/3\mathbb{Z},d} = j - \lfloor j/3 \rfloor$. Thus,

$$\begin{aligned} \dim \Delta^\circ_{3,d,\mathrm{Gal}} - \frac{d}{2} &= j - \left\lfloor \frac{j}{3} \right\rfloor - \frac{2(j+1)}{2} \\ &= -\left\lfloor \frac{j}{3} \right\rfloor - 1 \\ &\le -1. \end{aligned}$$

□

Proposition 3.10. *Suppose that G contains no transposition. Then, for any positive integer d,*

$$\dim \Delta_{G,d} - \frac{d}{2} \le -1.$$

Proof. Let $\nu = (\nu_1, \dots, \nu_l)$ be a partition of n by positive integers satisfying $\nu_1 \ge \nu_2 \ge \cdots \ge \nu_l$, and let $\delta = (\delta_1, \dots, \delta_l)$ be a partition of d by non-negative integers such that

$$\delta_i > 0 \Longleftrightarrow \nu_i > 1.$$

We write $\Delta_{\nu,\delta} := \operatorname{Im}(\eta_{\nu,\delta})$. Then, $\Delta_n = \bigcup_{(\nu,\delta)} \Delta_{\nu,\delta}$, where (ν,δ) runs over pairs of partitions as above. Moreover, if $o \in \Delta_n$ denotes the point corresponding to the trivial cover

$$\overbrace{\operatorname{Spec} K \amalg \cdots \amalg \operatorname{Spec} K}^{n \text{ copies}} \longrightarrow \operatorname{Spec} K,$$

then $\{o\} = \Delta_{(1,\dots,1),(0,\dots,0)}$ and

$$\Delta_n \setminus \{o\} = \bigcup_{(\nu,\delta)\neq((1,\dots,1),(0,\dots,0))} \Delta_{\nu,\delta}.$$

We define $\Delta^{(G)}_{\nu,\delta} := \operatorname{Im}(\psi_G) \cap \Delta_{\nu,\delta}$, where ψ_G is the map $\Delta_G \to \Delta_n$ introduced at the beginning of Section 3.2. To prove the proposition, it suffices to show that for every pair (ν,δ) as above other than $((1,\dots,1),(0,\dots,0))$, we have

$$\dim \Delta^{(G)}_{\nu,\delta} - \frac{\sum_i \delta_i}{2} \leq -1. \tag{3.3}$$

Note that

$$\begin{aligned}
&\dim \Delta^{(G)}_{\nu,\delta} - \frac{\sum_i \delta_i}{2} \\
&\leq \dim \Delta_{\nu,\delta} - \frac{\sum_i \delta_i}{2}. \\
&= \sum_i \left(\dim \Delta^{\circ}_{\nu_i,\delta_i} - \frac{\delta_i}{2}\right).
\end{aligned}$$

Case $\nu_1 \geq 4$: From Corollary 3.6, Inequality (3.3) holds in this case.

Case $\nu_1 = 3$ *and* $p \neq 3$: Again, from Corollary 3.6, Inequality (3.3) holds.

Case $\nu_1 = 3$ *and* $p = 3$: If ν has at least two entries and $\nu_2 \geq 2$, then

$$\begin{aligned}
\sum_i \left(\dim \Delta^{\circ}_{\nu_i,\delta_i} - \frac{\delta_i}{2}\right) &\leq \left(\dim \Delta^{\circ}_{\nu_1,\delta_1} - \frac{\delta_1}{2}\right) + \left(\dim \Delta^{\circ}_{\nu_2,\delta_2} - \frac{\delta_2}{2}\right) \\
&\leq -\frac{1}{2} - \frac{1}{2} \\
&= -1.
\end{aligned}$$

If ν is of the form $(3,1,\dots,1)$ (including the case $\nu = (3)$), then from Lemma 3.8, we have

$$\Delta^{(G)}_{\nu,\delta} \subset \eta_\nu\left(\Delta^{\circ}_{3,d,\mathrm{Gal}} \times \left(\Delta^{\circ}_{1,0}\right)^{d-3}\right).$$

Since $\dim(\Delta^{\circ}_{1,0})^{d-3} = 0$, from Lemma 3.9,

$$\dim \Delta^{(G)}_{\nu,\delta} - \frac{d}{2} \leq \dim \Delta^{\circ}_{3,d,\mathrm{Gal}} - \frac{d}{2} \leq -1.$$

Case $\nu_1 = 2$ *and* $p \neq 2$: For a quadratic extension L'/K, the map $\Gamma_K \to S_n$ corresponding to $L = L' \times K^{d-2}$ has image $\langle (1,2) \rangle \subset S_n$, which contains the transposition $(1,2)$. From the assumption that G contains no transposition, $\Delta^{(G)}_{\nu,\delta}$ is non-empty only when ν has at least two entries and $\nu_1 = \nu_2 = 2$. Then,

$$\begin{aligned}
\sum_i \left(\dim \Delta^{\circ}_{\nu_i,\delta_i} - \frac{d_i}{2}\right) &\leq \left(\dim \Delta^{\circ}_{\nu_1,\delta_1} - \frac{d_1}{2}\right) + \left(\dim \Delta^{\circ}_{\nu_2,\delta_2} - \frac{\delta_2}{2}\right) \\
&\leq -\frac{1}{2} - \frac{1}{2} \\
&= -1.
\end{aligned}$$

Case $\nu_1 = 2$ *and* $p = 2$: From Lemma 3.7, if an étale algebra of the form

$$L_1 \times \cdots \times L_m \times K^{n-2m}$$

with $L_1, \dots, L_m$ quadratic extensions of K corresponds to a point of $\Delta^{(G)}_{\nu,\delta}$, then the classes $[L_1], \dots, [L_m]$ in $K/\wp(K)$ are linearly dependent over $\mathbb{F}_2$. Using this fact, we will give an upper bound of the dimension of $\Delta^{(G)}_{\nu,\delta}$.

Let $\nu = (\nu_1, \dots, \nu_l)$ be a partition of n such that $\nu_1 = \cdots = \nu_r = 2$ and $\nu_{r+1} = \cdots = \nu_l = 1$, and let $\delta = (\delta_1, \dots, \delta_l)$ be a partition of d. For $i \in \{1, \dots, r\}$ and $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{F}_2^r$ with $a_i = 0$, we consider the map

$$\theta_{i,\mathbf{a}} \colon \prod_{\substack{1 \le j \le r \\ j \ne i}} \Delta^{\circ}_{2,\delta_j} \longrightarrow \Delta_n$$

$$(L_1, \dots, \check{L_i}, \dots, L_r) \longmapsto L_1 \times \cdots \times \check{L_i} \times \cdots \times L_r \times \left(\sum_{\substack{1 \le j \le r \\ j \ne i}} a_j L_j \right) \times K^{d-2r}.$$

Here the symbol $\check{}$ means that the designated entry is omitted, and $\sum_{j \ne i} a_j L_j$ means the quadratic extension of K corresponding to $\sum_{j \ne i} a_j [L_j] \in K/\wp K$. The fact mentioned above implies that $\Delta^{(G)}_{\nu,\delta}$ is contained in $\bigcup_{i,\mathbf{a}} \mathrm{Im}(\theta_{i,\mathbf{a}})$. It follows that if $\Delta^{(G)}_{\nu,\delta} \ne \emptyset$, then

$$\dim \Delta^{(G)}_{\nu,\delta} \le \max_{\substack{i \\ \Delta^{\circ}_{\nu_i,\delta_i} \ne \emptyset}} \sum_{\substack{1 \le j \le r \\ j \ne i}} \dim \Delta^{\circ}_{\nu_j,\delta_j}.$$

Suppose that the maximum on the right side is attained at $i = i_0$. Then, $\Delta^{\circ}_{\nu_{i_0},\delta_{i_0}} \ne \emptyset$, and from Theorem 3.4, $\dim \Delta^{\circ}_{\nu_{i_0},\delta_{i_0}} > 0$. We conclude that

$$\begin{aligned}
\dim \Delta^{(G)}_{\nu,\delta} - \frac{\sum_i \delta_i}{2} &\le \sum_{\substack{1 \le j \le r \\ j \ne i_0}} \dim \Delta^{\circ}_{\nu_j,\delta_j} - \frac{\sum_i \delta_i}{2} \\
&\le \sum_{1 \le i \le r} \left(\dim \Delta^{\circ}_{\nu_i,\delta_i} - \frac{d}{2} \right) - \dim \Delta^{\circ}_{\nu_{i_0},\delta_{i_0}} \\
&\le -\dim \Delta^{\circ}_{\nu_{i_0},\delta_{i_0}} \\
&\le -1.
\end{aligned}$$

□

4. Proofs of Theorems 1.2 and 1.3

4.1. Proof of Theorem 1.3

We first prove Theorem 1.3. We follow the notation from Section 3. From Corollary 3.6(4) and Proposition 2.5, the log pair (X, B) is log canonical. From Theorem 3.4,

$$\begin{aligned}
\dim \Delta^{(G)}_{\nu,\delta} - \frac{d}{2} &\le \sum_i \left(\dim \Delta^{\circ}_{\nu_i,\delta_i} - \frac{\delta_i}{2} \right) \\
&\le \sum_i \left(\left\lceil \frac{\delta_i - \nu_i + 1}{p} \right\rceil - \frac{\delta_i}{2} \right).
\end{aligned}$$

If $p > 2$, since
$$\lim_{\delta_i} \left\lceil \frac{\delta_i - \nu_i + 1}{p} \right\rceil - \frac{\delta_i}{2} = -\infty,$$
we have
$$\lim_{\nu,\delta} \left(\dim \Delta^{(G)}_{\nu,\delta} - \frac{d}{2} \right) = -\infty.$$
Proposition 2.5 shows (X,B) is Kawamata log terminal if $p \neq 2$. We have completed the proof of Theorem 1.3.

4.2. Proof of Theorem 1.2

If G contains no transpositions, then Theorem 1.2 follows from Propositions 2.4 and 3.10. In the general case, we will prove Theorem 1.2 by reducing it to Theorem 1.3 and the case without transpositions.

Case $p \neq 2$.— Let $f\colon Y \to X$ be a proper birational morphism from a normal variety Y, and let E be a prime divisor on Y. If $f(E) \not\subset \mathrm{Supp}(B)$, then X has quotient singularities associated to permutation actions without transpositions at general points of $f(E)$. Thus, from the case without transposition, which was considered at the beginning of this subsection, we have
$$\mathrm{discrep}\,(E;X,B) = \mathrm{discrep}\,(E;X) \geq 0.$$
If $f(E) \subset \mathrm{Supp}(B)$, then since B is a 2-Cartier effective divisor, f^*B has multiplicity at least $1/2$ along E. From Theorem 1.3, (X,B) is Kawamata log terminal. Since $K_X + B$ is 2-Cartier, $\mathrm{discrep}\,(E;X,B) \geq -1/2$. Thus,
$$\begin{aligned}\mathrm{discrep}\,(E;X) &= \mathrm{discrep}\,(E;X,B) + \mathrm{mult}_E(f^*B).\\ &\geq -1/2 + 1/2\\ &\geq 0.\end{aligned}$$
We have proved Theorem (1.2) when $p \neq 2$.

Case $p = 2$.— Let $f\colon Y \to X$ be a proper birational morphism from a normal variety Y, and let E be a prime divisor on Y. If $f(E) \not\subset \mathrm{Supp}(B)$, then the same argument as in the case $p \neq 2$ shows that $\mathrm{discrep}\,(E;X) \geq 0$. If $f(E) \subset \mathrm{Supp}(B)$, then since B is a Cartier effective divisor, f^*B has multiplicity at least 1 along E. From Theorem 1.3, (X,B) is log canonical and hence $\mathrm{discrep}\,(E;X,B) \geq -1$. Thus,
$$\begin{aligned}\mathrm{discrep}\,(E;X) &= \mathrm{discrep}\,(E;X,B) + \mathrm{mult}_E(f^*B).\\ &\geq -1 + 1\\ &\geq 0.\end{aligned}$$
We have proved Theorem (1.2) in the case $p = 2$.

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