

On a theorem of Campana and Păun

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Abstract. Let X be a smooth projective variety over the complex numbers, and $\Delta \subseteq X$ a reduced divisor with normal crossings. We present a slightly simplified proof for the following theorem of Campana and Păun: If some tensor power of the bundle $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, then (X, Δ) is of log general type. This result is a key step in the recent proof of Viehweg’s hyperbolicity conjecture.

Keywords. Viehweg’s hyperbolicity conjecture; log general type; log cotangent bundle; foliation; movable curve class; slope semi-stability

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[Français]

Titre. Sur un théorème de Campana et Păun

Résumé. Soit X une variété projective complexe lisse et $\Delta \subseteq X$ un diviseur réduit à croisements normaux. Nous présentons une démonstration légèrement simplifiée du théorème suivant de Campana et Păun : si une puissance tensorielle du fibré $\Omega_X^1(\log(\Delta))$ contient un faisceau dont le déterminant est *big*, la paire (X, Δ) est alors de log-type général. Ce résultat est une étape clé dans la récente démonstration de la conjecture d’hyperbolicité de Viehweg.

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1. Introduction

The purpose of this paper is to present a slightly simplified proof for the following result by Campana and Păun [CP15, Theorem 7.6]. It is a crucial step in the proof of Viehweg’s hyperbolicity conjecture for families of canonically polarized manifolds [CP15, Theorem 7.13], and more generally, for smooth families of varieties of general type [PS17, Theorem A].

Theorem 1. *Let X be a smooth projective variety, and $\Delta \subseteq X$ a reduced divisor with at worst normal crossing singularities. If some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, then $K_X + \Delta$ is big.*

The simplification is that I have substituted an inductive procedure for the arguments involving Campana’s “orbifold cotangent bundle”; otherwise, the proof of Theorem 1 that I present here is essentially the same as in the one in [CP15]. My reason for writing this paper is that it gives me a chance to draw attention to some of the beautiful ideas involved in the proof by Campana and Păun: slope stability with respect to movable classes; a criterion for the leaves of a foliation to be algebraic subvarieties; and positivity results for relative canonical bundles.

Remark 2. The most recent arXiv version of the paper by Campana and Păun (from June 14, 2017) also contains a brief summary of our proof; see [CP15, Section 8.1].

2. Strategy of the proof

Let (X, Δ) be a pair, consisting of a smooth projective variety X and a reduced divisor $\Delta \subseteq X$ with at worst normal crossing singularities. We denote the logarithmic cotangent bundle by the symbol $\Omega_X^1(\log \Delta)$, and its dual, the logarithmic tangent bundle, by the symbol $\mathcal{T}_X(-\log \Delta)$. Recall that $\mathcal{T}_X(-\log \Delta)$ is naturally a subsheaf of the tangent bundle \mathcal{T}_X , and that it is closed under the Lie bracket on \mathcal{T}_X . Indeed, suppose that Δ is given, in suitable local coordinates x_1, x_2, \dots, x_n , by the equation $x_1 x_2 \cdots x_k = 0$; then $\mathcal{T}_X(-\log \Delta)$ is generated by the n commuting vector fields

$$x_1 \frac{\partial}{\partial x_1}, \dots, x_k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n},$$

and is therefore closed under the Lie bracket.

Suppose that $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a subsheaf with big determinant, for some $N \geq 1$. The following observation reduces the problem to the case of line bundles.

Lemma 3. *If $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a subsheaf of generic rank $r \geq 1$ and with big determinant, then $\Omega_X^1(\log \Delta)^{\otimes Nr}$ contains a big line bundle.*

Proof. Let $\mathcal{B} \subseteq \Omega_X^1(\log \Delta)^{\otimes N}$ be a subsheaf of generic rank $r \geq 1$, with the property that $\det \mathcal{B}$ is big. After replacing \mathcal{B} by its saturation, whose determinant is of course still big, we may assume that the quotient sheaf

$$\Omega_X^1(\log \Delta)^{\otimes N} / \mathcal{B}$$

is torsion-free, hence locally free outside a closed subvariety $Z \subseteq X$ of codimension ≥ 2 . On $X \setminus Z$, we have an inclusion of locally free sheaves

$$\det \mathcal{B} \hookrightarrow \mathcal{B}^{\otimes r} \hookrightarrow \Omega_X^1(\log \Delta)^{\otimes Nr},$$

which remains valid on X by Hartog's theorem. \square

For the purpose of proving Theorem 1, we are therefore allowed to assume that $\Omega_X^1(\log \Delta)^{\otimes N}$ contains a big line bundle L as a subsheaf. Let \mathcal{Q} denote the quotient sheaf, and consider the resulting short exact sequence

$$0 \rightarrow L \rightarrow \Omega_X^1(\log \Delta)^{\otimes N} \rightarrow \mathcal{Q} \rightarrow 0. \quad (2.1)$$

Since $K_X + \Delta$ represents the first Chern class of $\Omega_X^1(\log \Delta)$, we obtain

$$N \cdot (\dim X)^{N-1} \cdot (K_X + \Delta) = c_1(L) + c_1(\mathcal{Q})$$

in $N^1(X)_{\mathbb{R}}$, the \mathbb{R} -linear span of codimension-one cycles modulo numerical equivalence. By assumption, the class $c_1(L)$ is big; Theorem 1 will therefore be proved if we manage to show that the class $c_1(\mathcal{Q})$ is pseudo-effective. In fact, we are going to prove the following more general result, which is of course just a special case of [CP15, Theorem 7.6 and Theorem 1.2].

Theorem 4. *Let X be a smooth projective variety, and $\Delta \subseteq X$ a reduced divisor with at worst normal crossing singularities. Suppose that some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant. Then the first Chern class of every quotient sheaf of every tensor power of $\Omega_X^1(\log \Delta)$ is pseudo-effective.*

3. Slopes and foliations

To simplify the presentation, we will prove Theorem 4 by contradiction. Suppose then that, for some integer $N \geq 1$, and for some quotient sheaf \mathcal{Q} of $\Omega_X^1(\log \Delta)^{\otimes N}$, the class $c_1(\mathcal{Q})$ was *not* pseudo-effective. Let $\mathcal{Q}_{tor} \subseteq \mathcal{Q}$ denote the torsion subsheaf. Since

$$c_1(\mathcal{Q}) = c_1(\mathcal{Q}_{tor}) + c_1(\mathcal{Q}/\mathcal{Q}_{tor}),$$

and since $c_1(\mathcal{Q}_{tor})$ is effective, we may replace \mathcal{Q} by $\mathcal{Q}/\mathcal{Q}_{tor}$, and assume without any loss of generality that \mathcal{Q} is torsion-free (and nonzero).

By the characterization of the pseudo-effective cone in [BDPP13, Theorem 2.2], there is a movable class $\alpha \in N_1(X)_{\mathbb{R}}$ such that $c_1(\mathcal{Q}) \cdot \alpha < 0$. As shown in [CP11, GKP16], there is a good theory of α -semistability for torsion-free sheaves, with almost all the properties that are familiar from the case of complete intersection curves. We use this theory freely in what follows. By assumption,

$$\mu_{\alpha}(\mathcal{Q}) = \frac{c_1(\mathcal{Q}) \cdot \alpha}{\operatorname{rk} \mathcal{Q}} < 0,$$

and so \mathcal{Q} is a torsion-free quotient sheaf of $\Omega_X^1(\log \Delta)^{\otimes N}$ with negative α -slope. The dual sheaf \mathcal{Q}^* is therefore a saturated subsheaf of $\mathcal{T}_X(-\log \Delta)^{\otimes N}$ with positive α -slope. At this point, we recall the following result about tensor products.

Theorem 5. *Let $\alpha \in N_1(X)_{\mathbb{R}}$ be a movable class. If \mathcal{F} and \mathcal{G} are torsion-free and α -semistable coherent sheaves on X , then their tensor product*

$$\mathcal{F} \hat{\otimes} \mathcal{G} = (\mathcal{F} \otimes \mathcal{G}) / (\mathcal{F} \otimes \mathcal{G})_{\text{tor}},$$

modulo torsion, is again α -semistable, and $\mu_{\alpha}(\mathcal{F} \hat{\otimes} \mathcal{G}) = \mu_{\alpha}(\mathcal{F}) + \mu_{\alpha}(\mathcal{G})$.

Proof. For the reflexive hull of the tensor product, this is proved in [GKP16, Theorem 4.2 and Proposition 4.4], based on analytic results by Toma [CP11, Appendix]. Since $\mathcal{F} \hat{\otimes} \mathcal{G}$ and its reflexive hull are isomorphic outside a closed subvariety of codimension ≥ 2 , the assertion follows. (The formula for the α -slope of $\mathcal{F} \hat{\otimes} \mathcal{G}$ is of course valid for arbitrary nonzero torsion-free coherent sheaves \mathcal{F} and \mathcal{G} .) \square

Similarly, the fact that $\mathcal{I}_X(-\log \Delta)^{\otimes N}$ has a subsheaf with positive α -slope implies, again by [GKP16, Theorem 4.2 and Proposition 4.4], that $\mathcal{I}_X(-\log \Delta)$ must also contain a subsheaf with positive α -slope. Let $\mathcal{F}_{\Delta} \subseteq \mathcal{I}_X(-\log \Delta)$ be the maximal α -destabilizing subsheaf [GKP16, Corollary 2.24].

Lemma 6. *\mathcal{F}_{Δ} is a saturated, α -semistable subsheaf of $\mathcal{I}_X(-\log \Delta)$, of positive α -slope. Every subsheaf of $\mathcal{I}_X(-\log \Delta) / \mathcal{F}_{\Delta}$ has α -slope less than $\mu_{\alpha}(\mathcal{F}_{\Delta})$.*

Proof. This is clear from the construction of the maximal destabilizing subsheaf in [GKP16, Corollary 2.4]. Note that \mathcal{F}_{Δ} is the first step in the Harder-Narasimhan filtration of $\mathcal{I}_X(-\log \Delta)$, see [GKP16, Corollary 2.26]. \square

Recall that we have an inclusion $\mathcal{I}_X(-\log \Delta) \subseteq \mathcal{I}_X$. We define another coherent subsheaf $\mathcal{F} \subseteq \mathcal{I}_X$ as the saturation of \mathcal{F}_{Δ} in \mathcal{I}_X ; then $\mathcal{I}_X / \mathcal{F}$ is torsion-free, and

$$\mathcal{F} \cap \mathcal{I}_X(-\log \Delta) = \mathcal{F}_{\Delta}. \quad (3.2)$$

We will see in a moment that \mathcal{F} is actually a (typically, singular) foliation on X . Recall that, in general, a *foliation* on a smooth projective variety is a saturated subsheaf $\mathcal{F} \subseteq \mathcal{I}_X$ that is closed under the Lie bracket on \mathcal{I}_X . From the Lie bracket, one constructs an \mathcal{O}_X -linear mapping

$$N: \mathcal{F} \hat{\otimes} \mathcal{F} \rightarrow \mathcal{I}_X / \mathcal{F},$$

called the *O'Neil tensor* of \mathcal{F} ; evidently, \mathcal{F} is a foliation if and only if its O'Neil tensor vanishes.

Lemma 7. *The O'Neil tensor*

$$N: \mathcal{F} \hat{\otimes} \mathcal{F} \rightarrow \mathcal{I}_X / \mathcal{F}$$

vanishes, and \mathcal{F} is therefore a foliation on X .

Proof. The Lie bracket of two sections of $\mathcal{I}_X(-\log \Delta)$ is a section of $\mathcal{I}_X(-\log \Delta)$, and so we get a logarithmic O'Neil tensor

$$N_{\Delta}: \mathcal{F}_{\Delta} \hat{\otimes} \mathcal{F}_{\Delta} \rightarrow \mathcal{I}_X(-\log \Delta) / \mathcal{F}_{\Delta}.$$

The key point is that $N_{\Delta} = 0$. Indeed, by Theorem 5, the tensor product $\mathcal{F}_{\Delta} \hat{\otimes} \mathcal{F}_{\Delta}$, modulo torsion, is again α -semistable of slope

$$\mu_{\alpha}(\mathcal{F}_{\Delta} \hat{\otimes} \mathcal{F}_{\Delta}) = 2 \cdot \mu_{\alpha}(\mathcal{F}_{\Delta}) > \mu_{\alpha}(\mathcal{F}_{\Delta}),$$

which is strictly greater than the slope of any nonzero subsheaf of $\mathcal{I}_X(-\log \Delta) / \mathcal{F}_{\Delta}$ by Lemma 6. This inequality among slopes implies that $N_{\Delta} = 0$, see for instance [GKP16, Proposition 2.16 and Corollary 2.17].

The O'Neil tensor N and the logarithmic O'Neil tensor N_Δ are both induced by the Lie bracket on \mathcal{T}_X , and so we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}_\Delta \hat{\otimes} \mathcal{F}_\Delta & \xrightarrow{N_\Delta} & \mathcal{T}_X(-\log \Delta)/\mathcal{F}_\Delta \\ \downarrow & & \downarrow \\ \mathcal{F} \hat{\otimes} \mathcal{F} & \xrightarrow{N} & \mathcal{T}_X/\mathcal{F} \end{array}$$

The vertical arrow on the right is injective by (3.2). Now $N_\Delta = 0$ implies that N factors through the cokernel of the vertical arrow on the left; but the cokernel is a torsion sheaf, whereas $\mathcal{T}_X/\mathcal{F}$ is torsion-free. The conclusion is that $N = 0$. \square

The next step in the proof is to show that the foliation \mathcal{F} is actually algebraic. This is a simple consequence of the powerful algebraicity theorem of Campana and Păun [CP15, Theorem 1.1], which generalizes a well-known result by Bogomolov and McQuillan [BM16] and Bost [Bos01, §3.3] from complete intersection curves to movable classes. (See also the paper [KST07] by Kebekus, Solà Conde, and Toma.)

Theorem 8. *Let X be a smooth projective variety over the complex numbers, and let $\mathcal{F} \subseteq \mathcal{T}_X$ be a foliation. Suppose that there exists a movable class $\alpha \in N_1(X)_\mathbb{R}$, such that every nonzero quotient sheaf of \mathcal{F} has positive α -slope. Then \mathcal{F} is an algebraic foliation, and its leaves are rationally connected.*

To apply this in our setting, we observe that every quotient sheaf of \mathcal{F} is, at least over the open subset $X \setminus \Delta$, also a quotient sheaf of \mathcal{F}_Δ , because \mathcal{F} and \mathcal{F}_Δ agree outside the divisor Δ . As \mathcal{F}_Δ is α -semistable with $\mu_\alpha(\mathcal{F}) > 0$, it follows easily that every quotient sheaf of \mathcal{F} has positive α -slope. We can now invoke Theorem 8 and conclude that the foliation \mathcal{F} is algebraic. In other words [CP15, §4], there exists a dominant rational mapping

$$p: X \dashrightarrow Z$$

to a smooth projective variety Z , such that

$$\mathcal{F} = \ker(dp: \mathcal{T}_X \rightarrow p^* \mathcal{T}_Z)$$

outside a subset of codimension ≥ 2 . More precisely, let us follow [CKT16, Construction 2.29] and denote by the symbol $\mathcal{T}_{X/Z}$ the unique reflexive sheaf on X that agrees with $\ker(dp: \mathcal{T}_X \rightarrow p^* \mathcal{T}_Z)$ on the big open subset where p is a morphism. Using this notation, the algebraicity of \mathcal{F} may be expressed as

$$\mathcal{F} = \mathcal{T}_{X/Z}; \tag{3.3}$$

indeed, \mathcal{F} is reflexive, due to the fact that $\mathcal{T}_X/\mathcal{F}$ is torsion-free.

Remark 9. Theorem 8 also says that the fibers of p are rationally connected, but we are not going to make any use of this extra information. This means that the proof of Theorem 4 only uses characteristic zero methods.

4. Pseudo-effectivity

Let us first convince ourselves that Z cannot be a point. This will later allow us to argue by induction on the dimension, because the general fiber of p has dimension less than $\dim X$.

Lemma 10. *With notation as above, we must have $\dim Z \geq 1$.*

Proof. If $\dim Z = 0$, then $\mathcal{F} = \mathcal{T}_X$ and $\mathcal{F}_\Delta = \mathcal{T}_X(-\log \Delta)$, and consequently, the logarithmic tangent bundle $\mathcal{T}_X(-\log \Delta)$ is α -semistable of positive slope. Since the tensor product of α -semistable sheaves remains α -semistable [GKP16, Proposition 4.4], this means that any tensor power of $\Omega_X^1(\log \Delta)$ is α -semistable of negative slope. But that contradicts the hypothesis of Theorem 4, namely that some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, because the α -slope of such a subsheaf is obviously positive. \square

The only properties of \mathcal{F}_Δ that we are still going to use in the proof of Theorem 4 are the identity in (3.2), and the fact that $c_1(\mathcal{F}_\Delta) \cdot \alpha > 0$ for a movable class $\alpha \in N_1(X)_\mathbb{R}$. In return, we are allowed to assume that $p: X \rightarrow Z$ is a morphism.

Lemma 11. *Without loss of generality, $p: X \rightarrow Z$ is a morphism.*

Proof. Choose a birational morphism $f: \tilde{X} \rightarrow X$, for example by resolving the singularities of the closure of the graph of $p: X \dashrightarrow Z$ inside $X \times Z$, with the following properties: the rational mapping $p \circ f$ extends to a morphism $\tilde{p}: \tilde{X} \rightarrow Z$; both $K_{\tilde{X}/X}$ and $\tilde{p}^*\Delta$ are normal crossing divisors; and f is an isomorphism over the open subset where p is already a morphism.

Let $\tilde{\Delta}$ be the reduced normal crossing divisor whose support is equal to the preimage of Δ in \tilde{X} . Then

$$\Omega_{\tilde{X}}^1(\log \tilde{\Delta}) \cong \tilde{p}^*\Omega_X^1(\log \Delta),$$

and since the pullback of a big line bundle by \tilde{p} stays big, it is still true that some tensor power of $\Omega_{\tilde{X}}^1(\log \tilde{\Delta})$ contains a big line bundle as a subsheaf. Now define

$$\tilde{\mathcal{F}} = \mathcal{T}_{\tilde{X}/Z} = \ker(\tilde{p}^*: \mathcal{T}_{\tilde{X}} \rightarrow \tilde{p}^*\mathcal{T}_Z),$$

which is a saturated subsheaf of $\mathcal{T}_{\tilde{X}}$. The intersection

$$\tilde{\mathcal{F}} \cap \mathcal{T}_{\tilde{X}}(-\log \tilde{\Delta})$$

is a saturated (and hence reflexive) subsheaf of $\mathcal{T}_{\tilde{X}}(-\log \tilde{\Delta})$, whose pushforward to X is isomorphic to \mathcal{F}_Δ , by (3.2) and the fact that \mathcal{F}_Δ is reflexive. Consequently,

$$c_1\left(\tilde{\mathcal{F}} \cap \mathcal{T}_{\tilde{X}}(-\log \tilde{\Delta})\right) \cdot \tilde{\alpha} = c_1(\mathcal{F}_\Delta) \cdot \alpha > 0,$$

where the class $\tilde{\alpha} = \tilde{p}^*\alpha \in N_1(\tilde{X})_\mathbb{R}$ is of course still movable. Nothing essential is therefore changed if we replace the rational mapping $p: X \dashrightarrow Z$ by the morphism $\tilde{p}: \tilde{X} \rightarrow Z$; the divisor $\Delta \subseteq X$ by $\tilde{\Delta} \subseteq \tilde{X}$; the sheaf \mathcal{F}_Δ by the intersection

$$\mathcal{T}_{\tilde{X}/Z} \cap \mathcal{T}_{\tilde{X}}(-\log \tilde{\Delta}) \subseteq \mathcal{T}_{\tilde{X}}$$

and the movable class $\alpha \in N_1(X)_\mathbb{R}$ by its pullback $\tilde{\alpha} = \tilde{p}^*\alpha$. \square

Let $R(p)$ denote the ramification divisor of the morphism $p: X \rightarrow Z$; see [CKT16, Definition 2.16] for the precise definition. Recall from [CKT16, Lemma 2.31] the following formula for the first Chern class of our foliation $\mathcal{F} \subseteq \mathcal{T}_X$, in $N^1(X)_\mathbb{R}$:

$$c_1(\mathcal{F}) = c_1(\mathcal{T}_{X/Z}) = -K_{X/Z} + R(p) \tag{4.4}$$

Computing the first Chern class of \mathcal{F}_Δ is a little tricky [CP15, Proposition 5.1], but at least we can use the fact that $\mathcal{F} = \mathcal{T}_{X/Z}$ to estimate the difference

$$c_1(\mathcal{F}) - c_1(\mathcal{F}_\Delta) = c_1(\mathcal{F}/\mathcal{F}_\Delta).$$

Recall that the *horizontal part* $\Delta^{hor} \subseteq \Delta$ is the union of all irreducible components of Δ that map onto Z ; evidently, Δ^{hor} is again a reduced divisor on X with at worst normal crossing singularities.

Lemma 12. *The class $c_1(\mathcal{F}) - c_1(\mathcal{F}_\Delta) - \Delta^{hor}$ is effective.*

Proof. It is easy to see from (3.2) that we have an inclusion of sheaves

$$\mathcal{F}/\mathcal{F}_\Delta \hookrightarrow \mathcal{T}_X/\mathcal{T}_X(-\log \Delta).$$

The sheaf on the right-hand side is supported on the divisor Δ , and a brief computation shows that

$$\mathcal{T}_X/\mathcal{T}_X(-\log \Delta) \cong \bigoplus_{D \subseteq \Delta} \mathcal{N}_{D|X}$$

is isomorphic to the direct sum of the normal bundles of the irreducible components of Δ . The rank of $\mathcal{F}/\mathcal{F}_\Delta$ at the generic point of D is thus either 0 or 1, and

$$c_1(\mathcal{F}/\mathcal{F}_\Delta) = \sum_{D \subseteq \Delta} a_D D,$$

where $a_D = 0$ if $\mathcal{F} = \mathcal{F}_\Delta$ at the generic point of D , and $a_D = 1$ otherwise. To prove that $c_1(\mathcal{F}/\mathcal{F}_\Delta) - \Delta^{hor}$ is effective, we only have to argue that $\mathcal{F} \neq \mathcal{F}_\Delta$ at the generic point of each irreducible component of Δ^{hor} . This is a consequence of the fact that $\mathcal{F} = \mathcal{T}_{X/Z}$, as we now explain.

Fix an irreducible component D of the horizontal part Δ^{hor} . At the generic point of D , the morphism $p: X \rightarrow Z$ is smooth. After choosing suitable local coordinates x_1, \dots, x_n in a neighborhood of a sufficiently general point of D , we may therefore assume that p is locally given by

$$p(x_1, \dots, x_n) = (x_1, \dots, x_d),$$

where $d = \dim Z$, and that the divisor Δ is defined by the equation $x_n = 0$. In these local coordinates, $\mathcal{F} = \mathcal{T}_{X/Z}$ is the subbundle of \mathcal{T}_X spanned by

$$\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{d+1}}.$$

On the other hand, the subsheaf $\mathcal{T}_X(-\log \Delta)$ is spanned by the vector fields

$$x_n \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{d+1}}, \dots, \frac{\partial}{\partial x_1},$$

and so it is clear from (3.2) that $\mathcal{F} \neq \mathcal{F}_\Delta$ in a neighborhood of the given point. \square

From Lemma 12, we draw the conclusion that

$$-(K_{X/Z} + \Delta^{hor} - R(p)) \cdot \alpha = (c_1(\mathcal{F}) - \Delta^{hor}) \cdot \alpha \geq c_1(\mathcal{F}_\Delta) \cdot \alpha > 0, \quad (4.5)$$

where $\alpha \in N_1(X)_{\mathbb{R}}$ is the movable class from above. We will therefore reach the desired contradiction if we manage to prove that the divisor class $K_{X/Z} + \Delta^{hor} - R(p)$ is pseudo-effective. According to [CP15, Theorem 3.3] or to [CKT16, Theorem 7.1], it is actually enough to check that $K_F + \Delta_F$ is pseudo-effective for a general fiber F of the morphism p ; and we can prove, by induction on the dimension, that $K_F + \Delta_F$ is not only pseudo-effective, but even big. The results that we use here are slight improvements of [Cam04, Theorem 4.13], which is itself a generalization of Viehweg's weak positivity theorem.

5. Induction on the dimension

In this section, we use induction on the dimension to finish the proof of Theorem 4 and Theorem 1.

Proposition 13. *Suppose that Theorem 1 is true in dimension less than $\dim X$. If some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with big determinant, then $K_{X/Z} + \Delta^{hor}$ is pseudo-effective.*

Proof. Let F be a general fiber of the morphism $p: X \rightarrow Z$; since $\dim Z \geq 1$, we have $\dim F \leq \dim X - 1$. Denote by Δ_F the restriction of Δ ; since F is a general fiber, Δ_F is still a normal crossing divisor. Clearly

$$(K_{X/Z} + \Delta^{hor})|_F = K_F + \Delta_F,$$

and according to [CKT16, Theorem 7.3], the pseudo-effectivity of $K_{X/Z} + \Delta^{hor}$ will follow if we manage to show that $K_F + \Delta_F$ is pseudo-effective.

By hypothesis and by Lemma 3, there is a nonzero morphism

$$L \rightarrow \Omega_X^1(\log \Delta)^{\otimes k}$$

from a big line bundle L to some tensor power of $\Omega_X^1(\log \Delta)$. Since F is a general fiber of $p: X \rightarrow Z$, we can restrict this morphism to F to obtain a nonzero morphism

$$L_F \rightarrow \left(\Omega_X^1(\log \Delta)|_F \right)^{\otimes k}.$$

Here L_F denotes the restriction of L to the fiber; since L is big, L_F is also big.

The inclusion of F into X gives rise to a short exact sequence

$$0 \rightarrow \mathcal{N}_{F|X} \rightarrow \Omega_X^1(\log \Delta)|_F \rightarrow \Omega_F^1(\log \Delta_F) \rightarrow 0,$$

which induces a filtration on the k -th tensor power of the locally free sheaf in the middle. Since the normal bundle $\mathcal{N}_{F|X}$ is trivial of rank $\dim Z$, we find, by looking at the subquotients of this filtration, that there is a nonzero morphism

$$L_F \rightarrow \Omega_F^1(\log \Delta_F)^{\otimes j}$$

for some $0 \leq j \leq k$. Because L_F is big, we actually have $1 \leq j \leq k$. Since we are assuming that Theorem 1 is true for the pair (F, Δ_F) , the class $K_F + \Delta_F$ is big on F , hence pseudo-effective. Appealing to [CKT16, Theorem 7.3], we deduce that the class $K_{X/Z} + \Delta^{hor}$ is pseudo-effective on X . \square

By induction on the dimension, the two assumptions of Proposition 13 are met in our case, and the class $K_{X/Z} + \Delta^{hor}$ is therefore pseudo-effective. According to [CKT16, Theorem 7.1], this implies that $K_{X/Z} + \Delta^{hor} - R(p)$ is also pseudo-effective.¹ Going back to the inequality in (4.5), we find that

$$0 \geq -(K_{X/Z} + \Delta^{hor} - R(p)) \cdot \alpha \geq c_1(\mathcal{F}_\Delta) \cdot \alpha > 0,$$

and so we have reached the desired contradiction. The conclusion is that $c_1(\mathcal{Q})$ is indeed pseudo-effective, and so Theorem 4 and Theorem 1 are proved.

Remark 14. Most of the argument, for example the proof of Lemma 10, goes through when some tensor power of $\Omega_X^1(\log \Delta)$ contains a subsheaf with *pseudo-effective* determinant. But Theorem 4 is obviously not true under this weaker hypothesis: for example, on the product $E \times \mathbb{P}^1$ of an elliptic curve and \mathbb{P}^1 , there are nontrivial one-forms, yet the canonical bundle is not pseudo-effective. What happens is that the last step in the proof of Proposition 13 breaks down: when L is not big, it may be that $j = 0$ (and L_F is then trivial).

¹ \uparrow As stated, both [CP15, Theorem 3.3] and [CKT16, Theorem 7.1] actually assume that $K_X + \Delta$ is pseudo-effective, but in the case of a morphism $p: X \rightarrow Z$, the proofs go through under the weaker hypothesis that $K_{X/Z} + \Delta^{hor}$ is pseudo-effective.

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