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Smoothing cones over K3 surfaces

Stephen Coughlan and Taro Sano

Abstract. We prove that the affine cone over a general primitively polarised K3 surface of genus g is smoothable if and only if $g \leq 10$ or $g = 12$. We also give several examples of singularities with special behaviour, such as surfaces whose affine cone is smoothable even though the projective cone is not.

Keywords. Deformations; affine cones; K3 surfaces; Fano 3-folds

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[Français]

Titre. Sur la lissabilité des cônes sur les surfaces K3

Résumé. Nous montrons que le cône affine sur une surface K3 primitivement polarisée générale de genre g est lissable si et seulement si ≤ 10 ou $g = 12$. Nous exhibons également plusieurs exemples de singularités affichant des comportements spécifiques, tels que des surfaces dont le cône affine est lissable alors méme que le cône projectif ne l'est pas.

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1. Introduction

1.1. In this paper, we study deformations and smoothability of the affine cone over a polarized manifold. See 2.1 for basic notions.

The cone over a normal elliptic curve is smoothable if and only if the curve has degree ≤ 9 [16], the cone over a projectively normal abelian variety of dimension ≥ 2 is never smoothable [19], and the cone over a curve of genus ≥ 2 embedded in degree at least $4g-3$ is not smoothable, if the curve is not hyperelliptic, trigonal, or a plane quintic cf. [22, §15] and references therein. Aside from the case of elliptic curves, in all of the above situations, the only deformations are again cones.

The cone over a K3 surface is a natural 3-dimensional generalisation of the cone over an elliptic curve; it is a normal, Gorenstein, isolated log canonical singularity. One of the main results of this present work is the following theorem.

Theorem 1.2. *Let S be a general K3 surface with primitive polarisation of genus g. Then the affine cone over S is smoothable if and only if $g \leq 10$ or $g = 12$. Indeed, if $g = 11$ or $g \geq 13$ then the only deformations of the affine cone over S are conical.*

A deformation of the affine cone is called *conical* when the conclusion of Proposition 5.2(ii) holds. Cones over non-general K3 surfaces of genus $g = 11$ or $g > 12$ may still be smoothable (see §4).

1.3. The theorem is proved in Section 3. The "only if" part follows from the vanishing of all graded parts of T^1 of the affine cone which have non-zero degree. It is proved by using a deep theorem of Beauville [3] and its slight modification (see 3.7). A weaker result can be derived from Green's conjecture, but with a precise condition on the polarization (see 3.6). The "if" part is proved by sweeping out the cone because for $g \leq 10$ and $g = 12$, the projective cone over S deforms to a smooth Fano 3-fold, (see 3.2).

1.4. Pinkham [17] gave an example of a 0-dimensional variety whose affine cone is smoothable, even though the projective cone is not. The cone is Cohen–Macaulay but not Gorenstein or normal. In section 2, we prove the following:

Theorem 1.5. *There exists a smooth, projectively normal surface S such that the affine cone over S is smoothable but the projective cone is not.*

The example is a particular surface of general type in its canonical model. We do not know of any example where S is a K3 surface. In light of Pinkham's theorem on elliptic curves and Theorem 1.2 above, we ask:

> If S is a K3 surface, is the affine cone over S smoothable if and only if the projective cone is smoothable?

By [4, 5], the projective cone over a general K3 surface of Picard rank 1 is smoothable if and only if $g \leq 10$ or $g = 12$.

1.6. We also construct K3 surfaces whose affine cone has several smoothing components:

Theorem 1.7. *There exist primitively polarised K3 surfaces (S, L) of genus 7, such that $C_a(S, L)$ has at least two topologically distinct smoothings.*

The proof is in §4, along with an analysis of cones over imprimitively embedded K3 surfaces, and cones over special K3 surfaces of large genus.

1.8. Given a very ample line bundle L on a smooth projective variety V which induces a projectively normal embedding $V \hookrightarrow \mathbb{P}^N$, we have the "classical" projective cone $C_p(V) \subset \mathbb{P}^{N+1}$ and affine cone $C_a(V) \subset \mathbb{A}^{N+1}$. Pinkham ([16, Theorem 5.1]) showed that, if the eigenspace $T^1_{C_a(V)}(k) = 0$ for $k > 0$, then the restriction homomorphism $\mathrm{Hilb}_{C_p(V)\subset\mathbb{P}^{N+1}} \to \mathrm{Def}_{C_a(V)}$ is formally smooth, where $\mathrm{Hilb}_{C_p(V)\subset\mathbb{P}^{N+1}}$ is the Hilbert functor and $\mathrm{Def}_{C_a(V)}$ is the usual deformation functor. Moreover, Schlessinger ([19, §4.3]) showed that, if $T^1_{C_a(V)}(k) = 0$ for $k \neq 0$, then we can define a morphism $\mathrm{Hilb}_{V\subset\mathbb{P}^N} \to \mathrm{Def}_{C_a(V)}$ and it is formally smooth.

In Section 5, we generalise these results to the case where L is only assumed to be ample. This is probably known to the experts, but a proof has not been written down, so we give one in 5.2. As an application, in 5.3, we show:

Theorem 1.9. *The affine cone over any polarised abelian variety of dimension ≥ 2 has only conical deformations.*

1.10. We work over the complex numbers unless otherwise stated.

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2. Affine and projective cones

2.1. Basic properties of cones. We use standard notation from deformation theory, see for example [10], [20]. Let X be an algebraic scheme. A *deformation* of X over a scheme $B = \operatorname{Spec} A$ of finite type with a closed point $0 \in B$ is a flat morphism $\pi\colon \mathcal{X} \to B$ together with a closed immersion $X \hookrightarrow \mathcal{X}$ which identifies X with the closed fibre over 0. A deformation is called *infinitesimal* if A is local Artinian. We say that X is *smoothable* if there exists a deformation $\pi\colon \mathcal{X} \to B$ of X over an integral scheme B of finite type whose fibre $\mathcal{X}_b$ is smooth for general $b \in B$ (cf. [10, §29]).

Let (X, L) be a *polarised manifold*, that is, X is a smooth projective variety such that $\dim X \geq 1$ and L is an ample line bundle. Let

$$R(X, L) := \bigoplus_{k\geq 0} H^0(X, L^{\otimes k}).$$

The *affine cone* over (X, L) is $C_a(X, L) := \operatorname{Spec} R(X, L)$ and the *projective cone* over (X, L) is $C_p(X, L) := \operatorname{Spec} R(X, L)[x]$, where x has degree 1. By [9, 8.8.6], $C_a(X, L)$ is normal. We recall the following property.

Proposition 2.2. *Let (X, L) be a polarised manifold such that $\dim X \geq 1$. Then we have the following:*

(i) *The cone $C_a(X, L)$ is Cohen–Macaulay if and only if $H^i(X, L^{\otimes k}) = 0$ for all $0 < i < \dim X$ and $k \in \mathbb{Z}$.*

(ii) *The cone $C_a(X, L)$ is Gorenstein if and only if it is Cohen–Macaulay and $\omega_X \simeq L^{\otimes m}$ for some $m \in \mathbb{Z}$.*

Proof. For (i), it is enough to check the conditions (a) and (b) in [7, 5.1.6(ii)]. We can check (a) by the construction of $C_a(X, L)$. The condition (b) is nothing but our assumption. Part (ii) follows from [7, 5.1.9]. □

2.3. A smoothable affine cone with a non-smoothable projectivization. The following example proves Theorem 1.5.

Example 2.4. Let S be a divisor of bidegree $(3,4)$ in $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^2$. If S has at worst ordinary double points, then S is a regular surface of general type with $p_g = 6$ and $K^2 = 11$. Indeed, by adjunction, $\omega_S = \mathcal{O}_S(1,1)$. From the standard short exact sequence $0 \to \mathcal{I}_S \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_S \to 0$ and vanishing of $H^1(\mathcal{O}_{\mathbb{P}}(-2,-3))$, it follows that $p_g(S) = h^0(\mathcal{O}_{\mathbb{P}}(1,1)) = 6$. Similarly, $q(S) = h^1(\mathcal{O}_{\mathbb{P}}) = 0$ so S is regular. Writing H_1, H_2 for the generators of $\operatorname{Pic}\mathbb{P}$, we compute $K_S^2 = (3H_1+4H_2)(H_1+H_2)^2 = 11H_1H_2^2 = 11$.

The above discussion shows that the canonical model of S is induced by the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. We next describe the defining equations of the canonical model. Let s_1, s_2, t_1, t_2, t_3 be the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^2$ and let $F \in H^0(\mathcal{O}_{\mathbb{P}}(3,4))$ be the defining equation of S. We choose $f_1, f_2, f_3 \in H^0(\mathcal{O}_{\mathbb{P}}(3,3))$ such that $F = t_1f_1 + t_2f_2 + t_3f_3$. Then the coordinates giving the Segre embedding are $x_1 = s_1t_1, x_2 = s_1t_2, x_3 = s_1t_3, x_4 = s_2t_1, x_5 = s_2t_2, x_6 = s_2t_3$ and f_1, f_2, f_3 may be written as cubics in the x_i. The canonical model of S in $\mathbb{P}^5$ is thus defined by the following five equations:

$$\mathrm{Pf}_5 = x_2x_4 - x_1x_5,\ \mathrm{Pf}_4 = x_3x_4 - x_1x_6,\ \mathrm{Pf}_3 = x_3x_5 - x_2x_6,$$
$$\mathrm{Pf}_2 = s_1F = x_1f_1 + x_2f_2 + x_3f_3,\ \mathrm{Pf}_1 = s_2F = x_4f_1 + x_5f_2 + x_6f_3,$$

where Pf_2, Pf_1 are obtained by writing s_1F (respectively s_2F) in terms of the x_i. According to the Buchsbaum–Eisenbud theorem on Gorenstein codimension 3 ideals, these equations may be written as 4×4 Pfaffians of the skew matrix M of the form

$$M = \begin{pmatrix} 0 & 0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_4 & x_5 & x_6 \\ -x_1 & -x_4 & 0 & f_3 & -f_2 \\ -x_2 & -x_5 & -f_3 & 0 & f_1 \\ -x_3 & -x_6 & f_2 & -f_1 & 0 \end{pmatrix},$$

where Pf_i is the Pfaffian of the skew symmetric matrix M_i obtained from M by deleting i-th row and column. The first three equations define the Segre embedding, and the last two cut out the divisor S.

Let $X = C_a(S, K_S) \subset \mathbb{A}^6$ be the affine cone over the canonical model of S. Then by construction, X is a Gorenstein normal 3-dimensional singularity. The equations defining X are still the 4×4 Pfaffians of M, and the coordinates on $\mathbb{A}^6$ are $x_1, \ldots, x_6$.

All deformations of X are obtained by varying the entries of M [12, 27] or [10, Theorem 9.7]. Thus after coordinate changes, the general fibre X' of any deformation of X is defined by the Pfaffians of

$$M' = \begin{pmatrix} 0 & g & x_1 & x_2 & x_3 \\ -g & 0 & x_4 & x_5 & x_6 \\ -x_1 & -x_4 & 0 & f_3' & -f_2' \\ -x_2 & -x_5 & -f_3' & 0 & f_1' \\ -x_3 & -x_6 & f_2' & -f_1' & 0 \end{pmatrix},$$

where $f_i' = f_i + h_i$ for some polynomials h_i, and g is an arbitrary polynomial. Then the 4×4 Pfaffians of M' are

$$\mathrm{Pf}_5 = gf_3' + x_2x_4 - x_1x_5,\ \mathrm{Pf}_4 = -gf_2' + x_3x_4 - x_1x_6,\ \mathrm{Pf}_3 = gf_1' + x_3x_5 - x_2x_6,$$
$$\mathrm{Pf}_2 = x_1f_1' + x_2f_2' + x_3f_3',\ \mathrm{Pf}_1 = x_4f_1' + x_5f_2' + x_6f_3'.$$

The smoothability of X is well known (cf. [12, Section 5]). Let g be a nonzero constant, and choose h_i sufficiently general with some terms of degree ≤ 1. Since g is constant, Pfaffians 1 and 2 are redundant, and X' is a nonsingular complete intersection for suitably chosen h_i.

Now restrict to deformations X' that are induced by a deformation of the projective cone $C_p(S, K_S) \subset \mathbb{P}^6$. Then $g \equiv 0$ for degree reasons, and h_i must have degree ≤ 3 — in particular, the above smoothing is not induced by $C_p(S, K_S)$. Since $g = 0$, X' passes through the origin, and a computation of the partial derivatives of Pfaffians 3, 4 and 5 shows that the Jacobian matrix of X' must have rank ≤ 2 there. Thus X' is singular at the origin. As pointed out by the referee, an analysis of the tangent cone shows that, at best, the singularity of X' is given by taking two hyperplane sections through the vertex of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. These hyperplane sections are defined by some perturbations of Pf_1 and Pf_2 respectively.

Remark 2.5. For any $k \geq 3$, we get 3-fold singularities with similar properties by taking a divisor S_k in $\mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(k, k+1)$.

3. Proof of Theorem 1.2

In this section we prove

Theorem 3.1. *Let S be a general K3 surface with primitive polarisation L of genus g and write $X = C_a(S, L)$. Then T^1_X is concentrated in degree 0 if and only if $g = 11$ or $g \geq 13$, and X is smoothable if and only if $g \leq 10$ or $g = 12$.*

Indeed, if $T^1_X(k) = 0$ for all $k \neq 0$, then X has only conical deformations by Schlessinger [2, Theorem 12.1] (cf. Proposition 5.2).

3.2. A Fano 3-fold with $b_2 = 1$ and genus g exists when $2 \leq g \leq 10$ or $g = 12$ (cf. [15, §4]). Then by [3, Corollary 4.1], a general primitively polarized K3 surface (S, L) is obtained from $S \in |-K_W|$ for W a Fano 3-fold with $b_2 = 1$ and $L = -K_W|_S$. Let $\sigma \in H^0(W, -K_W)$ be the defining section of S. Then we may regard $C_a(S, L)$ as a divisor in $C_a(W, -K_W)$. Now let $\mathcal{X} \subset C_a(W, -K_W) \times \mathbb{A}^1$ be the zero locus of $\sigma + \lambda$, where λ is the parameter of the affine line $\mathbb{A}^1$. This induces a smoothing $\mathcal{X} \to \mathbb{A}^1$ of X, which is called *sweeping out the cone.*

3.3. Computing graded pieces of T^1_X. Let (V, L) be a polarized manifold. By [19, 16], the $\mathbb{C}^*$-action on $X = C_a(V, L)$ induces a grading on T^1_X, the space of isomorphism classes of first order infinitesimal deformations of X:

$$T^1_X = \bigoplus_{k \in \mathbb{Z}} T^1_X(k).$$

By [25, Theorem 3.7] we have

$$T^1_X(k) \subset H^1(V, \mathcal{E}_L \otimes L^{\otimes k}),$$

with equality when $H^1(V, L^{\otimes k}) = 0$ for all k in $\mathbb{Z}$, where $\mathcal{E}_L$ is the extension

$$0 \to \mathcal{O}_V \to \mathcal{E}_L \to \mathcal{T}_V \to 0 \tag{1}$$

corresponding to $c_1(L) \in H^1(V, \Omega^1_V) \cong \mathrm{Ext}^1(\mathcal{T}_V, \mathcal{O}_V)$. When $V = S$ is a polarized K3 surface, $H^1(S, L^{\otimes k}) = 0$ for all k in $\mathbb{Z}$, and so $T^1_X(k) \cong H^1(S, \mathcal{T}_S \otimes L^{\otimes k})$.

3.4. Vanishing for $|k| \geq 2$. We recall the following criterion of Wahl for vanishing of $T^1(k)$:

Theorem 3.5. (Wahl [26, Corollary 2.8]) *Suppose the free resolution of $\mathcal{O}_S$ begins with*

$$\mathcal{O}_S \leftarrow \mathcal{O}_{\mathbb{P}} \leftarrow \mathcal{O}_{\mathbb{P}}(-2)^a \leftarrow \mathcal{O}_{\mathbb{P}}(-3)^b \leftarrow \dots. \tag{2}$$

Then $T^1_{C_a(S)}(k) = 0$ for $k \leq -2$.

Theorem 3.6. *Let S be a K3 surface with primitive polarisation L of Clifford index > 2. Let X be the affine cone over (S, L), then $T^1_X(k)$ vanishes for $|k| \geq 2$.*

Proof. By [18], we can choose $C \in |L|$ a nonsingular irreducible curve. Since C is a hyperplane section of $S \subset \mathbb{P}^g$ and the coordinate ring of S is Gorenstein, the free resolution of $\mathcal{O}_S$ is inherited from that of $\mathcal{O}_C$. According to Green's conjecture [8], the resolution of $\mathcal{O}_C$ has the form required by Wahl's criterion if and only if $\operatorname{Cliff} C > 2$. Since Green's conjecture holds for canonical curves on any K3 surface by Voisin [23, 24] and Aprodu–Farkas [1], the theorem is proved. □

3.7. Vanishing for $|k| = 1$. We recall the following theorem of Beauville and Mori–Mukai.

Theorem 3.8. (cf. Beauville [3, §5.2], Mukai [15, §4]) *Let S be a general K3 surface with primitive polarisation L of genus $g = 11$ or $g \geq 13$. Then $H^1(S, \Omega^1_S \otimes L) = 0$.*

Since $T^1_X(-1) \cong H^1(S, \Omega^1_S \otimes L)$ and $T^1(k) \cong T^1(-k)$ because S is a K3 surface, we have the required vanishing.

We briefly explain the proof of Theorem 3.8. Let $\mathcal{P}_g$ be the moduli stack of pairs (S, C), where (S, L) is a primitively polarized K3 surface with $c_1(L)^2 = 2g - 2$, and C is a stable curve in $|L|$. Beauville [3, (5.1)] shows that the vanishing in Theorem 3.8 is equivalent to generic finiteness of the forgetful morphism of smooth irreducible Deligne–Mumford stacks $\varphi_g \colon \mathcal{P}_g \to \overline{\mathcal{M}}_g$ defined by $(S, C) \mapsto C$. Mori and Mukai [14], [15, Theorem 7] prove that φ_g is generically finite when $g = 11$ and $g \geq 13$ by constructing explicit pairs (S, C) for which the fibre $\varphi_g^{-1}(C)$ is finite.

4. Cones over some special K3 surfaces

In this section we examine the behaviour of $C_a(S, L)$ in some situations where S is a non-general K3 surface.

4.1. Cones over imprimitively polarized K3 surfaces.

Proposition 4.2. *Let (S, L) be a general primitively polarized K3 surface of genus g and fix an integer $n > 1$. Then $X = C_a(S, L^{\otimes n})$ is smoothable if and only if one of the following holds: $2 \leq g \leq 6$ and $n = 2$, or $g = 3, 4$ and $n = 3$, or $(g, n) = (3, 4)$.*

Proof. First note that $T^1_X(k) \cong T^1_Y(kn)$, where $Y = C_a(S, L)$. Since the Clifford index of L is $\leq \lfloor \frac{g-1}{2} \rfloor$, it follows from Green's conjecture and explicit computations for $g \leq 6$, that $T^1_X(k)$ vanishes for all $k \neq 0$ when (g, n) lies outside the stated values. For the converse, if $(g, n) \neq (3, 3)$ then the smoothing is given by sweeping out the cone in the Fano 3-fold $(W, -K_W)$ with $-K_W = nA$, chosen so that $S \in |-K_W|$ and $A|_S = L$. In the special case $(g, n) = (3, 3)$, the 3-fold $W_4 \subset \mathbb{P}(1, 1, 1, 1, 3)$ inducing the smoothing of X has a quotient singularity. □

4.3. The cone over a K3 surface with $g = 11$ or $g \geq 13$ can nevertheless be smoothable. If W is a Fano 3-fold and S is an anticanonical section of W with polarization $\mathcal{O}_S(1) := -K_W|_S$, then $C_a(S, \mathcal{O}_S(1))$ is smoothable. From the Mori–Mukai classification [13] of Fano 3-folds with $b_2 \geq 2$, we see that such S, W exist for $g = 11$, $13 \leq g \leq 29$ and $g = 32$. The case $g = 33$ also occurs (see 4.1 with $(g, n) = (3, 4)$). If $g > 33$, then any smoothing of $C_a(S, \mathcal{O}_S(1))$ does not lift to the projective cone $C_p(S, \mathcal{O}_S(1))$. In spite of Example 2.4, we expect that $C_a(S, \mathcal{O}_S(1))$ is not smoothable for *any* S of genus > 33.

4.4. K3 surfaces whose affine cone has at least two distinct smoothings. In this section, we prove Theorem 1.7. First recall the following example:

Example 4.5. The degree 6 del Pezzo surface Y is a hyperplane section of $V_1 = V \colon (1, 1) \subset \mathbb{P}^2 \times \mathbb{P}^2$ and $V_2 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Thus $C_a(Y, -K_Y)$ has two distinct smoothings.

Inspired by this, we found the following example:

Example 4.6. Let $\pi\colon S \to Y$ a double cover of the degree 6 del Pezzo surface Y, branched in $B \in |-2K_Y|$. Let $L := \pi^*(-K_Y)$ so that (S,L) is a primitively polarised K3 surface of degree 12 in $\mathbb{P}^7$. By Example 4.5, $Y = V_i \cap H_i$ for some $H_i \in |-\frac{1}{2}K_{V_i}|$. Take $\pi_i\colon W_i \to V_i$ a double cover branched in $X_i \in |-K_{V_i}|$, where X_i are chosen so that $X_i \cap H_i = B$ since $H^0(V_i, -K_{V_i}) \to H^0(Y, -2K_Y)$ is surjective. The W_i are Fano 3-folds with distinct topology. Indeed, W_1 (respectively W_2) is number 2.6b (resp. 3.1) of the classification [13]. Moreover, $W_i \cap \pi_i^* H_i = S$, so the affine cone $C_a(S, \mathcal{O}_S(1))$ is a hyperplane section of $C_a(W_i, -K_{W_i}) \subset \mathbb{A}^9$ for each i, and so $C_a(S, \mathcal{O}_S(1))$ has two topologically distinct smoothings.

5. On quasihomogeneous cones

Let X be a projective manifold polarised by an ample line bundle L. We generalise Pinkham and Schlessinger's criteria on $T^1(k)$ [2, Theorem 12.1], to the case where L is not necessarily very ample. Choose generators $x_1, \dots, x_n$ of degrees $w_1, \dots, w_n$ for $R(X,L) = \bigoplus_{k\geq 0} H^0(X, L^{\otimes k})$ and let $\bar{X} = \operatorname{Proj} R(X,L)$ be the image of X in weighted projective space $\mathbb{P}(w_1, \dots, w_n)$.

Lemma 5.1. *The image $\bar{X}$ in $\mathbb{P}(w_1, \dots, w_n)$ is nonsingular and avoids the singular locus of $\mathbb{P}(w_1, \dots, w_n)$.*

Proof. The isomorphism $X \simeq \operatorname{Proj} R(X,L)$ is elementary and X is embedded into $\mathbb{P}(w_1, \dots, w_n)$ by the surjection $\mathbb{C}[z_1, \dots, z_n] \to R(X,L)$ sending z_i to x_i for $i = 1, \dots, n$.

Assume that $X \cap \operatorname{Sing} \mathbb{P} \neq \emptyset$. Then there exists $I := \{i_1, \dots, i_l\} \subset \{1, \dots, n\}$ such that $w_I := \gcd(w_{i_1}, \dots, w_{i_l}) > 1$ and the corresponding stratum $\Pi_I \subset \operatorname{Sing} \mathbb{P}$ of index w_I satisfies $X \cap \Pi_I \neq \emptyset$. Let $m > 0$ be a sufficiently large integer such that $L^{\otimes m}$ is very ample and $\gcd(m, w_I) = 1$. Since we have a surjection $\mathbb{C}[z_1, \dots, z_n] \to R(X,L)$ and it induces a surjection $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \to H^0(X, L^{\otimes m})$, we obtain $V(s) \supset \Pi_I \cap X$ for nonzero $s \in H^0(X, L^{\otimes m})$. This contradicts the base point freeness of $|L^{\otimes m}|$. ☐

Let $\mathrm{Art}_{\mathbb{C}}$ be the category of Artinian local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. We denote by $\mathrm{Hilb}^w_Y\colon \mathrm{Art}_{\mathbb{C}} \to (\mathrm{Sets})$, the weighted Hilbert functor parametrizing embedded deformations of $Y \hookrightarrow \mathbb{P}^n(w)$ in the weighted projective space $\mathbb{P}^n(w)$.

Proposition 5.2. *Let X be a projective manifold, L an ample line bundle on X and $X \hookrightarrow \mathbb{P}(w_1, \dots, w_n)$ be the embedding determined by generators $x_1, \dots, x_n \in R(X,L)$.*

(i) *(Negative graded case) Suppose that $T^1_{C_a}(k) = 0$ for all $k > 0$. Then the restriction map*

$$\Phi\colon \mathrm{Hilb}^w_{C_p(X,L)} \to \mathrm{Def}_{C_a(X,L)}$$

is formally smooth.

(ii) *(Conical deformations) Suppose that $T^1_{C_a}(k) = 0$ for all $k \neq 0$. Then we have a canonical morphism of functors*

$$\Psi\colon \mathrm{Hilb}^w_X \to \mathrm{Def}_{C_a(X,L)}$$

and it is formally smooth, that is, $C_a(X,L)$ has only conical deformations.

A weaker version of part (i) can be extracted from [16]: the restriction map $\Phi\colon \mathrm{Hilb}^w_{C_p(X,L)} \to \mathrm{Def}_{C_a(X,L)}$ has a section. Indeed, Pinkham and Schlessinger [16, Proposition 2.3] showed that a quasihomogeneous cone has a versal deformation, and a small modification of the argument used in [16, Theorem 4.2] shows the claim.

Proof. We generalise the approach of [2, Theorem 12.1] to the weighted setting. For part (i), we need to show that the following two properties hold:

(1) $d\Phi\colon \mathrm{Hilb}^w_{C_p}(k[\epsilon]) \to \mathrm{Def}_{C_a}(k[\epsilon])$ is surjective.

(2) Define $\xi_A := C_{p,A} \in \mathrm{Hilb}^w_{C_p}(A)$. Let $\bar{\xi}_A := \Phi(\xi_A) \in \mathrm{Def}_{C_a}(A)$ be its image and assume that $\bar{\xi}_A$ can be lifted over a small extension $A' \in \mathrm{Art}_{\mathbb{C}}$ of A. Then there exists a lift $\xi_{A'} \in \mathrm{Hilb}^w_{C_p}(A')$ of ξ_A over A'.

We prove (1). Let $C_p \subset \mathbb{P}(1, w_1, \dots, w_n)$ denote the projective cone over X and let $\pi_p \colon C'_p \to X$ be the $\mathbb{C}$-bundle over X arising from the punctured projective cone. Since C_p is normal at the vertex, we have

$$\begin{aligned}\mathrm{Hilb}^w_{C_p}(k[\epsilon]) = H^0(N_{C_p/\mathbb{P}(1,w)}) = H^0(C'_p, \pi_p^* N_{X/\mathbb{P}(w)}) &= H^0(X, \pi_{p_*}\pi_p^* N_{X/\mathbb{P}(w)}) \\ &= \bigoplus_{j \geq 0} H^0(X, N_{X/\mathbb{P}(w)} \otimes L^{\otimes -j}).\end{aligned}$$

Now, $\mathrm{Def}_{C_a}(k[\epsilon]) = T^1_{C_a}$, and according to [25, Theorem 3.7], the graded pieces of $T^1_{C_a}$ are

$$T^1_{C_a}(k) = \mathrm{coker}\left(H^0(X, \bigoplus_{i=1}^n L^{\otimes(k+w_i)}) \to H^0(X, Q \otimes L^{\otimes k}) \right)$$

where Q is the cokernel of $\mathcal{E}_L \to \bigoplus_i L^{\otimes w_i}$. Now, Q is simply the normal bundle to X in $\mathbb{P}(w_1, \dots, w_n)$. Since $T^1_{C_a}(k) = 0$ for $k > 0$, we see that

$$d\Phi \colon \bigoplus_{k \geq 0} H^0(N_{X/\mathbb{P}(w)} \otimes L^{\otimes -k}) \to T^1_{C_a} = \bigoplus_{k \geq 0} T^1_{C_a}(-k)$$

is surjective.

Next we prove (2). The obstruction to lifting $\bar{\xi}_{A'}$ to $\xi_{A'}$ lives in $H^1(C_p, N_{C_p/\mathbb{P}(1,w)})$. As before, since C_p is normal at the vertex, we have an inclusion $H^1(C_p, N_{C_p/\mathbb{P}(1,w)}) \subset \bigoplus_{k \geq 0} H^1(X, N_{X/\mathbb{P}(w)} \otimes L^{\otimes -k})$. Thus we have an inclusion

$$H^1(C_p, N_{C_p/\mathbb{P}(1,w)}) \subset H^1(C'_a, N_{C'_a}) = \bigoplus_{k=-\infty}^{\infty} H^1(X, N_{X/\mathbb{P}(w)} \otimes L^{\otimes k}),$$

where C'_a denotes the punctured affine cone, which is a $\mathbb{C}^*$-bundle over X. By assumption, $\bar{\xi}_A|_{C'_a}$ lifts to $\bar{\xi}_{A'}|_{C'_a}$ so the image of this obstruction in $H^1(C'_a, N_{C'_a})$ vanishes. This implies the existence of $\xi_{A'}$ in $\mathrm{Hilb}^w_{C_p}(A')$ lifting ξ_A.

Proof of part (ii). We define the canonical functor $\Psi \colon \mathrm{Hilb}^w_X \to \mathrm{Def}_{C_a}$ as follows: let $\xi_A = (X_A \subset \mathbb{P}_A(w))$ in $\mathrm{Hilb}^w_{C_p}(A)$ be an embedded deformation of $X \subset \mathbb{P}(w)$. Define $R_A = A[\tilde{x}_1, \dots, \tilde{x}_n]/I_{X_A}$, where generators $\tilde{x}_i$ are chosen so that $\mathbb{P}_A(w) = \mathrm{Proj}_A A[\tilde{x}_1, \dots, \tilde{x}_n]$, and I_{X_A} is the ideal defining $X_A \subset \mathbb{P}_A(w)$, i.e. $X_A = \mathrm{Proj}_A R_A$. By construction, R_A is a flat deformation of $R(X, L)$. Indeed, the generators $\tilde{x}_i$ clearly extend x_i because X_A is embedded in $\mathbb{P}_A(w)$, and the relations and syzygies of R_A lift those of $R(X, L)$ because X_A is flat. Thus we define $\Psi(\xi_A) = \bar{\xi}_A$ in $\mathrm{Def}_{C_a}(A)$ by $\mathrm{Spec}_A R_A$. The central fibre is $\mathrm{Spec}\, R(X, L)$ and $\bar{\xi}_A$ is automatically flat. Moreover, Ψ is functorial because Hilb^w_X is.

Formal smoothness is proved in a similar way to part (i). This time we have $\mathrm{Hilb}^w_X(k[\epsilon]) = H^0(X, N_{X/\mathbb{P}(w)})$ and by assumption, $T^1_{C_a} = T^1_{C_a}(0)$ so we again have a surjection $d\Psi \colon \mathrm{Hilb}^w_X(k[\epsilon]) \to \mathrm{Def}_{C_a}(k[\epsilon])$.

Let $\bar{\xi}_{A'}$ in $\mathrm{Def}_{C_a}(A')$ be an extension of $\Psi(\xi_A) = \bar{\xi}_A$. The obstruction to lifting $\bar{\xi}_{A'}$ to $\mathrm{Hilb}^w_X(A')$ lies in $H^1(N_{X/\mathbb{P}(w)})$. Now, $H^1(N_{X/\mathbb{P}(w)}) \subset H^1(C'_a, N_{C'_a})$ and we know that $\bar{\xi}_A|_{C'_a}$ lifts to $\bar{\xi}_{A'}|_{C'_a}$ by assumption. Thus the obstruction vanishes, and $\xi_{A'}$ in $\mathrm{Hilb}^w_X(A')$ exists. □

We think that the following result is known to experts, but we could not find it in the literature. We prove it as an application of Proposition 5.2.

Corollary 5.3. *Let X be an abelian variety of dimension $n \geq 2$ and L an ample line bundle on X. Then the affine cone $C_a = C_a(X, L)$ has only conical deformations.*

Proof. Since $\mathcal{T}_X \simeq \mathcal{O}_X^{\oplus n}$, we have $H^1(X, \mathcal{T}_X \otimes L^{\otimes k}) = 0$ because $H^1(X, L^{\otimes k}) = 0$ for any $k \neq 0$ by Serre duality, Kodaira vanishing and $n \geq 2$. Hence $H^1(X, \mathcal{E}_L \otimes L^{\otimes k}) = 0$ and thus $T^1_{C_a}(k) = 0$ for $k \neq 0$ by 3.3. Now apply Proposition 5.2(ii). □

Indeed, embedded deformations of the projective cone over an abelian variety were shown to be conical in [21]. A recent preprint [11, Cor. 8.7] contains a proof of Corollary 5.3 which works in positive characteristic.

5.4. Quasismooth K3 surfaces. It would be very interesting to generalise Theorem 1.2 to the case of affine cones over quasismooth K3 surfaces embedded in weighted projective space. Some applications of this are worked out in [6], and we have already made some progress in this direction with Proposition 5.2. We believe it is possible to further extend the Proposition to the quasismooth case. This motivates future work.

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