Épijournal de Géométrie Algébrique
epiga.episciences.org
Volume 10 (2026), Article No. 14

arXiv:2412.07483v3 [math.AG] 24 Aug 2026

Lines, twisted cubics on cubic fourfolds, and the monodromy of the Voisin map

Franco Giovenzana and Luca Giovenzana

Abstract. For a general cubic fourfold Y with associated Fano variety of lines F, we show that the monodromy group of the finite degree 16 rational Voisin self-map $\psi\colon F \dashrightarrow F$ is maximal. To achieve this, we investigate the intriguing interplay between ψ and the fixed locus of the antisymplectic involution on the LLSvS variety Z, examined via the degree 6 Voisin map $F \times F \dashrightarrow Z$.

Keywords. Irreducible symplectic varieties, cubic fourfolds

2020 Mathematics Subject Classification. 14J42, 14J35, 14J70

Received by the Editors on February 4, 2025, and in revised form on Novembre 19, 2025.
Accepted on February 26, 2026.

Franco Giovenzana
Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Rue Michel Magat, Bât. 307, 91405, Orsay, France
e-mail: franco.giovenzana@universite-paris-saclay.fr
Luca Giovenzana
Department of Mathematics, Universiteit Antwerpen, Middelheimlaan 1, 2020 Antwerpen, Belgium
e-mail: luca.giovenzana@uantwerpen.be

Franco Giovenzana was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Projektnummer 509501007, and partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC-2020- SyG-854361- HyperK). Both authors are members of INdAM GNSAGA.

© by the author(s) This work is licensed under http://creativecommons.org/licenses/by-sa/4.0/

Contents

1. Introduction 2
2. Nodal quintic surfaces with 16 nodes 4
3. On a Lagrangian submanifold of Z and singular cubic surfaces 7
4. The monodromy group of ψ is maximal 13
Appendix. Singular cubic surfaces 15
References 18

1. Introduction

Let Y be a smooth cubic fourfold and F be its (Fano) variety of lines. The rich geometry of these varieties has attracted great attention from the mathematical community for several reasons. Notably, the Fano variety F is one of the earliest examples of a locally complete family of projective hyperkähler manifolds. It is deformation equivalent to the Hilbert square of a K3 surface, and it has a natural polarisation of degree 6 and divisibility 2; see [BD85]. The duo $Y - F$ between a Fano variety and a hyperkähler manifold has inspired numerous other constructions.

One distinguishing feature of F is the degree 16 rational Voisin self-map $\psi\colon F \dashrightarrow F$. Remarkably, ψ is the only known finite, non-birational self-map defined on a locally complete family of projective hyperkähler manifolds. Finite maps such as ψ are particularly subtle to study as, unlike birational maps, they cannot be detected through their action on the second cohomology group and lack a comprehensive classification result like the Hodge Torelli theorem that would allow precise control over their behaviour. Since its construction, see [Voi04], this map has continued to attract significant interest in the mathematical community, as demonstrated by the numerous works on the subject [Ame09, GK23, GK25, GK24].

In this paper we focus on the monodromy of the Voisin map ψ. The monodromy group is a discrete invariant that encodes the symmetries and intrinsic geometry of finite maps. Determining the monodromy group of branched coverings is a classical problem, originating with Jordan in the 1870s. This topic was revitalised by Harris, who provided a modern framework by proving that the monodromy and Galois groups coincide, and who developed tools to establish when the monodromy group is maximal. Significant progress was made by Vakil, who introduced innovative techniques to study monodromy in Schubert problems on Grassmannians; see [Vak06].

Building on these developments, we take a step further by investigating linear spaces on a cubic hypersurface of dimension 4. Our main result shows that given a general cubic fourfold, the monodromy group of the Voisin map ψ is maximal, meaning that it is the full symmetric group on 16 elements. To achieve this, we leverage another Voisin map involving twisted cubics on the cubic fourfold, which unveils a fascinating connection with the fixed locus of the natural anti-symplectic involution acting on the LLSvS variety.

We now introduce the necessary notation and recall some useful results in order to state our main theorems. Lines on a cubic fourfold fall into two cases: for a general line L, the linear space

$$\Lambda_L := \bigcap_{p \in L} T_p Y$$

is 2-dimensional, and in this case the line is said to be of *type I*. For special lines, called of *type II*, the dimension of Λ_L is 3; see Proposition 2.1 and Definition 2.2 for more details. For the general line we have $\Lambda_L \cap Y = 2L + R$ for a line $R \in F$, and one sets $\psi(L) := R$. This map has been studied in detail in [Ame09], its indeterminacy locus consists of the lines of type II, which for a general cubic fourfold form a smooth surface S_{II}. Blowing up S_{II} provides a resolution of the indeterminacy

$$\begin{array}{ccc} & \widehat{F} & \\ & \swarrow \quad \searrow^{\widehat{\psi}} & \\ F & \overset{\psi}{- - - \to} & F. \end{array}$$

Gounelas and Kouvidakis recently computed that the restriction of $\widehat{\psi}$ to the exceptional divisor E of the blow-up morphism, which coincides with the ramification divisor of $\widehat{\psi}$, is birational onto the image.

In Section 2 we study the map $\widehat{\psi}$ by considering the projection of Y from a general line R in Y. Its resolution is a conic bundle over $\mathbb{P}^3$ with discriminant locus a quintic surface S_R with exactly 16 nodes. These nodes correspond to the preimage of R under $\widehat{\psi}$. Nodal quintic surfaces have been classically studied by Beauville and Catanese [Bea80, Cat81] and more recently studied in [Huy24, BFK+24, CCC+24]. First we study the singularities of S_R for a special line R.

Theorem 1.1 (see Theorem 2.8). *Let $R \in F$ be a general line in the branch divisor of $\widehat{\psi}$. Then the quintic surface S_R has one singularity of type A_3, corresponding to the unique line L of type* II *over R.*

In Section 3 we shift our focus to the geometry of twisted cubics. With any smooth cubic fourfold Y not containing a plane, Lehn, Lehn, Sorger, and van Straten associated an 8-dimensional hyperkähler variety Z parametrising families of twisted cubics and their flat degenerations. The variety Z is equipped with a natural antisymplectic involution τ; see [Leh15]. Its fixed locus is a smooth Lagrangian submanifold with two connected components: one is isomorphic to the cubic fourfold Y; the other one, W, is of general type, see [FMOS26], and remains somewhat mysterious. Using the degree 6 rational map $\varphi\colon F \times F \dashrightarrow Z$ constructed by Voisin [Voi16], we offer an alternative description of W. We define the variety P as the closure in $F \times F$ of

$$\{(L_1, L_2) \in F \times F \mid L_i \text{ are of type I, } L_1 \neq L_2, \text{ and } \psi(L_1) = \psi(L_2)\},$$

which is birational to an irreducible component of the self-product of $\widehat{F}$ over F.

Theorem 1.2 (see Theorem 3.4). *The variety P is mapped onto W under the Voisin map $\varphi\colon F \times F \dashrightarrow Z$.*

In Section 4, after revising the basic notions of monodromy, we tackle the study of $\widehat{\psi}$. Despite extensive study and numerous works concerning $\widehat{\psi}$, for example about its entropy, see [Ame09], many of its properties remain elusive. Through an investigation of the restriction of the map φ to the variety P, we prove the following.

Theorem 1.3 (see Theorem 4.4). *The monodromy group of ψ is the entire symmetric group on* 16 *elements.*

Acknowledgments

This project began a long time ago, and over the years we benefited from conversations with many people. It is our pleasure to thank everybody who expressed interest and shared their point of view, especially Enrico Fatighenti, Frank Gounelas, Christian Lehn, Emanuele Macrì, Giovanni Mongardi, Alan Thompson, and Yilong Zhang. We also express our gratitude to the anonymous referees for their careful reading.

2. Nodal quintic surfaces with 16 nodes

In this section we recall basic facts about lines on cubic fourfolds and various properties of the Voisin map on F. Then we move on to studying degenerations of nodal quintic surfaces with an even set of 16 nodes naturally arising in this context.

Recall that the Gauss map associates to any point of the smooth cubic fourfold $Y \subset \mathbb{P}(V)$ its projective tangent space:

$$\mathscr{G}\colon Y \longrightarrow \mathbb{P}\left(V^\vee\right), \quad P \longmapsto T_P Y \simeq \mathbb{P}^4.$$

Clemens and Griffiths distinguished lines on cubic hypersurfaces into two types. We recall here the definition for cubic fourfolds; see [CG72].

Proposition 2.1. *Given a line L on a smooth cubic fourfold Y, either the following equivalent conditions hold:*

(1) $N_{L|Y} \simeq \mathcal{O}_L^{\oplus 2} \oplus \mathcal{O}_L(1)$,
(2) $\mathscr{G}|_L\colon L \to \mathscr{G}(L)$ *is* $1:1$,
(3) $\mathscr{G}(L)$ *is a smooth conic in* $\mathbb{P}(V^\vee)$,
(4) $\bigcap_{P\in L} T_P Y$ *is isomorphic to* $\mathbb{P}^2$;

or the following equivalent conditions hold:

(5) $N_{L|Y} \simeq \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)^{\oplus 2}$,
(6) $\mathscr{G}|_L\colon L \to \mathscr{G}(L)$ *is* $2:1$,
(7) $\mathscr{G}(L)$ *is a line in* $\mathbb{P}(V^\vee)$,
(8) $\bigcap_{P\in L} T_P Y$ *is isomorphic to* $\mathbb{P}^3$.

Definition 2.2 (*cf.* [CG72, Definition 6.6]). Given a line L, we say that L is a line of type I if the equivalent conditions (1)–(4) hold, whereas if (5)–(8) hold, we say that L is a line of type II. We set $\Lambda_L := \cap_{P\in L} T_P Y$.

We record here elementary observations for future reference.

Remark 2.3. If $L \in F$ is of type II, then the line $\mathscr{G}(L) \subset \mathbb{P}(V^\vee)$ is the projective dual of $\Lambda_L \subset \mathbb{P}(V)$. In the case of a line of type I, $\mathscr{G}(L)$ spans a $\mathbb{P}^2$ in $\mathbb{P}(V^\vee)$ which is dual to Λ_L.

Remark 2.4 (*cf.* [Huy23, Remark 2.2.2]). Let $Y = V(f) \subset \mathbb{P}^5$ be a smooth cubic fourfold, and let L be a line in Y. Then L is of type II if and only if the partial derivatives $\partial_0 f|_L, \dots, \partial_5 f|_L$ span a vector space of dimension 2 in $H^0(L, \mathcal{O}_{\mathbb{P}^5}(2))$.

Voisin observed that if L is a line of type I, then $\Lambda_L \cap Y$ is a cubic curve, which, as it contains L with multiplicity 2, consists of L and a residual line R. As the general line is of type I, one gets the following.

Definition 2.5. The Voisin map ψ is the rational map defined by mapping a line L of type I to its residual R

$$\psi\colon F \dashrightarrow F, \quad L \longmapsto R.$$

In general, we say that the line R is *residual* to L if there exists a projective plane Λ such that $\Lambda \cap Y = 2L + R$. Following Gounelas and Kouvidakis, if a line is residual to itself, we refer to it as a *triple* line. For the general cubic fourfold, the locus of triple lines is an irreducible surface of general type; see [GK24, Theorem A]. We denote it by V.

The map ψ has been studied in [Voi04] and [Ame09], where it is proven to be finite of degree 16. For a general cubic fourfold, lines of type II form a smooth surface S_{II} in F, and blowing up F in this surface resolves the indeterminacy of ψ:

(2.1)
$$\begin{array}{ccc} & \widehat{F} & \\ {\scriptstyle \mathrm{pr}}\swarrow & & \searrow{\scriptstyle \widehat{\psi}} \\ F & \overset{\psi}{\dashrightarrow} & F. \end{array}$$

The blow-up $\widehat{F}$ can be identified as the closure of the following rational map (see [GK24, Lemma 4.1] and [Huy23, Remark 2.2.19]):

$$F \dashrightarrow \mathrm{Gr}(3,6), \quad L \longmapsto \Lambda_L.$$

Elements in the exceptional locus are then just pairs (L, Ξ), where L is a line of type II and Ξ is a projective plane such that $L \subset \Xi \subset \Lambda_L$. For such 2-dimensional spaces Ξ, we have that $\Xi \cap Y = 2L + R$; in particular, R is residual to L.

Given a general line $R \in F$, we consider the diagram

$$\begin{array}{ccc} & \widetilde{Y} := \mathrm{Bl}_R\, Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \widetilde{\pi}} \\ Y & \xrightarrow{\quad\pi_R\quad} & \mathbb{P}^3, \end{array}$$

where π_R is the projection from R and $\widetilde{Y}$ is the blow-up of Y in R. The morphism $\widetilde{\pi}$ is a conic bundle with discriminant a quintic surface S_R, whose singular locus consists of 16 nodes. The 16 nodes correspond to the preimage of R under $\widehat{\psi}$. Indeed, if p_i denote the nodes of S_R for $i = 1, \dots, 16$, then the $L_i := p(\widetilde{\pi}^{-1}(p_i))$ are the 16 lines for which $\psi(L_i) = R$ (see [Huy23, Section 6.4.5] for an account of the various results in the literature about this surface).

Let E be the exceptional divisor of the blow-up morphism $\widehat{F} \to F$. As F has trivial canonical bundle, the divisor E coincides with the ramification locus of the map $\widehat{\psi}$.

Theorem 2.6 (*cf.* [GK25, Theorem B]). *Let Y be a general cubic fourfold. The restriction $\widehat{\psi}|_E \colon E \to F$ is generically* 1*-to-*1 *onto the image.*

Proposition 2.7. *Let Y be a general cubic fourfold. The ramification at the general point of the ramification locus of $\widehat{\psi}$ is simple.*

Proof. Comparing the pullbacks of the canonical of F in Diagram (2.1), one gets

$$E = \mathrm{pr}^*(K_F) + E = K_{\widehat{F}} = \widehat{\psi}^*(K_F) + R = R,$$

where R is the ramification locus with the natural schematic structure. Hence the ramification is simple by [BHP+04, Lemma I.16.1] □

In other words, over the general point R in the branch divisor of $\widehat{\psi}$, there is exactly one point of ramification; *i.e.* there exists exactly one line L of type II with residual R, meaning that $\widehat{\psi}(L, \langle L, R\rangle) = R$.

Theorem 2.8. *Let $R \in F$ be a general line in the branch divisor of $\widehat{\psi}$. Then the quintic surface S_R has exactly* 14 *singularities of type A_1 and exactly* 1 *singularity of type A_3, corresponding to the unique line L of type* II *with residual line R.*

Proof. Let R be a general line in the branch locus of $\widehat{\psi}$, and let L be the unique line of type II with residual R. We may assume that R is of type I and that it is given as the zero set $V(x_2, x_3, x_4, x_5)$. Following Clemens and Griffiths [CG72, Equation 6.9], we write the equation of the cubic fourfold as

$$f := x_4 x_0^2 + x_5 x_0 x_1 + x_3 x_1^2 + x_0 Q_0 + x_1 Q_1 + P = 0, \tag{2.2}$$

where Q_0, Q_1, and P are homogeneous polynomials in the variables x_2, x_3 ,x_4, x_5 of degree 2, 2, and 3, respectively. The quintic surface $S_R \subset V(x_0, x_1) \simeq \mathbb{P}^3$ is given by the determinant of the matrix

$$M := \begin{pmatrix} P & Q_0 & Q_1 \\ Q_0 & x_4 & x_5 \\ Q_1 & x_5 & x_3 \end{pmatrix}.$$

We assume that the line L is given by $V(x_1, x_2, x_4, x_5)$; then $p := (1 : 0 : 0 : 0 : 0 : 0)$ is the intersection of L and R, and $q := (0 : 0 : 0 : 1 : 0 : 0)$ is the point of intersection with $\mathbb{P}^3$.

We translate these assumptions in conditions on the coefficients of Q_0, Q_1, and P. For this, we introduce the notation $Q_0 = \sum a_I \underline{x}^I$, $Q_1 = \sum b_I \underline{x}^I$, and $P = \sum c_I \underline{x}^I$.

The cubic fourfold Y contains the line L if $f(x_0, 0, 0, x_3, 0, 0, 0)$ is the zero form in $\mathbb{C}[x_0, x_3]$; hence we get the condition

$$a_{33} = c_{333} = 0. \tag{2.3}$$

The fact that the line R is residual to L, *i.e.* $\langle L, R\rangle \cap Y = 2L + R$, translates into the further condition

$$b_{33} = 0. \tag{2.4}$$

By Remark 2.4 the fact that the line L is of type II can be rephrased in terms of the derivatives $\partial_i f|_L$, and we get the additional condition

$$a_{23}c_{335} - a_{35}c_{233} = 0. \tag{2.5}$$

To conclude the proof, we check with the aid of Macaulay2 that the surface defined by $\det(M)$ for polynomials Q_0, Q_1, and P with coefficients satisfying conditions (2.3), (2.4), and (2.5) has a singularity of type A_3 at the point q. See the ancillary file for the code. □

As a consequence, we obtain an alternative proof of the simpleness of the ramification of $\widehat{\psi}$.

Proof of Proposition 2.7. Let b be a general point in the branch divisor of ψ. Then by Theorem 2.6 the preimage $\widehat{\psi}^{-1}(b)$ consists of a finite number of points $p_1, \dots, p_k$, where only p_1 is of ramification, while $\widehat{\psi}$ is étale at the points $p_2, \dots, p_k$. All the points p_i correspond to singularities of the quintic surface S_b: in particular, $p_2, \dots, p_k$ are singularities of type A_1; in contrast, p_1 is a singular point of type A_3 by Theorem 2.8. The surface S_b is a specialization of the surface S_R for the general line $R \in F$, which has exactly 16 nodes. Thus we deduce that 2 points among these 16 specialize to the singularity of type A_3, whereas the remaining 14 specialize to $p_2, \dots, p_k$, deducing that S_b has 14 singularities of type A_1 apart from the A_3 singularity. □

Let R be any line on the cubic fourfold. We denote by F_R the locus of lines meeting R:

$$F_R := \{L \in F : L \cap R \neq \emptyset\}.$$

If R is general, then F_R is a smooth surface; see [Voi86, Lemma 3.1]. As an application of our analysis, we study the singularities of F_R for a general point R in the branch locus of $\widehat{\psi}$.

Corollary 2.9. *Let $R \in F$ be a general line in the branch locus of $\widehat{\psi}$. If the surface F_R is normal, then it has exactly one singular point, which is of type A_1 and corresponds to the unique line L of type* II *with residual line R.*

Proof. Let $\pi_R \colon Y \dashrightarrow \mathbb{P}^3$ be the projection from R. Then there is a map

$$f : F_R \dashrightarrow S_R, \quad L \longmapsto \pi_R(L).$$

The map clearly extends to the point R by mapping R to $\pi_R(\Lambda_R)$, when the line R is of type I. The map f is 2-to-1 and can be regarded as the quotient of an involution, which, in case R is general, has 16 fixed points mapped to the 16 singular points of S_R. If R is as in the hypothesis, then the fixed points of this involution are in bijection with the singularities of S_R: 14 singularities of type A_1 and 1 singularity of type A_3 corresponding to the unique line L of type II with residual R by Theorem 2.8.

As F_R is normal, the morphism f around each singular point is the unique morphism characterized by [GKP16, Proposition 3.13]. Locally around the A_1 singular point, it is isomorphic to the quotient presentation $\mathbb{C}^2 \to \mathbb{C}^2/\mu_2$ of the A_1 singularity, and around the A_3 singular point, it is isomorphic to the presentation $\mathbb{C}^2/\mu_2 \to \mathbb{C}^2/\mu_4$ of the A_3 singularity as quotient of the A_1 singularity. □

3. On a Lagrangian submanifold of Z and singular cubic surfaces

In Section 3.1 we recall the construction of the hyperkähler variety $Z = Z(Y)$ associated with any smooth cubic fourfold Y not containing a plane, often referred to as the LLSvS variety, together with some of its basic properties. It carries the action of an antisymplectic involution which has been studied in [FMOS22, FMOS26], whose fixed locus consists of two connected components, one isomorphic to the cubic fourfold, which we hence denote by Y, and a second one which we denote by W. The goal of this section is to study the less-understood variety W. In order to do that, in Section 3.2 we define a subvariety P of $F \times F$ which is related to W via a rational map $\varphi\colon F \times F \dashrightarrow Z$ introduced by Voisin which we recall in Section 3.1. A key tool in our strategy is to reduce statements to the study of cubic surfaces S which are given by intersecting the cubic Y with a projective space $\mathbb{P}^3$ and consider lines and twisted cubics on S. We do so in Section 3.3. This analysis provides the key tools to prove the main result of the section, Theorem 3.4.

3.1. The LLSvS variety Z

Given a smooth cubic fourfold Y not containing any plane, Lehn, Lehn, Sorger, and van Straten constructed in [LLS+17] a hyperkähler variety Z of dimension 8 which parametrises families of twisted cubics on Y and their flat degenerations. The LLSvS manifold is deformation equivalent to the Hilbert scheme of four points on a K3 surface; see [AL17, Leh18].

As any twisted cubic has $3t+1$ as its Hilbert polynomial, the starting point is to consider the compactification $\mathrm{Hilb}^{\mathrm{gtc}}(Y)$ of the space of twisted cubics inside the Hilbert scheme $\mathrm{Hilb}^{3t+1}(Y)$. Curves parametrised by this variety are called *generalised twisted cubics*. The variety $\mathrm{Hilb}^{\mathrm{gtc}}(Y)$ is smooth, of dimension 10, and admits a morphism to the Grassmannian $G := \mathrm{Gr}(4,6)$, which maps any generalised twisted cubic to its linear span; see [LLS+17, Theorem A]. This morphism factors through a $\mathbb{P}^2$-fibration onto a variety Z' followed by a generically finite morphism $a'\colon Z' \to G$. The hyperkähler projective manifold Z is obtained from Z' by contracting the divisor parametrising linear systems of curves which are not arithmetically Cohen–Macaulay. The image of the divisor under the contraction is a Lagrangian subvariety isomorphic to the cubic fourfold Y. The variety Z comes with a generically finite rational map $a\colon Z \dashrightarrow G$ and thus a natural polarisation $\mathcal{L}$ induced by the Plücker polarisation of the Grassmannian G; see [LLS+17, Theorem B]. The family parametrised by cubic fourfolds form a projective locally complete family of polarised hyperkähler manifolds with polarisation of degree 2 and divisibility 2; see [AG23, LPZ23].

The hyperkähler manifold Z has a biregular antisymplectic involution $\tau\colon Z \to Z$, as pointed out in [Leh15]. It can be defined as follows: let $C \subset Y$ be a generalised twisted cubic, and let Q be any quadric (surface) in the linear span $\langle C \rangle$, which contains C. The intersection $Q \cap Y$ is a sextic curve consisting of C and the residual generalised twisted cubic C'. The involution τ is defined by mapping C to C'. Antisymplectic involutions of $\mathrm{K3}^{[n]}$-type manifolds and this particular example have been studied in [FMOS22, FMOS26].

Theorem 3.1 (*cf.* [FMOS26, Theorem 1.4]). *The fixed locus of the antisymplectic involution has two connected components, one of which is the Lagrangian subvariety Y. The other component W is a smooth fourfold of general type; in particular, $\omega_W = 3\mathcal{L}|_W$, where $\mathcal{L}$ is the primitive polarisation on Z.*

Voisin constructed a degree 6 rational map $\varphi\colon F \times F \dashrightarrow Z$ in the following way; see [Voi16, Theorem 4.8]. Given a general pair of skew lines (L_1, L_2) and a point $p \in L_1$, the union of L_1 and the conic residual to L_2 in the intersection $\langle L_2,\ p\rangle \cap Y$ is a generalised twisted cubic C. The map φ is defined by the association $(L_1, L_2) \mapsto [C]$. The indeterminacy locus, its resolution, and its resolution's branch locus have been studied in [Mur20, Che21, Gio26].

The involution τ fits in the following commutative diagram:

$$\begin{array}{ccc} F\times F & \xrightarrow{\varphi} & Z \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ F\times F & \xrightarrow{\varphi} & Z, \end{array}$$

where $\sigma\colon (L_1, L_2) \mapsto (L_2, L_1)$ switches the factors.

3.2. A variety dominating W

We denote by P the closure in $F \times F$ of

$$\{(L_1, L_2) \in F \times F \mid L_i \text{ are of type I, } L_1 \neq L_2, \text{ and } \psi(L_1) = \psi(L_2)\}.$$

The variety P turns out to have a central role in our study: the following intriguing lemma is the starting point which allows us to relate properties of ψ and ϕ.

Lemma 3.2. *Let $(L_1, L_2) \in P$ be a general pair of lines, satisfying the conditions*

(1) *L_1 and L_2 are skew,*
(2) *L_1 and L_2 are both of type I,*
(3) *and $\psi(L_1) = \psi(L_2)$.*

Then the linear span of L_1 and L_2 intersects Y in a cubic surface with four singular points of type A_1.

Cubic surfaces with four singularities of type A_1 are called Cayley cubics; we analyse them in Section 3.3.

Proof. We claim that the linear span $\Lambda := \langle L_1, L_2\rangle$ intersects Y in a cubic surface with two singular points on L_1. We denote by R the residual line $\psi(L_1)$; then R and L_1 are coplanar, and we consider the line T in $\mathbb{P}(V^*)$ of hyperplanes which contain Λ.

As both T and $\mathcal{G}(L_1)$ lie in the 2-dimensional linear space $\langle L_1, R\rangle^* \subset \mathbb{P}(V^*)$, the line T intersects the smooth conic $\mathcal{G}(L_1)$ in two points, which correspond to the image under the Gauss map of two singular points of $S := \Lambda \cap Y$ lying on the line L_1.

Repeating the argument for L_2, we find two singular points of S on L_2, and we deduce that the cubic surface S is Cayley by [Doll2, Corollary 9.2.3]. □

Following the construction of the Voisin map, one gets the following.

Lemma 3.3. *Let L_1 and L_2 be two lines that satisfy the following general conditions:*

(1) *L_1 and L_2 are of type I,*
(2) *L_1 and L_2 are skew.*

If $\psi(L_1) = \psi(L_2) =: R$, then $\varphi(L_1, L_2) = \varphi(L_2, L_1)$.

Proof. By hypothesis we have $\Lambda_{L_1} \cap Y = 2L_1 + R$ and $\Lambda_{L_2} \cap Y = 2L_2 + R$. Since L_1 and L_2 are skew, each meets R in a single point, say $L_i \cap R = \{p_i\}$ for $i = 1, 2$. We compute $\varphi(L_1, L_2)$ using the point p_1, as described in Section 3.1. Because the construction there applies to a general point of L_1, we must extend it to this special point p_1.

The plane $\langle p_1, L_2\rangle$ coincides with Λ_{L_2}; hence its intersection with Y is the cubic curve $2L_2 + R$. The union $L_2 \cup R$ is the conic residual to L_2 in this cubic: this agrees with the definition of residual conic obtained by taking the flat limit. Indeed, set $S := Y \cap \langle L_1, L_2\rangle$, and consider the morphism $L_1 \to \mathrm{Hilb}^{2t+1}(S)$ mapping a point P of L_1 to the conic in $\langle P, L_2\rangle \cap S$ residual to L_2. Then the limiting residual conic corresponding to p_1 is precisely $L_2 \cup R$. Finally, one checks that $L_1 \cup L_2 \cup R$ is a point in $\mathrm{Hilb}^{3t+1}(S)$, so $\varphi(L_1, L_2) = L_1 \cup R \cup L_2$. The analogous computation for $\varphi(L_2, L_1)$ yields the same conclusion, proving the claim. □

We report the following theorem, which is a key result for our study, but we defer its proof to later, because it relies on some observations concerning Cayley cubics. We recall from Section 3.1 that the variety Z' admits a morphism a' onto the Grassmannian $G = \mathrm{Gr}(4,6)$, which descends to a rational map $a\colon Z \dashrightarrow G$ whose locus of indeterminacy is the connected component of the fixed locus of the involution τ isomorphic to Y.

Theorem 3.4. *The Voisin map $\varphi\colon F \times F \dashrightarrow Z$ restricts to a dominant rational map $\varphi|_P\colon P \dashrightarrow W$.*

The regular map $a|_W\colon W \to G$ is birational onto the image, which consists of the projective spaces that intersect Y in a Cayley cubic or a degeneration of it.

In the following we analyse the maps $P \dashrightarrow W \to G$ in greater detail. In order to do so, we describe singular cubic surfaces, focusing on lines on them.

3.3. Cayley cubics and their degenerations

The stratification of the space of cubic surfaces given by the type of singularities has been studied by Bruce in [Bru80]. Here we are interested only in cubic surfaces with exactly four singular points of type A_1, called Cayley cubics, and their degenerations. From Bruce's analysis, degenerations of Cayley cubics can be summed up by the following diagram:

$$4A_1 \;\geq\; 2A_1A_3 \;\geq\; A_1A_5 \;\geq\; \widetilde{E}_6 \;\geq X_6 \;\geq X_7 \;\geq X_8 \;\geq\; X_9, \tag{3.1}$$

where $\widetilde{E}_6$ denotes the cubics with one simple elliptic singularity and the X_i are non-normal cubic surfaces. We say that a cubic surface S is of type $2A_1A_3$ if S has exactly two singularities of type A_1 and one singularity of type A_3, and analogously for the other singularities.

In the following we focus on lines and twisted cubics on singular cubic surfaces to study the maps φ and ψ. We notice that there is not an analogue of the map ψ for the variety of lines on a cubic surface S because a line L need not have a plane tangent at every point. However, if $\mathbb{P}^2 \simeq \pi \subset \langle S \rangle$ is tangent at every point of $L \subset S$ with $\pi \cap S = 2L + R$ for a line R, we say that the line R is *residual* and that L is *triple* if $R = L$. Generalised twisted cubics on singular cubic surfaces have already been carefully analysed in [LLS+17, Section 2], which we closely follow. Every cubic surface S with at most rational double points has a minimal resolution $\widetilde{S} \to S$ with $\widetilde{S}$ isomorphic to the blow-up of $\mathbb{P}^2$ in six points. The orthogonal $K_{\widetilde{S}}^{\perp} \subset H^2(\widetilde{S}, \mathbb{Z})$ is isomorphic to the negative-definite root lattice E_6. We denote by Γ the roots in $K_{\widetilde{S}}^{\perp}$ and by Γ_0 the subset of effective roots. The latter correspond to irreducible components of exceptional curves for the resolution. We also denote by $W(\Gamma_0)$ the subgroup of the Weyl group of E_6 generated by reflections in the effective roots Γ_0. The moduli space of generalised twisted cubics on S with reduced structure is then given by a union of projective planes; see [LLS+17, Theorem 2.1]:

$$\mathrm{Hilb}^{\mathrm{gtc}}(S)_{\mathrm{red}} = (\Gamma / W(\Gamma_0)) \times \mathbb{P}^2. \tag{3.2}$$

In particular, linear systems of generalised twisted cubics on S are in bijection with the orbits of Γ under the action of $W(\Gamma_0)$. Along this bijection, families of arithmetically Cohen–Macaulay (aCM) curves correspond to orbits containing an effective root.

Going again through the construction of Z from [LLS+17], let Y be a cubic fourfold not containing a plane. Mapping a generalised twisted cubic to its linear span defines a morphism from $\mathrm{Hilb}^{\mathrm{gtc}}(Y)$ to the Grassmannian $G = \mathrm{Gr}(4,6)$ which factors through a $\mathbb{P}^2$-fibration followed by a generically finite morphism $a'\colon Z' \to G$. Let C be a twisted cubic and E be its linear span; the intersection $S := Y \cap E$ is a cubic surface. If S has ADE singularities, the preimage $Z'(S) := (a')^{-1}(E)$ is in bijection with the set $\Gamma / W(\Gamma_0)$ in Equation (3.2) and the points $Z'_{\mathrm{aCM}}(S)$ corresponding to aCM curves are in bijection with $(\Gamma \setminus \Gamma_0)/W(\Gamma_0)$. By its definition the involution τ restricts to an action on $Z'_{\mathrm{aCM}}(S)$ and is given by $\alpha \mapsto -\alpha$ for $\alpha \in \Gamma$; see [Leh15].

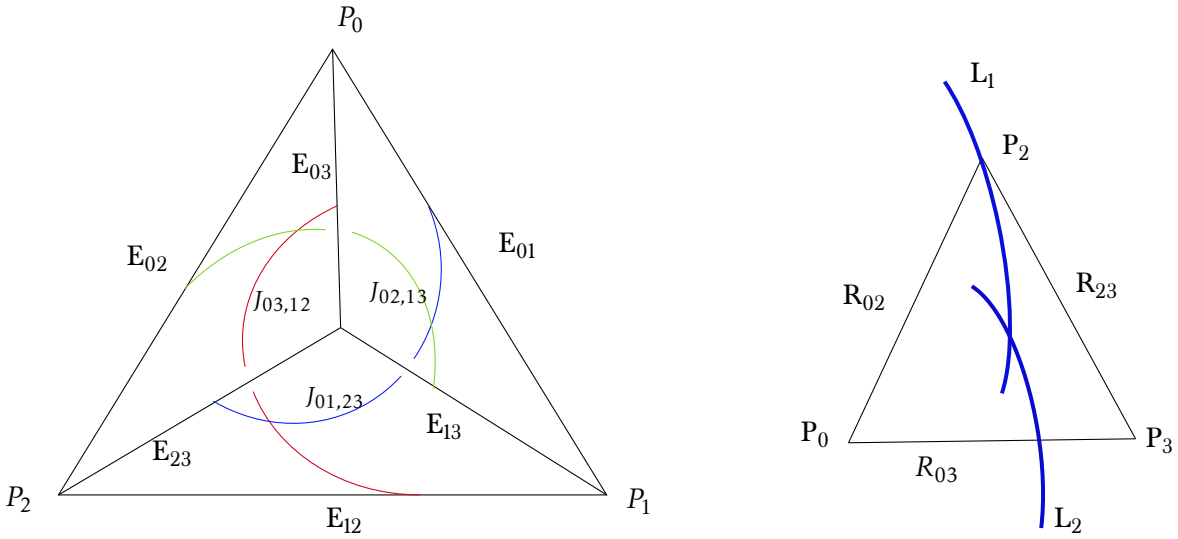
We defer to the appendix details about these singular surfaces; here we recall only the main facts we need to analyse the Voisin maps.

We also introduce a piece of notation. We denote by G_{4A1} the closure in the Grassmannian G of

$$\{E \in G \mid E \cap Y \text{ is a singular cubic surface of type } 4A_1\}.$$

Analogously, we define the loci $G_{2A_1A_3}$, $G_{A_1A_5}$, $G_{\widetilde{E}_6}$, and for the non-normal surfaces, G_{X_6}, G_{X_7}, G_{X_8}, G_{X_9}.

Singular cubic surfaces of type $4A_1$.— Any Cayley cubic, *i.e.* a cubic surface with exactly four A_1 singular points, contains exactly nine lines whose schematic representation resembles a tetrahedron with vertices at the singular points. We name *edges* the six of them connecting two singular points. In particular, if P_0, P_1, P_2, P_3 are the singular points, we denote E_{ij} the line containing P_i and P_j. Each of the other three lines intersects exactly two opposite edges; we denote by $J_{ij,kl}$ the line joining the edges E_{ij} and E_{kl} for $\{i,j,k,l\} = \{0,1,2,3\}$. See Figure 1a.



(a) Lines on a singular cubic surface of type $4A_1$

(b) Lines on a $2A_1A_3$ surface

Figure 1

We denote by K the linear system of generalised twisted cubics containing the reducible elements given by the lines through any singular point P_i:

$$K = [E_{ij} \cup E_{ik} \cup E_{il}] \in Z'_{aCM}(S)$$

for $\{i,j,k,l\} = \{0,1,2,3\}$.

Lemma 3.5. *A pair of opposite edges maps to K under φ; that is,*

$$\varphi(E_{ij}, E_{kl}) = K \quad \text{for indices such that } \{i,j,k,l\} = \{1,2,3,4\}.$$

Proof. See the appendix. □

Lemma 3.6. *Let S be a Cayley cubic. Then*

$$\#Z'_{aCM}(S) = 13 \quad \text{and} \quad \#Z'_{aCM}(S)^\tau = 1,$$

where $Z'_{aCM}(S)^\tau$ denotes the locus of fixed points of the involution τ.

Proof. The first equality follows from [LLS+17, Table 1]. The second one can be deduced from the description of τ in terms of roots. □

Proof of Theorem 3.4. By Lemma 3.2 the general point (L_1, L_2) in P corresponds to opposite edges on a Cayley cubic surface. By Lemma 3.5 its image $\varphi((L_1, L_2))$ is an aCM generalised twisted cubic fixed by the involution on Z, *i.e.* a point in W. In particular, φ restricts to a dominant map $\varphi : P \dashrightarrow W$. Finally, the morphism $a|_W : W \to G$ is generically 1:1 onto its image because, by Lemma 3.6, given a projective space $\Lambda \simeq \mathbb{P}^3$ which intersects Y in a Cayley cubic S, K is the unique linear system of *aCM* generalised twisted cubics in S which is fixed by τ. □

Singular cubic surfaces of type $2A_1A_3$.— Let S be a cubic surface with exactly one singular point P_2 of type A_3 and two singular points P_0, P_3 of type A_1. It contains exactly five lines as depicted in Figure 1b: the line R_{02} through P_0, P_2; the line R_{23} through P_2, P_3; the line R_{03} through P_0, P_3; the line L_1 containing the only singular point P_2; and the line L_2 contained in the smooth locus.

Proposition 3.7. *Let S be a cubic surface of type $2A_1A_3$. Then*

$$L_1 \text{ is residual to } R_{23}, \quad L_1 \text{ is residual to } R_{02},$$
$$L_2 \text{ is residual to } R_{03}, \quad L_2 \text{ is residual to } L_1.$$

Moreover, there is no plane in $\langle S \rangle \simeq \mathbb{P}^3$ tangent to S at every point of L_2.

Proof. See the appendix. □

Lemma 3.8. *Let S be a cubic surface of type $2A_1A_3$. Then*

$$\#Z'_{aCM}(S) = 5 \quad \text{and} \quad \#Z'_{aCM}(S)^\tau = 1.$$

Proof. The first equality follows from [LLS+17, Table 1]. The second one can be deduced from the description of τ in terms of roots. □

Singular cubic surfaces of type A_1A_5.— Let S be a cubic surface with an A_1 and an A_5 singularity. It contains exactly two lines, say L_1 passing through both singular points and L_2 which intersects the singular locus in the A_5 point.

Proposition 3.9. *Let S be a cubic surface of type A_1A_5. Then*

$$L_2 \text{ is residual to itself (i.e. triple on } S), \text{ and } L_2 \text{ is residual to } L_1.$$

Proof. See the appendix. □

Lemma 3.10. *Let S be a cubic surface of type A_1A_5. Then*

$$\#Z'_{aCM}(S) = \#Z'_{aCM}(S)^\tau = 1.$$

Proof. This is clear from [LLS+17, Table 1]. □

3.3.1. Simple elliptic cubic surfaces.— Any cubic surface S with a simple elliptic singularity is a cone over a planar elliptic curve E. Let v be the vertex. The Fano variety $F(S)$ of S can be naturally identified with the embedded elliptic curve E; indeed, to any point P of E corresponds the line through P and v. The scheme of generalised twisted cubics is isomorphic to $\mathrm{Sym}^3(E)$, and the summation map to E is a $\mathbb{P}^2$-bundle giving the fibration $\mathrm{Hilb}^{\mathrm{gtc}}(S) \to Z'(S)$; see [LLS+17, Proposition 2.7]. An aCM generalised twisted cubic corresponds to the union of three lines, which join the vertex with three non-collinear points on E. If the three points are collinear, then they are the support of a non-aCM curve with an embedded point at the vertex. Via the isomorphism $F(S) \simeq E$, the residual line can be expressed in terms of the group law of E, and the analogue of P for lines on the surface S,

$$P(S) := \{(p_1, p_2) \in F(S) \times F(S) \mid p_1 \text{ and } p_2 \text{ have the same residual}\}$$

admits a nice description, as follows.

Proposition 3.11. *Let S be a simple elliptic cubic surface. Then*

- *the residual to the line $p \in E \simeq F(S)$ is given by $-2p$,*
- *$P(S) = \{(p, p+\xi) \mid \xi \in E[2]\}$,*
- *the triple lines on S are given by the 3-torsion points $E[3] \subset E$; in particular, every simple elliptic cubic surface has nine triple lines.*

We now explore some connections of our study of singular cubic surfaces with results of Gounelas and Kouvidakis [GK24]. As a consequence, we deduce the existence of a surface of type A_1A_5, which is a fundamental little ingredient for the main theorem.

Let Y be a smooth cubic fourfold. The natural map $\gamma\colon \mathbb{L} \to Y$ from the universal family of lines to the cubic fourfold is a fibration in (2,3)-complete intersection curves. Given a point $y \in Y$, its fibre $\gamma^{-1}(y)$ consists of the lines through the point y. Gounelas and Kouvidakis studied the g^1_3 of these curves coming from the rulings of the corresponding quadric and their relation with the Voisin map. They proved that for the general cubic fourfolds, the surface V of triple lines intersects S_{II} in an irreducible curve with 3780 nodes. In particular, the general triple line is of type I; see [GK24, Theorem B].

Corollary 3.12. *Let Y be a general cubic fourfold.*

(1) *For any $L \in S_{\mathrm{II}}$, the intersection $\Lambda_L \cap Y$ is a non-normal singular cubic surface.*
(2) *If $L \in V \cap S_{\mathrm{II}}$, then $\Lambda_L \cap Y$ is of type X_7 or worse.*
(3) *If $L \in (V \cap S_{\mathrm{II}})_{\mathrm{sing}}$ is a node, then $\Lambda_L \cap Y$ is of type X_8 or worse. In particular, there are 3780 projective spaces of dimension 3 intersecting Y in a singular surface of type X_8.*
(4) *For any $L \in S_{\mathrm{II}}$, the surface $\Lambda_L \cap Y$ is not of type X_9.*

Proof. Given a line L of type II, the cubic surface $\Lambda_L \cap Y$ is singular along the entire L. By the study of Gounelas and Kouvidakis, a line as in item (2) corresponds to a triple line of type II, and one as in item (3) corresponds to a type II line with two triple tangent 2-planes; see [GK24, Comments right above Remark 4.20]. The distinction in cases comes from an elementary analysis of lines on non-normal cubic surfaces (see the appendix). The second statement in item (3) follows from [GK24, Theorem B].

For the last item, let L be a line of type II such that $\Lambda_L \cap Y$ is a singular cubic surface of type X_9, that is, a cone over a cuspidal cubic curve. Then the lines passing through the vertex of the cone form a curve, which contains a cuspidal cubic. This is in contradiction with [GK24, Proposition 3.5]. □

Proposition 3.13. *Let Y be a general cubic fourfold, and let L_1 and L_2 be general distinct lines such that L_2 is residual to L_1 and L_2 is of type I and triple. Then $\langle L_1, \Lambda_{L_2}\rangle$ is a 3-dimensional space which cuts Y in a cubic surface of type either A_1A_5 or $\widetilde{E}_6$.*

Proof. By Lemma 3.2 a general point of P consists of a pair of lines whose linear span intersects Y in a Cayley cubic. Thus, if L_1 and L_2 are as in the hypothesis, then the 3-dimensional linear space $\langle L_1, \Lambda_{L_2}\rangle$ intersects Y in a cubic surface S which is a degeneration of a Cayley cubic and contains L_2 as a triple line on the surface. By the analysis of lines on cubic surfaces, S is either of type A_1A_5, of type $\widetilde{E}_6$, or non-normal. As both V and S_{II} are surfaces, both the families of non-normal cubic surfaces and of surfaces containing a triple line, which is triple on the surface, have dimension 2. By Corollary 3.12 and the appendix, the general non-normal cubic surface (which is a linear section of Y) does not contain a triple line; thus the general surface containing a triple line is of type $\widetilde{E}_6$ or A_1A_5. □

Corollary 3.14. *Let Y be a general cubic fourfold. Then there exists a $\Lambda \in G$ such that $\Lambda \cap Y$ is a cubic surface of type A_1A_5.*

Proof. Let L_1, L_2 be general distinct lines such that L_2 is a triple line of type I and L_2 is residual to L_1. We consider the projective space $\Lambda := \langle \Lambda_{L_2}, L_1\rangle$ and the singular cubic surface $S := \Lambda \cap Y$ it defines. In

particular, the line L_2 is triple on S and is residual to L_1. By Proposition 3.13 the surface S is either singular of type A_1A_5 or isomorphic to a cone over an elliptic curve.

In other words, the association $(L,R) \mapsto \Lambda$ for pairs of lines as above defines a rational map $\widehat{\psi}^{-1}(V) \dashrightarrow G$ whose image lies in the union of loci $G_{A_1A_5} \cup G_{\widetilde{E}_6}$. This is generically finite by Propositions 3.11 and 3.9. Since $G_{\widetilde{E}_6}$ is 1-dimensional by the discussion in [NO21, Section 1.4], the locus $G_{A_1A_5}$ is not empty. □

4. The monodromy group of ψ is maximal

In this section we prove that the monodromy group of $\widehat{\psi}$ is maximal. In order to do so, we use that the ramification of $\widehat{\psi}$ is simple, as proven in Section 2 and the results of Section 3 about the variety P.

Given a generically finite dominant morphism $f\colon X \to Y$ of degree d between irreducible varieties (necessarily of the same dimension), we obtain a degree d extension of function fields $k(X)/k(Y)$, and taking the Galois closure $K/k(Y)$ of this extension, we set $\mathrm{Gal}_f = \mathrm{Gal}(K/k(Y))$. This agrees with the usual monodromy group Mon_f (see [Har79, SY25]), which is defined as the image in the symmetric group S_d of the group of deck transformations of the unramified (*i.e.* topological) cover $X \setminus f^{-1}(\mathrm{Branch}(f)) \to U$, where $U := Y \setminus \mathrm{Branch}(f)$. Recall the following classical results for $f\colon X \to Y$ and U as above.

Proposition 4.1 (*cf.* [Har79, Section II.3, p. 698]). *If there exists a fibre of f with a unique point of ramification index 2 and all other points unramified, then $\mathrm{Mon}_f \subset S_d$ contains a transposition.*

Let $X_U^{(s)}$ be the complement of the big diagonal in the fibre product of X_U s-times with itself over U. In other words, the fibre of the natural morphism $X_U^{(s)} \to U$ consists of s distinct points in the fibre of f.

Lemma 4.2 (*cf.* [SY25, Proposition 2]). *The variety $X_U^{(s)}$ is irreducible if and only if Mon_f is an s-transitive subgroup of S_d.*

We now want to apply these results to the resolution of the Voisin map $\widehat{\psi}\colon \widehat{F} \to F$. For the remainder of the section, let Y be a general cubic fourfold, F be its Fano variety of lines, and $U \subset F$ be the complement of the branch locus of $\widehat{\psi}$. We recall that $P \subset F \times F$, introduced in Section 3.2, is mapped onto W under the map $\varphi\colon F \times F \dashrightarrow Z$. Thanks to Proposition 2.7, we can readily apply Proposition 4.1 and deduce that the monodromy group $\mathrm{Mon}_{\widehat{\psi}}$ contains a transposition. In order to apply Lemma 4.2, we are led to study the irreducibility of $\widehat{F}_U^{(2)}$, which is strictly related to the variety P. The rest of the section is then devoted to proving the following, which is the crucial ingredient to conclude the maximality of monodromy.

Proposition 4.3. *The variety P is irreducible.*

We postpone the proof and illustrate right away how to get the maximality of monodromy.

Theorem 4.4. *The monodromy of $\widehat{\psi}$ is maximal; i.e.* $\mathrm{Mon}_{\widehat{\psi}} = S_{16}$.

Proof. From Propositions 2.7 and 4.1, we deduce that $\mathrm{Mon}_{\widehat{\psi}}$ contains a transposition, and by Proposition 4.3 and Lemma 4.2, the action of $\mathrm{Mon}_{\widehat{\psi}}$ is 2-transitive. Thus $\mathrm{Mon}_{\widehat{\psi}}$ contains all transpositions: indeed, up to renumbering, we can assume the transposition is $(1,2) \in \mathrm{Mon}_{\widehat{\psi}}$. By 2-transitivity, there exists a $\sigma \in \mathrm{Mon}_{\widehat{\psi}}$ such that $\sigma(1) = a$ and $\sigma(2) = b$ for any a and b and we have

$$\sigma \circ (1,2) \circ \sigma^{-1} = (a,b).$$

Hence $\mathrm{Mon}_{\widehat{\psi}}$ contains any transposition and coincides with the entire symmetric group on 16 elements. □

In order to prove the irreducibility of P, we take advantage of the map $\varphi\colon P \dashrightarrow W$ and use its various properties we have showed in Section 3 to compute the ramification of φ at some special points. Clearly, there is an open subset $U_{\mathrm{et}} \subset W$ for which the base change $P_{U_{\mathrm{et}}} \to U_{\mathrm{et}}$ is étale of degree 6 and proper,

but as we are interested in the ramification of φ, we need to study a slightly bigger open subset; thus we provide the following technical proposition. Recall that Z admits a generically finite map $a\colon Z \dashrightarrow G$ whose indeterminacy locus is Y.

Proposition 4.5. *There is an open $U_{\text{flat}} \subset W$ such that*

(1) *the pullback $\varphi\colon P_{U_{\text{flat}}} \to U_{\text{flat}}$ is a flat finite morphism,*
(2) *there exists a point $w \in U_{\text{flat}}$ such that the 3-dimensional linear space $a(w)$ intersects Y in a cubic surface of type A_1A_5.*

Proof. Let $P \xleftarrow{\pi} \widehat{P} \xrightarrow{\widehat{\varphi}} W$ be any resolution of indeterminacy of φ and $\widehat{P} \xrightarrow{\varphi_1} P_{\text{Stein}} \xrightarrow{\varphi_2} W$ be its Stein factorization. We denote by E_π and E_1 the exceptional loci of π and φ_1, respectively. Then the morphism

$$\widehat{P} \setminus \widehat{\varphi}^{-1}(\widehat{\varphi}(E_1 \cup E_\pi)) \longrightarrow W \setminus \widehat{\varphi}(E_1 \cup E_\pi) =: U$$

is proper, as it is the base change of the proper morphism $\widehat{\varphi}$. It has finite fibres, as we removed the exceptional divisor E_1. Thus it is a finite morphism. As we removed E_π, the source space is actually a subset of P. Hence, we have proven that the restriction of φ to $P_U \to U$ is a finite morphism.

Moreover, the further restriction of φ to $U \setminus \varphi(P_{\text{sing}})$ is a finite morphism of non-singular varieties, hence flat by [Liu02, Remark 4.3.11]. This open set satisfies point (1) of the claim.

We now construct explicitly such an open set for which we show point (2) of the claim. In order to do so, we consider the map φ and the auxiliary map φ_G:

$$P \overset{\varphi}{\dashrightarrow} W \xrightarrow{a} G, \qquad P \xrightarrow{\varphi_G} G.$$

Their graph closures are going to serve as an explicit resolution of indeterminacy:

$$\widehat{P} := \text{closure}\{((L_1, L_2), w) \in P \times W \mid \varphi(L_1, L_2) = w\},$$
$$\widehat{P_G} := \text{closure}\{((L_1, L_2), g) \in P \times G \mid \varphi_G(L_1, L_2) = g\}.$$

As $\varphi_G = a \circ \varphi$, by continuity we have a natural map $\widehat{P} \to \widehat{P_G}$ given by

$$\widehat{P} \longrightarrow \widehat{P_G}, \quad (L_1, L_2, w) \longmapsto (L_1, L_2, a(w)). \tag{4.1}$$

As before, we factor the resolution of indeterminacy $\widehat{\varphi}\colon \widehat{P} \to W$ in $\widehat{\varphi} = \varphi_2 \circ \varphi_1$ using Stein factorization, and we denote by E_1 the exceptional divisor of φ_1 and by E_π the exceptional divisor of $\pi\colon \widehat{P} \to P$.

By Corollary 3.14 there exists a point $w \in W$ such that $a(w) \in G$ is a point that intersects Y in a surface S with singularities A_1A_5. By Proposition 3.9 the surface S contains exactly two lines L_1 and L_2, which by generality are of type I, such that $\psi(L_1) = L_2 = \psi(L_2)$. By the very definition of $\widehat{P_G}$, its points consist of a pair of lines and a 3-dimensional linear subspace, which contains the two lines; hence

$$\widehat{\varphi}_G^{-1}(a(w)) = \{((L_1, L_2), a(w)),\ ((L_2, L_1), a(w))\}.$$

Now, as there is just one linear system of arithmetically Cohen–Macaulay generalised twisted cubics on a cubic surface of type A_1A_5 by Lemma 3.10, we get

$$\widehat{\varphi}^{-1}(w) = \{((L_1, L_2), w),\ ((L_2, L_1), w)\}.$$

As $\widehat{\varphi}^{-1}(w)$ is finite, we have $w \notin \widehat{\varphi}(E_1)$.

Notice that $\varphi_G|_P\colon P \dashrightarrow G$ is defined by the association $(L_1, L_2) \mapsto \langle L_1, L_2\rangle = \langle L_1, L_2, R\rangle$, where $R = \psi(L_1) = \psi(L_2)$. Moreover, if both L_1 and L_2 are of type I, then we have $\varphi_G((L_1, L_2)) = \langle \Lambda_{L_1}, \Lambda_{L_2}\rangle$, hence pairs of distinct lines of type I in P are not in the indeterminacy locus of φ. Since $Z_{aCM}(S)$ is a singleton for a cubic surface S of type A_1A_5 by Lemma 3.10, the map φ is defined at the point (L_1, L_2); thus we conclude that $w \notin \widehat{\varphi}(E_\pi)$.

Now we are left to prove that (L_1, L_2) is a regular point of P so that φ is flat at the points (L_1, L_2) and (L_2, L_1) in P. We prove the regularity of P at the point (L_1, L_2) in the lemma below. □

Lemma 4.6. *The singular locus of $\widehat{F} \times_{\widehat{\psi}} \widehat{F}$ is contained in $E \times_{\widehat{\psi}} E$.*

Proof. Let $(f_1, f_2) \in \widehat{F} \times_{\widehat{\psi}} \widehat{F}$ be a point with $\widehat{\psi}(f_1) = \widehat{\psi}(f_2) = R$. Then its tangent space is the fibre product over the diagram

$$T_{f_1}\widehat{F} \xrightarrow{d\widehat{\psi}} T_R F \xleftarrow{d\widehat{\psi}} T_{f_2}\widehat{F},$$

where $T_{f_1}\widehat{F}$ and $T_{f_2}\widehat{F}$ have dimension 4 as $\widehat{F}$ is a non-singular fourfold. If one among f_1 and f_2 is not in the ramification of $\widehat{\psi}$, which coincides with the exceptional divisor E, then $d\widehat{\psi}$ is an isomorphism at that point and the fibre product is a vector space of dimension 4, hence the claim. □

Proof of Proposition 4.3. First we are going to prove that P modulo the switching involution σ is irreducible. In order to do so, we start with a point $q \in U_{\text{flat}} \cap W_{A_1A_5}$, which exists by Proposition 4.5. As exactly two lines lie on a surface with singularities A_1A_5 (see Proposition 3.9), the preimage $\varphi^{-1}(q)$ consists of two points, say

$$(L_1, L_2) \quad \text{and} \quad (L_2, L_1).$$

Using the involution σ on $F \times F$ switching the factors, which clearly restrict to an involution of P, we see that P at (L_1, L_2) is locally isomorphic to P at $\sigma((L_1, L_2)) = (L_2, L_1)$. As φ is flat, proper, and of degree 6 at the point q, we conclude that the schematic preimage $\varphi^{-1}(q)$ is a finite scheme with multiplicity 3 at both (L_1, L_2) and (L_2, L_1). We now pick a point $p \in U_{\text{et}}$ close enough to q and focus on the preimages $\varphi^{-1}(p) = \{p_1, \dots, p_6\}$: if we let p tend to q, then three points out of the six in $\varphi^{-1}(p)$, say p_1, p_2, and p_3, tend to (L_1, L_2), and the other three points, *i.e.* p_4, p_5, and p_6, tend to (L_2, L_1). Moreover, σ maps the first triple to the second one; *i.e.* $\sigma(\{p_1, p_2, p_3\}) = \{p_4, p_5, p_6\}$. Thus p_1, p_2, and p_3 lie in the same connected component of P; in other words, the quotient of P by the switching involution is path-connected. By Lemma 4.6 we deduce that P/σ is irreducible.

Hence, it is now sufficient to show that for some point $p_1 \in P_{U_{\text{et}}}$, the points p_1 and $\sigma(p_1)$ lie in the same connected component of $P_{U_{\text{et}}}$. To do so, we consider one transposition τ in Mon_{ψ}, whose existence is granted by Propositions 4.1 and 2.7. If we consider it as an element of deck transformations, this means that there exist a point $R \in F \setminus \text{Branch}(\widehat{\psi})$ and a loop γ centered at R such that γ acts on the fibre of R exchanging two points, say $L_1 \mapsto L_2$ and $L_2 \mapsto L_1$, and as the identity on the remaining points of the fibre. We now consider the action under deck transformation of γ for the natural map $P \dashrightarrow F$ defined by ψ. A moment's thought tells us that under the action of γ, the point $p = (L_1, L_2) \in P$, with $\psi(L_1) = \psi(L_2) = R$, is mapped to (L_2, L_1). Thus P is irreducible, as we claimed. □

Appendix. Singular cubic surfaces

In this appendix we treat degenerations of Cayley cubics we introduced in Diagram (3.1), focusing on lines, generalised twisted cubics, and the restrictions of the Voisin maps φ and ψ to a cubic surface.

Singular cubic surfaces of type $4A_1$

Under a suitable change of coordinate, every cubic surface with four singular points of type A_1 can be expressed as zero locus of the polynomial

$$t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3.$$

Its singular locus consists of the coordinate points

$$p_0 = (1:0:0:0), \quad p_1 = (0:1:0:0), \quad p_2 = (0:0:1:0), \quad p_3 = (0:0:0:1).$$

Such a surface can be obtained as the anticanonical model of the blow-up of $\mathbb{P}^2$ in a special configuration of points. Indeed, let $\widetilde{S}$ be the blow-up of $\mathbb{P}^2$ in the six points given by the intersection of four general lines L_0, L_1, L_2, L_3. Then the anticanonical linear system of $\widetilde{S}$ maps to a cubic surfaces S and contracts the strict transforms of the L_i to four singular A_1 points. The exceptional divisors map to lines in a tetrahedron-like arrangement; we refer to them as edges and denote by E_{ij} the line connecting the i^{th} and the j^{th} coordinate points. In particular, the edge E_{ij} is cut by the equations $(t_k = t_l = 0)$, where $\{i,j,k,l\} = \{0,1,2,3\}$. Besides the edges, there are three more lines on S joining opposite edges; we denote by $J_{ij,kl}$ the line intersecting the edges E_{ij} and E_{kl}, where $\{i,j,k,l\} = \{0,1,2,3\}$. These are images of the strict transforms of lines through the blown-up points corresponding to the edges they intersect. For example, the line $J_{01,23}$ is the zero locus $(t_0 + t_1 = t_2 + t_3 = 0)$.

Generalised twisted cubics on singular cubic surfaces have been studied in [LLS+17, Section 2]. Any generalised twisted cubic on an integral normal cubic surface moves in a 2-dimensional linear system and can be described as the image of the pullback of a line in $\mathbb{P}^2$ along the diagram of a determinantal representation of S. The determinantal representation for the Cayley cubic described above defines $K \in Z'(S)$, which contains, as reducible elements of the family, the union of all the edges passing through a fixed vertex of the tetrahedron:

$$K = [E_{ij} \cup E_{ik} \cup E_{il}] \in Z'(S).$$

Proof of Lemma 3.5. The proof is elementary and can be checked directly by the definition of the Voisin map φ. For example to compute $\varphi(E_{02}, E_{13})$, consider the plane $\pi = (t_1 = 0)$ containing the line E_{02} and the point $p \in E_{13}$. Its intersection with S is given by $\pi \cap S = E_{02} \cup E_{03} \cup E_{23}$; hence $\varphi(E_{02}, E_{13}) = [E_{03} \cup E_{13} \cup E_{23}]$ and the claim is shown.

The other cases are completely similar. □

Singular cubic surfaces of type $2A_1A_3$

Let S be a cubic surface with a singular point P_2 of type A_3 and two singular points P_0, P_3 of type A_1. It contains exactly five lines as depicted in Figure 1b: the line R_{23} through P_2, P_3; the line R_{02} through P_0, P_2; the line R_{03} through P_0, P_3; the line L_1 containing the only singular point P_2; and the line L_2 contained in the smooth locus.

Such a surface is given in an opportune coordinate system by the equation

$$t_0t_2t_3 - t_1^2t_3 - t_0t_1^2 = 0.$$

Its singular locus consists of the point $P_2 := (0:0:1:0)$ of type A_3 and of the points $P_0 := (1:0:0:0)$ and $P_3 := (0:0:0:1)$ of type A_1.
The lines are then described by the equations $R_{23} = (t_0 = t_1 = 0)$, $R_{02} = (t_1 = t_3 = 0)$, $R_{03} = (t_1 = t_2 = 0)$, $L_1 = (t_0 = t_3 = 0)$, and $L_2 = (t_2 = t_0 + t_3 = 0)$.

Proof of Proposition 3.7. The proof can be checked through straightforward computations. For example, intersecting the surface S with the plane $\pi = (t_0 = 0)$, one gets

$$\begin{aligned} \pi \cap S &= (t_0 = 0) \cap \left(t_0t_2t_3 - t_1^2t_3 - t_0t_1^2 = 0\right) \\ &= (t_0 = 0) \cap \left(t_1^2t_3 = 0\right) = 2R_{23} + L_1, \end{aligned}$$

which shows that L_1 is residual to R_{23}. □

Singular cubic surfaces of type A_1A_5

A singular cubic surface S with exactly one A_1 and one A_5 singularity is given in an opportune of coordinate system by the equation

$$t_3\left(t_0t_2 - t_1^2\right) + t_0^3 = 0,$$

whose singular locus consists of the two points $P_1 := (0:0:0:1)$ of type A_5 and $P_2 := (0:0:1:0)$ of type A_1. There are exactly two lines on S; they have equations $L_1 = (t_0 = t_1 = 0)$ and $L_2 = (t_0 = t_3 = 0)$.

Proof of Proposition 3.9. This follows from the straightforward computation

$$(t_3 = 0) \cap S = 3L_2, \quad (t_0 = 0) \cap S = L_1 + 2L_2. \qquad \square$$

We now study non-normal cubic surfaces, whose analysis is necessary to prove Corollary 3.12.

The cubic surface X_6

In a suitable system of coordinates, the surface X_6 is given by the equation

$$t_0^2t_2 + t_1^2t_3 = 0.$$

It is a non-normal surface whose singular locus consists of the line $L = (t_0 = t_1 = 0)$.

Proposition A.1. *The line L is not triple on X_6.*

Proof. The proof is an elementary computation: each plane containing L is of the form $\pi_{(a:b)} = (at_0 + bt_1 = 0)$ with $(a:b) \in \mathbb{P}^1$. By considering the intersection with such a plane $\pi_{(-1:b)}$, one has

$$X_6 \cap (-t_0 + bt_1 = 0) \;=\; \left(-t_0 + bt_1 = t_1^2\left(b^2t_2 + t_3\right) = 0\right) \;=\; 2L + R_b$$

with $R_b = \left(b^2t_2 + t_3 = bt_1 + t_0 = 0\right)$ different from L, and

$$X_6 \cap (t_1 = 0) \;=\; \left(t_1 = t_0^2t_2 = 0\right) \;=\; 2L + R$$

with $R = (t_1 = t_2 = 0)$. This shows that L is not triple. $\square$

The cubic surface X_7

In a suitable system of coordinates, the surface X_7 is given by the equation

$$t_0t_1t_2 + t_0^2t_3 + t_1^3 = 0.$$

It is a non-normal surface whose singular locus consists of the line $L = (t_0 = t_1 = 0)$.

Proposition A.2. *There is a unique plane π making L triple, i.e. such that $\pi \cap X_7 = 3L$.*

Proof. The proof is an elementary computation. Each plane containing L is of the form $\pi_{(a:b)} = (at_0 + bt_1 = 0)$ with $(a:b) \in \mathbb{P}^1$. By considering the intersection with such a plane $\pi_{(a:-1)}$, one has

$$X_7 \cap (at_0 - t_1 = 0) \;=\; \left(at_0 - t_1 = t_0^2\left(a^3t_0 + at_2 + t_3\right) = 0\right) \;=\; 2L + R_a$$

with $R_a = (a^3t_0 + at_2 + t_3 = t_1 - at_0 = 0)$. Also, one has

$$X_7 \cap (t_0 = 0) \;=\; \left(t_1^3 = t_0 = 0\right) \;= 3L.$$

Hence $\pi = (t_0 = 0)$ is the only plane as in the claim. $\square$

The cubic surface X_8

In a suitable system of coordinates, the surface X_8 is given by the equation

$$t_1^3 + t_2^3 + t_1 t_2 t_3 = 0.$$

It is a non-normal surface isomorphic to the cone over a nodal planar cubic whose singular locus consists of the line $L = (t_1 = t_2 = 0)$.

Proposition A.3. *There are two planes π and π' making L triple, i.e. such that $\pi \cap X_8 = \pi' \cap X_8 = 3L$.*

Proof. The proof is an elementary computation: each plane containing L is of the form $\pi_{(a:b)} = (at_1 + bt_2 = 0)$ with $(a : b) \in \mathbb{P}^1$. By considering the intersection with such a plane $\pi_{(1:-b)}$, one has

$$X_8 \cap (t_1 - bt_2 = 0) \;=\; \left(t_1 - bt_2 = t_2^2\left(\left(b^3+1\right)t_2 + bt_3\right) = 0\right) \;=\; 2L + R_b$$

with $R_b = (t_1 - bt_2 = (b^3+1)t_2 + bt_3 = 0)$. Also, one has

$$X_8 \cap (t_2 = 0) \;=\; \left(t_1^3 = t_2 = 0\right) \;=\; 3L.$$

The planes $\pi = (t_1 = 0)$ and $\pi' = (t_2 = 0)$ are hence the only planes as in the claim. □

References

[AG23] N. Addington and F. Giovenzana, *On the period of Lehn, Lehn, Sorger, and van Straten's symplectic eightfold*, Kyoto J. Math. **63** (2023), no. 1, 71–86, doi:10.1215/21562261-2022-0033.

[AL17] N. Addington and M. Lehn, *On the symplectic eightfold associated to a Pfaffian cubic fourfold*, J. reine angew. Math. **731** (2017), 129–137, doi:10.1515/crelle-2014-0145.

[Ame09] E. Amerik, *A computation of invariants of a rational self-map*, Ann. Fac. Sci. Toulouse Math. (6) **18** (2009), no. 3, 445-457, doi:10.5802/afst.1211.

[BHP+04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Ergeb. Math. Grenzgeb. (3), vol. 4, Springer-Verlag, Berlin, 2004, doi:10.1007/978-3-642-57739-0.

[Bea80] A. Beauville, *Sur le nombre maximum de points doubles d'une surface dans $\mathbb{P}^3$ ($\mu(5) = 31$)*, in: *Journées de Géométrie Algébrique d'Angers, Juillet 1979*, pp. 207–215, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[BD85] A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.

[BFK+24] M. Bernardara, E. Fatighenti, G. Kapustka, M. Kapustka, L. Manivel, G. Mongardi, and F. Tanturri, *Even nodal surfaces of K3 type*, preprint arXiv:2402.08528 (2024).

[Bru80] J. W. Bruce, *A stratification of the space of cubic surfaces*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 427–441, doi:10.1017/s0305004100056863.

[Cat81] F. Catanese, *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Invent. Math. **63** (1981), no. 3, 433–465, doi:10.1007/bf01389064.

[CCC+24] F. Catanese, in collaboration with Y. Cho, S. Coughlan, D. Frapporti, A. Verra, M. Kiermaier, and S. Kurz, *Varieties of nodal surfaces, coding theory and discriminants of cubic hypersurfaces. Part 1: Generalities and nodal K3 surfaces. Part 2: Cubic hypersurfaces, associated discriminants. Part 3: Nodal quintics. Part 4: Nodal sextics*, preprint arXiv:2206.05492v3 (2024).

[Che21] H. Chen, *The Voisin map via families of extensions*, Math. Z. **299** (2021), no. 3-4, 1987–2003, doi:10.1007/s00209-021-02747-1.

[CG72] C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356, doi:10.2307/1970801.

[Dol12] I. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge Univ. Press, Cambridge, 2012, doi:10.1017/CBO9781139084437.

[FMOS22] L. Flapan, E. Macrì, K. G. O'Grady, and G. Saccà, *The geometry of antisymplectic involutions, I*, Math. Z. **300** (2022), no. 4, 3457–3495, doi:10.1007/s00209-021-02909-1.

[FMOS26] ———, *The geometry of antisymplectic involutions, II*, J. Éc. polytech. Math. **13** (2026), 809–851, doi:10.5802/jep.337.

[Gio26] F. Giovenzana, *On the uniruled Voisin divisor on the LLSvS variety*, Beitr. Algebra Geom., published online on May 6, 2026, to appear in print, doi:10.1007/s13366-026-00851-z.

[GK23] F. Gounelas and A. Kouvidakis, *On some invariants of cubic fourfolds*, Eur. J. Math. **9** (2023), no. 3, Paper No. 58, doi:10.1007/s40879-023-00651-y.

[GK24] ———, F. Gounelas and A. Kouvidakis, *Geometry of Lines on a Cubic Four-Fold*, Int. Math. Res. Not. IMRN **2024**, no. 2, 1373–1421, doi:10.1093/imrn/rnac160.

[GK25] ———, *The Fermat cubic and monodromy of lines*, New York J. Math. **31** (2025), 650–667.

[GKP16] D. Greb, S. Kebekus, and T. Peternell, *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties*, Duke Math. J. **165** (2016), no. 10, 1965–2004, doi:10.1215/00127094-3450859.

[Har79] J. Harris, *Galois groups of enumerative problems*, Duke Math. J. **46** (1979), no. 4, 685–724, doi:10.1215/S0012-7094-79-04635-0.

[Huy23] D. Huybrechts, *The Geometry of Cubic Hypersurfaces*, Cambridge Stud. Adv. Math., vol. 206, Cambridge Univ. Press, Cambridge, 2023, doi:10.1017/9781009280020.

[Huy24] ———, *Nodal quintic surfaces and lines on cubic fourfolds* (with an appendix by J. C. Ottem), Enseign. Math., **70** (2024), no. 3-4, 499–532, doi:10.4171/lem/1063.

[Leh18] C. Lehn, *Twisted cubics on singular cubic fourfolds—on Starr's fibration*, Math. Z. **290** (2018), no. 1-2, 379–388, doi:10.1007/s00209-017-2021-x.

[LLS+17] C. Lehn, M. Lehn, C. Sorger, and D. van Straten, *Twisted cubics on cubic fourfolds*, J. reine angew. math. **731** (2017), 87–128, doi:10.1515/crelle-2014-0144.

[Leh15] M. Lehn, *Twisted cubics on a cubic fourfold and in involution on the associated 8-dimensional symplectic manifold*, in: *Mini-workshop: Singular curves on $K3$ surfaces and hyperkähler manifolds*, pp. 2962–2964, Oberwolfach Rep. **12** (2015), no. 4. doi:doi.org/10.4171/OWR/2015/51.

[LPZ23] C. Li, L. Pertusi, and X. Zhao, *Twisted cubics on cubic fourfolds and stability conditions*, Algebr. Geom. **10** (2023), no. 5, 620–642, doi:10.14231/AG-2023-022.

[Liu02] Q. Liu, *Algebraic geometry and arithmetic curves* (translated from the French by R. Erné), Oxf. Grad. Texts Math., vol 6, Oxford Univ. Press, Oxford, 2002.

[Mur20] G. E. Muratore, *The indeterminacy locus of the Voisin map*, Beitr. Algebra Geom. **61** (2020), no. 1, 73–88, doi:10.1007/s13366-019-00454-x.

[NO21] D. Nesterov and G. Oberdieck, *Elliptic curves in hyper-Kähler varieties*, Int. Math. Res. Not. IMRN **2021**, no. 4, 2962–2990, doi:10.1093/imrn/rnaa016.

[SY25] F. Sottile and T. Yahl, *Galois groups in enumerative geometry and applications*, Expo. Math. **43** (2025), no. 6, Paper No. 125731, doi:10.1016/j.exmath.2025.125731.

[Vak06] R. Vakil, *Schubert induction*, Ann. of Math. (2) **164** (2006), no. 2, 489–512, doi:10.4007/annals.2006.164.489.

[Voi86] C. Voisin, *Théorème de Torelli pour les cubiques de* $\mathbb{P}^5$, Invent. Math. **86** (1986), no. 3, 577–602, doi:10.1007/BF01389270.

[Voi04] ———, *Intrinsic pseudo-volume forms and K-correspondences*, in: *The Fano Conference*, pp. 761–792. Univ. Torino, Turin, 2004.

[Voi16] ———, *Remarks And Questions On Coisotropic Subvarieties and 0-Cycles of Hyper-Kähler Varieties*, in: *K3 Surfaces and their Moduli*, pp. 365–399, Progr. Math., vol. 315, Birkhäuser/Springer, Cham, 2016, doi:10.1007/978-3-319-29959-4_14.