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Crepant Resolutions and Open Strings II

Andrea Brini and Renzo Cavalieri

Abstract. We recently formulated a number of Crepant Resolution Conjectures (CRC) for open Gromov–Witten invariants of Aganagic–Vafa Lagrangian branes and verified them for the family of threefold type A-singularities. In this paper we enlarge the body of evidence in favor of our open CRCs, along two different strands. In one direction, we consider non-hard Lefschetz targets and verify the disk CRC for local weighted projective planes. In the other, we complete the proof of the quantum (all-genus) open CRC for hard Lefschetz toric Calabi–Yau three dimensional representations by a detailed study of the G-Hilb resolution of $[\mathbb{C}^3/G]$ for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Our results have implications for closed-string CRCs of Coates–Iritani–Tseng, Iritani, and Ruan for this class of examples.

Keywords. Crepant resolution conjecture; Gromov-Witten theory; open invariants; quantum cohomology; orbifold cohomology; mirror symmetry

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[Français]

Titre. Résolutions crépantes et cordes ouvertes II

Résumé. Nous avons récemment formulé un ensemble de Conjectures de Résolutions Crépantes (CRC) pour les invariants de Gromov–Witten ouverts des branes lagrangiennes de Aganagic–Vafa, et nous les avons vérifiées pour la famille des singularités transverses de type A en dimension trois. Dans cet article, nous élargissons le faisceau de preuves en faveur de nos CRC ouvertes, et ce dans deux directions. Dans la première, nous considérons des cibles satisfiant la condition dite de "Lefschetz forte" et vérifions la CRC du disque pour des plans projectifs à poids locaux. Dans l'autre, nous complétons la démonstration de toutes les CRC ouvertes quantiques (en tout genre) pour les représentations tridimensionnelles toriques de type Calabi–Yau et vérifiant la condition de Lefschetz forte, ceci se faisant à travers une étude détaillée de la résolution G-Hilb de $[\mathbb{C}^3/G]$ pour $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Nos résultats ont des conséquences sur les CRC pour les cordes fermées de Coates–Iritani–Tseng, Iritani et Ruan pour cette classe d'exemples.

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1. Introduction

In a recent paper [2], we proposed two versions of a Crepant Resolution Conjecture for open Gromov–Witten invariants of Aganagic–Vafa orbi-branes inside semi-projective toric Calabi–Yau 3-orbifolds:

- a general Bryan–Graber-type comparison between disk potentials after analytic continuation (**the disk CRC**);
- a stronger identification of the full open string partition function at all genera and arbitrary boundary components for hard Lefschetz targets (**the quantized open CRC**).

We recall these statements more precisely in Section 2. Both conjectures were proved in [2] for the case of the crepant resolutions of type A threefold singularities, but they are expected to hold in wider generality. In particular, the disk CRC should hold true for general (non-hard Lefschetz) toric CY3 that are projective over their affinization; moreover, the proof of the quantized open CRC in [2] left out one exceptional example of (toric) hard Lefschetz crepant resolution. The purpose of this paper is to offer further evidence of the general validity of the disk CRC, as well as to conclude the proof of the quantized open CRC for hard Lefschetz toric three dimensional representations.

The first problem we tackle is the disk CRC for non-hard Lefschetz targets. We concentrate our attention to local weighted projective planes: our poster-child is the partial crepant resolution $\pi : K_{\mathbb{P}(1,1,n)} \to \mathbb{C}^3/\mathbb{Z}_{n+2}$, where π contracts the image of the zero section to give the quotient singularity $\frac{1}{n+2}(1,1,-2)$. In particular, we establish the following

Theorem 1 [(Theorem 3.6 and Corollary 3.7)]: the disk CRC holds for $Y = K_{\mathbb{P}(n,1,1)}$ and $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{n+2}]$.

On a somewhat orthogonal direction, we complete the study of hard Lefschetz crepant resolutions of three dimensional representations by considering the G-Hilb resolution of $[\mathbb{C}^3/G]$ for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ – the so-called *closed topological vertex* geometry studied in [4].

Theorem 2 [(Theorem 4.7 and Corollary 4.8)]: the quantized CRC holds for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$ and Y its canonical G-Hilb resolution.

In [5], it was shown in detail in the specific example of the A_1 threefold singularity that the local CRC for $[\mathbb{C}^3/\mathbb{Z}_2]$ glues to a crepant resolution statement for $K_{\mathbb{P}^1\times\mathbb{P}^1} \to [\mathcal{O}(-1)_{\mathbb{P}^1} \oplus \mathcal{O}(-1)_{\mathbb{P}^1}/\mathbb{Z}_2]$. Theorem 2, the results of [2], and a suitable generalization of the gluing theorem of [5] would together imply the all genus open CRC for all toric hard Lefschetz CY3 targets.

Context and further discussion

Good part of the proof of Theorem 1 relies on the well-established mirror symmetry framework of [10, 6]: we construct twisted I-functions as hypergeometric modifications of the untwisted ones and then study their analytic continuation corresponding to a change of chamber in the Kähler moduli space of the target. The first step is standard [27, 9, 8]; for the second, we overcome the technical intricacies of the Mellin–Barnes method [6] through a combined use of hypergeometric resummation and a generalized Kummer-type connection formula for the analytic continuation across a single wall. This technique has a number of features of independent interest: it turns out to be significantly more powerful than the usual Mellin–Barnes method, and it is applicable to the study of wall-crossings in toric Gromov–Witten theory in quite large generality. In particular, it might be applied in combination with the mirror theorem of [7] for the study, and hopefully the proof, of the closed-string CRC in the toric setting.

As for Theorem 2, our strategy to prove it follows closely ideas of [2] for the case of $[\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}]$. In [2, 1], the Gromov–Witten/Integrable Systems was employed to exhibit a one-dimensional Landau–Ginzburg mirror model for the equivariant quantum cohomology of type A resolutions: the relevant superpotential was identified with the dispersionless Lax function of the q-deformed $(n+1)$-KdV hierarchy. For the case of $[\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$, the relevant Frobenius manifold turns out to be the coefficient space of a particular reduction of the genus-zero Whitham hierarchy with three marked points [24]; a detailed study of this system and its bihamiltonian structure will appear elsewhere. As was the case in [2], this has two main upshots: in genus zero, it allows a one-step study of wall-crossing beyond multiple walls; and in higher genus, it significantly reduces the complexity of the proof of the quantized version of the open CRC, which turns into an exercise in all-order classical Laplace asymptotics.

Limited to the class of examples considered here, our results also have implications for ordinary (closed) Crepant Resolution Conjectures of Iritani [21] and Coates–Iritani–Tseng/Ruan [10, 11]. The proof of the disk CRC in Section 3 establishes in particular a natural fully-equivariant version of Iritani's K-theoretic Crepant Resolution Conjecture for the examples at hand[1], whereas the study of the quantized OCRC in Section 4 leads us to verify the all-genus closed CRC with descendents for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$.

Plan of the paper

The paper is organized as follows. In Section 2, we concisely review our setup in [2] for the disk and the quantized open CRC. We then furnish a proof of the disk CRC in Section 3, and study its implications at the level of scalar potentials for each of the two brane setups allowed by the geometry. In Section 4 we study the closed topological vertex geometry: we first present a mirror description in terms of a one-dimensional logarithmic Landau–Ginzburg model, which is then used in the analytic continuation relevant for the disk CRC and the all-order asymptotic analysis necessary to establish the quantized OCRC.

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2. Crepant Resolution Conjectures: a review

Given $\mathcal{X}$ a Gorenstein algebraic orbifold and $Y \to X$ a crepant resolution of its coarse moduli space, Ruan conjectured [26] that the small quantum cohomologies of Y and $\mathcal{X}$ should be isomorphic after analytic continuation and a suitable identification of the quantum parameters. More recently, Coates–Iritani–Tseng

[1] ↑ A much more general proof for semi-projective toric orbifolds has been announced by Coates–Iritani–Jiang.

shaped – and generalized – Ruan's original Crepant Resolution Conjecture (CRC) into a comparison of Lagrangian cones via a symplectic isomorphism $\mathbb{U}_\rho^{\mathcal{X},Y} : \mathcal{H}_\mathcal{X} \to \mathcal{H}_Y$ between the Givental spaces of $\mathcal{X}$ and Y [10]; here ρ denotes a choice of analytic continuation path. Further, Iritani's theory of integral structures [21] makes a prediction for $\mathbb{U}_\rho^{\mathcal{X},Y}$ based exclusively on the classical geometry of the targets. In this section we briefly summarize some of the recent extensions of the Coates–Iritani–Tseng CRC that this work relates to, and that are relevant for our formulation of the CRC for open Gromov–Witten invariants. Background, motivation, and extensive discussions of the setup presented here can be found in our previous paper [2, Sec. 2 and App. A]; the reader who is not familiar with the closed string CRC and its higher genus analogues is referred to the survey papers [11, 22].

2.A. The disk CRC

In [2], the authors formulate an Open Crepant Resolution Conjecture (OCRC) as a comparison diagram relating geometric objects in the Givental spaces of the targets, following the philosophy of [10]. Let $\mathcal{W}$ be a three-dimensional CY toric orbifold, p a fixed point such that a neighborhood is isomorphic to $[\mathbb{C}^3/G]$, with $G \cong \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_l}$. The local group action is defined by the character vectors $(\vec{m}^1, \vec{m}^2, \vec{m}^3)$ and a Calabi–Yau 2-torus action $T \simeq (\mathbb{C}^*)^2$ is specified by weights $(w_1, w_2, w_3) \in H_T^\bullet(\mathrm{pt})$. Fix a Lagrangian boundary condition L which we assume to be on the first coordinate axis in this local chart. Define $n_{\mathrm{eff}} = \mathrm{lcm}\{n_j/\gcd(m_j^1, n_j) \mid j = 1, \ldots, l\}$ to be the size of the effective part of the action along the first coordinate axis. There exist a map from an orbi-disk mapping to the first coordinate axis with winding d and twisting[2] $\vec{k}$ if the compatibility condition

$$\frac{d}{n_{\mathrm{eff}}} - \sum_{j=1}^{l} \frac{k_j m_j^1}{n_j} \in \mathbb{Z} \tag{1}$$

is satisfied. Via the Atiyah–Bott isomorphism, the Chen–Ruan cohomology ring of $[\mathbb{C}^3/G]$ is naturally identified with a part of $H_T^\bullet(\mathcal{W})$, with generators $\mathbf{1}_{\mathbf{p},\mathbf{k}}$. Denoting by $\mathbf{1}_\mathbf{p}^\mathbf{k}$ the Poincaré dual of $\mathbf{1}_{\mathbf{p},\mathbf{k}}$, we define the disk tensor at p as:

$$\overline{\mathcal{D}}^+_{\mathcal{W},p}(z;\vec{w}) \triangleq \frac{\pi}{w_1 |G| \sin\left(\pi\left(\left\langle \sum_{j=1}^{l} \frac{k_j m_j^3}{n_j}\right\rangle - \frac{w_3}{z}\right)\right)} \frac{1}{\overline{\Gamma}^k_{\mathcal{W}}} \mathbf{1}_\mathbf{p}^\mathbf{k} \otimes \mathbf{1}_\mathbf{p}^\mathbf{k}, \tag{2}$$

where $\overline{\Gamma}^k_{\mathcal{W}}$ is the $\mathbf{1}_{\mathbf{p},\mathbf{k}}$ coefficient of Iritani's homogenized Gamma function ([2], Eqn. (27)). The global ***disk tensor*** for $\mathcal{W}$ is then defined as the sum of the disk tensors at the points adjacent to the Lagrangian L in the toric diagram of $\mathcal{W}$. Note that z is thought of as the descendant parameter and hence $\overline{\mathcal{D}}^+_{\mathcal{W}}(z;\vec{w})$ is naturally a tensor on $\mathcal{H}_\mathcal{W}$, the Givental space of $\mathcal{W}$.

The ***winding neutral disk potential*** is defined to be the contraction of the J function of $\mathcal{W}$ with the disk tensor. Lowering indices in the J function with the Poincaré pairing, we can write this as the composition:

$$\mathbb{F}_L^{\mathrm{disk}}(\tau, z, \vec{w}) \triangleq \overline{\mathcal{D}}^+_{\mathcal{W}} \circ J^{\mathcal{W}}(\tau, z; \vec{w}). \tag{3}$$

The winding neutral disk potential is a section of Givental space that contains information about disk invariants at all winding, in the sense that disk invariants of winding d appear in the specialization of $\mathbb{F}_L^{\mathrm{disk}}(t, z, \vec{w})$ at $z = n_{\mathrm{eff}} w_1/d$, as coefficients in front of monomials where the compatibility condition (1) is satisfied. Rather then performing the specialization of the variable z to construct a generating function for open invariants, we formulate the OCRC as a comparison diagram of winding neutral disk potentials, i.e. a comparison among sections of Givental space.

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[2] ↑ Here twisting refers to the image of the center of the disk in the evaluation map to the inertia orbifold.

Proposal 1. (The OCRC) *For $\mathcal{W}$ either $\mathcal{X}$ or Y, let $\Delta_{\mathcal{W}}$ denote the free module in the cohomology of $\mathcal{W}$ over $H(BT)$ spanned by the T-equivariant lifts of Chen–Ruan cohomology classes having age-shifted degree at most two. There exists a $\mathbb{C}((z^{-1}))$-linear map of Givental spaces $\mathbb{O}: \mathcal{H}_{\mathcal{X}} \to \mathcal{H}_Y$ and analytic functions $\mathfrak{h}_{\mathcal{W}}: \Delta_{\mathcal{W}} \to \mathbb{C}$ such that*

$$\mathfrak{h}_Y^{1/z} \mathbb{F}^{\text{disk}}_{L,Y}\Big|_{\Delta_Y} = \mathfrak{h}_{\mathcal{X}}^{1/z} \mathbb{O} \circ \mathbb{F}^{\text{disk}}_{L,\mathcal{X}}\Big|_{\Delta_{\mathcal{X}}} \tag{4}$$

upon analytic continuation of quantum cohomology parameters.

The analytic functions $\mathfrak{h}_{\mathcal{W}}$ arise from the discrepancy between the small J-function and the canonical basis-vector of solutions of the Picard–Fuchs system: a precise definition and discussion appears in [2, App. A.1.1]. Here we only remark that the functions $\mathfrak{h}_{\mathcal{W}}$ are completely determined by classical geometric data. Because of the close relationship between the disk tensor and the Gamma factors of the central charge in Iritani's theory of integral structures [21, 2], we have a prediction for the transformation $\mathbb{O}$ in terms of the toric geometry of the targets.

Proposal 2. (The transformation $\mathbb{O}$) *Choose a grade restriction window $\mathfrak{W}$ in the GIT problem to identify the K-theory lattices of $\mathcal{X}$ and Y, and for $\mathcal{W} = \mathcal{X}, Y$, define:*

$$\Theta_{\mathcal{W}}(\mathbf{1}_{\mathbf{p},\mathbf{k}}) \triangleq \frac{1}{\sin\left(\pi\left(\left\langle \sum_{j=1}^{l} \frac{k_j m_j^3}{n_j}\right\rangle - \frac{w_3}{z}\right)\right)} \mathbf{1}_{\mathbf{p}}^{\mathbf{k}} \tag{5}$$

Then the transformation $\mathbb{O}$ in Proposal 1 has the form:

$$\mathbb{O} = \Theta_Y \circ \overline{\mathrm{CH}}_Y \circ \overline{\mathrm{CH}}_{\mathcal{X}}^{-1} \circ \Theta_{\mathcal{X}}{}^{-1}, \tag{6}$$

where we denote by $\overline{\mathrm{CH}}_{\mathcal{W}} = z^{-\frac{1}{2}\deg} \mathrm{CH}_{\mathcal{W}}$ the matrix of Chern characters (homogenized with respect to the cohomological degree "$\deg$") in the bases given by $\mathfrak{W}$.

In [2], we show that Proposal 1 follows from the Coates–Iritani–Tseng's CRC. Proposal 2 coincides with $\mathbb{U}_\rho^{\mathcal{X},Y}$ being predicted by a natural equivariant version[3] of Iritani's K-theoretic Crepant Transformation Conjecture [21]:

Conjecture 2.1. *For $\mathcal{W} = \mathcal{X}, Y$, denote by $\overline{\Gamma}_{\mathcal{W}}$ the diagonal matrix whose kk entry is $\overline{\Gamma}^k_{\mathcal{W}}$. Then, for every choice $\mathfrak{M}$ of grade restriction window, there exists a choice of analytic continuation path ρ such that*

$$\mathbb{U}_\rho^{\mathcal{X},Y} = \overline{\Gamma}_Y \circ \overline{\mathrm{CH}}_Y \circ \overline{\mathrm{CH}}_{\mathcal{X}}^{-1} \circ \overline{\Gamma}_{\mathcal{X}}^{-1}. \tag{7}$$

From Proposal 1 one can extract comparison statements about generating functions for disk invariants. The strongest statement can be made when the Lagrangian boundary condition intersects a leg whose isotropy is preserved in the crepant transformation.

Proposal 3. (Scalar disk potentials) *Let L be a Lagrangian boundary condition on $\mathcal{X}$ that intersects a torus invariant line whose generic point has isotropy group G_L, and such that if we denote L' be the corresponding boundary condition in Y, then L' also intersects a torus invariant line with generic isotropy group G_L. For $\mathcal{W} = \mathcal{X}, Y$, define the scalar disk potential[4] :*

$$F^{\text{disk}}_{\mathcal{W}}(\tau, y, \vec{w}) = \sum_d \frac{y^d}{d!} \sum_n \frac{1}{n!} \left| \langle \tau, \dots, \tau \rangle_{0,n}^{\mathcal{W},L,d} \right| \triangleq \sum_d \frac{y^d}{d!} \left| \left(\mathcal{D}^+_{\mathcal{W}}(d;\vec{w}), J^{\mathcal{W}}\left(\tau, \frac{n_{\text{eff}} w_1}{d}\right)\right)_{\mathcal{W}} \right|. \tag{8}$$

[3] ↑ The fact that Γ-integral structures match with the natural B-model integral structures under mirror symmetry was proved in [21] for compact toric orbifolds. A general proof of the fully equivariant version of Iritani's K-theoretic CRC has been announced by Coates–Iritani–Jiang.

[4] ↑ We choose to define the scalar disk potential as a generating function for the *absolute value* of disk invariants. In the course of the verifications of Proposal 3, one may observe that the scalar potentials could be matched on the nose with the use of appropriate matrices of roots of unity - that in the end contribute just signs, albeit with some non-trivial pattern. We have deliberately forgone to keep track of these phenomena, especially in light of the choice-of-signs the theory of open invariants is everywhere laden with.

Then, upon identifying the insertion variables via the change of variable prescribed by the closed CRC, we have:

$$F_{L',Y}^{\mathrm{disk}}(\tau, \mathfrak{h}_Y^{\frac{1}{n_{\mathrm{eff}}w_1}} y, \vec{w}) = F_{L,\mathcal{X}}^{\mathrm{disk}}(\tau, \mathfrak{h}_{\mathcal{X}}^{\frac{1}{n_{\mathrm{eff}}w_1}} y, \vec{w}). \tag{9}$$

2.B. Hard Lefschetz targets: the quantized OCRC

When $\mathcal{X}$ satisfies the hard Lefschetz condition[5], a natural generalization of the CRC to higher genus GW invariants is achieved by canonical quantization [10, 11]: the all-genus Gromov–Witten partition functions are viewed as elements of the respective Fock spaces [19, 18], conjecturally matched by the Weyl-quantization of the classical canonical transformation $\mathbb{U}_\rho^{\mathcal{X},Y}$.

Conjecture 2.2. (The hard Lefschetz quantized CRC, from [10, 11]) *Let $\mathcal{X} \to X \leftarrow Y$ be a Hard Lefschetz crepant resolution diagram for which the Coates–Iritani–Tseng CRC holds. For $\mathcal{W}$ either $\mathcal{X}$ or Y, let $Z_{\mathcal{W}}$ denote the generating function of disconnected Gromov–Witten invariants of $\mathcal{W}$ viewed as an element of the Fock space of $H_{\mathrm{orb}}(\mathcal{W}) \otimes \mathbb{C}((z))$, and $\mathbb{U}_\rho^{\mathcal{X},Y}$ the Coates–Iritani–Tseng morphism of Givental spaces identifying the Lagrangian cones of $\mathcal{X}$ and Y. Then*

$$Z_Y = \widehat{\mathbb{U}_\rho^{\mathcal{X},Y}} Z_{\mathcal{X}} \tag{10}$$

In the context of torus-equivariant Gromov–Witten theory of orbifolds with zero-dimensional fixed loci, the hard Lefschetz quantized CRC can be proven in two steps [2, Prop. 6.3], as follows.

(1) Combining the Coates–Givental/Tseng quantum Riemann–Roch theorem [9, 27] with Givental's quantization formula in a neighborhood of the large radius points of $\mathcal{W}$ identifies a "canonical" R-calibration defined locally by the genus 0 GW theory of $\mathcal{W}$;

(2) Conjecture 2.2 then follows from establishing the equality, upon analytic continuation, of the canonical R-calibrations of $\mathcal{X}$ and Y on the locus where the quantum product is semi-simple.

The main consequence drawn in [2] for open Gromov–Witten invariants is a CRC statement for all genera and number of holes.

Proposal 4. (The quantized OCRC [2]) *Let $\mathcal{X} \to X \leftarrow Y$ be a Hard Lefschetz diagram for which the higher genus closed CRC holds. Define the genus g, ℓ-holes winding neutral potential $\mathbb{F}_{\mathcal{W},L}^{g,\ell} : H(\mathcal{W}) \to \mathcal{H}_{\mathcal{W}}^{\otimes \ell}$ as*

$$\mathbb{F}_{\mathcal{W},L}^{g,\ell}(\tau, z_1, \ldots, z_\ell, \vec{w}) \triangleq \overline{\mathcal{D}}_{\mathcal{W}}^{+\otimes \ell} \circ J_{g,\ell}^{\mathcal{W}}(\tau, z_1, \ldots, z_\ell; \vec{w}), \tag{11}$$

where $J_{g,\ell}^{\mathcal{W}}$ encodes genus g, ℓ-point descendent invariants:

$$J_{g,\ell}^{\mathcal{W}}(\tau, \mathbf{z}; \vec{w}) \triangleq \left\langle\left\langle \frac{\phi_{\alpha_1}}{z_1 - \psi_1}, \ldots, \frac{\phi_{\alpha_\ell}}{z_\ell - \psi_\ell} \right\rangle\right\rangle_{g,\ell} \phi^{\alpha_1} \otimes \cdots \otimes \phi^{\alpha_\ell}. \tag{12}$$

Further, let $\mathbb{O}^{\otimes \ell} = \mathbb{O}(z_1) \otimes \ldots \otimes \mathbb{O}(z_\ell)$. Then,

$$\mathbb{F}_{L',Y}^{g,\ell} = \mathbb{O}^{\otimes \ell} \circ \mathbb{F}_{L,\mathcal{X}}^{g,\ell}. \tag{13}$$

[5] ↑ This is $\mathrm{age}(\phi) = \mathrm{age}(I^*(\phi))$ for all $\phi \in H_{\mathrm{orb}}(\mathcal{X})$, where $I : I\mathcal{X} \to I\mathcal{X}$ is the canonical involution on the inertia stack.

3. Example 1: local weighted projective planes

3.A. Classical geometry

The family of geometries we study arises as the GIT quotient

$$\mathbb{C}^4 /\!\!/_{\chi} \mathbb{C}^{\star}, \tag{14}$$

with torus action on the coordinates (x_1, x_2, x_3, x_4) specified by the charge matrix

$$M = \begin{pmatrix} n & 1 & 1 & -2-n \end{pmatrix}. \tag{15}$$

The quotients obtained as the character χ varies are the toric varieties whose fans are represented in Figure 1. The right hand side of Figure 1 corresponds to $\chi > 0$. The irrelevant ideal is

$$\mathcal{I}_{\mathrm{LR}} \triangleq \langle x_1, x_2, x_3 \rangle \tag{16}$$

and the resulting geometry Y is the total space of $\mathcal{O}(-n-2)_{\mathbb{P}(n,1,1)}$; $[x_1 : x_2 : x_3]$ serve as (quasi)-homogeneous coordinates for the base, and x_4 is an affine fiber coordinate. Torus fixed points and invariant lines are:

$$L_1 = V(x_1, x_4), \qquad L_2 = V(x_2, x_4), \qquad L_3 = V(x_3, x_4), \tag{17}$$

$$P_1 = V(x_2, x_3, x_4), \qquad P_2 = V(x_1, x_3, x_4), \qquad P_3 = V(x_1, x_2, x_4). \tag{18}$$

We have $L_1 \simeq \mathbb{P}^1$, $L_2, L_3 \simeq \mathbb{P}(1,n)$, P_2, $P_3 \simeq [\mathrm{pt}]$, $P_1 \simeq B\mathbb{Z}_n$. The fibers over the fixed points P_2 and P_3 are non-gerby. The fiber over P_1 is non-gerby when when n is odd; when n is even, it has a $\mathbb{Z}_2$-subgroup as a stabilizer.

When χ is negative we have the fan on left hand side of Figure 1, which gives the irrelevant ideal

$$\mathcal{I}_{\mathrm{OP}} \triangleq \langle x_4 \rangle. \tag{19}$$

Quotienting by $x_4 \neq 0$ gives a residual $\mathbb{Z}_{n+2}$ action on $\mathbb{C}^3$ with weights $(n, 1, 1)$; the resulting orbifold $[\mathbb{C}^3/\mathbb{Z}_{n+2}]$ will be denoted by $\mathcal{X}$. Moving across $\chi = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{C}^4 /\!\!/ \mathbb{C}^* \to \begin{bmatrix} x_1 x_4^{\frac{n}{n+2}} \\ x_2 x_4^{\frac{1}{n+2}} \\ x_3 x_4^{\frac{1}{n+2}} \end{bmatrix} \in \mathbb{C}^3/\mathbb{Z}_{n+2} \tag{20}$$

where we denoted by $[x_1, \ldots, x_n]$ the equivalence class in the appropriate quotient, is a birational contraction of the image of the zero section $s : \mathbb{P}(n,1,1) \hookrightarrow K_{\mathbb{P}(n,1,1)}$.

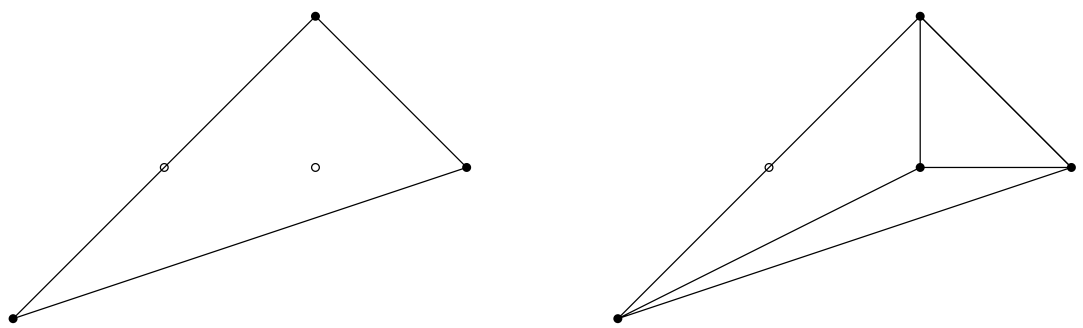

Figure 1: A height 1 slice of the fans of $[\mathbb{C}^3/\mathbb{Z}_{n+2}]$ (left) and local $\mathbb{P}(n,1,1)$ (right) for $n = 2$.

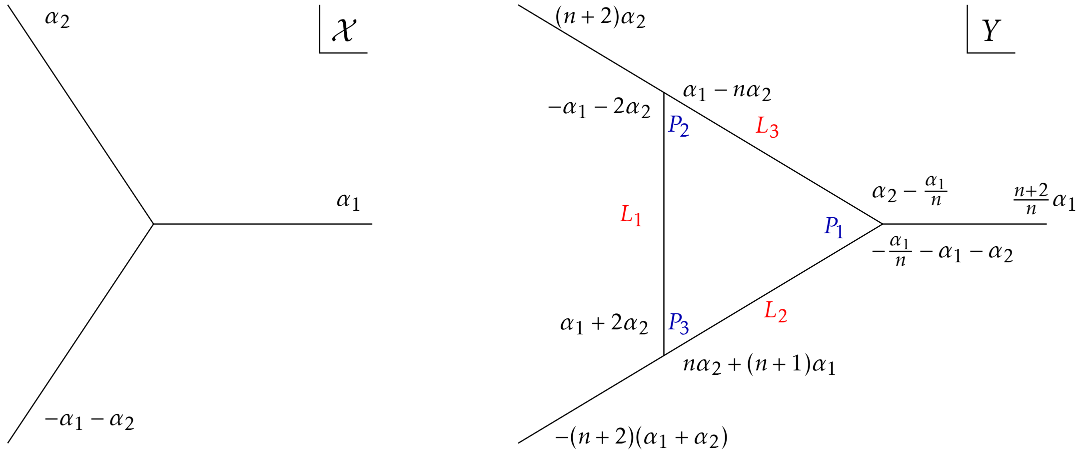

Figure 2: Toric web diagrams and weights at the fixed points for $\mathcal{X}$ and Y.

3.A.a. Bases for cohomology

We consider a Calabi–Yau 2-torus action on Y and $\mathcal{X}$, descending from an action on $\mathbb{C}^4$ with geometric weights $(\alpha_1, \alpha_2, -(\alpha_1+\alpha_2), 0)$. Note that we consider the geometric weights as elements of $H^2(BT)$: an integer α corresponds to the first Chern class of the representation $t \mapsto t^\alpha$. The tangent weights at the torus fixed points are depicted in the toric diagrams in Figure 2.

Let $p = \pi^* c_1(\mathcal{O}_{\mathbb{P}(n,1,1)}(1)) \in H_T(K_{\mathbb{P}(n,1,1)})$, where $\pi : K_{\mathbb{P}(n,1,1)} \to \mathbb{P}(n,1,1)$ is the bundle projection and the torus action on $\mathcal{O}_{\mathbb{P}(n,1,1)}(1)$ is linearized canonically by identifying $\mathbb{C}^4$ with the tautological bundle $\mathcal{O}_{\mathbb{P}(n,1,1)}(-1)$. Via the Atiyah–Bott isomorphism we have:

$$p = -\frac{\alpha_1}{n} P_1 - \alpha_2 P_2 + (\alpha_1+\alpha_2) P_3 \in H^2_T(K_{\mathbb{P}(n,1,1)}). \tag{21}$$

The products w_i of the three normal (tangent) weights at the fixed points P_i read

$$\begin{aligned} w_1 &= -\frac{n+2}{n}\alpha_1\left(\alpha_2 - \frac{\alpha_1}{n}\right)\left(\alpha_1+\alpha_2+\frac{\alpha_1}{n}\right), \\ w_2 &= -(n+2)\alpha_2(\alpha_1 - n\alpha_2)(\alpha_1+2\alpha_2), \\ w_3 &= -(n+2)(\alpha_1+\alpha_2)(\alpha_1+n(\alpha_1+\alpha_2))(\alpha_1+2\alpha_2). \end{aligned} \tag{22}$$

As a module over $H(BT)$, the equivariant Chen–Ruan cohomology ring of $Y = K_{\mathbb{P}(n,1,1)}$ is spanned by $\{\mathbf{1}_Y, p, p^2, \mathbf{1}_{\frac{1}{n}}, \ldots, \mathbf{1}_{\frac{n-1}{n}}\}$. On $\mathcal{X}$, we have cohomology classes $\mathbf{1}_g$, labeled by the corresponding group elements $g = 1, \mathrm{e}^{2\pi\mathrm{i}/n+2}, \ldots, \mathrm{e}^{2\pi\mathrm{i}(n+1)/(n+2)}$; the involution on the inertia stack exchanges $\mathbf{1}_{\frac{k}{n+2}} \leftrightarrow \mathbf{1}_{1-\frac{k}{n+2}}$.

3.B. Quantum geometry

Genus-zero Gromov–Witten invariants of $\mathcal{X}$ and Y can be computed using the quantum Riemann–Roch theorems of Coates–Givental [9] and Tseng [27] applied to the Gromov–Witten theories of $B\mathbb{Z}_{n+2}$ and $\mathbb{P}(n,1,1)$, respectively. We have the following

Proposition 3.1. **([9, 27, 8])** *For* $|y| < n^n(n+2)^{-2-n}$,$|x| < (n+2)n^{-n/(n+2)}$, *define the* I-functions

$$I^Y(y,z) \triangleq z y^{p/z} \sum_{nd\in\mathbb{Z}^+} y^d \frac{\prod_{\substack{\langle m\rangle = \langle (n+2)d\rangle \\ 0\le m<(n+2)d-1}} (-(n+2)p - mz)}{\prod_{\substack{\langle m\rangle=\langle d\rangle \\ 0<m\le d}} (p+\alpha_2+mz)(p-\alpha_1-\alpha_2+mz)} \frac{1}{\prod_{m=1}^{nd}(np+\alpha_1+mz)}, \tag{23}$$

$$I^{\mathcal{X}}(x,z) \triangleq \sum_{k\geq 0} \frac{\prod_{\substack{\langle b\rangle=\langle k/(n+2)\rangle \\ 0\leq b<\frac{k}{n+2}}}\left(\frac{\alpha_2}{n+2}-bz\right)\left(-\frac{\alpha_1+\alpha_2}{n+2}-bz\right)\prod_{\substack{\langle b\rangle=\langle kn/(n+2)\rangle \\ 0\leq b<\frac{kn}{n+2}}}\left(\frac{n\alpha_1}{n+2}-bz\right)}{z^k}\frac{x^k}{k!}\mathbf{1}_{\langle k/n+2\rangle}. \tag{24}$$

Then, for $\mathcal{W}$ either $\mathcal{X}$ or Y and w either x or y, $I^{\mathcal{W}}(w,-z) \in -z + H_T(\mathcal{W})\otimes\mathbb{C}[[z^{-1}]]\cap\mathcal{L}_{\mathcal{W}}$ identically in w.

Proof. This is [6, Theorem 3.5 and 3.7]. □

Since the I-functions of $\mathcal{X}$ and Y belong to the cone and behave like $z+\mathcal{O}(1)$ at large z, they coincide with suitable restrictions of the respective big J-functions to a subfamily of quantum cohomology parameters.

Corollary 3.2. *Denote by q the Novikov variable associated to p and write $\phi = \sum_{k=0}^{n+1}\tau_{\frac{k}{n+2}}\mathbf{1}_{\frac{k}{n+2}}$ for an orbifold cohomology class $\phi \in H_T^{\mathrm{orb}}(\mathcal{X})$. Then the following equalities hold:*

$$J^Y_{\mathrm{small}}(q,z) = I^Y(y(q),z), \tag{25}$$

$$J^{\mathcal{X}}_{\mathrm{big}}(\phi,z)\Big|_{\tau_{k/(n+2)}=\delta_{k1}\tau} = I^Y(x(\tau),z), \tag{26}$$

where $\log q = \lim_{z\to\infty}(I^Y(y,z)-z)$, $\tau = \lim_{z\to\infty}(I^{\mathcal{X}}(x,z)-z)$. *In particular,*

$$\mathfrak{h}_Y = \mathfrak{h}_{\mathcal{X}} = 1. \tag{27}$$

3.B.a. Analytic continuation and $\mathbb{U}_\rho^{\mathcal{X},Y}$

A standard method [10, 8] to relate the Lagrangian cones of $\mathcal{X}$ and Y upon analytic continuation hinges on the following three-step procedure:

(1) find a holonomic linear differential system of rank equal to $\dim H^\bullet(Y) = \dim H^\bullet_{\mathrm{orb}}(\mathcal{X})$ jointly satisfied, upon appropriate identification of the quantum parameters, by the components of the I-functions of $\mathcal{X}$ and Y as convergent power series around the respective boundary point;

(2) determine the relation between the I-functions upon analytic continuation along a path ρ connecting the two boundary points;

(3) invoke a reconstruction theorem to recover from the latter the content of big quantum cohomology and the full-descendent theory in genus zero [7, 13].

Step (3) has been achieved in full generality for toric Deligne–Mumford stacks in [7]. The first step is also standard [17]; we spell out the details below for the sake of completeness. The main intricacy here lies in Step (2), as the rank of the system is parametrically large in n and the usual Mellin–Barnes method [6, 20] is technically more subtle to apply; we present a workaround in the discussion leading to Lemma 3.4.

Lemma 3.3. *Let $\mathcal{D}_Y$ the $(n+2)^{\mathrm{th}}$ order linear differential operator*

$$\mathcal{D}_Y \triangleq (\theta_y+\alpha_2)(\theta_y-\alpha_1-\alpha_2)\prod_{m=0}^{n}(n\theta_y+\alpha_1-mz) - y\prod_{m=0}^{n+1}(-(n+2)\theta_y-mz) \tag{28}$$

where $\theta_y = zy\partial_y$ and define $\mathcal{D}_{\mathcal{X}}$ to be the differential operator obtained by replacing $y = x^{-n-2}$ in Eq. (28). *Then,*

$$\mathcal{D}_\bullet I^\bullet = 0 \tag{29}$$

Proof. The statement follows from a straightforward calculation from Eqs. (23) and (24). □

The linear operator $\mathcal{D}_{\mathcal{W}}$ is the Picard–Fuchs operator of $\mathcal{W} = \mathcal{X}, Y$: Lemma 3.3 establishes that the torus-localized components of the I-functions of $\mathcal{X}$ and Y furnish two bases solutions of the linear system $\mathcal{D}_{\mathcal{W}} f = 0$, respectively in the neighbourhood of the Fuchsian points $y = 0$ and ∞. Relating the cones of $\mathcal{X}$ and Y thus boils down to finding the change-of-basis matrix connecting the two set of solutions upon analytic continuation from one boundary point to the other. Let $I_k^{\mathcal{X}}(x,z)$ denote the coefficient of $\mathbf{1}_{k/(n+2)}$ in Eq. (24), and define in the same vein

$$I_k^Y(y,z) = \mathrm{Coeff}_{\mathbf{1}_{P_{k+1}}} I^Y(y,z), \qquad k = 0,1,2, \tag{30}$$

$$I_{\frac{j}{n}}^Y(y,z) = \mathrm{Coeff}_{\mathbf{1}_{\frac{j}{n}}} I^Y(y,z), \qquad j = 1,\dots,n-1. \tag{31}$$

It is immediately noticed that $I_k^{\mathcal{X}}(x,z) = x^k(z^{1-k}/k! + \mathcal{O}(x^{n+2}))$: this uniquely characterizes $\{I_k^{\mathcal{X}}\}_{k=0}^{n+1}$ as a basis of solutions of $\mathcal{D}_{\mathcal{X}} f = 0$. On the other hand, localizing Eq. (23) to the T-fixed points and resumming in d for $|y| < \frac{n^n}{(n+2)^{n+2}}$ we obtain

$$I_k^Y = i_{P_k}^* \left[z y^{p/z} {}_{n+3}F_{n+2}\left(\{A_n\};\{B_n\};(-n-2)^{n+2} n^{-n} y\right)\right], \tag{32}$$

$$I_{\frac{j}{n}}^Y = \frac{z^{1-j} y^{j/n}}{j!} {}_{n+2}F_{n+1}\left(\{C_{n,j}\};\{D_{n,j}\};(-n-2)^{n+2} n^{-n} y\right), \tag{33}$$

where

$$\begin{aligned}
A_n &= \left(1, \frac{1}{n+2} + \frac{p}{z}, \dots, \frac{n+1}{n+2} + \frac{p}{z}, \frac{p}{z}\right),\\
B_n &= \left(\frac{1}{n} + \frac{np+\alpha_1}{nz}, \dots, \frac{n-1}{n} + \frac{np+\alpha_1}{nz}, 1 + \frac{np+\alpha_1}{nz}, 1 + \frac{p-\alpha_1-\alpha_2}{z}, 1 + \frac{p+\alpha_2}{z}\right),\\
C_{n,j} &= \left(1, \frac{1}{n+2} - \frac{j}{n}, \dots, \frac{n+1}{n+2} - \frac{j}{n}\right), D_{n,j} = \left(\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+n-1}{n}, 1 + \frac{j}{n}\right),
\end{aligned} \tag{34}$$

and ${}_pF_q(\{\mathcal{A}\};\{\mathcal{B}\};y)$ denotes the generalized hypergeometric series

$${}_pF_q(\{\mathcal{A}\};\{\mathcal{B}\};w) \triangleq \frac{\prod_{i=1}^q \Gamma(\mathcal{B}_i)}{\prod_{j=1}^p \Gamma(\mathcal{A}_j)} \sum_{n=0}^\infty \frac{\prod_{i=1}^p \Gamma(\mathcal{A}_i + n)}{\prod_{j=1}^q \Gamma(\mathcal{B}_j + n)} \frac{w^n}{n!}, \tag{35}$$

which is convergent for $|w| < 1$.

In order to continue to $x = y^{-n-2} \ll 1$ we will need the following analytic continuation theorem for ${}_pF_q(\{\mathcal{A}\};\{\mathcal{B}\};y)$, which generalizes the classical Kummer continuation formula at infinity for the Gauss function.

Lemma 3.4. *Let $p = q+1$, $\mathcal{B}_j \notin \mathbb{N}$, $\mathcal{A}_i - \mathcal{A}_j \notin \mathbb{Z}$ for $i \neq j$ and let $\rho : \mathbb{R} \to \mathbb{C}$ be a path in the complex y-plane from $y = 0$ to $y = \infty$ having trivial winding number around both $y = 0$ and $y = 1$. Then the analytic continuation of Eq. (35) to $|y| \gg 1$ along ρ satisfies*

$${}_{q+1}F_q(\{\mathcal{A}\};\{\mathcal{B}\};y) \sim \sum_{k=1}^{q+1} \prod_{j=1}^q \frac{\Gamma(\mathcal{B}_j)}{\Gamma(\mathcal{B}_j - \mathcal{A}_k)} \prod_{j\neq k} \frac{\Gamma(\mathcal{A}_j - \mathcal{A}_k)}{\Gamma(\mathcal{A}_j)} (-y)^{-\mathcal{A}_k} \left(1 + \mathcal{O}\left(\frac{1}{y}\right)\right). \tag{36}$$

Proof. The argument follows almost verbatim the steps leading to the well-known result for $q = 1$. $\Phi(w) \triangleq_{q+1} F_q(\{\mathcal{A}\};\{\mathcal{B}\};w)$ satisfies the generalized hypergeometric equation

$$\left[\theta \prod_{j=1}^q (\theta + \mathcal{B}_j - 1) - w \prod_{j=1}^q (\theta + \mathcal{A}_j)\right] \Phi(w) = 0. \tag{37}$$

with $\theta = w\partial_w$. The same analysis at $w=\infty$ as for the Gauss equation reveals that $\mathcal{A}_i$ are local exponents of Eq. (37),

$$\widetilde{\Phi}(w) \sim \sum_{j=1}^{q+1} c_j\left(\{\mathcal{A}\};\{\mathcal{B}\}\right)(-w)^{-\mathcal{A}_j} \tag{38}$$

for some $c_j\left(\{\mathcal{A}\};\{\mathcal{B}\}\right) \in \mathbb{C}$. Let now k be such that $\mathfrak{Re}(\mathcal{A}_k - \mathcal{A}_j) < 0$ for all $j \neq k$; then

$$c_k\left(\{\mathcal{A}\};\{\mathcal{B}\}\right) = \lim_{w\to\infty}(-w)^{\mathcal{A}_k}\widetilde{\Phi}(w) \tag{39}$$

Now, $\Phi(w)$ can be represented as the multiple Euler–Pochhammer integral [16]

$$\Phi_j(w) = \prod_{i=1}^{q} \frac{\Gamma(\mathcal{B}_i)}{\Gamma(\mathcal{A}_i)\Gamma(\mathcal{B}_i-\mathcal{A}_i)} \frac{1}{(1-\mathrm{e}^{2\pi\mathrm{i}\mathcal{A}_i})(1-\mathrm{e}^{2\pi\mathrm{i}(\mathcal{B}_i-\mathcal{A}_i)})} \int_\gamma \cdots \int_\gamma \frac{t_i^{\mathcal{A}_i}(1-t_i)^{\mathcal{B}_i-\mathcal{A}_i}}{(1-w\prod_i t_i)} \prod_{i=1}^{q}\frac{\mathrm{d}t_i}{t_i(1-t_i)}, \tag{40}$$

where $\gamma = [C_0, C_1]$ is the commutator of simple loops around $t=0$ and $t=1$. Taking the limit $w\to\infty$ along ρ and using the Euler Beta integral,

$$\frac{1}{(1-\mathrm{e}^{2\pi\mathrm{i}\mathcal{A}_i})(1-\mathrm{e}^{2\pi\mathrm{i}(\mathcal{B}_i-\mathcal{A}_i)})} \int_\gamma t_i^{\mathcal{A}_i-1}(1-t_i)^{\mathcal{B}_i-\mathcal{A}_i-1}\prod_{i=1}^{q}\mathrm{d}t_i = \frac{\Gamma(\mathcal{A}_i)\Gamma(\mathcal{B}_i-\mathcal{A}_i)}{\Gamma(\mathcal{B}_i)}, \tag{41}$$

gives

$$c_k(\mathcal{A},\mathcal{B}) = \prod_{i=1}^{q}\frac{\Gamma(\mathcal{B}_i)}{\Gamma(\mathcal{B}_i-\mathcal{A}_k)}\prod_{i\neq k}\frac{\Gamma(\mathcal{A}_i-\mathcal{A}_k)}{\Gamma(\mathcal{A}_i)}. \tag{42}$$

from which Eq. (36) follows by the invariance of Eq. (35) under permutation of $\mathcal{A}_i$ and analytic continuation to $\mathfrak{Re}(\mathcal{A}_j - \mathcal{A}_i) < 0$, $j \neq k \neq i$. □

Denote by $\widetilde{I^Y}(y,z)$ the analytic continuation of $I^Y(y,z)$ along ρ as in Lemma 3.4. The matrix expression of the symplectomorphism $\mathbb{U}_\rho^{\mathcal{X},Y} : \mathcal{H}_\mathcal{X} \to \mathcal{H}_Y$ of Conjecture 2.1 in the bases $\{\mathbf{1}_{\frac{k}{n+2}}\}_{k=0,1,\dots,n-1}$ for $H_T^\bullet(\mathcal{X})$ and $\{P_1, \mathbf{1}_{\frac{1}{n}}, \dots, \mathbf{1}_{\frac{n-1}{n}}, P_2, P_3\}$ for $H_T^\bullet(Y)$ can then be read off upon applying Eq. (36) to Eqs. (32)–(34),

$$\widetilde{I}_i^Y(x^{-n-2}, z) = \sum_{k=0}^{n+1}(\mathbb{U}_\rho^{\mathcal{X},Y})_{ik} I^{\mathcal{X}}_{\frac{k}{n+2}}(x,z). \tag{43}$$

Example 3.1. ($n=1$) We have, from Eq. (23) and Eq. (36) for $q=2$,

$$\begin{aligned}
(\mathbb{U}_\rho^{\mathcal{X},Y})_{0,0} &= \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)27^{\frac{\alpha_2}{z}}\Gamma\left(\frac{z+\alpha_1-\alpha_2}{z}\right)\Gamma\left(\frac{z-\alpha_2+\alpha_3}{z}\right)}{\Gamma\left(\frac{z+\alpha_1}{z}\right)\Gamma\left(\frac{1}{3}-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{2}{3}-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{z+\alpha_3}{z}\right)},\\
(\mathbb{U}_\rho^{\mathcal{X},Y})_{0,\frac{1}{3}} &= \frac{z\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)3^{\frac{3\alpha_2}{z}-1}\Gamma\left(\frac{z+\alpha_1-\alpha_2}{z}\right)\Gamma\left(\frac{z-\alpha_2+\alpha_3}{z}\right)}{\Gamma\left(\frac{\alpha_1}{z}+\frac{2}{3}\right)\Gamma\left(-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{2}{3}-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{\alpha_3}{z}+\frac{2}{3}\right)},\\
(\mathbb{U}_\rho^{\mathcal{X},Y})_{0,\frac{2}{3}} &= \frac{2z^2\Gamma\left(-\frac{2}{3}\right)\Gamma\left(-\frac{1}{3}\right)3^{\frac{3\alpha_2}{z}-2}\Gamma\left(\frac{z+\alpha_1-\alpha_2}{z}\right)\Gamma\left(\frac{z-\alpha_2+\alpha_3}{z}\right)}{\Gamma\left(\frac{\alpha_1}{z}+\frac{1}{3}\right)\Gamma\left(-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{1}{3}-\frac{\alpha_2}{z}\right)\Gamma\left(\frac{\alpha_3}{z}+\frac{1}{3}\right)},
\end{aligned} \tag{44}$$

where $\alpha_3 = -\alpha_1-\alpha_2$, and $(\mathbb{U}_\rho^{\mathcal{X},Y})_{ik}(\alpha_{(1,2,3)}) = (\mathbb{U}_\rho^{\mathcal{X},Y})_{0k}(\alpha_{\chi^i(1,2,3)})$, where $\chi \in S_3$ is the cyclic permutation $1\to 2$, $2\to 3$, $3\to 1$.

Remark 3.5. (On general toric wall-crossings) The arguments we used for the examples of this Section have a wider applicability to wall-crossings in toric Gromov–Witten theory, including the multi-parameter case. On general grounds, I-functions - and their extended versions [7] - are multiple hypergeometric functions of Horn type [20, 21]. When crossing a *single* wall in the B-model moduli space, however, the analytic continuation is effectively taking place in *one* parameter only. Restricting to the sublocus where all the spectator variables are set to zero reduces the multiple Horn series to a single-variable series which, upon manipulations of Gamma factors in the summand as in the next section, can always be cast in the form of a generalized hypergeometric function ${}_pF_q(\{\mathcal{A}\},\{\mathcal{B}\},w)$ with $q \geq p-1$. Whenever the series has a finite radius of convergence as in the Calabi–Yau case, we have $p = q+1$, for which Lemma 3.4 applies. The general case is obtained similarly.

3.B.b. Grade restriction window and the K-theoretic CRC

Let us now turn to Conjecture 2.1 for this family of geometries. Throughout this section, we work with the natural basis $\{\mathbf{1}_{\frac{k}{n+2}}\}_{k=0,1,\dots,n-1}$ for $H^\bullet_T(\mathcal{X})$ and with the localized basis $\{P_1, \mathbf{1}_{\frac{1}{n}}, \dots, \mathbf{1}_{\frac{n-1}{n}}, P_2, P_3\}$ for $H^\bullet_T(Y)$. The grade restriction window $\mathfrak{W} = \{L_j\}_{j=0,\dots,n+1}$, where L_j is a $\mathbb{C}^*$ equivariant line bundle on $\mathbb{C}^4$ with character χ_j given by

$$\chi_j = \begin{cases} j & j < 1 + \frac{n}{2}, \\ j-n-2 & \text{else}, \end{cases} \tag{45}$$

yields a natural bijection between the K-lattices of $\mathcal{X}$ and Y. We make the notational convention of taking all indexing sets to range from 0 to $n+1$, with the sole purpose of leaving the coefficients corresponding to identities/trivial objects in the first row/column of any matrix we write. With these choices the matrices representing the (homogenized, involution pulled-back) Chern characters for $\mathcal{X}$ and Y are

$$[\overline{\mathrm{CH}}_{\mathcal{X}}]^k_j = \left(\frac{2\pi \mathrm{i}}{z}\right)^{\frac{1}{2}\deg} \mathrm{inv}^* \mathrm{CH}_{\mathcal{X}} = \mathrm{e}^{-jk\frac{2\pi\mathrm{i}}{n+2}}, \tag{46}$$

$$[\overline{\mathrm{CH}}_Y]^l_j = \begin{cases} \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\chi_j\left(l-\frac{\alpha_1}{z}\right)} & \text{for } l = 0,\dots,n-1. \\ \mathrm{e}^{-2\pi\mathrm{i}\frac{\chi_j\alpha_2}{z}} & \text{for } l = n. \\ \mathrm{e}^{-2\pi\mathrm{i}\frac{\chi_j\alpha_3}{z}} & \text{for } l = n+1. \end{cases} \tag{47}$$

Theorem 3.6. *Conjecture 2.1 holds with the restriction window $\mathfrak{W}$ above and the analytic continuation path ρ as in Lemma 3.4.*

Proof. Consider the linear map $\mathbb{V} : \mathcal{H}_{\mathcal{X}} \to \mathcal{H}_Y$ defined by

$$\mathbb{V} = \Gamma_Y^{-1} \mathbb{U}_\rho^{\mathcal{X},Y} \Gamma_{\mathcal{X}}, \tag{48}$$

in the bases above for $H^\bullet_T(\mathcal{X})$ and $H^\bullet_T(Y)$. The Gamma factors in Eqs. (36) and (48) telescope away by virtue of Eq. (34), the multiplication formula

$$\Gamma(b+mz) = (2\pi)^{\frac{1-m}{2}} m^{b+mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{b+k}{m} + z\right); \quad m \in \mathbb{Z} \wedge m > 0, \tag{49}$$

and Euler's identity, $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$; the final result is a trigonometric matrix with coefficients $[\mathbb{V}]^i_j$ being Laurent polynomials in $\mathrm{e}^{2\pi\mathrm{i}\alpha_k}$, $k = 1,2,3$. Right-multiplication by the Chern character matrix of $\mathcal{X}$ and telescoping the resulting sums over roots of unity returns $\overline{\mathrm{CH}}_Y$, as given in Eq. (47). □

3.C. The OCRC

As discussed in Section 2.A, the first implication we draw from Theorem 3.6 is a comparison theorem for winding neutral disk potentials.

Corollary 3.7. *Proposals 1 and 2 hold for* $Y = K_{\mathbb{P}(n,1,1)}$ *and* $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{n+2}]$.

This can be employed to obtain more concrete identifications of scalar disk potentials, as we now show.

3.C.a. Scalar disk potentials: non-special legs

In the case where the Lagrangian on Y is on a leg that attached to a non-stacky point, the equality of scalar disk potentials follows in a simple fashion for all n. When the Lagrangian is on the leg that attached to the stacky point, we need to consider separately the case n-odd, where the quotient on the leg is effective, and n-even, where there is a residual $\mathbb{Z}_2$ isotropy.

We consider non-special legs first. We have the following

Theorem 3.8. *Consider a Lagrangian boundary condition* L *on* $\mathcal{X}$ *which intersects the second coordinate axis, and denote by* L' *the proper transform in* Y. *Then, upon identifying the insertion variables via the change of variable prescribed by the closed CRC, we have the equality of scalar disk potentials:*

$$F^{\rm disk}_{L',Y}(\tau, y, \vec{w}) = F^{\rm disk}_{L,\mathcal{X}}(\tau, y, \vec{w}). \tag{50}$$

Proof. In this case the tensors Θ from (5) are:

$$\left[\Theta_{\mathcal{X}}^{-1}\right]^{kk} = \sin\left(\pi\left(\frac{-\alpha_1}{z} + \left\langle\frac{nk}{n+2}\right\rangle\right)\right), \tag{51}$$

$$[\Theta_Y]_{ll} = \frac{1}{\sin\left(\pi\left(\frac{n\alpha_2-\alpha_1}{z}\right)\right)}\delta_{l,n}. \tag{52}$$

We compute the transformation $\mathbb{O}$ as in Eq. (6); note it has nonzero coefficients only for $l = n$. We then specialize $z = \frac{(n+2)\alpha_2}{d}$ to obtain a map we denote $\mathbb{O}_d$,

$$\mathbb{O}^k_{d,n} = \frac{\sin\left(\pi\left(-\frac{\alpha_1 d}{(n+2)\alpha_2} + \left\langle\frac{nk}{n+2}\right\rangle\right)\right)}{\sin\left(\pi\left(-\frac{\alpha_1 d}{(n+2)\alpha_2} + \frac{nd}{(n+2)}\right)\right)}\frac{1}{n+2}\mathrm{e}^{\frac{2\pi i j}{n+2}(k-d)}. \tag{53}$$

The expression in Eq. (53) is summed over the index j ranging from 0 to $n+1$. When k is not congruent to d modulo $n+2$, the exponential part is a sum of roots of unity that adds to 0. When $k \equiv d$ modulo $n+2$, $\mathbb{O}^k_{d,n} = \pm 1$. Hence our OCRC, Corollary 3.7, together with Eq. (53) gives

$$\pm\,\mathbb{F}^{\rm disk}_{L,\mathcal{X}|z=\frac{(n+2)\alpha_2}{d}}(\mathbf{1}_{\langle\frac{d}{n+2}\rangle}) = \mathbb{F}^{\rm disk}_{L',Y|z=\frac{(n+2)\alpha_2}{d}}(P_2). \tag{54}$$

Disk invariants of winding d for $\mathcal{X}$ are the coefficients of the classes $\mathbf{1}_{\frac{k}{n+2}}$ with $k \equiv d$ modulo $n+2$ after specializing $z = \frac{(n+2)\alpha_2}{d}$ in $\mathbb{F}^{\rm disk}_{L,\mathcal{X}}$. Summing over all d, we obtain the equality of scalar potentials as stated in Theorem 3.8. □

3.C.b. Scalar disk potentials for the special leg: n odd

Theorem 3.9. *Let n be an odd integer. Consider a Lagrangian boundary condition L on $\mathcal{X}$ which intersects the first coordinate axis, and denote by L' the proper transform in Y. Then, upon identifying the insertion variables via the change of variable prescribed by the closed CRC, we have the equality of scalar disk potentials:*

$$F^{\text{disk}}_{L',Y}(\tau, y, \vec{w}) = F^{\text{disk}}_{L,\mathcal{X}}(\tau, y, \vec{w}). \tag{55}$$

Proof. In this case the tensors Θ from (5) are:

$$\left[\Theta^{-1}_{\mathcal{X}}\right]^{kk} = \sin\left(\pi\left(\frac{\alpha_1+\alpha_2}{z} + \left\langle\frac{k}{n+2}\right\rangle\right)\right), \tag{56}$$

$$[\Theta_Y]_{ll} = \frac{1}{\sin\left(\pi\left(\frac{\alpha_1+\alpha_2}{z} + \frac{\alpha_1}{nz} + \left\langle-\frac{l}{n}\right\rangle\right)\right)}. \tag{57}$$

We compute the transformation $\mathbb{O}$ as in Eq. (6). We then specialize $z = \frac{(n+2)\alpha_1}{d}$ to obtain $\mathbb{O}_d$.

$$\mathbb{O}^k_{d,l} = \frac{\sin\left(\pi\left(\frac{d(\alpha_1+\alpha_2)}{(n+2)\alpha_1} + \left\langle\frac{k}{n+2}\right\rangle\right)\right)}{\sin\left(\pi\left(\frac{d(\alpha_1+\alpha_2)}{(n+2)\alpha_1} + \frac{d}{n(n+2)} + \left\langle-\frac{l}{n}\right\rangle\right)\right)} \frac{1}{n+2} \mathrm{e}^{\frac{2\pi i j}{n(n+2)}(kn+l(n+2)-d)}. \tag{58}$$

The expression in Eq. (58) is summed over the index j ranging from 0 to $n+1$. The degree-twisting compatibilities are:

$\mathcal{X}$: $d \equiv kn \mod n+2,$

Y: $d \equiv 2l \mod n.$

The Chinese remainder theorem then states that both compatibilities are satisfied when

$$d \equiv kn + l(n+2) \mod n(n+2). \tag{59}$$

When (59) is satisfied, the difference in the arguments in the sine functions is an integer multiple of π, hence $\mathbb{O}^k_{d,l} = \pm 1$. When only the compatibility for Y is satisfied, then the exponential part of Eq. (58) consists of a sum of $(n+2)$ roots of unity that add to 0. All other entries of the matrix representing $\mathbb{O}_d$ do not matter for our purposes. For a fixed d, there is a unique pair $(\bar{k}, \bar{l})$ satisfying both twisting conditions, and Eq. (58) gives:

$$\mathbb{F}^{\text{disk}}_{L,\mathcal{X}|z=\frac{(n+2)\alpha_1}{d}}(\mathbf{1}_{\frac{\bar{k}}{n+2}}) = \pm\mathbb{F}^{\text{disk}}_{L',Y|z=\frac{(n+2)\alpha_1}{d}}(\mathbf{1}_{\frac{\bar{l}}{n}}). \tag{60}$$

Disk invariants of winding d for $\mathcal{X}$ are the coefficients of the class $\mathbf{1}_{\frac{\bar{k}}{n+2}}$ after specializing $z = \frac{(n+2)\alpha_1}{d}$ in $\mathbb{F}^{\text{disk}}_{L,\mathcal{X}}$, whereas for Y they are obtained as the coefficients of the class $\mathbf{1}_{\frac{\bar{l}}{n}}$ after the same specialization of z in $\mathbb{F}^{\text{disk}}_{L,Y}$. Hence, summing over all d, Eq. (60) yields the equality of scalar potentials as stated in Theorem 3.9. □

3.C.c. Scalar disk potentials for the special leg: n even

Theorem 3.10. *Let n be an even integer. Consider a Lagrangian boundary condition L on $\mathcal{X}$ which intersects the first coordinate axis, and denote by L' the proper transform in Y. Then, upon identifying the insertion variables via the change of variable prescribed by the closed CRC, we have the equality of scalar disk potentials:*

$$F^{\text{disk}}_{L',Y}(\tau, y, \vec{w}) = F^{\text{disk}}_{L,\mathcal{X}}(\tau, y, \vec{w}). \tag{61}$$

Proof. The transformation $\mathbb{O}$ in this case is the same as in Section 3.C.c. However we specialize to $z = \frac{(n+2)\alpha_1}{2d}$ to obtain $\mathbb{O}_d$:

$$\mathbb{O}^k_{d,l} = \frac{\sin\left(\pi\left(\frac{2d(\alpha_1+\alpha_2)}{(n+2)\alpha_1} + \left\langle \frac{k}{n+2}\right\rangle\right)\right)}{\sin\left(\pi\left(\frac{2d(\alpha_1+\alpha_2)}{(n+2)\alpha_1} + \frac{2d}{n(n+2)} + \left\langle -\frac{l}{n}\right\rangle\right)\right)} \frac{1}{n+2} e^{\frac{2\pi i j}{n(n+2)}(kn+l(n+2)-2d)}. \tag{62}$$

The expression in Eq. (62) is summed over the index j ranging from 0 to $n+1$. The degree-twisting compatibilities are:

$\mathcal{X}$: $2d \equiv kn \mod n+2,$

Y: $2d \equiv 2l \mod n.$

Modular arithmetic again tells us that for any d there are four pairs of solutions to the above system of congruences, corresponding to the solutions to:

$$2d \equiv kn + l(n+2) \mod \frac{n(n+2)}{2}. \tag{63}$$

Note that if (k_0, l_0) is a solution of (63), then the other solutions are $(k_0, l_1), (k_1, l_0), (k_1, l_1)$, where $k_1 = k_0 + \frac{n+2}{2}$ and $l_1 = l_0 + \frac{n}{2}$. Without loss of generality we denote (k_0, l_0) and (k_1, l_1) the solutions such that $2d \equiv kn + l(n+2) \mod n(n+2)$ and we observe that $\mathbb{O}^{k_0}_{d,l_0} = \mathbb{O}^{k_1}_{d,l_1} = \pm 1$, whereas $\mathbb{O}^{k_0}_{d,l_1} = \mathbb{O}^{k_1}_{d,l_0} = 0$.

Just as before, for $l = l_0, l_1$ and all other k's, the corresponding coefficients in the matrix $\mathbb{O}_d$ vanish. This gives the equalities:

$$\mathbb{F}^{\rm disk}_{L,\mathcal{X}|z=\frac{(n+2)\alpha_1}{2d}}(\mathbf{1}_{\frac{k_0}{n+2}}) = \pm\mathbb{F}^{\rm disk}_{L',Y|z=\frac{(n+2)\alpha_1}{2d}}(\mathbf{1}_{\frac{l_0}{n}}), \tag{64}$$

$$\mathbb{F}^{\rm disk}_{L,\mathcal{X}|z=\frac{(n+2)\alpha_1}{2d}}(\mathbf{1}_{\frac{k_1}{n+2}}) = \pm\mathbb{F}^{\rm disk}_{L',Y|z=\frac{(n+2)\alpha_1}{2d}}(\mathbf{1}_{\frac{l_1}{n}}). \tag{65}$$

We recognize the disk invariants of winding d for $\mathcal{X}$ (resp. Y) in the sum of the left hand sides (resp. right hand sides) of Eq. (64) and Eq. (65). Hence, summing over all d, Eq. (60) yields the equality of scalar potentials as stated in Theorem 3.10. □

4. Example 2: the closed topological vertex

4.A. Classical geometry

The closed topological vertex arises from the GIT quotient construction [12]

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{M^T} \mathbb{Z}^6 \xrightarrow{N} \mathbb{Z}^3 \longrightarrow 0\,, \tag{66}$$

where

$$M = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{67}$$

The resulting geometry is a quotient $\mathbb{C}^6/\!/_\chi(\mathbb{C}^\star)^3$, where the characters of the torus action on the affine coordinates $x_1, \dots, x_6$ of $\mathbb{C}^6$ are encoded in the rows of M.

In two distinct chambers, the GIT quotient yields the toric varieties whose fans are given by cones over Figure 3. The picture on the left hand side corresponds to the orbifold chamber: we delete the unstable locus

$$\Delta_{\rm OP} \triangleq V(\langle x_4 x_5 x_6 \rangle). \tag{68}$$

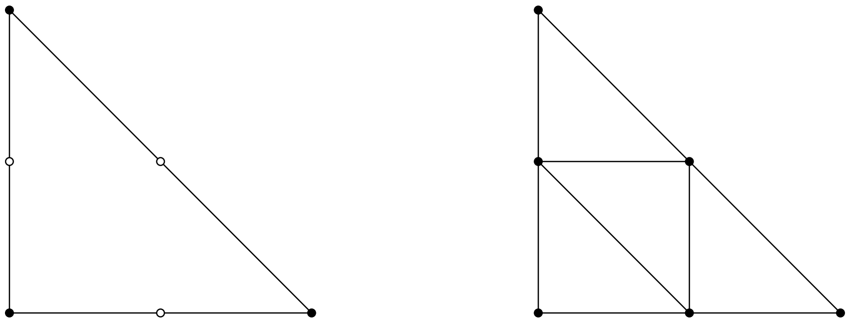

Figure 3: Fans of $[\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)]$ (left) and its G-Hilb canonical resolution (right), depicting a slice of the three dimensional picture with a horizontal hyperplane at height 1.

and then quotient by Eq. (67): using the torus action to make x_4, x_5 and x_6 equal to 1 gives a residual effective $\mu_2^3/\mu_2 \cong \mathbb{Z}_2\times\mathbb{Z}_2$ action[6] on $\mathbb{C}^3$ with coordinates x_1, x_2, x_3. We denote by $\mathcal{X} \triangleq [\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)]$ the resulting orbifold, and by X its coarse moduli space.

The picture on the right hand side corresponds instead to the distinguished large radius chamber that gives rise to Nakamura's Hilbert scheme of $(\mathbb{Z}_2\times\mathbb{Z}_2)$-clusters: we delete the set

$$\Delta_{\mathrm{LR}} \triangleq V\left(\prod_{(i,j,k)\neq(1,4,5),(2,4,6),(3,5,6),(4,5,6)} \left\langle x_i, x_j, x_k\right\rangle\right) \tag{69}$$

and then quotient by the $(\mathbb{C}^\star)^3$ action in Eq. (67); we will denote by Y the corresponding smooth toric variety. This is the trivalent geometry on the right-hand-side of Figure 4: the local geometry of three $(-1,-1)$ curves inside a Calabi–Yau threefold intersecting at a point.

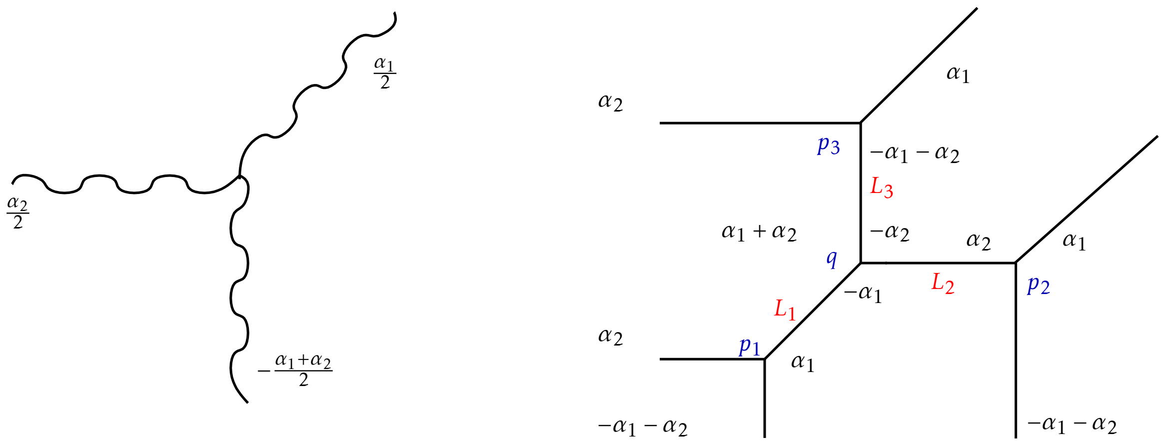

Figure 4: Toric web diagrams and weights at the fixed points of $[\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)]$ (left) and its G-Hilb canonical resolution (right).

4.A.a. Bases for cohomology

We equip Y and $\mathcal{X}$ with a Calabi–Yau 2-torus action descending from the action on $\mathbb{C}^6$ with geometric weights $(\alpha_1,\alpha_2,-\alpha_1-\alpha_2,0,0,0)$. This descends to give an effective $T\simeq(\mathbb{C}^*)^2$ action on Y and $\mathcal{X}$ which

[6] ↑ We choose the isomorphism given by $(0,1)$ being the element whose representation fixes z, $(1,0)$ fixing y and $(1,1)$ fixing x.

preserves their canonical bundle; the resolution diagram

$$
\begin{tikzcd}
Y \arrow[dr, "\rho"] & & \mathcal{X} \arrow[dl, "\pi"'] \\
& X &
\end{tikzcd}
\tag{70}
$$

is T-equivariant.

Bases for the equivariant cohomology of Y and $\mathcal{X}$ can be presented as follows. Let $L_i \subset Y$, $i = 1, 2, 3$ denote the torus-invariant projective lines

$$L_1 = V(x_4, x_5), \tag{71}$$
$$L_2 = V(x_4, x_6), \tag{72}$$
$$L_3 = V(x_5, x_6). \tag{73}$$

The cohomology of Y is generated as a module by the duals $\omega_i = [L_i]^\vee \in H_2(Y)$ of the fundamental classes in Eqs. (71)–(73), plus the identity class $\mathbf{1}_Y \in H_0(Y)$. The action on $\mathbb{C}^6$ above yields canonical lifts of $i^*_{L_j}\omega_i = c_1(\mathcal{O}_{L_j}(\delta_{ij}))$ to equivariant cohomology. Denoting by q the intersection of the three fixed lines, p_i the other torus fixed point of L_i, and by capital letters the corresponding cohomology classes, the Atiyah–Bott isomorphism sends:

$$\omega_1 \to \frac{\alpha_1}{2}(Q - P_1 + P_2 + P_3), \tag{74}$$
$$\omega_2 \to \frac{\alpha_2}{2}(Q + P_1 - P_2 + P_3), \tag{75}$$
$$\omega_3 \to -\frac{\alpha_1 + \alpha_2}{2}(Q + P_1 + P_2 - P_3). \tag{76}$$

The T-equivariant Poincaré pairing $\eta^Y(\phi_1, \phi_2) = \sum_{P_i} \phi_1|_{P_i}\phi_2|_{P_i} \mathrm{e}^{-1}(N_{P_i/Y})$, in the basis (Q, P_1, P_2, P_3) for $H^\bullet_T(Y)$, takes the block-diagonal form

$$\eta^Y = \begin{pmatrix} \frac{2}{\alpha_2\alpha_1^2+\alpha_2^2\alpha_1} & 0 & 0 & 0 \\ 0 & \frac{\alpha_1}{2\alpha_2^2+2\alpha_1\alpha_2} & -\frac{1}{2(\alpha_1+\alpha_2)} & \frac{1}{2\alpha_2} \\ 0 & -\frac{1}{2(\alpha_1+\alpha_2)} & \frac{\alpha_2}{2\alpha_1^2+2\alpha_2\alpha_1} & \frac{1}{2\alpha_1} \\ 0 & \frac{1}{2\alpha_2} & \frac{1}{2\alpha_1} & \frac{1}{2}\left(\frac{1}{\alpha_2}+\frac{1}{\alpha_1}\right) \end{pmatrix}. \tag{77}$$

On $\mathcal{X}$, the torus equivariant cohomology is spanned by the T-equivariant cohomology classes $\mathbf{1}_g$, labeled by the corresponding group elements $g = (0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.

4.A.b. The grade restriction window

Consider the natural restriction window $\mathfrak{W}$ consisting of the trivial representation of $(\mathbb{C}^*)^3$ and the three one dimensional representations whose characters are given by the first three columns of the matrix M in Eq. (67). These descend to the four irreducible representations of $\mathcal{X}$, whose nontrivial characters are still encoded by the first three columns of M via $\mathrm{i}\pi$-exponentiation; and to the bundles $\mathcal{O}$ and $\mathcal{O}_{L_j}(\delta_{ij})$ on Y. Using $\mathfrak{W}$ to identify the K-lattices, the natural basis of irreducible representations for $H^\bullet_T(\mathcal{X})$ and the fixed point basis for $H^\bullet_T(Y)$, the matrix representing the (homogenized, involution pulled-back) Chern character for $\mathcal{X}$ and Y are

$$(\overline{\mathrm{CH}}_{\mathcal{X}})^k_j \triangleq \left(\frac{2\pi\mathrm{i}}{z}\right)^{\frac{1}{2}\deg} \mathrm{inv}^*\mathrm{CH}_{\mathcal{X}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \tag{78}$$

$$(\overline{\mathrm{CH}}_Y)^l_j = \begin{pmatrix} 1 & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_1}{z}} & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_2}{z}} & \mathrm{e}^{-\frac{\pi\mathrm{i}(\alpha_1+\alpha_2)}{z}} \\ 1 & \mathrm{e}^{-\frac{\pi\mathrm{i}\alpha_1}{z}} & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_2}{z}} & \mathrm{e}^{-\frac{\pi\mathrm{i}(\alpha_1+\alpha_2)}{z}} \\ 1 & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_1}{z}} & \mathrm{e}^{-\frac{\pi\mathrm{i}\alpha_2}{z}} & \mathrm{e}^{-\frac{\pi\mathrm{i}(\alpha_1+\alpha_2)}{z}} \\ 1 & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_1}{z}} & \mathrm{e}^{\frac{\pi\mathrm{i}\alpha_2}{z}} & \mathrm{e}^{\frac{\pi\mathrm{i}(\alpha_1+\alpha_2)}{z}} \end{pmatrix}. \tag{79}$$

4.B. Quantum geometry

The primary T-equivariant Gromov–Witten invariants of Y were computed for all genera and degrees in [23]. Let d_i, $i = 1, 2, 3$ be the degrees of the image of a stable map to Y measured with respect to the basis L_i, $i = 1, 2, 3$ of $H_2(Y, \mathbb{Z})$, and suppose that $d_1 + d_2 + d_3 \neq 0$. Then [23, Prop. 11–15]

$$\int_{\overline{\mathcal{M}_{g,0}(Y;d_1,d_2,d_3)}} 1 = \frac{|B_{2g}|(2g-1)}{(2g)!(d_1+d_2+d_3)^{3-2g}} \begin{cases} 1 & d_1 = d_2 = d_3, \\ 1 & d_i = d_j = 0, d_k > 0, i \neq j \neq k, \\ -1 & d_1 = d_j > 0, d_k = 0, i \neq j \neq k, \\ 0 & \text{else.} \end{cases} \tag{80}$$

The genus-zero Gromov–Witten potential then takes the form

$$\begin{aligned} F^Y(t) &\triangleq \frac{1}{3!}\eta^Y(\phi, \phi \cup \phi) + \sum_{n\geq 0}\sum_{d_1,d_2,d_3} \int_{\overline{\mathcal{M}_{0,n}(Y;d_1,d_2,d_3)}} \frac{\prod_{i=1}^n \mathrm{ev}_i^*\phi}{n!} \\ &= \frac{1}{6}\left(\frac{t_0^3}{\alpha_1(-\alpha_1-\alpha_2)\alpha_2} + \frac{(t_0-\alpha_2 t_2)^3}{\alpha_1\alpha_2(\alpha_1+\alpha_2)} + \frac{((\alpha_1+\alpha_2)t_3+t_0)^3}{\alpha_1\alpha_2(\alpha_1+\alpha_2)} + \frac{(t_0-\alpha_1 t_1)^3}{\alpha_1\alpha_2(\alpha_1+\alpha_2)}\right) \\ &+ \mathrm{Li}_3\left(\mathrm{e}^{t_1}\right) + \mathrm{Li}_3\left(\mathrm{e}^{t_2}\right) - \mathrm{Li}_3\left(\mathrm{e}^{t_1+t_2}\right) + \mathrm{Li}_3\left(\mathrm{e}^{t_3}\right) - \mathrm{Li}_3\left(\mathrm{e}^{t_1+t_3}\right) - \mathrm{Li}_3\left(\mathrm{e}^{t_2+t_3}\right) + \mathrm{Li}_3\left(\mathrm{e}^{t_1+t_2+t_3}\right) \end{aligned} \tag{81}$$

where we denoted $H_T(Y) \ni \phi := \sum_{i=0}^3 t_i\omega_i$ and $\mathrm{Li}_3(x)$ is the polylogarithm function of order 3:

$$\mathrm{Li}_n(y) = \sum_{k>0} \frac{y^k}{k^n}. \tag{82}$$

As far as $\mathcal{X}$ is concerned, its quantum cohomology was determined in [3] by an explicit calculation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ Hurwitz–Hodge integrals. Introduce linear coordinates $x_{i,j}$ on the T-equivariant Chen–Ruan cohomology of $\mathcal{X}$ by $H_T^{\mathrm{orb}}(\mathcal{X}) \ni \varphi := \sum_{i,j\in 0,1} x_{i,j}\mathbf{1}_{(i,j)}$. Then [3, Thm. 2],

$$F^{\mathcal{X}}(x) = F^Y(t(x)) \tag{83}$$

where the Bryan–Graber change of variables $t(x)$ reads

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\mathrm{i}\alpha_1 & \frac{1}{2}\mathrm{i}\alpha_2 & -\frac{1}{2}\mathrm{i}(\alpha_1+\alpha_2) \\ 0 & \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} \\ 0 & -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} \\ 0 & -\frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} \end{pmatrix}\begin{pmatrix} x_{0,0} \\ x_{1,0} \\ x_{0,1} \\ x_{1,1} \end{pmatrix} + \frac{\mathrm{i}\pi}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{84}$$

4.C. One-dimensional mirror symmetry

In the analysis of the disk and quantized CRC for the type A resolutions in [2], a prominent role was played by a realization of the D-modules underlying quantum cohomology in terms of a single-field logarithmic Landau–Ginzburg model, or, in the language of [25], the Frobenius dual-type structure on a genus-zero double Hurwitz space. This was motivated by a connection of the Gromov–Witten theory for these targets with a class of reductions of the 2-Toda hierarchy [1]. A similar connection with integrable systems holds for the closed topological vertex as well; the general story will appear elsewhere, but its consequences for the purposes of the paper are discussed below.

Define

$$Z_1 \triangleq -\frac{e^{\frac{t_2}{2}}\left(e^{t_1}-1\right)\left(e^{t_3}-1\right)}{(e^{t_1+t_2}-1)^2}, \qquad Z_2 \triangleq \frac{e^{-\frac{t_2}{2}}\left(e^{t_2}-1\right)\left(e^{t_1+t_2+t_3}-1\right)}{(e^{t_1+t_2}-1)^2},$$
$$Z_3 \triangleq \frac{e^{t_1+\frac{t_2}{2}}\left(e^{t_2}-1\right)\left(e^{t_3}-1\right)}{(e^{t_1+t_2}-1)^2}, \qquad Z_4 \triangleq -\frac{e^{\frac{t_2}{2}}\left(e^{t_1}-1\right)\left(e^{t_1+t_2+t_3}-1\right)}{(e^{t_1+t_2}-1)^2}. \tag{85}$$

Fix now a branch $\mathcal{C}$ of the logarithm and denote by $\mathcal{M}_{\alpha_1,\alpha_2} \simeq M_{0,6}\times\mathbb{C}^*$ the smooth complex four-dimensional manifold of multi-valued functions $\lambda(q)$ of the form

$$\mathcal{M}_{\alpha_1,\alpha_2} = \Big\{\lambda(q) = t_0 + \frac{(\alpha_1-\alpha_2)t_2}{2} + \alpha_1\log(Z_1-q)(Z_2-q) + \alpha_2\log(Z_3-q)(Z_4-q)$$
$$-(\alpha_1+\alpha_2)\log q; \quad Z_i \neq 0,1,Z_j\Big\}. \tag{86}$$

A given point $\lambda \in \mathcal{M}_{\alpha_1,\alpha_2}$ is a perfect Morse function in q with four critical points $q_i^{\rm cr}$, $i = 1,\dots,4$; its critical values,

$$u^i = \log\lambda(q_i^{\rm cr}), \tag{87}$$

give a system of local coordinates on $\mathcal{M}_{\alpha_1,\alpha_2}$, which is canonical up to permutation. Define now holomorphic tensors $\eta \in \Gamma(\mathrm{Sym}^2 T^*\mathcal{M}_{\alpha_1,\alpha_2})$, $c \in \Gamma(\mathrm{Sym}^3 T^*\mathcal{M}_{\alpha_1,\alpha_2})$ on $\mathcal{M}_{\alpha_1,\alpha_2}$ via

$$\eta(\partial,\partial') = \sum_{i=1}^4 \mathrm{Res}_{q=q_i^{\rm cr}} \frac{\partial(\lambda)\partial'(\lambda)}{\lambda'(q)}\psi(q)\mathrm{d}q, \tag{88}$$

$$c(\partial,\partial',\partial'') = \sum_{i=1}^4 \mathrm{Res}_{q=q_i^{\rm cr}} \frac{\partial(\lambda)\partial'(\lambda)\partial''(\lambda)}{\lambda'(q)}\psi(q)\mathrm{d}q \tag{89}$$

for holomorphic vector fields ∂, ∂', ∂'' on $\mathcal{M}_{\alpha_1,\alpha_2}$, where

$$\psi(q) = \frac{1}{\alpha_2}\left[\frac{1}{q-Z_1} + \frac{1}{q-Z_2} - \frac{1}{q}\right]. \tag{90}$$

Whenever η is non-degenerate, this defines a commutative, unital product $\partial\circ\partial'$ on $\Gamma(T\mathcal{M}_{\alpha_1,\alpha_2})$ by "raising the indices": $\eta(\partial,\partial'\circ\partial'') = c(\partial,\partial',\partial'')$.

Theorem 4.1. *Eqs. (88) and (89) define a semi-simple Frobenius manifold structure $\mathcal{F}_{\alpha_1,\alpha_2} \triangleq (\mathcal{M}_{\alpha_1,\alpha_2},\eta,\circ)$ on $\mathcal{M}_{\alpha_1,\alpha_2}$ with covariantly constant unit. Moreover,*

$$\mathcal{F}_{\alpha_1,\alpha_2} = QH_T(Y) \simeq QH_T(\mathcal{X}) \tag{91}$$

Proof. Associativity and semi-simplicity of the product follow immediately from the fact that the canonical coordinate fields, ∂_{u^i}, are a basis of idempotents of Eq. (89). A straightforward computation of the residues in Eq. (88) in the coordinate chart t_i shows that Eq. (88) is a flat metric and the variables t_i are a flat coordinate system for η; similarly, a direct evaluation of Eq. (89) shows that the algebra admits a potential function, which coincides with Eq. (81). □

Corollary 4.2. *Let $\nabla^{(z)}_X Y = \mathrm{d}_X Y + zX \circ Y$ be the Dubrovin connection on $\mathcal{F}_{\alpha_1,\alpha_2}$. Then a system of flat coordinates for $\nabla^{(z)}_X$ is given by the periods*

$$\Pi_i = \frac{z}{(1-\mathrm{e}^{2\pi\mathrm{i}\alpha_1/z})(1-\mathrm{e}^{(-1)^i 2\pi\mathrm{i}(\alpha_1+\alpha_2)/z})}\int_{\gamma_i}\mathrm{e}^{\lambda/z}\psi(q)\mathrm{d}q \tag{92}$$

where $\gamma_1 = [C_{Z_1}, C_\infty]$, $\gamma_2 = [C_0, C_{Z_2}]$, $\gamma_3 = [C_{Z_2}, C_\infty]$, $\gamma_4 = [C_0, C_{Z_1}]$ and we denoted by C_x a simple loop encircling counterclockwise the point $q = x$.

This is [2, Prop. 5.2], where the superpotential and primitive differential λ and ϕ there are identified respectively with e^λ and $\psi(q)\mathrm{d}q$ here: the contours γ_i give a basis of the first homology of the complex line twisted by a set of local coefficients given by the algebraic monodromy of $\mathrm{e}^{\lambda/z}$ around the singular points Z_i, 0 and ∞. The reason behind this particular choice of basis, as well as the normalization factor in front of the integral, will be apparent in the course of the asymptotic analysis of Section 4.D.d.

Remark 4.3. In the language of [25], the Frobenius manifold $\mathcal{F}_{\alpha_1,\alpha_2}$ is the Frobenius dual-type structure on the genus zero double Hurwitz space $H_{0,\kappa}$ with ramification profile $\kappa = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, -\alpha_1 - \alpha_2, \alpha_1 - \alpha_2)$, with e^λ as its superpotential and the third kind differential $\psi(q)\mathrm{d}q$ as its primitive one-form; the integrals Eq. (92) were called the *twisted periods* of $\mathcal{F}_{\alpha_1,\alpha_2}$ in [2]. The corresponding Principal Hierarchy [13] is a four-component reduction of the genus-zero Whitham hierarchy with three punctures [24]. The special case $\alpha_1 = \alpha_2 = \alpha$ is particularly interesting, as in that case $\mathcal{F}_{\alpha,\alpha}$ is the dual (in the sense of Dubrovin [14]) of a conformal charge one Frobenius manifold with non-covariantly constant identity; flat coordinates for the two Frobenius structures are in bijection with Darboux coordinates for a pair of compatible Poisson brackets for the Principal Hierarchy, which thus gives rise to a (new) bihamiltonian integrable system of independent interest. We will report on it in a forthcoming work.

4.C.a. Computing $\mathbb{U}^{\mathcal{X},Y}_\rho$

Encoding the coefficients of $\overline{\Gamma}_{\mathcal{X}}(z)$ and $\overline{\Gamma}_Y(z)$ as entries of diagonal matrices, the prediction for the symplectomorphism $\mathbb{U}^{\mathcal{X},Y}_\rho$ from Iritani's theory of integral structure is obtained by composing

$$\mathbb{U}^{\mathcal{X},Y}_\rho = \overline{\Gamma}_Y \circ \overline{\mathrm{CH}}_Y \circ \overline{\mathrm{CH}}^{-1}_{\mathcal{X}} \circ \overline{\Gamma}^{-1}_{\mathcal{X}}, \tag{93}$$

as we now turn to verify. Let $\mathcal{Y}_\epsilon$ be the ball of radius ϵ around $\mathrm{e}^t = 0$, measured w.r.t. the Euclidean metric $(\mathrm{d}s^2) = \sum_i (\mathrm{d}\mathrm{e}^{t_i})^2$ in exponentiated flat coordinates, and define the path in $\mathcal{Y}_1$

$$\begin{array}{cccc} \rho: & [0,1] & \to & \mathcal{Y}_1, \\ & y & \to & (\rho(y))_j = \mathrm{i}y. \end{array} \tag{94}$$

Beside Π_i, systems of flat coordinates for the deformed flat connection $\nabla^{(z)}$ are given by the components of the J-functions of $\mathcal{X}$ and Y; the discrepancy between them encodes the morphism of Givental spaces that identifies the Lagrangian cones of $\mathcal{X}$ and Y under analytic continuation along the path ρ:

$$J^Y = \mathbb{U}^{\mathcal{X},Y}_\rho J^{\mathcal{X}}. \tag{95}$$

As in [2], $\mathbb{U}^{\mathcal{X},Y}_\rho$ can be computed in two steps, by expressing $J^\bullet$ in terms of the periods Π,

$$\Pi_i = \sum_{\alpha=0}^{3} \mathcal{B}_{i\alpha} J^{\mathcal{X}}_\alpha, \tag{96}$$

$$\Pi_i = \sum_{j=1}^{r} \mathcal{A}^{-1}_{ij} J^Y_j, \tag{97}$$

where $J^{\mathcal{X}}_\alpha$ and J^Y_j are the components of the J-functions of $\mathcal{X}$ and Y respectively in the inertia basis of $\mathcal{X}$ and in the localized basis of Y; we have labeled elements of $\mathbb{Z}_2\times\mathbb{Z}_2$ by a single index $\alpha = 0,1,2,3$ for $g=(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ respectively. Throughout the rest of this Section, in order to simplify formulas, we define $\mu_i \triangleq \alpha_i/z$.

Proposition 4.4. *We have*

$$\mathcal{A}^{-1}\mathcal{D}_0^{-1} = \begin{pmatrix} -e^{i\pi(\mu_1+\mu_2)}\frac{\sin(\pi\mu_1)}{\sin(\pi\mu_2)} & 0 & e^{i\pi(\mu_1+\mu_2)}\frac{\sin(\pi\mu_1)}{\sin(\pi\mu_2)} & -1 \\ -(-1)^{\mu_1}\frac{\sin(\pi(\mu_1+\mu_2))}{\sin(\pi\mu_2)} & (-1)^{2\mu_1} & (-1)^{\mu_1}\frac{\sin(\pi(\mu_1+\mu_2))}{\sin(\pi\mu_1)} & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \tag{98}$$

$$\mathcal{B}_{i\alpha} = (\mathcal{D}_1\mathcal{I}\mathcal{D}_2)_{i\alpha} \tag{99}$$

where

$$\mathcal{D}_0 = \operatorname{diag}\mu_2^{-1}\left(-B(\mu_1,-\mu_1-\mu_2), B(-\mu_1,\mu_1+\mu_2), B(-\mu_1-\mu_2,1+\mu_2), -B(\mu_1,-\mu_1-\mu_2)\right), \tag{100}$$

$$\mathcal{D}_1 = \operatorname{diag}\left(e^{\frac{1}{2}i\pi(2\mu_1+\mu_2)}, e^{\frac{1}{2}i\pi(2\mu_1+3\mu_2)}, e^{-\frac{1}{2}i\pi\mu_2}, e^{\frac{1}{2}i\pi\mu_2}\right), \tag{101}$$

$$\begin{aligned}\mathcal{D}_2 = \operatorname{diag}\Bigg[&-\frac{2}{\mu_2}B\left(\frac{\mu_1}{2},-\frac{\mu_1+\mu_2}{2}\right), iB\left(\frac{\mu_1}{2},\frac{1}{2}(1-\mu_1-\mu_2)\right),\\ &-B\left(\frac{1}{2}(\mu_1+1),\frac{1}{2}(1-\mu_1-\mu_2)\right), iB\left(\frac{1}{2}(\mu_1+1),-\frac{\mu_1+\mu_2}{2}\right)\Bigg],\end{aligned} \tag{102}$$

$$\mathcal{I} = \frac{1}{4}\begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{103}$$

and $B(x,y)$ *denotes Euler's* β*-function*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{104}$$

Proof. $J^{\mathcal{X}}_\alpha(x,z)$ is characterized as the unique system of flat coordinates of $\nabla^{(z)}$ which is linear with no inhomogeneous term in $e^{x_0/z}$ and satisfies

$$\partial_\alpha J_\beta(0,z) = \delta_{\alpha,\beta} \tag{105}$$

at the orbifold point $x=0$. Then,

$$\mathcal{B}_{i,\alpha} = \partial_\alpha \Pi_i(0,z). \tag{106}$$

The integrals appearing on the r.h.s. of Eq. (106) can be explicitly evaluated in terms of the Euler β-integral; this is illustrated in detail in Appendix A.A, and returns Eqs. (101)–(103). Similarly, $J^Y_j(t,z)$ is characterized as the unique system of flat coordinates of $\nabla^{(z)}$ (linear with vanishing inhomogeneous term in $e^{t_0/z}$) that diagonalizes the monodromy of $\nabla^{(z)}$ at large radius as

$$\begin{aligned} J^Y_j(t,z)P_j =& z\left(i^*_{p_j}e^{t\cdot\omega/z}\right)\left(1+\mathcal{O}(e^t)\right) \\ \sim& ze^{t_0/z}\begin{cases} e^{-\mu_1 t_1}P_1 & j=1, \\ Q & j=2, \\ e^{-\mu_2 t_2}P_2 & j=3, \\ e^{(\mu_1+\mu_2)t_3}P_3 & j=4, \end{cases}\end{aligned} \tag{107}$$

where the r.h.s. is determined by the localization of ω_i at p_j as in Eqs. (74)–(76). Then $\mathcal{A}$ is determined by the decomposition of the periods in terms of eigenvectors of the monodromy at large radius, that is, by their asymptotic behavior as $\mathfrak{Re}(t) \to -\infty$. The details of the large radius asymptotics of Π_i are quite involved and are deferred to Appendix A.B; the final result is Eq. (98). □

Corollary 4.5. *Conjecture 2.1 holds for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$ and $Y \to X$ its G-Hilb resolution with grade restriction window $\mathfrak{W}$ and analytic continuation path ρ as in Eqs.* (78), (79), *and* (94).

4.D. Quantization and the all-genus CRC

For $j = 1, \dots, 4$, define 1-forms formally analytic in z, $\mathcal{R}_j = R_{ij}(u,z)\mathrm{e}^{u^j/z}\mathrm{d}u^i$, satisfying the following set of conditions:

R1: $R_{ij}(u,z) \in \mathcal{O}_{\mathcal{M}_{\alpha_1,\alpha_2}} \otimes \mathbb{C}[[z]]$,

R2: $\nabla^{(z)}\mathcal{R}_j = 0$ as a formal Taylor series in z,

R3: $\sum_j R_{ij}(u,z)R_{kj}(u,-z) = \delta_{ik}$.

By condition **R2** and their prescribed singular behavior at $z = 0$, $\mathcal{R}_j$ are formal (asymptotic) flat sections of the Dubrovin connection uniquely defined up to right multiplication by constants, $R_{ij}(u,z) \to R_{ij}(u,z)\mathcal{N}_j(z)$; picking a choice of $\mathcal{R}$ is said to endow $\mathcal{F}_{\alpha_1,\alpha_2}$ with an *R-calibration.* Write B_k for the k^{th} Bernoulli number,

$$\sum_{k\geq 0} B_k \frac{t^k}{k!} \triangleq \frac{t}{\mathrm{e}^t - 1}, \tag{108}$$

and let $\Delta_i(u)$ be the normalized inverse-square-length of the coordinate vector field ∂_{u^i} in the Frobenius metric, Eq. (88). We will also denote by $\psi^{\mathcal{W}}$ the Jacobian matrix of the change-of-variables from the canonical frame, Eq. (87), to the flat coordinate systems $\mathbf{t}$ and $\mathbf{x}$ for $\mathcal{W} = Y$ and $\mathcal{X}$ respectively, with columns normalized by $\sqrt{\Delta}$.

Definition 4.1. The Gromov–Witten R-calibration $(\mathcal{R}_Y)_j = (R_Y)_{ij}(u,z)\mathrm{e}^{u^j/z}\mathrm{d}u^i$ of Y is the unique R-calibration on $QH_T(Y) \simeq \mathcal{F}_{\alpha_1,\alpha_2}$ such that

$$\lim_{\mathfrak{Re}(t)\to-\infty} (R_Y)_{ij}(u,z) = \mathscr{D}_i^Y(z)\delta_{ij}, \tag{109}$$

where

$$\mathscr{D}_i^Y(z) = \begin{cases} \exp\left[\sum_{k>0} \frac{B_{2k}}{2k(2k-1)}\left(-\mu_1^{1-2k} - \mu_2^{1-2k} + (\mu_1+\mu_2)^{1-2k}\right)\right] & i = 1, \\ \exp\left[\sum_{k>0} \frac{B_{2k}}{2k(2k-1)}\left(\mu_1^{1-2k} + \mu_2^{1-2k} - (\mu_1+\mu_2)^{1-2k}\right)\right] & \text{else.} \end{cases} \tag{110}$$

The Gromov–Witten R-calibration $(\mathcal{R}_{\mathcal{X}})_j = (R_{\mathcal{X}})_{ij}(u,z)\mathrm{e}^{u^j/z}\mathrm{d}u^i$ of $\mathcal{X}$ is the unique R-calibration on $QH_T(\mathcal{X}) \simeq \mathcal{F}_{\alpha_1,\alpha_2}$ satisfying

$$\sum_i \psi^{\mathcal{X}}_{\alpha i} R^{\mathcal{X}}_{ij}(u,z)\Big|_{x=0} = \left(e^{\mathrm{eq}}(V^{(0)})\right)^{-1/2} \mathscr{D}^{\mathcal{X}}_\alpha(z)\chi_{\alpha j}, \tag{111}$$

where $\chi_{\alpha j}$ is the character table of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $V^{(0)}$ is the trivial part of the representation V (thought of as a vector bundle on the classifying stack), and

$$\mathscr{D}_\alpha^{\mathcal{X}} = \begin{cases} z\exp\left[\sum_{k>0} \frac{B_{2k}z^{2k-1}}{2k(2k-1)}\left(\mu_1^{1-2k} + \mu_2^{1-2k} - (\mu_1+\mu_2)^{1-2k}\right)\right] & \alpha = 0, \\ \frac{\mathrm{i}}{\sqrt{\mu_2(\mu_1+\mu_2)}}\exp\left[\sum_{k>0} \frac{B_{2k}z^{2k-1}}{2k(2k-1)}\left(\frac{1}{\mu_1^{2k-1}} + \frac{2^{1-2k}-1}{\mu_2^{2k-1}} + \frac{1-2^{1-2k}}{(\mu_1+\mu_2)^{2k-1}}\right)\right] & \alpha = 1, \\ -\frac{1}{\sqrt{-\mu_1(\mu_1+\mu_2)}}\exp\left[\sum_{k>0} \frac{B_{2k}z^{2k-1}}{2k(2k-1)}\left(\frac{1}{\mu_2^{2k-1}} + \frac{2^{1-2k}-1}{\mu_1^{2k-1}} + \frac{1-2^{1-2k}}{(\mu_1+\mu_2)^{2k-1}}\right)\right] & \alpha = 2, \\ \frac{\mathrm{i}}{\sqrt{-\mu_1\mu_2}}\exp\left[\sum_{k>0} \frac{B_{2k}z^{2k-1}}{2k(2k-1)}\left(\frac{2^{1-2k}-1}{\mu_1^{2k-1}} + \frac{2^{1-2k}-1}{\mu_2^{2k-1}} - \frac{1}{(\mu_1+\mu_2)^{2k-1}}\right)\right] & \alpha = 3. \end{cases} \tag{112}$$

For either $\mathcal{X}$ or Y, Eqs. (109) and (111) together with conditions **R1-R3** above determine the Gromov–Witten R-calibration uniquely. Existence of an R-calibration $\mathcal{R}_Y$ compatible with Eq. (109) follows from the general theory of semi-simple quantum cohomology of manifolds; the existence of an asymptotic solution $\mathcal{R}_{\mathcal{X}}$ of the deformed flatness equations satisfying the (a priori over-constrained) normalization condition Eq. (111) will be shown in the course of the proof of Theorem 4.7.

The relevance of Definition 4.1 is encoded in the following statement, which condenses [18, Thm. 9.1] and [2, Lem. 6.3, 6.5].

Proposition 4.6. *Givental's quantization formula holds for $\mathcal{W} = \mathcal{X}$ or Y in any path-connected domain containing the large radius point of $\mathcal{W}$,*

$$Z^{\mathcal{W}}(\mathrm{t}_u) = \widehat{S_{\mathcal{W}}^{-1}}\widehat{\psi_{\mathcal{W}}}\widehat{R_{\mathcal{W}}}\widehat{\mathrm{e}^{u/z}}\prod_{i=1}^{4} Z_{i,\mathrm{pt}}. \tag{113}$$

where t_u denotes the shifted descendent times $\mathrm{t}_u^p = t^p + \tau_{\mathcal{W}}(u)\delta_{p0}$. Moreover, the Coates–Iritani–Tseng/Ruan quantized CRC,

$$Z^Y(\mathrm{t}_u) = \widehat{\mathbb{U}_\rho^{\mathcal{X},Y}} Z^{\mathcal{X}}(\mathrm{t}_u), \tag{114}$$

holds if and only if the Gromov–Witten R-calibrations agree on the semi-simple locus,

$$R^{\mathcal{X}}(u,z) = R^Y(u,z). \tag{115}$$

4.D.a. Saddle-point asymptotics

Formal power series solutions in z of $\nabla^{(z)}\mathcal{R} = 0$ are obtained from the saddle-point asymptotics of Eq. (92) at $z = 0$. The latter is an essential singularity of the horizontal sections of the Dubrovin connection, and their asymptotic analysis at $z = 0$ relies on a choice of phase for the parameters α_1, α_2, z – namely, a choice of Stokes sector. A technically convenient choice is to restrict our study to the wedge $\mathcal{S}_+ = \{(\mu_1,\mu_2) | \mathfrak{Re}(\mu_1) > 0, \mathfrak{Re}(\mu_2) < -\mathfrak{Re}(\mu_1)\}$; as individual correlators depend rationally on μ_1, μ_2, our statements will hold in full generality by analytic continuation in the space of equivariant parameters.

Theorem 4.7. *The all-genus, full-descendent Crepant Resolution Conjecture (Conjecture 2.2) holds with $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2]$, $Y \to X$ its G-Hilb resolution and ρ the analytic continuation path of Eq. (94).*

Proof. Asymptotic horizontal sections $\mathcal{R}_i(u,z)$ are given by the classical Laplace asymptotics of the integrals

$$\mathcal{I}_i = z\int_{\mathfrak{L}_i} \mathrm{e}^{\lambda/z}\phi(q)\mathrm{d}q \tag{116}$$

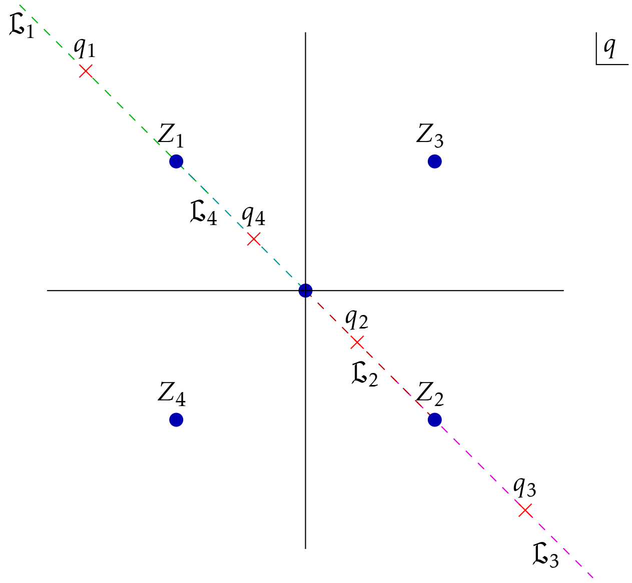

Figure 5: Singular and critical points of the superpotential at the orbifold point. Z_1, Z_2 are negative log-infinities of the superpotential. Z_3 and Z_4 are positive log-infinities. q_i, $i = 1, 2, 3, 4$ are the critical points.

where the Lefschetz thimble $\mathfrak{L}_i$ is given by the union of the downward gradient lines of $\mathfrak{Re}(\lambda)$ emerging from its i^{th} critical point. Let us first consider the situation at the orbifold point, which is schematized in Figure 5. We compute from Eq. (86)

$$q_i^{\text{cr}}\Big|_{x=0} = \frac{(-1)^{1/4+\sigma(i)}}{2}\bar{q}^{(-1)^i}, \quad \bar{q} = \sqrt{\frac{\sqrt{\mu_1}+\sqrt{-\mu_2}}{\sqrt{\mu_1}-\sqrt{-\mu_2}}}, \tag{117}$$

$$\{Z_1, Z_2, Z_3, Z_4\}\Big|_{x=0} = \frac{\mathrm{e}^{\pi \mathrm{i}/4}}{2}\{\mathrm{i}, -\mathrm{i}, -1, 1\}, \tag{118}$$

with $\sigma(1) = \sigma(4) = 0$, $\sigma(3) = \sigma(2) = 1$. It is straightforward to check that the constant phase paths of $\mathrm{e}^{\lambda/z}$ emerging from q_i^{cr} are the straight lines $\arg(q) = \pm\pi(\sigma(i)+1/4)$ that terminate at the nearest algebraic zero of $\mathrm{e}^{\lambda/z}$ or at infinity, as in Figure 5. Moreover, for our choice of phases of the weights in $\mathcal{S}_+$, the contour integrals of $\mathrm{e}^{\lambda/z}\psi$ around the Pochhammer contours γ_i retract [2, Rmk 5.5] to line integrals on the straight line segments connecting the zeroes of $\mathrm{e}^{\lambda/z}$ inside γ_i. At the orbifold point, these are precisely the Lefschetz thimbles $\mathfrak{L}_i$: then, the saddle-point expansion of the differentials $\mathcal{R}_i = \psi^{\mathcal{X}}_{\alpha j}R_{ji}(u,z)\mathrm{e}^{u^i/z}\mathrm{d}x^\alpha \triangleq \mathrm{d}\mathcal{I}_i = \mathrm{d}\Pi_i$ satisfies conditions **R1-R2** above. We claim that up to right multiplication by $\mathcal{N}_i \in \mathbb{C}[[z]]$, $\mathcal{R}_i$ this satisfies **R3** and coincides with the Gromov–Witten R-calibration of $\mathcal{X}$. Indeed, as shown in Appendix A.A, in the trivialization given by x^α the differential of the periods of $\mathrm{e}^{\lambda/z}$ at $x = 0$ reduce to Euler Beta integrals, whose steepest-descent asymptotics is determined by Stirling's expansion for the Γ function:

$$\Gamma(x+y)x^{-x}\mathrm{e}^x x^{1/2-y} \simeq \sqrt{2\pi}\exp\left(\sum_{k>0}\frac{B_{k+1}(1-y)}{k(k+1)}x^k\right), \quad \mathfrak{Re}(x) \gg 0. \tag{119}$$

Then:

$$\begin{aligned} \mathrm{e}^{-u^i/z}\partial_{x^\alpha}\Pi_i\Big|_{x=0} &= \mathrm{e}^{-u^i/z}\big|_{x=0}\mathcal{B}^{-1}_{i\alpha} \\ &\simeq \psi^{\mathcal{X}}_{aj}R_{ji}\Big|_{x=0} \end{aligned} \tag{120}$$

and by Eqs. (106), (119), and (112) we obtain

$$\psi^{\mathcal{X}}_{aj} R_{ji}\Big|_{x=0} = \sqrt{\frac{2\pi}{e^{\mathrm{eq}}(V^{(0)})_\alpha}} \mathscr{D}^{\mathcal{X}}_a \chi_{ai} \tag{121}$$

so that $\mathcal{R} = \sqrt{2\pi}\mathcal{R}^{\mathcal{X}}$. In particular, since by Eq. (112) $\mathcal{R}$ satisfies the unitarity condition at $x = 0$, and because parallel transport under the Dubrovin connection is an isometry of the pairing in **R3**, it satisfies condition **R3** for all u. At large radius, by condition **R1** and the asymptotic behavior of $J^Y(t,z)$ around $\mathfrak{Re}(t) \to -\infty$ (Eq. (107)), we must have that

$$\mathcal{R} \simeq \mathrm{d}J^Y \mathcal{N}^Y \tag{122}$$

for some $\mathcal{N}^Y = \lim_{\mathfrak{Re}(t)\to-\infty} \mathrm{e}^{-u/z}\mathcal{I} \in \mathbb{C}[[z]]$. Its calculation via the steepest descent analysis of Eq. (116) at large radius requires extra care since $\mathrm{e}^t = 0$ is a singular point for $\nabla^{(z)}$: in this limit, the critical points of the superpotential either coalesce at zero or drift off to infinity,

$$\begin{aligned} q_1^{\mathrm{cr}} &\sim \frac{\alpha_1}{\alpha_2}\mathrm{e}^{t_2/2}, & q_2^{\mathrm{cr}} &\sim \left(1+\frac{\alpha_1}{\alpha_2}\right)\mathrm{e}^{t_1+t_2/2}, \\ q_3^{\mathrm{cr}} &\sim \frac{\alpha_2}{\alpha_1+\alpha_2}\mathrm{e}^{-t_2/2}, & q_4^{\mathrm{cr}} &\sim -\mathrm{e}^{t_2/2}. \end{aligned} \tag{123}$$

The essential divergences in the saddle-point computation of $\mathcal{N}^Y$ from Eq. (116) can be treated as follows: first rescale the integration variables in Eq. (116) by $\mathrm{e}^{-t_2/2}$, $\mathrm{e}^{-t_1-t_2/2}$, $\mathrm{e}^{t_2/2}$ and $\mathrm{e}^{-t_2/2}$ for $i = 1,2,3,4$ respectively; then integrate over the steepest descent path, isolating the essential divergence at the large radius point, and finally take the resulting (finite) limit $\mathfrak{Re}(t) \to -\infty$: notice that the last two steps do not commute in general, as poles are generally created along the integration contour in the large radius limit. The final result reduces, for all i, to the computation of the saddle-point asymptotics of Beta integrals. Explicitly, we get

$$\begin{aligned} \sqrt{\Delta^{\mathrm{cl}}}\mathcal{N}^Y &= \lim_{\mathfrak{Re}(t)\to-\infty} \sqrt{\Delta_i(u)}\mathrm{e}^{-u_i/z}\mathcal{I}_i \\ &= \begin{cases} \frac{2\pi\mu_1^{\mu_1-1/2}(-\mu_2)^{\mu_2+1/2}(-\mu_1-\mu_2)^{-\mu_1-\mu_2-1/2}}{B^{\mathrm{as}}(\mu_1,-\mu_1-\mu_2)} & i=1, \\ \frac{B^{\mathrm{as}}(\mu_1,-\mu_2-\mu_1)}{\mu_1^{\mu_1-1/2}(-\mu_2)^{\mu_2+1/2}(-\mu_1-\mu_2)^{-\mu_1-\mu_2-1/2}} & \text{else}, \end{cases} \end{aligned} \tag{124}$$

where $\Delta^{\mathrm{cl}} = \lim_{\mathfrak{Re}(t)\to-\infty}\Delta(u)$ and $B^{\mathrm{as}}(x,y)$ denotes the Stirling expansion of the Euler Beta function. Then,

$$\lim_{\mathfrak{Re}(t)\to-\infty} R_{ij}(u,z) = \sqrt{2\pi}\mathscr{D}^Y_i \delta_{ij}, \tag{125}$$

and thus $R^{\mathcal{X}} = R^Y$, concluding the proof. □

Corollary 4.8. *The quantized OCRC, Proposal 4, holds for $\mathcal{X}$ and Y as in Theorem 4.7.*

Appendix. Boundary behavior of periods

For $|x_i| < 1$, $i = 1,2,3$, and $\mathfrak{Re}(c) > \mathfrak{Re}(a) > 0$ let $F_D^{(3)}(a,b_1,b_2,b_3,c,x_1,x_2,x_3)$ denote the Lauricella hypergeometric function of type D [15],

$$F_D^{(3)}(a,b_1,b_2,b_3,c,x_1,x_2,x_3) \triangleq \sum_{d_1,d_2,d_3\geq 0} \frac{(a)_{d_1+d_2+d_3}}{(c)_{d_1+d_2+d_3}} \prod_{i=1}^{3} \frac{(b_i)_{d_i} x_i^{d_i}}{d_i!}, \tag{126}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \prod_{i=1}^{3} (1-x_i t)^{-b_i} \mathrm{d}t. \tag{127}$$

The last line analytically continues outside the unit polydisc the power-series definition of $F_D^{(3)}$. Furthermore, the continuation to arbitrary parameters a and c is obtainted through the use of the Pochhammer contour:

$$\int_0^1 \to \frac{1}{(1-\mathrm{e}^{2\pi \mathrm{i} a})(1-\mathrm{e}^{2\pi \mathrm{i} c})} \int_{[C_0, C_1]} . \tag{128}$$

Eqs. (127) and (128) can then be used to express Eq. (92) in the form of a sum of generalized hypergeometric functions. Explicitly, we have

$$\begin{aligned} \Pi_4 = & -\mathrm{e}^{\frac{t_0}{z} + \frac{(\alpha_1-\alpha_2)t_2}{2}} \frac{Z_2^{\alpha_1} Z_3^{\alpha_2} Z_4^{\alpha_2}}{Z_1^{\alpha_2}} \\ & \left\{ \frac{\Gamma(-\alpha_1-\alpha_2)\Gamma(1+\alpha_1)}{\Gamma(1-\alpha_2)} F_D^{(3)} \left(-\alpha_1-\alpha_2, -\alpha_1, -\alpha_2, -\alpha_2, 1-\alpha_2, \frac{Z_1}{Z_2}, \frac{Z_1}{Z_3}, \frac{Z_1}{Z_4} \right) \right. \\ & \frac{\Gamma(1-\alpha_1-\alpha_2)\Gamma(\alpha_1)}{\Gamma(1-\alpha_2)} F_D^{(3)} \left(1-\alpha_1-\alpha_2, -\alpha_1, -\alpha_2, -\alpha_2, 1-\alpha_2, \frac{Z_1}{Z_2}, \frac{Z_1}{Z_3}, \frac{Z_1}{Z_4} \right) \\ & \left. \frac{\Gamma(1-\alpha_1-\alpha_2)\Gamma(1+\alpha_1)}{\Gamma(2-\alpha_2)} \frac{Z_1}{Z_2} F_D^{(3)} \left(1-\alpha_1-\alpha_2, 1-\alpha_1, -\alpha_2, -\alpha_2, 2-\alpha_2, \frac{Z_1}{Z_2}, \frac{Z_1}{Z_3}, \frac{Z_1}{Z_4} \right) \right\}, \end{aligned} \tag{129}$$

$$\Pi_1 = \Pi_4 \quad (Z_1 \leftrightarrow Z_2), \tag{130}$$

$$\begin{aligned} \Pi_2 = & \mathrm{e}^{\frac{t_0}{z} + \frac{(\alpha_1-\alpha_2)t_2}{2}} Z_1^{\alpha_1+\alpha_2} \\ & \left\{ \frac{\Gamma(-\alpha_1-\alpha_2)\Gamma(\alpha_1)}{\Gamma(-\alpha_2)} F_D^{(3)} \left(-\alpha_1-\alpha_2; -\alpha_1, -\alpha_2, -\alpha_2; -\alpha_2, \frac{Z_2}{Z_1}, \frac{Z_3}{Z_1}, \frac{Z_4}{Z_1} \right) \right. \\ & + \frac{\Gamma(-\alpha_1-\alpha_2)\Gamma(\alpha_1+1)}{\Gamma(1-\alpha_2)} F_D^{(3)} \left(-\alpha_1-\alpha_2; 1-\alpha_1, -\alpha_2, -\alpha_2; 1-\alpha_2, \frac{Z_2}{Z_1}, \frac{Z_3}{Z_1}, \frac{Z_4}{Z_1} \right) \\ & \left. - \frac{\Gamma(-\alpha_1-\alpha_2)\Gamma(\alpha_1+1)}{\Gamma(1-\alpha_2)} F_D^{(3)} \left(-\alpha_1-\alpha_2; -\alpha_1, -\alpha_2, -\alpha_2; 1-\alpha_2, \frac{Z_2}{Z_1}, \frac{Z_3}{Z_1}, \frac{Z_4}{Z_1} \right) \right\} \end{aligned} \tag{131}$$

$$\Pi_3 = \Pi_2 \quad (Z_1 \leftrightarrow Z_2), \tag{132}$$

where $Z_i(t)$, $i = 1, 2, 3, 4$ were defined in Eq. (85).

A.A. Orbifold point

By Eq. (106), the matrix $\mathcal{B}$ in Eq. (106) is computed by evaluating the derivatives of Π_i at the orbifold point $x = 0$. Consider for simplicity the case $\alpha = 0$. We have

$$\begin{aligned} & Z_1 Z_2^{-1}|_{x=0} = -1, && Z_1 Z_3^{-1}|_{x=0} = \mathrm{i}, && Z_1 Z_4^{-1}|_{x=0} = -\mathrm{i}, \\ & Z_2 Z_3^{-1}|_{x=0} = -\mathrm{i}, && Z_2 Z_4^{-1}|_{x=0} = \mathrm{i}, && Z_3 Z_4^{-1}|_{x=0} = -1. \end{aligned} \tag{133}$$

The value of the Lauricella function, Eq. (126), for arguments equal to distinct roots of unity different from one can be computed explicitly using the integral representation of Eq. (127): the symmetry of the Gauss function ${}_2F_1(a,b,c,x)$ under transposition of a and b and simple manipulations with the products over roots of unity allow to simplify the integrands down to $t^\beta(1-t)^\gamma$ for parameters β and γ depending linearly on μ_1,μ_2. The integrals are in turn evaluated with the aid of the Euler Beta integral, Eq. (41). For example, for $i=4$, we have

$$\begin{aligned}
\partial_{x_0}\Pi_4|_{x=0} =& \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi\mu_2}}{\mu_2 2^{\mu_1+\mu_2+1}}\int_0^1 \frac{(1-q)^{\mu_1-1}(1+q)^{\mu_2+1}}{q^{\frac{\mu_1+\mu_2}{2}}}\frac{\mathrm{d}q}{q} \\
=& \frac{\Gamma(\mu_1)\Gamma(-\frac{\mu_1+\mu_2}{2})}{\Gamma(\frac{\mu_1-\mu_2}{2})}\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi\mu_2}}{\mu_2 2^{\mu_1+\mu_2+1}}\,{}_2F_1\left(-\frac{\mu_1+\mu_2}{2},-1-\mu_2,\frac{\mu_1-\mu_2}{2},-1\right) \\
=& \frac{\Gamma(\mu_1)\Gamma(-\frac{\mu_1+\mu_2}{2})}{\Gamma(\frac{\mu_1-\mu_2}{2})}\frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi\mu_2}}{2^{\mu_1+\mu_2+2}}\frac{\Gamma(\frac{\mu_1-\mu_2}{2})}{\mu_2\Gamma(-1-\mu_2)\Gamma(1+\frac{\mu_1+\mu_2}{2})}\int_0^1\frac{(1-q)^{\frac{\mu_1+\mu_2}{2}}}{q^{1/2+\mu_2/2}}\frac{\mathrm{d}q}{q} \\
=& \frac{\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}\mu_2}}{4}\frac{\Gamma\left(\frac{\mu_1}{2}\right)\Gamma\left(-\frac{\mu_1+\mu_2}{2}\right)}{\Gamma(1-\mu_2)}
\end{aligned} \tag{134}$$

The value of the derivatives with respect to x_α for $\alpha>0$ are computed in the same way; the final result is Eqs. (99)–(103).

A.B. Large radius

By the discussion at the end of the proof of Proposition 4.4, twisted periods behave around large radius as

$$\Pi_i(t,\alpha)\sim z\left(\mathcal{A}_{i,1}^{-1}+\mathcal{A}_{i2}^{-1}\mathrm{e}^{-t_1\mu_1}+\mathcal{A}_{i3}^{-1}\mathrm{e}^{-t_2\mu_2}+\mathcal{A}_{i,4}^{-1}\mathrm{e}^{t_3(\mu_1+\mu_2)}\right). \tag{135}$$

When $\mathfrak{Re}(t)\to-\infty$, the arguments of the Lauricella functions appearing in the expression of Π_i behave like

$$(Z_2Z_1^{-1},Z_2Z_3^{-1},Z_2Z_4^{-1})\sim(-\infty,\infty,\infty), \tag{136}$$
$$(Z_2Z_1^{-1},Z_3Z_1^{-1},Z_4Z_1^{-1})\sim(-\infty,0,1), \tag{137}$$
$$(Z_1Z_2^{-1},Z_3Z_2^{-1},Z_4Z_2^{-1})\sim(0,0,0), \tag{138}$$
$$(Z_1Z_2^{-1},Z_1Z_3^{-1},Z_1Z_4^{-1})\sim(0,-\infty,1). \tag{139}$$

The simplest asymptotics is for $i=3$, as it is dictated by the convergent power series expansion of Eq. (126):

$$\begin{aligned}
\Pi_3 \sim& \mathrm{e}^{\frac{t_0}{z}+\frac{(\mu_1-\mu_2)t_2}{2}}Z_1^{\mu_1+\mu_2}\frac{\Gamma(-\mu_1-\mu_2)\Gamma(\mu_1-1)}{\Gamma(-\mu_2)} \\
\sim& -\mathrm{e}^{\frac{t_0}{z}-\mu_2t_2}\frac{\Gamma(-\mu_1-\mu_2)\Gamma(1+\mu_2)}{\Gamma(1-\mu_1)}.
\end{aligned} \tag{140}$$

This sets $\mathcal{A}_{3,j}=\delta_{j,3}\frac{\Gamma(-\mu_1-\mu_2)\Gamma(1+\mu_2)}{\Gamma(1-\mu_1)}$.

The other cases are more delicate. One strategy to treat them, as in [2], is to resum Eq. (126) in one of the variables and then apply the Kummer formulas to the summand, which in all cases has the form of a Gauss function in the resummed variable. For Π_2 and Π_4, we use that, when $(x_1,x_2,x_3)\sim(0,\infty,1)$, $F_D^{(3)}(a,b_1,b_2,b_3,c,x_1,x_2,x_3)\sim F_1(a,b_2,b_3,c,x_2,x_3)$, where

$$F_1(a,b_2,b_3,c,x_2,x_3)=\sum_{m,n\geq0}\frac{(a)_{m+n}(b_2)_m(b_3)_n}{(c)_{m+n}m!n!}x_2^mx_3^n \tag{141}$$

is the Appell F_1 function. Performing the summation on n for fixed m in Eq. (141) gives

$$F_1(a,b_2,b_3,c,x_2,x_3)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{m\geq 0}\frac{x_2^m(b_2)_m\Gamma(a+m)}{m!\Gamma(c+m)}\;{}_2F_1(a+m,b_3,c+m,x_3)$$

$$\frac{\Gamma(c)(1-x_3)^{-a-b_3+c}\Gamma(a-c+b_3)}{\Gamma(a)\Gamma(b_3)}\sum_{k=0}^{\infty}\frac{x_2^k(b_2)_k\;{}_2F_1(c-a,c+k-b_3;-a+c-b_3+1;1-x_3)}{k!}$$

$$+\frac{\Gamma(c)\Gamma(-a+c-b_3)}{\Gamma(c-a)}\sum_{k=0}^{\infty}{}_2F_1(a+k,b_3;a-c+b_3+1;1-x_3)\frac{x_2^k(a)_k(b_2)_k}{k!\Gamma(c+k-b_3)} \tag{142}$$

The leading asymptotics at $x_3\sim 1$ is therefore given by

$$F_1(a,b_2,b_3,c,x_2,x_3)$$
$$\sim\frac{\Gamma(c)\Gamma(a-c+b_3)}{\Gamma(a)\Gamma(b_3)}(1-x_3)^{c-a-b_3}(1-x_2)^{-b_2}+\frac{\Gamma(c)\Gamma(-a+c-b_3)}{\Gamma(c-a)\Gamma(c-b_3)}\,{}_2F_1(a,b_2;c-b_3;x_2), \tag{143}$$

and further application of the Kummer formula at infinity on x_2 yields

$$F_1(a,b_2,b_3,c,x_2,x_3)\sim\frac{\Gamma(c)\Gamma(a-c+b_3)}{\Gamma(a)\Gamma(b_3)}(1-x_3)^{c-a-b_3}(-x_2)^{-b_2}+\frac{\Gamma(c)}{\Gamma(c-a)}\frac{\Gamma(b_2-a)}{\Gamma(b_2)}(-x_2)^{-a}$$
$$+\frac{\Gamma(c)\Gamma(c-a-b_3)\Gamma(a-b_2)}{\Gamma(c-a)\Gamma(a)\Gamma(c-b_3-b_2)}(-x_2)^{-b_2}. \tag{144}$$

Hence:

$$\mathrm{e}^{-\frac{t_0}{z}}\Pi_4\sim\frac{\Gamma(\mu_1)\Gamma(-\mu_1-\mu_2)\mathrm{e}^{-\mu_1 t_1}}{\Gamma(1-\mu_2)}-\frac{\Gamma(-\mu_1)\Gamma(\mu_1+\mu_2)}{\Gamma(1+\mu_2)}+\frac{\Gamma(\mu_1)\Gamma(-\mu_1-\mu_2)\mathrm{e}^{(\mu_1+\mu_2)t_3}}{\Gamma(1-\mu_2)}, \tag{145}$$

$$\mathrm{e}^{-\frac{t_0}{z}}\Pi_2\sim-\frac{\mathrm{e}^{\mathrm{i}\pi(\mu_1+\mu_2)}\Gamma(-\mu_2)\Gamma(\mu_1+\mu_2)}{\Gamma(1+\mu_1)}+\frac{\mathrm{e}^{\mathrm{i}\pi(\mu_1+\mu_2)}\Gamma(-\mu_1-\mu_2)\Gamma(\mu_2)\mathrm{e}^{-\mu_2 t_2}}{\Gamma(1-\mu_1)}$$
$$-\frac{\Gamma(\mu_1)\Gamma(-\mu_1-\mu_2)\mathrm{e}^{(\mu_1+\mu_2)t_3}}{\Gamma(1-\mu_2)}. \tag{146}$$

Finally, for Π_1 we use that

$$F_D^{(3)}(a;b_1,b_2,b_3;c;x_1,x_2,x_3)=(-x_2)^{-b_2}F_1(a-b_2,b_1,b_3,c-b_2,x_1,x_3)\left(1+\mathcal{O}\left(\frac{1}{x_2}\right)\right)$$
$$+(-x_2)^{-a}\frac{\Gamma(c)\Gamma(b_2-a)}{\Gamma(b_2)\Gamma(c-a)}\left(1+\mathcal{O}\left(\frac{1}{x_2}\right)\right) \tag{147}$$

where we have resummed w.r.t. x_2, applied Lemma 3.4 for $q=1$, and isolated the leading contribution in x_2 for $x_1/x_2\sim 0$, $x_3/x_2\sim 0$, as is the case when $\mathfrak{Re}(t)\sim-\infty$ by Eqs. (136)–(139). Setting now $x_1=x_3$ and further application of Lemma 3.4 gives

$$F_D^{(3)}(a;b_1,b_2,b_3;c;x_1,x_2,x_3)\sim(-x_2)^{-a}\frac{\Gamma(c)\Gamma(b_2-a)}{\Gamma(b_2)\Gamma(c-a)}$$
$$+(-x_2)^{-b_2}\frac{\Gamma(c)\Gamma(a-b_2)}{\Gamma(a)\Gamma(c-b_2)}{}_2F_1(a-b_2,b_1+b_3,c-b_2,x_1)$$
$$\sim(-x_2)^{-a}\frac{\Gamma(c)\Gamma(b_2-a)}{\Gamma(b_2)\Gamma(c-a)}$$
$$+(-x_2)^{-b_2}(-x_1)^{b_2-a}\frac{\Gamma(c)\Gamma(a-b_2)\Gamma(b_1+b_3+b_2-a)}{\Gamma(b_1+b_3)\Gamma(c-a)\Gamma(a)}$$

$$+(-x_2)^{-b_2}(-x_1)^{-b_1-b_3}\frac{\Gamma(c)\Gamma(a-b_2-b_1-b_3)}{\Gamma(a)\Gamma(c-b_1-b_2-b_3)}, \tag{148}$$

so that

$$e^{-\frac{t_0}{z}}\Pi_1 \sim \frac{(-1)^{\mu_1}\Gamma(\mu_1)\Gamma(-\mu_1-\mu_2)e^{-\mu_1 t_1}}{\Gamma(1-\mu_2)} - \frac{\Gamma(-\mu_1)\Gamma(-\mu_2)}{\Gamma(1-\mu_1-\mu_2)} - \frac{\Gamma(\mu_1)\Gamma(\mu_2)e^{-\mu_2 t_2}}{\Gamma(1+\mu_1+\mu_2)}, \tag{149}$$

which concludes the proof.

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