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Double spinor Calabi-Yau varieties

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Abstract. Consider the ten-dimensional spinor variety $\mathcal{S}$ in a projectivized half-spin representation of Spin_{10}. This variety is projectively isomorphic to its projective dual $\mathcal{S}^\vee$ in the dual projective space. The intersection $X = \mathcal{S}_1 \cap \mathcal{S}_2$ of two general translates of $\mathcal{S}$ is a smooth Calabi-Yau fivefold, as well as the intersection of their duals $Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee$. We prove that although X and Y are not birationally equivalent, they are deformation equivalent, Hodge equivalent, derived equivalent and L-equivalent.

Keywords. Spinor variety, Calabi-Yau variety, derived equivalence, birational equivalence, projective duality

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[Français]

Titre. Variétés de spineurs doubles de Calabi-Yau

Résumé. Soit $\mathcal{S}$ la variété spinorielle de dimension 10 dans le projectivisé d'une représentation semi-spin de Spin_{10}. Cette variété est projectivement isomorphe à son dual projectif $\mathcal{S}^\vee$ dans l'espace projectif dual. L'intersection $X = \mathcal{S}_1 \cap \mathcal{S}_2$ de deux translatés généraux de $\mathcal{S}$ est une variété lisse de Calabi-Yau de dimension 5, de même que l'intersection $Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee$ de leurs duaux. Bien que X et Y ne soient pas birationnellement équivalentes, nous montrons qu'elles sont équivalentes par déformations, par équivalence de Hodge, par équivalence dérivée et qu'elles sont également L-équivalentes.

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1. Introduction

There has been a lot of recent interest in the relations, for pairs of Calabi-Yau threefolds, between derived equivalence, Hodge equivalence and birationality. The Pfaffian-Grassmannian equivalence provided pairs of non birational Calabi-Yau threefolds which are derived equivalent, but with distinct topologies. Recently, examples were found of non birational Calabi-Yau threefolds which are derived equivalent, and also deformation and Hodge equivalent. They are constructed from the six-dimensional Grassmannian $G(2,V) \subset \mathbf{P}(\wedge^2 V)$, where V denotes a five-dimensional complex vector space. This Grassmannian is well-know to be projectively self dual, more precisely its projective dual is $G(2,V^\vee) \subset \mathbf{P}(\wedge^2 V^\vee)$, where $V^\vee$ denotes the dual space to V. Let Gr_1 and Gr_2 be two $\mathrm{PGL}(\wedge^2 V)$-translates of $G(2,V)$ in $\mathbf{P}(\wedge^2 V)$, and suppose they intersect transversely. Then $Gr_1^\vee$ and $Gr_2^\vee$ also intersect transversely in $\mathbf{P}(\wedge^2 V^\vee)$. Moreover

$$X = Gr_1 \cap Gr_2 \qquad \text{and} \qquad Y = Gr_1^\vee \cap Gr_2^\vee$$

are smooth Calabi-Yau threefolds with the required properties. This was established independently in [22] and [2], to which we refer for more details on the general background. We should note however that those remarkable threefolds had already appeared in the litterature, see [6, 11, 12].

The purpose of this note is to show that the very same phenomena occur if we replace the Grassmannian $G(2,V) \subset \mathbf{P}(\wedge^2 V)$ by the ten-dimensional spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$, where Δ denotes one of the two half-spin representations of Spin_{10}. These representations have dimension 16. Recall that $\mathcal{S}$ parametrizes one of the two families of maximal isotropic spaces in a ten-dimensional quadratic vector space. The projective dual of $\mathcal{S} \subset \mathbf{P}\Delta$ is the other such family $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$, which is projectively equivalent to $\mathcal{S}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two $\mathrm{PGL}(\Delta)$-translates of $\mathcal{S}$ in $\mathbf{P}\Delta$, and suppose that they intersect transversely. Then $\mathcal{S}_1^\vee$ and $\mathcal{S}_2^\vee$ also intersect transversely in $\mathbf{P}\Delta^\vee$. Moreover

$$X = \mathcal{S}_1 \cap \mathcal{S}_2 \qquad \text{and} \qquad Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee$$

are smooth Calabi-Yau fivefolds of Picard number one which are deformation equivalent, derived[1] equivalent (Proposition 4.2), Hodge equivalent (Corollary 4.3), but not birationally equivalent (Proposition 4.4). Also, the difference of their classes in the Grothendieck ring of varieties is annihilated by a power of the class of the affine line (Proposition 4.5): in the terminology of [16], X and Y are L-equivalent. This confirms their conjecture, which also appears as a question in [8], that (at least for simply connected projective varieties) D-equivalence should imply L-equivalence.

From the point of view of mirror symmetry, X and Y being D-equivalent should have the same mirror, and in this respect they form a double mirror. Of course, from the projective point of view they are also (projective) mirrors one of the other.

The close connection between the Grassmannian $G(2,5) \subset \mathbf{P}^9$ and the spinor variety $\mathcal{S} \subset \mathbf{P}^{15}$ is classical, and manifests itself at different levels.

[1] ↑ D-equivalent in the sequel.

1. They are the only two Hartshorne varieties among the rational homogeneous spaces, if we define a Hartshorne variety to be a smooth variety $Z \subset \mathbf{P}^N$ of dimension $n = \frac{2}{3}N$ which is not a complete intersection (recall that Hartshorne's conjecture predicts that $n > \frac{2}{3}N$ is impossible) [28, Corollary 2.16].

2. They are both prime Fano manifolds of index $\iota = \frac{N+1}{2}$, while their topological Euler characteristic is also equal to $N+1$; this allows their derived categories to admit rectangular Lefschetz decompositions of length ι, based on the similar pairs $\langle \mathcal{O}_Z, U^\vee \rangle$, where U denotes their tautological bundle [14].

3. The Grassmannian $G(2,5) \subset \mathbf{P}^9$ can be obtained from $\mathcal{S} \subset \mathbf{P}^{15}$ as parametrizing the lines in $\mathcal{S}$ through some given point [18]; conversely, $\mathcal{S} \subset \mathbf{P}^{15}$ can be reconstructed from $G(2,5) \subset \mathbf{P}^9$ by a simple quadratic birational map defined in terms of the quadratic equations of the Grassmannian [17].

That the story should be more or less the same for our double spinor varieties, as for the double Grassmannians, is therefore not a big surprise. We thought it would nevertheless be useful to check that everything was going through as expected. For that we essentially followed the ideas of [2] and [22], to which this note is of course heavily indebted.

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2. Spinor varieties

2.A. Pure spinors

We start with some basic facts about spin representations. See [20] for more details, and references therein. Our base field all over the paper will be the field of complex numbers. For convenience we will restrict to even dimensions, so we let $V = V_{2n}$ be a complex vector space of dimension $2n$, endowed with a non degenerate quadratic form. The variety of isotropic n-dimensional subspaces of V, considered as a subvariety of the Grassmannian $G(n, 2n)$, has two connected components

$$\mathcal{S}_+ = OG_+(n, 2n) \quad \text{and} \quad \mathcal{S}_- = OG_-(n, 2n),$$

called the spinor varieties, or varieties of pure spinors. Moreover the Plücker line bundle restricted to $\mathcal{S}_+$ or $\mathcal{S}_-$ has a square root L, which is still very ample and embeds the spinor varieties into the projectivizations of the two half-spin representations of Spin_{2n}, the simply connected double cover of SO_{2n}. We denote the half-spin representations by Δ_+ and Δ_-, in such a way that

$$\mathcal{S}_+ \subset \mathbf{P}\Delta_+ \qquad \text{and} \qquad \mathcal{S}_- \subset \mathbf{P}\Delta_-.$$

Since the two half-spin representations can be exchanged by an outer automorphism of Spin_{2n}, these two embeddings are projectively equivalent. Note that the spinor varieties have dimension $n(n-1)/2$, while the half-spin representations have dimension 2^{n-1}. The half-spin representations are self dual when n is even, and dual one to the other when n is odd.

It follows from the usual Bruhat decomposition that the Chow ring of $\mathcal{S}_\pm$ is free, and the dimension of its k-dimensional component is equal to the number of strict partitions of k with parts smaller than n. In particular the Picard group has rank one, and L is a generator; we therefore denote $L = \mathcal{O}_{\mathcal{S}_\pm}(1)$. We also let U be the rank n vector bundle obtained by restricting the tautological bundle of $G(n, 2n)$. Then the tangent bundle to $\mathcal{S}_\pm$ is isomorphic to $\wedge^2 U^\vee$. Since $\det(U^\vee) = L^2$, this implies that $\mathcal{S}_\pm$ is a prime Fano manifold of index $2n-2$.

2.B. The ten dimensional spinor variety

From now on we specialize to $n=5$, and we simply denote $\mathcal{S} \subset \mathbf{P}\Delta$ one of the spinor varieties. This is a ten-dimensional prime Fano manifold of index eight, embedded in codimension five. This case is specific for several reasons. First, it admits a very simple rational parametrization.

Proposition 2.1. *Let $E \subset V_{10}$ be a maximal isotropic subspace, that defines a point of $\mathcal{S}$. Then $\mathcal{S} \subset \mathbf{P}\Delta$ is projectively isomorphic to the image of the rational map from $\wedge^2 E^\vee$ to $\mathbf{P}(\mathbb{C} \oplus \wedge^2 E^\vee \oplus \wedge^4 E^\vee)$ that sends ω to $[1, \omega, \omega \wedge \omega]$.*

Note that this rational map is $\mathrm{GL}(E)$-equivariant, and that $\mathrm{GL}(E)$ is a Levi factor of the parabolic subgroup of Spin_{10} that stabilizes the base point of $\mathcal{S}$ defined by E.

Proof. Let us fix another isotropic subspace F of V_{10}, transverse to E. The quadratic form identifies F with the dual of E. A general point of $\mathcal{S}$ corresponds to a subspace of V_{10} defined by the graph of a map ω from E to $F \simeq E^\vee$; the isotropy condition translates to the skew-symmetry of this map. The embedding to $\mathbf{P}\Delta$ is then given by the Pfaffians of ω of any even size: that is 1 in size 0, ω itself in size 2, $\omega \wedge \omega$ in size four. See [20, Section 2.3] or [17, Theorem 2.4] for more details. □

A useful consequence is the following. In the preceding description of $\mathcal{S}$, its projectivized tangent space at the base point is $\mathbf{P}(\mathbb{C} \oplus \wedge^2 E^\vee)$ and the normal space identifies with the remaining factor $\wedge^4 E^\vee$. This identifies the representation of $\mathrm{GL}(E)$ that defines the normal bundle as a homogeneous bundle on $\mathcal{S}$ and we deduce our next statement. Recall that U denotes the tautological rank 5 vector bundle on $\mathcal{S}$, and that $\det(U) = \mathcal{O}_{\mathcal{S}}(-2)$, where $\mathcal{O}_{\mathcal{S}}(1)$ is the positive generator of $\mathrm{Pic}(\mathcal{S})$.

Proposition 2.2. *The normal bundle to $\mathcal{S}$ in $\mathbf{P}\Delta$ is $\wedge^4 U^\vee \simeq U(2)$.*

Another nice property that $\mathcal{S}$ shares with $G(2,5)$ is that its complement is homogeneous.

Proposition 2.3. *The action of Spin_{10} on $\mathbf{P}\Delta - \mathcal{S}$ is transitive.*

In particular Δ admits a prehomogeneous action, not of Spin_{10}, but of $\mathrm{GL}(1) \times \mathrm{Spin}_{10}$. This is discussed on page 121 of [24]. In fact, a much stronger result is proved in [7, Proposition 2]: over any field of characteristic different from two, there are only two orbits of non zero spinors.

The next important property also shared with $G(2,5)$ is that the quadratic equations allow to recover the natural representation.

Proposition 2.4. *The quadratic equations of $\mathcal{S}$ are parametrized by V_{10}.*

These equations can be described in terms of the Clifford product, that defines a map $V_{10} \otimes \Delta_\pm \to \Delta_\mp = \Delta_\pm^\vee$. In more down to earth terms, we can use the decomposition $\Delta = \mathbb{C} \oplus \wedge^2 E^\vee \oplus \wedge^4 E^\vee$ introduced in Proposition 2.1: a point $[\omega_0, \omega_2, \omega_4]$ belongs to $\mathcal{S}$, as is readily verified, if and only if

$$\omega_0 \omega_4 = \omega_2 \wedge \omega_2, \quad \text{and} \quad \omega_4 * \omega_2 = 0.$$

The first equation is in $\wedge^4 E^\vee \simeq E$. In the second equation, we used this isomorphism in order to identify ω_4 with an element of E; then contracting with ω_2 gives an element of $E^\vee$ that we denoted $\omega_4 * \omega_2$. We thus get, as expected, quadratic equations parametrized by $E \oplus E^\vee = V_{10}$. And this is necessarily an identification as Spin_{10}-modules since V_{10} is its only nontrivial ten dimensional representation.

As any equivariantly embedded rational homogeneous variety, the spinor variety is cut out by quadrics. Moreover the spinor variety $\mathcal{S}$ has a beautiful self-dual minimal resolution (necessarily equivariant), which appears in [13, 5.1]:

$$0 \to \mathcal{O}(-8) \to V_{10}(-6) \to \Delta^\vee(-5) \to \Delta(-3) \to V_{10}(-2) \to \mathcal{O} \to \mathcal{O}_{\mathcal{S}} \to 0. \tag{1}$$

For future use let us compute the Hilbert polynomial of $\mathcal{S} \subset \mathbf{P}\Delta$. Note that $H^0(\mathcal{O}_S(k))$ is, by the Borel-Weil theorem, the irreducible Spin_{10}-module of highest weight $k\omega_5$ (where ω_5 is the fundamental weight corresponding to the half-spin representation $\Delta^\vee$). Its dimension can thus be computed by a direct application of the Weyl dimension formula, and we get

$$H_{\mathcal{S}}(k) = \frac{1}{2^6 3^3 5^2 7}(k+1)(k+2)(k+3)^2(k+4)^2(k+5)^2(k+6)(k+7).$$

In particular, as is well-known, $\mathcal{S}$ has degree 12. Finally the Poincaré polynomial is also easy to compute; since the Betti numbers are given by numbers of strict partitions, as we already mentioned, we readily get that

$$P_{\mathcal{S}}(t) = (1+t^3)(1+t+t^2+t^3+t^4+t^5+t^6+t^7).$$

2.C. Self duality

Our next statement is a well-known direct consequence of Proposition 2.3:

Corollary 2.5. *The spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is projectively self dual.*

To be more precise, the dual variety of the spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is the other spinor variety $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$, in the other half-spin representation.

Note for future use that the self-duality of $\mathcal{S}$ is preserved at the categorical level, in the sense that $\mathcal{S} \subset \mathbf{P}\Delta$ and $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$ are homologically projectively dual [14, Section 6.2]. As already mentioned in the introduction, the derived category of coherent sheaves on the spinor variety $\mathcal{S}$ has a specially nice rectangular Lefschetz decomposition, defined by eight translates of the exceptional pair $\langle \mathcal{O}_S, U^\vee \rangle$.

Another consequence of Proposition 2.3 is that, up to the group action, there are only two kinds, up to projective equivalence, of hyperplane sections of $\mathcal{S}$: the smooth and the singular ones. Let us briefly describe their geometries.

Proposition 2.6. *A singular hyperplane section $\mathcal{HS}_{sing}$ of $\mathcal{S}$ is singular along a projective space of dimension four. Moreover $\mathcal{HS}_{sing}$ admits a cell decomposition and its Poincaré polynomial is*

$$P_{\mathcal{HS}_{sing}}(t) = 1+t+t^2+2t^3+2t^4+2t^5+2t^6+2t^7+t^8+t^9.$$

Proof. Recall that we may consider $\mathcal{S}$ and $\mathcal{S}^\vee$ as the two families of maximal isotropic subspaces of V_{10}. Moreover, if E and F are two maximal isotropic spaces, they belong to the same family if and only if their intersection has odd dimension. Given a point of $\mathcal{S}$, that we identify, with some abuse, to such an isotropic space E, the set of hyperplanes tangent to $\mathcal{S}$ at E defines a subvariety of $\mathcal{S}^\vee$.

Lemma 2.7. *A point $F \in \mathcal{S}^\vee$ defines a hyperplane in $\mathbf{P}\Delta$ which is tangent to $\mathcal{S}$ at E, if and only if* $\dim(E\cap F) = 4$.

Proof of Lemma 2.7. It is a consequence of Witt's theorem that the action of $\mathrm{Spin}(V_{10})$ on $\mathcal{S}\times\mathcal{S}^\vee$ has exactly three orbits, characterized by the three possible values for the dimension of the intersection (recall that this dimension must be even). Therefore the stabilizer of E in $\mathrm{Spin}(V_{10})$ has also three orbits in $\mathcal{S}^\vee$, defined by the three possible values for the dimension of the intersection with E. Geometrically, they have to correspond to the three possible positions with respect to E, of a hyperplane defined by a point $F \in \mathcal{S}^\vee$: tangent to $\mathcal{S}$ at E, containing E but not tangent, or not containing E. This implies the claim. □

Since an isotropic space $E \in \mathcal{S}$ such that $\dim(E\cap F) = 4$ is uniquely determined by $E\cap F$, the hyperplane defined by F is tangent to $\mathcal{S}$ along a subvariety of $\mathcal{S}$ isomorphic to $\mathbf{P}F^\vee$.

For the last assertions, note that a singular hyperplane section of $\mathcal{S}$ is just a Schubert divisor. By general results on the Bruhat decomposition, we know that its complement in $\mathcal{S}$ is precisely the big cell. So $\mathcal{HS}_{sing}$ has a cell decomposition given by all the cells of $\mathcal{S}$ except the big one, and $P_{\mathcal{HS}_{sing}}(t) = P_{\mathcal{S}}(t) - t^{10}$. □

Proposition 2.8. *A smooth hyperplane section $\mathcal{HS}_{reg}$ of $\mathcal{S}$ admits a quasi-homogeneous action of its automorphism group, which is isomorphic to* $(\mathrm{Spin}_7 \times \mathbb{G}_m) \ltimes \mathbb{G}_a^8$. *Moreover $\mathcal{HS}_{reg}$ admits a cell decomposition and its Poincaré polynomial is*

$$P_{\mathcal{HS}_{reg}}(t) = 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 + t^7 + t^8 + t^9.$$

Proof. For the first statement, we refer to [5, Proposition 3.9]. As observed in [5], $\mathcal{HS}_{reg}$ coincides with the horospherical variety that appears as case 4 of [23, Theorem 1.7]. In particular, being horospherical it admits an algebraic cell decomposition. Finally the Betti numbers are given by the Lefschetz hyperplane theorem. □

3. Double spinor varieties

In this section we introduce our main objects of interest, the *double spinor varieties*

$$X = \mathcal{S}_1 \cap \mathcal{S}_2,$$

where $\mathcal{S}_1 = g_1\mathcal{S}$ and $\mathcal{S}_2 = g_2\mathcal{S}$ are translates of $\mathcal{S}$ by $g_1, g_2 \in \mathrm{PGL}(\Delta)$. Up to projective equivalence, we can of course suppose that $X = \mathcal{S} \cap g\mathcal{S}$ for $g \in \mathrm{PGL}(\Delta)$.

By the Eagon-Northcott generic perfection theorem [3, Theorem 3.5], the resolution (1) gives a free resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathcal{S}_1}$-module:

$$0 \to \mathcal{O}_{\mathcal{S}_1}(-8) \to V_{10}(-6) \to \Delta^\vee(-5) \to \Delta(-3) \to V_{10}(-2) \to \mathcal{O}_{\mathcal{S}_1} \to \mathcal{O}_X \to 0. \tag{2}$$

3.A. Local completeness

Let $G = \mathrm{PGL}(\Delta)$, with its subgroup $H = \mathrm{Aut}(\mathcal{S}) \simeq \mathrm{PSO}_{10}$ (as follows from [4]). The family of double spinor varieties is by definition the image of a rational map

$$\xi : G/H \times G/H \dashrightarrow Hilb(\mathbf{P}\Delta),$$

where $\xi(g_1, g_2) = g_1\mathcal{S} \cap g_2\mathcal{S}$. Moreover the diagonal left action of G is by projective equivalence, hence factors out when we consider local deformations of a given X. At the global level, the quotient $[(G/H \times G/H)/G]$ should be thought of as the moduli stack of double spinor varieties. One could reproduce the analysis of the similar stack made in [2] for the double Grassmannians, but we will not do that. We will only check the local completeness of our family.

Proposition 3.1. *The family of smooth double spinor varieties is locally complete.*

Proof. We first observe that $H^1(X, T\mathcal{S}_{1|X}) = 0$. Because of (2), this is a direct consequence of the vanishing of $H^1(\mathcal{S}_1, T\mathcal{S}_1)$, of $H^q(\mathcal{S}_1, T\mathcal{S}_1(-k))$ for $0 < k < 8$ and $q > 1$, and of $H^6(\mathcal{S}_1, T\mathcal{S}_1(-8))$, which are all consequences of Bott's theorem. (Alternatively, the vanishing of $H^q(\mathcal{S}_1, T\mathcal{S}_1(-k))$ follows from the Kodaira-Nakano vanishing theorem, since this group is Serre dual to $H^{10-q}(\mathcal{S}_1, \Omega_{\mathcal{S}_1}(k-8))$.) Hence the map

$$H^0(X, N_{X/\mathcal{S}_1}) \simeq H^0(X, N_{\mathcal{S}_2/\mathbf{P}|X}) \longrightarrow H^1(X, TX)$$

is surjective. Here we abbreviated $\mathbf{P}\Delta$ by $\mathbf{P}$. What remains to prove is that the composition

$$H^0(\mathbf{P}, T\mathbf{P}) \xrightarrow{r} H^0(X, T\mathbf{P}_{|X}) \xrightarrow{s} H^0(X, N_{\mathcal{S}_2/\mathbf{P}|X})$$

is also surjective. In order to prove that r is surjective, it is convenient to use the Euler exact sequence on $\mathbf{P}$ and its restriction to X; from (2) we easily get that $H^1(X, \mathcal{O}_X) = 0$ and $H^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(1)) \simeq H^0(X, \mathcal{O}_X(1))$, and the surjectivity of r readily follows. The surjectivity of s follows from the vanishing of $H^1(X, T\mathcal{S}_{2|X})$, which we already verified. □

The following observation (already made in [12, Proposition 4.7] for the double Grassmannian varieties) will be useful: when g goes to identity, $X = \mathcal{S} \cap g\mathcal{S}$ deforms smoothly to the zero locus in $\mathcal{S}$ of a global section of its normal bundle $\wedge^4 U^\vee = U(2)$. In our situation the discussion of [22, section 5] applies almost verbatim.

The normal bundle is generated by global sections, being homogeneous and irreducible, with

$$H^0(\mathcal{S}, \wedge^4 U^\vee) = \wedge^4 V_{10}^\vee = V_{\omega_4+\omega_5}$$

by the Borel-Weil theorem. So the zero locus of a general section, which we call a normal degeneration, is a smooth Calabi-Yau fivefold which is deformation equivalent to the smooth double spinor varieties; in particular the family of those zero-loci is not locally complete, something that seems to be quite exceptional.

Remark. Note that other kinds of degenerations of double Grassmannians, this time singular, were considered in [6]: typically, they are joins of two elliptic quintics (which are linear sections of the Grassmannian) in two disjoint $\mathbf{P}^4$'s in $\mathbf{P}^9$. Such degenerations were studied in connection with the Horrocks-Mumford vector bundle, in order to describe the moduli space of $(1,10)$-polarized abelian surfaces. It would certainly be interesting to study the similar story in our setting. The analogous singular degenerations are of course the joins of two K3 surfaces of degree 12 (which are linear sections of the spinor variety) in two disjoint $\mathbf{P}^7$'s in $\mathbf{P}^{15}$.

3.B. Invariants

Proposition 3.2. *Any smooth double spinor variety $X = \mathcal{S}_1 \cap \mathcal{S}_2$ (of the expected dimension) is a Calabi-Yau fivefold. Moreover:*

1. $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{O}_X(1)$, *and* $H^p(X, \Omega_X^p) = \mathbb{C}$ *for* $0 \le p \le 5$;
2. $H^q(X, \Omega_X^p) = 0$ *for* $p \neq q$ *and* $p + q \neq 5$;
3. $H^5(X, \mathbb{Z})$ *is torsion free.*

Proof. The hypothesis that X is smooth of the expected dimension is equivalent to the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ meet transversely. Then X has dimension five. Suppose to simplify notations that $\mathcal{S}_1 = \mathcal{S}$.

Since $\omega_S = \mathcal{O}_\mathcal{S}(-8)$, the line bundles $\mathcal{O}_\mathcal{S}(-k)$ are acyclic for $0 < k < 8$. Moreover $h^q(\mathcal{O}_\mathcal{S}(-8)) = 0$ for $q \le 9$. Then (2) yields that $h^0(\mathcal{O}_X) = 1$, so that X is connected. Moreover, the relative dualizing sheaf

$$\omega_{X/\mathcal{S}} = \det N_{X/\mathcal{S}} = \det N_{\mathcal{S}_2/\mathbf{P}|X} = \mathcal{O}_X(8),$$

and since $\omega_\mathcal{S} = \mathcal{O}_\mathcal{S}(-8)$, we conclude that X has trivial canonical bundle.

As was done in [22, Lemma 3.3], we now apply [26, Corollary b)] to $A = \mathcal{S}_1$ and $B = \mathcal{S}_2$ in $\mathbf{P}^{15}$: we get that the relative homotopy groups $\pi_i(\mathcal{S}_1, X) = 0$ for $i \le 5$. In particular X is simply connected, and by the Bogomolov decomposition theorem, it is Calabi-Yau. Moreover, after passing from homotopy to homology, we deduce that $H_i(\mathcal{S}_1, X, \mathbb{Z}) = 0$ for $i \le 5$. Since the cohomology of $\mathcal{S}_1$ is pure and torsion free, this implies the remaining assertions. □

Knowing the Hilbert polynomial of $\mathcal{S}$, we also deduce that the Hilbert polynomial of X is

$$H_X(k) = \frac{2}{5}k(k^2+1)(3k^2+17).$$

Proposition 3.3. *The non zero Hodge numbers of a smooth double spinor variety are $h^{p,p} = 1$ for $0 \le p \le 5$, and*

$$h^{0,5} = h^{5,0} = 1, \quad h^{1,4} = h^{4,1} = 165, \quad h^{2,3} = h^{3,2} = 7708.$$

Proof. The missing Hodge number are $h^4(\Omega_X) = 1 + \chi(\Omega_X)$ and $h^3(\Omega^2_X) = 1 - \chi(\Omega^2_X)$. In order to compute them, we may suppose that X is a normal degeneration of a double spinor variety, that is, the zero locus of a general section of $\wedge^4 U^\vee = U(2)$ on $\mathcal{S}$. Then the bundle of forms on X is resolved by the conormal sequence, and the bundle of two-forms by its skew-symmetric square, that is

$$0 \to S^2 U_X^\vee(-4) \to \wedge^2 U_X \otimes U_X^\vee(-2) \to \wedge^2(\wedge^2 U_X) \to \Omega^2_X \to 0.$$

This allows to compute $\chi(\Omega_X)$ as $\chi(\wedge^2 U_X) - \chi(U_X^\vee(-2))$, and $\chi(\Omega^2_X)$ as $\chi(\wedge^2(\wedge^2 U_X)) - \chi(\wedge^2 U_X \otimes U_X^\vee(-2)) + \chi(S^2 U_X^\vee(-4))$. Using the Koszul complex, this reduces once again to computations on the spinor variety. Finally, on the latter we can use the Borel-Weil-Bott theorem (whose concrete application is illustrated in the proof of Lemma 3.5 below). □

3.C. Uniqueness

In this section we prove that the only translates of $\mathcal{S}$ that contain $X = \mathcal{S} \cap g\mathcal{S}$ are $\mathcal{S}$ itself, and $g\mathcal{S}$. In particular there is a unique way to represent X as an intersection of two translates of the spinor variety. We follow the approach of [2].

Proposition 3.4. *Let N denote the normal bundle to $\mathcal{S}$ in $\mathbf{P}\Delta$. Then the restriction of N to X is slope stable.*

Proof. Recall that we denoted by U the rank five tautological bundle on $\mathcal{S}$. By Proposition 2.2 there is an isomorphism $N \simeq \wedge^4 U^\vee = U(2)$, so we just need to prove that $U_X^\vee$ is stable. Since the Picard group of X is cyclic by Proposition 3.2, we can apply Hoppe's criterion [9, Proposition 1], following which it is enough to check that

$$H^0(X, U_X^\vee(-1)) = H^0(X, \wedge^2 U_X^\vee(-1)) = H^0(X, \wedge^3 U_X^\vee(-2)) = H^0(X, \wedge^4 U_X^\vee(-2)) = 0.$$

This easily follows from the resolution (2) and the following statement. □

Lemma 3.5. *Suppose that* $1 \le e \le 4$, $0 \le q \le 5$ *and* $t > 0$. *Then* $H^q(\mathcal{S}, \wedge^e U^\vee(-t)) = 0$, *except for the following cohomology groups:*

1. $H^1(\mathcal{S}, \wedge^3 U^\vee(-2)) = \mathbb{C}$,
2. $H^0(\mathcal{S}, \wedge^4 U^\vee(-1)) = \Delta$.

Proof. This is a straightforward application of the Borel-Weil-Bott theorem (see e.g. [1, Theorem 2.1] and references therein). Let us prove the first identity, to explain how this theorem applies in our setting. The root system D_5 can be described in terms of a lattice with orthonormal basis $\epsilon_1, \dots, \epsilon_5$. The simple roots of D_5 can be chosen to be

$$\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3,\ \alpha_3 = \epsilon_3 - \epsilon_4,\ \alpha_4 = \epsilon_4 - \epsilon_5,\ \alpha_5 = \epsilon_4 + \epsilon_5.$$

The fundamental weights are then $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_1 + \epsilon_2$, $\omega_3 = \epsilon_1 + \epsilon_2 + \epsilon_3$ and

$$\omega_4 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5), \qquad \omega_5 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5).$$

The sum of the fundamental weights is

$$\rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4.$$

The weights of $U^\vee$ are the ϵ_i's, so the weights of $\wedge^3 U^\vee$ are the sums of three distinct ϵ_i's, and the highest one is $\epsilon_1 + \epsilon_2 + \epsilon_3$. Thus the highest weight of $\wedge^3 U^\vee(-2)$ is $\omega_3 - 2\omega_5$. Bott's theorems states that in order to find the cohomology groups of our bundle, we first need to add ρ to this weight, which gives

$\tau = \omega_1 + \omega_2 + 2\omega_3 + \omega_4 - \omega_5$. No root of D_5 is orthogonal to τ, so there will be one non zero cohomology group. To find it, we choose a simple root with negative scalar product with τ; there is only one, α_5; so we apply the associated simple reflection, which yields $s_5(\tau) = \tau + \alpha_5 = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5$. Since no coefficient is negative we do not need to repeat this operation; we just subtract ρ. This yields the weight zero, so we get a cohomology group isomorphic to the trivial representation $\mathbb{C}$, in degree one because we just needed to apply one simple reflection. (Note by the way that $\wedge^3 U^\vee(-2) \simeq \wedge^2 U \simeq \Omega_{\mathcal{S}}$, which explains why we obtain $H^1(\mathcal{S}, \wedge^3 U^\vee(-2)) = H^1(\mathcal{S}, \Omega_{\mathcal{S}}) = \mathbb{C}$.) $\square$

The next step is to prove the following statement:

Proposition 3.6. *Suppose $X \subset \mathcal{S}_1$, where $\mathcal{S}_1$ is a translate of $\mathcal{S}$. Let N_1 be the normal bundle to $\mathcal{S}_1$ in $\mathbf{P}\Delta$. Then from its restriction to X one can reconstruct the embeddings $X \subset \mathcal{S}_1 \subset \mathbf{P}(\Delta)$.*

Proof. Suppose $\mathcal{S}_1 = \mathcal{S}$ to simplify the notations. Since $\mathcal{S}$ is cut out by quadrics, our strategy will be to reconstruct its quadratic equations from N_X, or equivalently, from U_X. Then we simply recover $\mathcal{S}$ as the zero locus of its quadratic equations.

Step 1. The key observation is that there is a natural isomorphism

$$H^0(\mathcal{S}, U(1)) \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^\vee. \tag{3}$$

Indeed, $U^\vee$ is the irreducible bundle associated to the representation of highest weight $\omega_1 = \epsilon_1$. The highest weight of its dual U is $-\epsilon_5 = \omega_4 - \omega_5$. Therefore the highest weight of $U(1)$ is ω_4, and the assertion follows from the Borel-Weil theorem.

Step 2. There are natural morphisms

$$H^0(\mathcal{S}, U^\vee) \otimes H^0(\mathcal{S}, U(1)) \longrightarrow H^0(\mathcal{S}, U^\vee \otimes U(1)) \longrightarrow H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1)),$$

the right hand side being induced by the trace map $U^\vee \otimes U \to \mathcal{O}_{\mathcal{S}}$. This determines the quadratic equations of $\mathcal{S}$, as the image of the induced map

$$V_{10} = V_{10}^\vee = H^0(\mathcal{S}, U^\vee) \longrightarrow H^0(\mathcal{S}, U(1))^\vee \otimes H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1)) \simeq H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^{\otimes 2} \longrightarrow Sym^2 H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1)).$$

(The composition is non zero because $\wedge^2 H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))$ does not contain any direct factor isomorphic to V_{10}.)

With the help of these observations, we now need to prove that we can recover the quadratic equations of $\mathcal{S}$ just starting from U_X. We will show we can follow exactly the same argument as above, using only spaces of sections of bundles defined on X only.

Step 3. We deduce from (2) and the Borel-Weil-Bott theorem that the restriction morphism

$$res_F : H^0(\mathcal{S}, F) \longrightarrow H^0(X, F_X)$$

is an isomorphism for either $F = \mathcal{O}_{\mathcal{S}}(1), U^\vee, U(1), U(2), Sym^2 U^\vee$.

Step 4. We recover the quadratic form (up to scalar) on $V_{10}^\vee = H^0(X, U_X^\vee)$ as a generator of the one dimensional kernel of the map $Sym^2 H^0(X, U_X^\vee) \longrightarrow H^0(X, Sym^2 U_X^\vee)$; hence the isomorphism $V_{10}^\vee \simeq V_{10}$.

Step 5. We have $\Delta_+^\vee = H^0(X, \mathcal{O}_X(1))$ and we want to identify its dual with $\Delta_-^\vee = H^0(X, U_X(1))$. First note that we have a natural map $\eta : V_{10}^\vee \otimes \Delta_-^\vee \longrightarrow \Delta^\vee$ obtained by multiplying sections and contracting:

$$\eta : H^0(X, U_X^\vee) \otimes H^0(X, U_X(1)) \longrightarrow H^0(X, U_X^\vee \otimes U_X(1)) \longrightarrow H^0(X, \mathcal{O}_X(1)).$$

Step 6. From η we get a map $\chi : \Delta_-^\vee \longrightarrow \Delta^\vee \otimes V_{10}$, hence also a map

$$\xi : \Delta^\vee \otimes \Delta_-^\vee \longrightarrow \wedge^2 \Delta^\vee \otimes V_{10}.$$

We claim that we can recover the duality between Δ and Δ_- as being given by the one-dimensional kernel $K \subset \Delta^\vee \otimes \Delta_-^\vee$ of the ξ. Because of the results of Step 3, it is enough to check this claim when we consider the same maps as defined on $\mathcal{S}$ rather than on X. We use the following decompositions into irreducible Spin_{10}-modules [19]:

$$\Delta^\vee \otimes \Delta_-^\vee = V_{\omega_4+\omega_5} \oplus V_{\omega_2} \oplus \mathbb{C},$$

$$\wedge^2 \Delta^\vee \otimes V_{10} = V_{\omega_4+\omega_5} \oplus V_{\omega_2} \oplus V_{\omega_1+\omega_3}.$$

There is a unique irreducible factor $K = \mathbb{C}$ that appears in the first decomposition and not in the second one. Elementary computations, left to the reader, allow to check that K is exactly the kernel of ξ. Note that since Δ and Δ_- are irreducible, K must define a perfect duality between these modules.

Step 7. Once we have the duality defined by $K \subset \Delta^\vee \otimes \Delta_-^\vee$, we can argue exactly as in Step 2 to realize V_{10}, just starting from X, as a system of quadrics on $\mathbf{P}(\Delta)$. We finally recover $\mathcal{S}$ as the base locus of this system. □

Remark. The key isomorphism (3) can be explained as follows. Recall that $\mathcal{S} \subset \mathbf{P}\Delta$ and $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$ parametrize the two families of maximal isotropic spaces of V_{10}. Two such spaces belong to different families if and only if they intersect in even dimension. Moreover, any isotropic four-plane is contained in exactly two maximal isotropic subspaces, one from each family. We can therefore identify the orthogonal Grassmannian $OG(4, V_{10})$ with the incidence variety $\mathcal{I} \subset \mathcal{S} \times \mathcal{S}^\vee$ of pairs (U, U') such that $L = U \cap U'$ has dimension four, and denote by $p, p_\vee$ the two projections. By the preceding observations, the projection p identifies $\mathcal{I}$ with the projective bundle $\mathbf{P}U^\vee$. Moreover, with the previous notations, we have $\det(U) = \det(L) \otimes U/L$, $\det(U') = \det(L) \otimes U'/L$, and the quadratic form induces a natural duality between U/L and U'/L. From this one easily deduces that $p_\vee^* \mathcal{O}_{\mathcal{S}^\vee}(1) = \mathcal{O}_{\mathbf{P}U^\vee}(1) \otimes p^*\mathcal{O}_{\mathcal{S}}(1)$. And then

$$H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^\vee = H^0(\mathcal{S}^\vee, \mathcal{O}_{\mathcal{S}^\vee}(1)) = H^0(\mathcal{I}, p_\vee^*\mathcal{O}_{\mathcal{S}^\vee}(1)) = H^0(\mathcal{I}, \mathcal{O}_{\mathbf{P}U^\vee}(1) \otimes p^*\mathcal{O}_{\mathcal{S}}(1)) = H^0(\mathcal{S}, U(1)).$$

By the same argument as in [2, Proposition 2.3], we deduce from the previous Proposition that:

Proposition 3.7. *If $X = \mathcal{S}_1 \cap \mathcal{S}_2$ is a transverse intersection, then the only translates of $\mathcal{S}$ that contain X are $\mathcal{S}_1$ and $\mathcal{S}_2$.*

4. Double mirrors

Recall that the spinor variety $\mathcal{S} \subset \mathbf{P}\Delta$ is projectively dual to the other spinor variety $\mathcal{S}^\vee \subset \mathbf{P}\Delta^\vee$. We may therefore associate to $X = \mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathbf{P}\Delta$, the other double spinor variety $Y = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee \subset \mathbf{P}\Delta^\vee$. When X is smooth, its presentation as the intersection of two translated spinor varieties is unique, and therefore Y is uniquely defined.

4.A. Derived equivalence

Proposition 4.1. *The double spinor varieties X and Y are simultaneously smooth of expected dimension.*

Proof. Suppose $\mathcal{S}_1 = g_1\mathcal{S}$ and $\mathcal{S}_2 = g_2\mathcal{S}$ and let $x \in X$. Then $x = g_1E_1 = g_2E_2$ for some E_1, E_2 in $\mathcal{S}$. The intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ fails to be transverse at x if and only if there is a point $y \in \mathbf{P}\Delta^\vee$ such that the corresponding hyperplane H_y in $\mathbf{P}\Delta$ is tangent to both $\mathcal{S}_1$ and $\mathcal{S}_2$ at x. By Lemma 2.7, this means that $y = g_1^tF_1 = g_2^tF_2$ for some F_1, F_2 in $\mathcal{S}^\vee$, such that $\dim(E_1 \cap F_1) = \dim(E_2 \cap F_2) = 4$. In particular y belongs to $g_1^t\mathcal{S}^\vee \cap g_2^t\mathcal{S}^\vee = \mathcal{S}_1^\vee \cap \mathcal{S}_2^\vee = Y$ and by symmetry, the intersection of $\mathcal{S}_1^\vee$ and $\mathcal{S}_2^\vee$ fails to be transverse at y. This implies the claim. □

Proposition 4.2. *When they are smooth, the double spinor varieties X and Y are derived equivalent.*

From the point of view of mirror symmetry, X and Y being D-equivalent should have the same mirror: they form an instance of a double mirror.

Proof. This is a direct application of the results of [15], or of the Main Theorem in [10]. As we already mentioned, the fact that $(\mathcal{S}_1, \mathcal{S}_1^\vee)$ and $(\mathcal{S}_2, \mathcal{S}_2^\vee)$ are pairs of homologically projectively dual varieties was established in [14, Section 6.2]. □

Remark. In fact the results of [10, 15] imply the stronger statement that X and Y are derived equivalent as soon as they have dimension five, even if they are singular. Since the smoothness of a variety can be detected at the level of its derived category, this provides another proof of Proposition 4.1.

Applying Proposition 2.1 of [22], we deduce (recall from Proposition 3.2 that $H^5(X,\mathbb{Z})$ and $H^5(Y,\mathbb{Z})$ are torsion free):

Corollary 4.3. *The polarized Hodge structures on $H^5(X,\mathbb{Z})$ and $H^5(Y,\mathbb{Z})$ are equivalent.*

4.B. Non birationality

Now we sketch a proof of the following result, according to the ideas of [22, Proof of Lemma 4.7].

Proposition 4.4. *Generically, the mirror double spinors X and Y are not birationally equivalent.*

Proof. By a standard argument, it is enough to prove that X and Y in $\mathbf{P}(\Delta)$ are not projectively equivalent. Indeed, suppose X and Y are birational. Since they are Calabi-Yau, X and Y are minimal models, and this implies that the birational equivalence must be an isomorphism in codimension two. Since their Picard groups are both cyclic, the birational equivalence identifies their (very ample) generators, and induces an isomorphism between their spaces of sections, yielding a projective equivalence as claimed.

So suppose that X and Y are projectively equivalent. Since they are both contained in a unique pair of translates $\mathcal{S}_1$ and $\mathcal{S}_2$ of the spinor variety, there would exist a projective isomorphism $u : \mathbf{P}(\Delta) \simeq \mathbf{P}(\Delta^\vee)$ such that either $u(\mathcal{S}_1) = \mathcal{S}_1^\vee$ and $u(\mathcal{S}_2) = \mathcal{S}_2^\vee$, or $u(\mathcal{S}_1) = \mathcal{S}_2^\vee$ and $u(\mathcal{S}_2) = \mathcal{S}_1^\vee$.

Let us fix once and for all a linear isomorphism $u_0 : \mathbf{P}(\Delta) \simeq \mathbf{P}(\Delta^\vee)$ such that $u_0(\mathcal{S}_1) = \mathcal{S}_1^\vee$. There is a linear automorphism g of $\mathbf{P}(\Delta)$ such that $\mathcal{S}_1 = g(\mathcal{S}_2)$. It is easy to check that the existence of u is equivalent to the existence of v, w in $\mathrm{Aut}(\mathcal{S}_1)$ such that either

$$u_0 g^t u_0^{-1} = v g^{-1} w \qquad \text{or} \qquad u_0 g^t u_0^{-1} = v g w.$$

We follow the approach of [22] to prove that for a general g, such elements of $H = \mathrm{Aut}(\mathcal{S}_1)$ do not exist.

First case. In order to exclude the possibility that $u_0 g^t u_0^{-1} = v g^{-1} w$, one might exhibit an $H \times H$-invariant function on $G = \mathrm{PGL}(\Delta)$ such that $F(g^{-1}) \neq F(u_0 g^t u_0^{-1})$. To do this, recall that the quadratic equations of the spinor variety $\mathcal{S} \subset \mathbf{P}(\Delta)$ are parametrized by $V_{10} \simeq V_{10}^\vee \subset S^2\Delta^\vee$. The invariant quadratic form $q \in S^2 V^\vee$ is thus mapped to an invariant element $Q \in S^2\Delta^\vee \otimes S^2\Delta^\vee$. (In fact this element belongs to the kernel of the product map to $S^4\Delta^\vee$, since the latter contains no invariant.) Dually there is an invariant element $Q^\vee \in S^2\Delta \otimes S^2\Delta$, and the function we use is $F(g) = \langle Q^\vee, gQ \rangle$. (This is actually a function on $\mathrm{SL}(\Delta)$, but a suitable power will descend to $G = \mathrm{PGL}(\Delta)$.) Indeed, restricted to a maximal torus of $\mathrm{SL}(\Delta)$, $F(u_0 g^t u_0^{-1})$ is a polynomial function of degree four, and it cannot coincide with $F(g^{-1})$ which is a polynomial of degree four in the inverses of the variables – even modulo the condition that the product of the sixteen variables is one.

Second case. As observed in [22, Proof of Lemma 4.7], it suffices to show that there exists some partition λ such that the space of H-invariants in $S_\lambda\Delta$ is at least two-dimensional (S_λ denotes the Schur functor associated to the partition λ). We provide an abstract argument for that. Suppose the contrary. Let $G = \mathrm{SL}(\Delta)$. By the Peter-Weyl theorem, the multiplicity of $S_\lambda\Delta$ inside $\mathbb{C}[G/H]$ is the dimension of its subspace of H-invariants. If this dimension is always smaller or equal to one, $\mathbb{C}[G/H]$ is multiplicity free, which means that H is a *spherical subgroup* of G. Then by [27, Theorem 1], H has an open orbit in the complete flag variety $Fl(\Delta)$. But the dimension of H is just too small for that to be true, and we get a contradiction. □

4.C. L-equivalence

Recall that $\mathbf{L}$ denotes the class of the affine line in the Grothendieck ring of complex varieties.

Proposition 4.5. *The double spinor varieties X and Y are such that*

$$([X]-[Y])\mathbf{L}^7 = 0$$

in the Grothendieck ring of varieties.

Note that when X and Y are not birational, $[X]-[Y]\neq 0$ in the Grothendieck ring (see [16, Proposition 2.2]).

Proof. The proof is the same as for Theorem 1.6 in [2]. We consider the incidence correspondence

$$\begin{tikzcd} & Q \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{S}_1 & & \mathcal{S}_2^\vee \end{tikzcd}$$

where Q is the variety of pairs $x\in\mathcal{S}_1$, $y\in\mathcal{S}_2^\vee$ such that x belongs to the hyperplane H_y. The fiber of p_2 over y is $\mathcal{S}_1\cap H_y$; it is singular if and only if y also belongs to $\mathcal{S}_1^\vee$, hence to Y. In this case the fiber is isomorphic to $\mathcal{HS}_{sing}$, otherwise it is isomorphic to $\mathcal{HS}_{reg}$. This yields two fibrations with constant fibers, which may not be Zariski locally trivial but must be ***piecewise trivial***, like in [21, Lemme 3.3]. Indeed, by [25, Theorem 4.2.3], this follows from the already mentioned result of Igusa that over any field (not of characteristic two) over which the spinor group splits, in particular over any field containing $\mathbb{C}$, there are only two orbits of non zero spinors [7, Proposition 2].

We deduce that in the Grothendieck ring of varieties,

$$[Q]=[Y][\mathcal{HS}_{sing}]+[\mathcal{S}_2^\vee - Y][\mathcal{HS}_{reg}].$$

The same analysis for the other projection yields the symmetric relation

$$[Q]=[X][\mathcal{HS}_{sing}]+[\mathcal{S}_1 - X][\mathcal{HS}_{reg}].$$

Taking the difference (recall that $\mathcal{S}_1$ and $\mathcal{S}_2^\vee$ are isomorphic varieties), we get

$$0=([X]-[Y])([\mathcal{HS}_{sing}]-[\mathcal{HS}_{reg}]).$$

But $\mathcal{HS}_{sing}$ and $\mathcal{HS}_{reg}$ both have cell decompositions (Propositions 2.6 and 2.8), with the same numbers of cells except that $\mathcal{HS}_{reg}$ has one less in dimension seven. Hence $[\mathcal{HS}_{sing}]-[\mathcal{HS}_{reg}]=\mathbf{L}^7$. □

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