Hamiltonian actions of unipotent groups on compact Kähler manifolds

Daniel Greb and Christian Miebach

Abstract. We study meromorphic actions of unipotent complex Lie groups on compact Kähler manifolds using moment map techniques. We introduce natural stability conditions and show that sets of semistable points are Zariski-open and admit geometric quotients that carry compactifiable Kähler structures obtained by symplectic reduction. The relation of our complex-analytic theory to the work of Doran–Kirwan regarding the Geometric Invariant Theory of unipotent group actions on projective varieties is discussed in detail.

[Français]

Titre. **Actions hamiltoniennes des groupes unipotents sur les variétés kählériennes compactes**

Résumé. Nous étudions les actions méromorphes de groupes de Lie complexes unipotents sur les variétés kählériennes compactes en utilisant des techniques de type application moment. Nous introduisons des conditions de stabilité naturelles et nous montrons que l'ensemble des points semi-stables forme un ouvert de Zariski et admet des quotients géométriques munis de structures kählériennes compactifiables obtenues par réduction symplectique. Le parallèle entre notre théorie analytique complexe et les travaux de Doran–Kirwan concernant la Théorie Géométrique des Invariants des actions de groupes unipotents sur les variétés projectives est discuté en détails.

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Introduction

Since the fundamental work of Mumford [MFK94], Kirwan [Kir84], Guillemin–Sternberg [GS82], and others, moment map geometry has become one of the most important tools for studying actions of *complex-reductive Lie groups* $G = K^{\mathbb{C}}$ on Kähler manifolds. Given a Hamiltonian G-manifold, i.e., a Kähler G-manifold (X, ω_X) admitting a moment map $\mu \colon X \to \mathrm{Lie}(K)^*$ for the K-action, by the work of Heinzner–Loose [HL94], Heinzner–Huckleberry–Loose [HHL94], and Sjamaar [Sj95], the set of μ-semistable points $X_G^{ss}(\mu) := \{x \in X \mid \overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}$ admits an analytic Hilbert quotient, i.e., a G-invariant holomorphic Stein map $\pi \colon X_G^{ss}(\mu) \to X_G^{ss}(\mu)/\!\!/ G$ onto a Kählerian complex space $X_G^{ss}(\mu)/\!\!/ G$ with structure sheaf $\mathscr{O}_{X_G^{ss}(\mu)/\!\!/ G} = (\pi_* \mathscr{O}_{X_G^{ss}(\mu)})^G$; see also the survey [HH99]. If X is projective algebraic, and if the Kähler form ω_X as well as the moment map μ are induced by an embedding of X into some projective space, both the set $X_G^{ss}(\mu)$ of semistable points and the quotient $X_G^{ss}(\mu)/\!\!/ G$ are the ones constructed via Geometric Invariant Theory (GIT). This theory crucially uses the fact that complex-analytic objects on X can be averaged over the compact group K to produce G-invariant objects, which then can be used to construct the quotient.

On the other hand, actions of *unipotent groups* on (compact) Kähler manifolds appear naturally in a number of contexts and play an important role in Kähler geometry. By a fundamental result of Lichnerowicz and Matsushima, a given compact Kähler manifold X can admit a constant scalar curvature Kähler metric only if the Lie sub-algebra of all holomorphic vector fields having a zero is reductive, see e.g. [Sze14, Proposition 4.18 and Remark 4.12]. In other words, unipotent subgroups of $\mathrm{Aut}(X)$ appear as obstructions to the existence of such metrics. In a related direction, the paper [CD16] proposes a way to produce canonical destabilising test-configurations (showing K-unstability of X) from non-reductive subgroups of the automorphism group.

Motivated by these and other moduli-theoretic questions, Doran and Kirwan in [DK07] started to study actions of unipotent algebraic groups N on projective manifolds X linearised in very ample line bundles, using invariant-theoretic methods on the one hand and Geometric Invariant Theory for related actions of reductive groups G containing N on twisted products $G \times_N X$ on the other hand. When thinking about the relation of their work to Kähler geometry and moment maps, one encounters three basic questions:

(a) What is the correct analogue of a "linear" action in Kähler geometry?

(b) Given a compact Kähler N-manifold X, how can one produce Kähler metrics on the non-compact twisted product $G \times_N X$?

(c) If (b) has a positive answer, can one use moment map geometry on the non-compact Hamiltonian G-manifold to produce quotients for the N-action on X with good geometric and complex-analytic properties?

With a different set of problems in mind, Question (a) has been solved a rather long time ago by Fujiki [Fuj78], Lieberman [Lie78], and Sommese [Som75]: meromorphic actions and more generally actions for which the induced action on the Albanese torus is trivial were already called "linear" or "projective"

in *loc. cit.*, and it turns out that also with a view towards moment map geometry and Kählerian quotient theory, these are the correct conditions to impose, see Remark 2.2. For actions of reductive Lie groups this was observed by Huckleberry–Wurzbacher [HW90, Remark on page 262] and Fujiki [Fuj96, Lemma 2.1]. Note that the Lie algebra of the group of all automorphisms acting trivially on Albanese consists exactly of those holomorphic vector fields having a zero, e.g. see [Fuj78, Proposition 6.8], which connects the question of "linearity" to the one concerning the existence of extremal Kähler metrics discussed above.

Main results

As our first contribution, using a criterion of Blanchard and and properties of unipotent groups, with respect to Question (b) we prove the following result.

Theorem (Theorem 3.7). *Let N be a unipotent subgroup of the simply-connected complex semisimple Lie group G. For a connected compact Kähler manifold (X, ω_X) endowed with a holomorphic N-action the following statements are equivalent.*

(1) *There exists a Hamiltonian G-extension (Z, ω_Z) of the N-action on X.*

(2) *The N-action on X is meromorphic.*

(3) *The twisted product $G \times_N X$ is Kähler.*

Moreover, given a meromorphic N-action on X we can always find a Hamiltonian G-extension that is a G-equivariant compactification of $G \times_N X$.

    Here, a *Hamiltonian G-extension* of the N-action on (X, ω_X) consists of a connected compact Hamiltonian G-manifold (Z, ω_Z) and an N-equivariant embedding $\iota\colon X \hookrightarrow Z$ such that for the de Rham cohomology classes associated with the Kähler forms we have $\iota^*[\omega_Z] = [\omega_X]$. This theorem links the condition "meromorphic" to moment map geometry of the G-action on $G \times_N X$, and therefore opens the door to using the complex-reductive theory for the construction of quotients of X with respect to the N-action.

    Once the existence of Hamiltonian G-extensions is established, these can be used to define the set of N-*semistable points* $X_N^{ss}[\omega_X]$ with respect to the given Kähler class $[\omega_X] \in H^2(X, \mathbb{R})$, after one has chosen a suitable Kähler form on the quasi-affine homogeneous space G/N. While we show in Theorem 4.4, using Hodge-theoretic arguments as well as the relation of G-semistability to K-invariant strictly plurisubharmonic exhaustion functions, that this definition does not depend on the chosen G-extension, the choice of metric on G/N influences semistability in a subtle way, as we explore in great detail in Section 4.C. The problems that occur are closely related to the ones encountered in the algebraic situation when searching for various kinds of "reductive envelopes", cf. [DK07, Sections 5.2 and 5.3], and can be traced back to the fact that in general there is no choice of metric on G/N so that the corresponding moment map is proper or admissible in the sense of [Sj95]. A detailed comparison of the moment map approach introduced here and the GIT approach of Doran–Kirwan is presented in Section 4.E.

    Finally, regarding Question (c), we establish that the set of N-semistable points indeed has a number of very desirable complex-geometric properties. The following result summarises the content of Theorem 5.3, Proposition 5.4, Theorem 5.5, and Theorem 5.6.

Theorem. *Let (X, ω_X) be a compact Kähler manifold endowed with a meromorphic N-action. Then, the following holds.*

(a) *The set $X_N^{ss}[\omega_X]$ of semistable points admits a geometric quotient $\pi\colon X_N^{ss}[\omega_X] \to X_N^{ss}[\omega_X]/N$ by the N-action. In fact, π is a principal N-fibre bundle and $X_N^{ss}[\omega_X]/N =: Q$ is smooth.*

(b) *The definition of semistability naturally induces an open embedding $\phi\colon Q = X_N^{ss}[\omega_X]/N \hookrightarrow \overline{Q}$ of Q into a compact complex space $\overline{Q}$ such that $\overline{Q} \setminus \phi(Q)$ is analytic.*

(c) *The set $X_N^{ss}[\omega_X]$ of semistable points is Zariski-open in X. Moreover, the quotient map $\pi\colon X_N^{ss}[\omega_X] \to Q$ extends to a meromorphic map $\pi\colon X \dashrightarrow \overline{Q}$.*

(d) *There exists a Kähler structure $\omega_{\overline{Q}}$ on $\overline{Q}$ whose restriction $\omega_Q = \omega_{\overline{Q}}|_Q$ to $Q \hookrightarrow \overline{Q}$ is smooth and fulfils*

$$[\pi^*\omega_Q] = [\omega_X|_{X_N^{ss}[\omega_X]}] \in H^2(X_N^{ss}[\omega_X], \mathbb{R}).$$

Future directions

With the fundamental results of a Kählerian quotient theory for meromorphic actions of unipotent groups established, interesting questions include whether despite the difficulties presented by the examples collected in Section 4.3 under certain additional conditions the set of semistable points can be shown to be independent of further choices, whether in certain applications there are natural Kähler metrics on the homogeneous space G/N leading to a quotient that is as well-adapted as possible to the given geometric situation at hand, and whether given a compact Kähler manifold (X, ω_X) with non-trivial non-reductive part in the automorphism group (obstructing the existence of special metrics) one can use the Kählerian quotient theory for this unipotent group in order to produce a Kählerian complex space/manifold where said obstruction vanishes and special metrics might exist.

Organisation of the paper

In the first two sections we review basic facts about meromorphic actions and GIT on Kähler manifolds via moment maps. In Section 3 we prove our first main result, Theorem 3.7. The difficulties one encounters in defining the set of semistable points for holomorphic actions of unipotent groups, the actual definition of semistability, as well as fundamental properties of the set of semistable points are discussed in Section 4, while in the final Section 5 we prove the second main result concerning the properties of semistable quotients for unipotent group actions.

Acknowledgements

Global conventions

We work over the field $\mathbb{C}$ of complex numbers. A *complex space* is a reduced complex space with countable topology. *Analytic subsets* are assumed to be closed. *Manifolds* are assumed to be connected.

1. Meromorphic group actions

Let us review some facts about meromorphic group actions from [Fuj78] that we will use freely in the following. Lieberman obtained essentially the same results in [Lie78]. In this section, G denotes a complex Lie group.

A *meromorphic structure* on the complex Lie group G is a compactification $\overline{G}$ together with a meromorphic mapping $\overline{\mu}\colon \overline{G} \times \overline{G} \to \overline{G}$ which extends the group multiplication of G such that $\overline{\mu}$ is holomorphic on $(G \times \overline{G}) \cup (\overline{G} \times G)$. In other words, the compactification $\overline{G}$ is $(G \times G)$-equivariant. Moreover, the map $G \to G$, $g \mapsto g^{-1}$, extends to a meromorphic map $\overline{G} \to \overline{G}$.

Let us fix a meromorphic structure on G. A complex subgroup H of G is *meromorphic* if the topological closure $\overline{H}$ of H in $\overline{G}$ is analytic.

Remark 1.1. If G is linear algebraic, we may and will choose $\overline{G}$ to be a projective manifold. The meromorphic subgroups of G are then precisely the algebraic subgroups of G.

Let X be a complex space endowed with a holomorphic G-action. This G-action is *meromorphic* if the action map $G \times X \to X$ extends to a meromorphic map $\overline{G} \times X \to X$. For meromorphic actions on compact Kähler manifolds (in fact, on reduced compact complex spaces of class $\mathcal{C}$) we have the following quotient theorem, see [Fuj78, Theorem 4.1].

Theorem 1.2. (Quotient Theorem) *Let X be a compact Kähler manifold on which G acts meromorphically. Then, there exist a compact complex space Y and a G-invariant surjective meromorphic map $\pi \colon X \dashrightarrow Y$ such that the following universal property is satisfied. If $\pi' \colon X \dashrightarrow Y'$ is another G-invariant meromorphic map to a compact complex space Y', then there exists a unique meromorphic map $m \colon Y \dashrightarrow Y'$ such that $m \circ \pi = \pi'$. In this situation we call $\pi \colon X \dashrightarrow Y$ a* meromorphic quotient *for the G-action on X.*

Example 1.3. Let H be an algebraic subgroup of G. Then there exists a G-equivariant smooth projective compactification $\overline{G/H}$ of G/H. Moreover, the H-principal bundle $\pi \colon G \to G/H$ extends to a rational map $\overline{\pi} \colon \overline{G} \dashrightarrow \overline{G/H}$. Then $\overline{\pi}$ is a meromorphic quotient for the H-action on $\overline{G}$.

In fact, by inspecting Fujiki's construction, one obtains the following more precise result that has appeared in several places in the literature; see for example [BBS85, Theorem 0.2.2], as well as [Gre10b, Proposition 3.1], [Hu05, Section 3], and the references given there for the analogous result in the algebraic category.

Proposition 1.4. *Let X be a compact Kähler manifold on which G acts meromorphically. Then, there exists an irreducible, compact analytic subset Q_F of the cycle space of X, a G-invariant meromorphic map $\pi_F \colon X \dashrightarrow Q_F$, and a G-invariant Zariski-open subset $U_F \subset \operatorname{dom}(\pi_F)$, called a* Fujiki set *of X, such that*

(1) $\pi_F \colon X \dashrightarrow Q_F$ *is a meromorphic quotient for the G-action on X,*

(2) $U_F \subset X_{\mathrm{gen}} := \{x \in X \mid \dim G \bullet x = m \text{ is maximal}\}$,

(3) *for all $u \in U_F$, we have $\pi_F(u) = \overline{G \bullet u}$, considered as a (reduced) cycle of X,*

(4) $\pi_F(U_F)$ *is smooth and Zariski-open in Y, and the restriction $\pi_F|_{U_F} \colon U_F \to \pi_F(U_F)$ is a geometric quotient.*

We call $\pi_F \colon X \dashrightarrow Q_F$ a *Fujiki quotient* of X by G.

2.   Hamiltonian G-spaces

Let $G = K^{\mathbb{C}}$ be a complex reductive group with maximal compact subgroup K. In this section we review the definition and some properties of Hamiltonian G-manifolds. A general reference for the complex-analytic theory of moment maps on Kähler manifolds is [HH99].

2.A.   Moment maps

Let (Z, ω_Z) be a Kähler manifold with Kähler form ω_Z. Suppose that G acts holomorphically on Z such that ω_Z is K-invariant. A *moment map* for the K-action on Z is a K-equivariant smooth map $\mu \colon Z \to \mathfrak{k}^*$, where K acts via the coadjoint representation on the dual $\mathfrak{k}^*$ of its Lie algebra, such that

$$d\mu^{\xi} = \iota_{\xi_Z} \omega_Z. \tag{2.1}$$

Here, for every $\xi \in \mathfrak{k}$, we write $\mu^\xi \in \mathcal{C}^\infty(Z)$ for the function defined by $\mu^\xi(z) = \mu(z)(\xi)$, and ξ_Z for the vector field on Z whose flow is given by $(t, z) \mapsto \exp(-t\xi) \cdot z$, and $\iota_{\xi_Z} \omega_Z$ for the contraction of ω_Z by ξ_Z.

If a moment map for the K-action on Z exists, we say that (Z, ω_Z) is a *Hamiltonian G-manifold*. The notions introduced above make sense for actions of G on Kählerian complex spaces, see for example [Gre10a, Sections 3.1 and 3.2] and the references given there; in this setup one speaks about *Hamiltonian G-spaces*.

Remark 2.1. If G is semisimple, then every Kähler manifold (Z, ω_Z) endowed with a holomorphic G-action such that ω_Z is K-invariant is Hamiltonian. Moreover, in this case the moment map $\mu\colon Z \to \mathfrak{k}^*$ is unique, see [GS84, Proposition 24.1] and the discussion following this proposition.

Remark 2.2. For a *compact* Kähler manifold (Z, ω_Z) endowed with a holomorphic action of the connected complex reductive Lie group $G = K^{\mathbb{C}}$ and K-invariant Kähler form ω_Z the following statements are equivalent:

(1) Z is a Hamiltonian G-manifold.

(2) G acts meromorphically on Z in the sense of [Fuj78], see also Section 1.

(3) G acts trivially on the Albanese torus $\mathrm{Alb}(Z)$ of Z.

The equivalence (1) $\Longleftrightarrow$ (3) was observed in [Fuj96, Lemma 2.1 and subsequent remark]. The implication (2) $\Longrightarrow$ (3) follows from [Fuj78, Lemma 3.8] since every complex reductive group is linear algebraic. The last implication (3) $\Longrightarrow$ (2) follows from [Som75, Proposition I], see also [Fuj78, Proposition 6.10].

For later use we record an elementary result on the moment image of a Hamiltonian G-manifold (Z, ω_Z) with moment map $\mu\colon Z \to \mathfrak{k}^*$: If the interior of $\mu(Z)$ relative to $\mathfrak{k}^*$ is non-empty, then, due to Sard's Theorem, there exists a point $z \in Z$ such that $d\mu_z$ is surjective. Since condition (2.1) implies

$$\ker d\mu_z = (\mathfrak{k} \cdot z)^{\perp_{\omega_Z}} = \{\xi_Z(z) \mid \xi \in \mathfrak{k}\}^{\perp_{\omega_Z}}, \tag{2.2}$$

cf. [HH99, Section 2.3], and therefore that the rank of μ in z coincides with $\dim K \cdot z$, we conclude that K_z is finite. Conversely, if K_z is finite, then μ is a submersion in z, hence the image of μ has interior points in $\mathfrak{k}^*$. We summarize this discussion in the following

Lemma 2.3. *Suppose that (Z, ω_Z) is a G-connected[1] Hamiltonian G-manifold. Then $\mu(Z)$ has non-empty interior in $\mathfrak{k}^*$ if and only if K acts with generically finite isotropy on Z.*

2.B. The set of semistable points

The set of semistable points is defined by

$$Z_G^{ss}(\mu) := \{z \in Z \mid \overline{G \cdot z} \cap \mu^{-1}(0) \neq \emptyset\}.$$

The G-invariant set $Z_G^{ss}(\mu)$ is open and can be characterized in the following way. For $z \in Z_G^{ss}(\mu)$ consider the inclusion $\iota\colon \overline{G \cdot z} \cap Z_G^{ss}(\mu) \hookrightarrow Z$. Then $\iota^* \omega_Z = i\partial\bar{\partial}\rho$ for some strictly plurisubharmonic exhaustion function ρ and $\iota^*\mu = \mu_\rho$ where $\mu_\rho\colon \overline{G \cdot z} \cap Z_G^{ss}(\mu) \to \mathfrak{k}^*$ is given by

$$\mu_\rho(z)(\xi) := d\rho_z(J\xi_Z(z)), \tag{2.3}$$

where J denotes the complex structure of Z, see [HH96, Section 3].

If Z is compact and if G is semisimple, then $Z_G^{ss}(\mu)$ depends only on the Kähler class $[\omega_Z]$, see [HH96, p. 71]. Since we generalize this result later on for Hamiltonian actions of unipotent groups, we repeat its proof here for the readers' convenience.

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[1] ↑ We say that Z is G-connected if it cannot be written as the disjoint union of two non-empty G-stable closed subsets.

Proposition 2.4. *Let (Z, ω_Z) be a compact Hamiltonian G-manifold with moment map μ and let ω_Z' be another K-invariant Kähler form on Z such that $[\omega_Z] = [\omega_Z'] \in H^2(Z, \mathbb{R})$. Then there exists a moment map μ' for the K-action on (Z, ω_Z') such that $Z_G^{ss}(\mu) = Z_G^{ss}(\mu')$.*

Proof. Since Z is compact Kähler, the $\partial\overline{\partial}$-lemma implies $\omega_Z' = \omega_Z + i\partial\overline{\partial}\varphi$ for some $\varphi \in \mathcal{C}^\infty(Z)$. Defining $\mu_\varphi \colon Z \to \mathfrak{k}^*$ in the same way as in Equation (2.3) one checks directly that the map $\mu' := \mu + \mu_\varphi$ is a moment map for the K-action on (Z, ω_Z').

Now suppose that $z \in Z_G^{ss}(\omega_Z)$. As noted above, we have $\iota^*\omega_Z = i\partial\overline{\partial}\rho$ and $\iota^*\mu = \mu_\rho$ for some strictly plurisubharmonic function ρ on $\overline{G \cdot z} \cap Z_G^{ss}(\mu)$ where $\iota \colon \overline{G \cdot z} \cap Z_G^{ss}(\mu) \hookrightarrow Z$ is the inclusion. Since Z is assumed to be compact, the function φ is bounded, hence $\rho + \iota^*\varphi$ is still an exhaustion function on $\overline{G \cdot z} \cap Z_G^{ss}(\mu)$. Consequently, $\iota^*\mu' = \mu_{\rho+\iota^*\varphi}$ has a zero on $\overline{G \cdot z} \cap Z_G^{ss}(\mu)$, which implies $z \in Z_G^{ss}(\mu')$. The converse inclusion follows by symmetry. $\qquad\square$

Remark 2.5. If G is semisimple and if (Z, ω_Z) is a compact Hamiltonian G-manifold, the moment map for the K-action on Z is unique, see Remark 2.1 above. Due to Proposition 2.4, the set of semistable points therefore depends only on $[\omega_Z] \in H^2(Z, \mathbb{R})$. In this case we thus write $Z_G^{ss}[\omega_Z]$ instead of $Z_G^{ss}(\mu)$.

2.C.   Analytic Hilbert quotients

The importance of semistability stems from the fact that the set of semistable points admits the analogue of a good quotient in the analytic category:

Let G be a complex reductive Lie group and Z a complex space endowed with a holomorphic G-action. A complex space Y together with a G-invariant surjective holomorphic map $\pi \colon Z \to Y$ is called an *analytic Hilbert quotient*[2] of Z by the action of G if

(1) π is a locally Stein map, and

(2) $(\pi_*\mathcal{O}_Z)^G = \mathcal{O}_Y$ holds.

Here, *locally Stein* means that there exists an open covering of Y by open Stein subspaces U_α such that $\pi^{-1}(U_\alpha)$ is a Stein subspace of Z for all α; by $(\pi_*\mathcal{O}_Z)^G$ we denote the sheaf $U \mapsto \mathcal{O}_Z(\pi^{-1}(U))^G = \{f \in \mathcal{O}_Z(\pi^{-1}(U)) \mid f \ G\text{-invariant}\}$, U open in Y.

An analytic Hilbert quotient of a holomorphic G-space Z is unique up to biholomorphism once it exists and we will denote it by $Z /\!\!/ G$. The following properties follow from the corresponding ones in the Stein case, where analytic Hilbert quotients always exist, see [Hei91]: Two points $x, x' \in Z$ have the same image in $Z /\!\!/ G$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$. For each $q \in Z /\!\!/ G$, the fibre $\pi^{-1}(q)$ contains a unique closed G-orbit $G \cdot x$. The stabiliser G_x of x in G is a complex reductive Lie group, see [Mat60]. If $A \subset X$ is a G-invariant analytic subset, then $\pi(A) \subset X /\!\!/ G$ is analytic, and $\pi|_A \colon A \to \pi(A)$ is an analytic Hilbert quotient.

The main results in the quotient theory for complex reductive group actions on Kähler spaces are summarised in the following theorem.

Theorem 2.6. (**[HL94]**, **[HHL94]**, **[HH99]**, **[Sj95]**) *Let Z be a Hamiltonian G-space with Kähler form ω_Z and moment map $\mu \colon Z \to \mathfrak{k}^*$. Then,*

(1) *$Z_G^{ss}(\mu)$ is open and G-invariant, and the analytic Hilbert quotient $\pi \colon Z_G^{ss}(\mu) \to Z_G^{ss}(\mu) /\!\!/ G$ exists,*

(2) *the inclusion $\mu^{-1}(0) \hookrightarrow Z_G^{ss}(\mu)$ induces a homeomorphism $\mu^{-1}(0)/K \simeq Z_G^{ss}(\mu) /\!\!/ G$,*

(3) *the complex space $Z_G^{ss}(\mu) /\!\!/ G$ carries a Kähler structure that is induced by symplectic reduction from ω_Z and that is smooth along a natural stratification of $Z_G^{ss}(\mu) /\!\!/ G$.*

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[2] $\uparrow$ In some places in the literature, the terminology *semistable quotient* is used for the same concept.

2.D. Moment maps associated with representations and their images

Let G be a connected semisimple complex Lie group acting linearly on a finite dimensional complex vector space V. If we equip V with the K-invariant flat Kähler metric ω_V given by a K-invariant hermitian inner product $\langle \cdot, \cdot \rangle$, then a moment map $\mu_V \colon V \to \mathfrak{k}^*$ for the K-action on V is given by $\mu_V^\xi(v) = -\frac{i}{2}\langle \xi.v, v \rangle$ for every $\xi \in \mathfrak{k}$. Note that $\omega_V = i\partial\overline\partial\rho$ where $\rho(v) = \|v\|^2$ for all $x \in V$ and that $\mu_V = \mu_\rho$ in this case.

For any $v \in V$ consider the affine G-variety $\overline{G \cdot v}$. The restriction of μ_V to $\overline{G \cdot v}$ yields the moment map for the K-action on $\overline{G \cdot v}$ associated with the strictly plurisubharmonic exhaustion function $\rho|_{\overline{G \cdot v}}$. By abuse of notation we will denote the restricted Kähler form and moment map again by ω_V and μ_V, respectively. For later use we record the following result of Sjamaar, see [Sj98, Theorem 4.9, Lemma 4.10]. For its statement we have to introduce a maximal torus T of K with Lie algebra t, the choice of a positive Weyl chamber $\mathfrak{t}_+^*$, and the corresponding set Λ^+ of dominant weights. For $\lambda \in \Lambda^+$ let V_λ denote the irreducible G-representation with highest weight λ.

Theorem 2.7. *The moment map $\mu_V \colon \overline{G \cdot v} \to \mathfrak{k}^*$ is proper and verifies*

$$\mu_V(\overline{G \cdot v}) \cap \mathfrak{t}_+^* = \mathrm{cone}\{\lambda \in \Lambda^+ \mid V_\lambda \text{ occurs in } \mathbb{C}[\overline{G \cdot v}]\}.$$

In general it may be rather difficult to decide for which dominant weight λ the irreducible representation V_λ occurs in $\mathbb{C}[G \cdot v]$. Note that the inclusion $G \cdot v \hookrightarrow \overline{G \cdot v}$ gives an injective homomorphism of algebras $\mathbb{C}[\overline{G \cdot v}] \to \mathbb{C}[G \cdot v] \cong \mathbb{C}[G]^{G_v}$. Therefore the situation is slightly easier for affine completions of quasi-affine homogeneous spaces G/G_v for which the map $\mathbb{C}[\overline{G \cdot v}] \to \mathbb{C}[G]^{G_v}$ is an isomorphism.

To follow this train of thought, recall that an algebraic subgroup H of G is called *Grosshans* if G/H is quasi-affine and if the algebra $\mathbb{C}[G]^H \cong \mathbb{C}[G/H]$ is finitely generated. This is equivalent to the existence of a finite-dimensional G-representation space W containing G/H as an orbit $G \cdot w$ such that the codimension of $\overline{G \cdot w} \setminus G \cdot w$ in $\overline{G \cdot w}$ is at least 2, see [Gr97, Theorem 4.3]. Recall that an algebraic subgroup of G is called *unipotent* if it consists entirely of unipotent elements. If N is a unipotent subgroup of G, then G/N is always quasi-affine, see [Gr97, Corollary 1.5], but not affine, see [Mat60]; however, not every such N is Grosshans. If N is the unipotent radical of a parabolic subgroup of G, then N is Grosshans, see [Gr83, Theorem 2.2].

Suppose that the unipotent subgroup N of G is Grosshans. Then we have a canonical affine completion $\overline{G/N}^a = \operatorname{Spec}\mathbb{C}[G]^N$. Since N is contained in a maximal unipotent subgroup of G, we can deduce from [Sj98, Example 4.19] that

$$\{\lambda \in \Lambda^+ \mid V_\lambda \text{ occurs in } \mathbb{C}[G]^N\} = \Lambda^+.$$

Consequently, by embedding $\overline{G/N}^a$ into any G-representation, we can find a Kähler form inducing a proper, surjective moment map for the K-action on $\overline{G/N}^a$. Combining this observation with an application of Lemma 2.3 to the free K-action on $G/N \subset \overline{G/N}^a$ we obtain the following result.

Lemma 2.8. *Suppose that N is a unipotent Grosshans subgroup of G. Then there exists a Kähler form ω_V on G/N such that the image of the corresponding moment map $\mu_V \colon G/N \to \mathfrak{k}^*$ is a K-invariant dense open subset of $\mathfrak{k}_{\mathrm{reg}}^*$. Moreover, ω_V and μ_V extend to the canonical affine completion $\overline{G/N}^a$.*

As the following example shows, this result depends on the choice of the G-representation into which we embed G/N.

Example 2.9. Let G be a simply-connected semisimple complex Lie group. For any choice of $\lambda_1, \ldots, \lambda_k \in \Lambda^+$ consider $v := v_1 + \cdots + v_k \in V_{\lambda_1}^* \oplus \cdots \oplus V_{\lambda_k}^*$ where $v_j \in V_{\lambda_j}^*$ is a highest weight vector. Then we have $\mathbb{C}[\overline{G \cdot v}] = \bigoplus_{\lambda \in M} V_\lambda$ where M is the submonoid of Λ^+ generated by $\lambda_1, \ldots, \lambda_k$, see [PV72, Theorem 6]. Consequently, $\mu_V(\overline{G \cdot v}) \cap \mathfrak{t}^*$ is the cone generated by $\lambda_1, \ldots, \lambda_k$.

To construct an example where the image of $\overline{G \cdot v}$ under the moment map is not the whole of $\mathfrak{k}^*$, let $G = \mathrm{SL}(3, \mathbb{C})$ and choose $\lambda_1 = \varpi_1 + \varpi_2$ and $\lambda_2 = 2\varpi_1 + \varpi_2$ where ϖ_1, ϖ_2 are the fundamental weights of G. Note that V_{λ_1} and hence also $V_{\lambda_1}^*$ are isomorphic to the adjoint representation of $G = \mathrm{SL}(3, \mathbb{C})$. It follows that G_{v_1} is the connected subgroup of G having as Lie algebra the semi-direct sum of the kernel of λ_1 in the chosen Cartan sub-algebra of $\mathfrak{g}$ and the positive maximal unipotent sub-algebra $\mathfrak{n}$ of $\mathfrak{g}$. Since the Lie algebra of G_{v_2} for $v_2 \in V_{\lambda_2}$ contains $\mathfrak{n}$ and since the kernels of λ_1 and λ_2 intersect only trivially, the Lie algebra of the stabiliser of $v = v_1 + v_2$ must coincide with $\mathfrak{n}$. In summary, we see that in the chosen setup G_v is the unipotent radical of a Borel subgroup of G, hence Grosshans, and that $\mu_V \colon \overline{G \cdot v} \to \mathfrak{k}^*$ is not surjective.

Remark 2.10. If N is the unipotent radical of a parabolic subgroup of G, a G-representation space E containing $\overline{G/N}^{\mathrm{a}}$ and a certain K-invariant hermitian inner product on E are described in great detail in [Kir11], extending [GJS02] which dealt with unipotent radicals of Borel subgroups. In this situation it is natural to equip $\overline{G/N}^{\mathrm{a}}$ with the restriction of the associated flat Kähler form ω_E as above, since the associated symplectic structure coincides with the one obtained via symplectic implosion from the cotangent bundle T^*K.

3.   Hamiltonian G-extensions

Let $G = K^{\mathbb{C}}$ be a complex reductive group with maximal compact subgroup K. Recall that a unipotent subgroup of G is by definition an algebraic subgroup of G consisting entirely of unipotent elements. Such groups are automatically nilpotent and connected, see [OV90, Chapter 3.2.2], hence simply-connected. Let N be such a unipotent subgroup of G. Since our focus lies on actions of N, *we suppose from now on that G is connected and semisimple.* Due to the simply-connectedness of N, by lifting to the universal cover if necessary, we may and will often assume that G is simply-connected as well.

3.A.   Meromorphic actions and Hamiltonian extensions

We will explore the relation between meromorphic N-actions and Hamiltonian G-actions.

Definition 3.1. Let (X, ω_X) be a connected compact Kähler manifold endowed with a holomorphic N-action. A *Hamiltonian G-extension of (the N-action on) (X, ω_X)* consists in a connected compact Hamiltonian G-manifold (Z, ω_Z) and an N-equivariant embedding $\iota \colon X \hookrightarrow Z$ such that for the de Rham cohomology classes associated with the Kähler forms we have

$$\iota^*[\omega_Z] = [\omega_X]. \tag{3.1}$$

Remark 3.2. As $G = K^{\mathbb{C}}$, and hence K, is assumed to be semisimple, it follows from the fact that integration over K does not change the cohomology class of a given Kähler form and from Remark 2.1 that any N-equivariant embedding of (X, ω_X) into a compact Kähler G-manifold (Z, ω_Z) satisfying Equation (3.1) is automatically a Hamiltonian G-extension.

The definition is motivated by the following example and the role it plays in the Geometric Invariant Theory of unipotent group actions on projective varieties, cf. [DK07, Section 5].

Example 3.3. Let N act effectively on a smooth projective variety X. Any N-equivariant embedding $N \hookrightarrow \mathbb{P}(W)$, where W is an N-representation space on which N acts via an embedding $N \hookrightarrow \mathrm{SL}(W)$ is a Hamiltonian $\mathrm{SL}(W)$-extension.

Example 3.4. Suppose that G acts on (X, ω_X), extending the N-action. Since integration over K does not change the cohomology class of ω_X, we may, and will, suppose that ω_X is K-invariant. Therefore we can simply take $Z = X$ with $\iota = \mathrm{id}_X$ as Hamiltonian G-extension of the N-action on X.

On the other hand, the twisted product[3] $G \times_N X$ is G-equivariantly isomorphic to $G/N \times X$ via the map $[g, x] \mapsto (gN, g \bullet x)$ and hence embeds into $Z := \overline{G/N} \times X$ where $\overline{G/N}$ is a smooth projective G-equivariant compactification of the quasi-affine homogeneous space G/N. Endowing Z with a direct product Kähler metric $\omega_0 \oplus \omega_X$ and considering $\iota \colon X \hookrightarrow Z$, $\iota(x) = (eN, x)$, we obtain another Hamiltonian G-extension of the N-action on (X, ω_X).

Remark 3.5. Although the automorphism group of a compact Kähler manifold has a natural structure of a meromorphic group acting meromorphically on X, see [Fuj78, Theorem 5.5], this cannot be used in order to find a natural embedding of a unipotent algebraic group N acting holomorphically on X into a complex reductive group G sitting inside $\mathrm{Aut}(X)$ as the following example shows.

Example 3.6. Consider the connected algebraic group

$$G = \left\{ \begin{pmatrix} (ad - bc)^{-1} & z & w \\ 0 & a & b \\ 0 & c & d \end{pmatrix}; \ ad - bc \neq 0 \right\} \cong \mathrm{GL}(2, \mathbb{C}) \ltimes \mathbb{C}^2.$$

According to [Br14, Theorem 1] there exists a 12-dimensional projective complex manifold X having $\mathrm{Aut}^0(X)$ isomorphic to G.

The group

$$N = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}; \ z, w \in \mathbb{C} \right\} \cong \mathbb{C}^2$$

is a unipotent subgroup of G which is not conjugate to a subgroup of the radical $R_u(G)$, nor to a subgroup of a Levi subgroup of G.

The group

$$N = \left\{ \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}; \ t \in \mathbb{C} \right\} \cong \mathbb{C}$$

is a non-algebraic subgroup of G. Consequently, N does not act meromorphically on X. Its Zariski closure is the group

$$\overline{N} = \left\{ \begin{pmatrix} t & s & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}; \ t \in \mathbb{C}^*, s \in \mathbb{C} \right\} \cong \mathbb{C}^* \times \mathbb{C}.$$

Note that $\overline{N}$ and hence N are not conjugate to subgroups of neither a Levi subgroup nor the radical of G.

Let us state the main result of this section.

Theorem 3.7. *Let N be a unipotent subgroup of the simply-connected complex semisimple Lie group G. For a connected compact Kähler manifold (X, ω_X) endowed with a holomorphic N-action the following statements are equivalent.*

(1) *There exists a Hamiltonian G-extension (Z, ω_Z) of the N-action on X.*

(2) *The N-action on X is meromorphic.*

(3) *The twisted product $G \times_N X$ is Kähler.*

Moreover, given a meromorphic N-action on X we can always find a Hamiltonian G-extension that is a G-equivariant compactification of $G \times_N X$.

---

[3] ↑ The twisted product $G \times_N X$ is by definition the quotient of $G \times X$ by the proper holomorphic N-action given by $n \bullet (g, x) = (gn^{-1}, n \bullet x)$. The N-orbit of (g, x) will be denoted by $[g, x] \in G \times_N X$.

Remark 3.8. Since the centre of G is finite and since N does not contain finite subgroups, it follows that the G-action on $G \times_N X$ is effective whenever N acts effectively on X.

Motivated by the above and by the equivalent conditions listed in Remark 2.2, we make the following definition that is central to our discussion.

Definition 3.9. Let N be a unipotent subgroup of the simply-connected complex semisimple Lie group G, and let X be a connected compact Kähler manifold endowed with a holomorphic N-action. We say that (X, ω_X) is a *Hamiltonian N-manifold* if there exists a Hamiltonian G-extension (Z, ω_Z) of the N-action on X.

The rest of this section is devoted to the proof of Theorem 3.7.

3.B. Necessary conditions for the existence of a G-extension

In this section, we will show the implications "(1) $\Rightarrow$ (3) $\Rightarrow$ (2)" of Theorem 3.7. Let (X, ω_X) be a connected compact Kähler manifold on which N acts holomorphically, and let $\alpha\colon X \to \mathrm{Alb}(X)$ be the natural map from X to its Albanese torus.

Lemma 3.10. ("(1) $\Rightarrow$ (3)") *If the N-action on X admits a Hamiltonian G-extension, then the twisted product $G \times_N X$ is Kähler.*

Proof. Let us consider the induced proper embedding $Y = G \times_N X \hookrightarrow G \times_N Z \cong G/N \times Z$. The claim follows from the fact G/N is quasi-affine, and hence Kähler. $\square$

Remark 3.11. In fact, if N is a connected nilpotent closed complex subgroup of G, then assuming that $G \times_N X$ be Kähler we can show that N is in fact algebraic: applying [Bla56, Proposition II.1] to the fibre bundle $G \times_N X \to G/N$, we see that G/N is Kähler, an application of [GMO11, Corollary 4.12] then yields the claim. This justifies our a priori algebraicity assumption on $N < G$.

Next, we embark on proving the implication "(3) $\Rightarrow$ (2)". As a first step, we prove the following

Lemma 3.12. *If $G \times_N X$ is Kähler, then the action of N on the Albanese torus $\mathrm{Alb}(X)$ is trivial.*

Proof. Suppose that the action of N on $\mathrm{Alb}(X)$ is non-trivial. Then, as N acts on $\mathrm{Alb}(X)$ by translations, we can find a closed one-parameter subgroup $\mathbb{C} \hookrightarrow N$ such that $\dim \mathbb{C} \cdot [0] = 1$ where $[0]$ denotes the base point of $\mathrm{Alb}(X)$. The topological closure $T := \overline{\mathbb{C} \cdot [0]} \subset \alpha(X) \subset \mathrm{Alb}(X)$ is a connected compact subgroup of $\mathrm{Alb}(X)$, hence a subtorus. Its pre-image $\alpha^{-1}(T)$ is a compact $\mathbb{C}$-invariant subvariety of X. According to the Jacobson-Morozov Theorem, see [Jac62, Theorem III.17], there is a closed subgroup S of G that is locally isomorphic to $\mathrm{SL}(2, \mathbb{C})$ and contains the one-parameter subgroup $\mathbb{C} \hookrightarrow N$ under discussion. Since $G \times_N X$ is Kähler and contains $Y' := S \times_{\mathbb{C}} \alpha^{-1}(T)$ as a closed S-invariant analytic subset, every S-orbit in Y' and therefore every $\mathbb{C}$-orbit in $\alpha^{-1}(T)$ is Zariski-open in its closure by [GMO11, Corollary 3.9].

There exists a 1-dimensional orbit $\mathbb{C} \cdot x$ in X such that $\alpha(x) = [0] \in \mathrm{Alb}(X)$. Consequently, going through the possible stabiliser subgroups, we see that there are two possibilities: either the orbit $\mathbb{C} \cdot x$ is compact, i.e., it is a 1-dimensional complex torus, or the normalisation of $\overline{\mathbb{C} \cdot x}$ is biholomorphic to $\mathbb{P}_1$. Both cases lead to contradictions. Indeed, in the first case, the isotropy of $\mathbb{C}$ and hence the isotropy in S would be infinite discrete, which contradicts [GMO11, Proposition 4.4], while in the second case the fact that $\alpha|_{\overline{\mathbb{C} \cdot x}}$ is non-constant by construction would produce a non-zero holomorphic 1-form on $\mathbb{P}^1$. $\square$

We remark that the fact that N acts trivially on $\mathrm{Alb}(X)$ alone does not imply the existence of a Hamiltonian G-extension of the N-action on X as the following example shows.

Example 3.13. Let $N = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix} \subset G = \mathrm{SL}(2, \mathbb{C})$ and consider its action on $X = \mathbb{P}_2$ given by

$$t \bullet [x_0 : x_1 : x_2] := [e^t x_0 : e^{it} x_1 : x_2].$$

The N-orbits in X are not locally Zariski closed, and there are elements having isotropy isomorphic to $\mathbb{Z}$. According to [GMO11, Theorem 3.6] $G \times_N X$ cannot be Kähler. Hence, there does not exist a Hamiltonian G-extension of the N-action on X although N acts trivially on $\mathrm{Alb}(X) = \{\mathrm{pt}\}$.

Our next goal is to show that, if there exists a Hamiltonian G-extension of the N-action on X, then N acts meromorphically on X. Due to Lemma 3.12 the claim is equivalent to the fact that the map $N \to \mathrm{Aut}_{\mathrm{aff}}(X)$ induced by the action has analytically Zariski-closed image, where $\mathrm{Aut}_{\mathrm{aff}}(X)$ denotes the kernel of the Jacobi homomorphism $\alpha_* \colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathrm{Alb}(X))$, cf. [Fuj78, §2]. Since by [Fuj78, Lemma 4.6 and Theorem 5.5] or [Lie78, Proposition 2.1] the analytic Zariski-topology on the meromorphic subgroup $\mathrm{Aut}_{\mathrm{aff}}(X) < \mathrm{Aut}^0(X)$ is obtained from the complex structure on the cycle space $\mathscr{C}_{\dim X}(X \times X)$, as a first step we prove a technical lemma about induced actions on cycle spaces.

Lemma 3.14. *Let N be a unipotent subgroup of G, let M be a Kähler manifold endowed with a holomorphic N-action, and let $\mathscr{C}_k(M)$ be the Barlet space of compact k-cycles. Then, there exists a natural G-equivariant isomorphism $G \times_N \mathscr{C}_k(M) \cong \mathscr{C}_k(G \times_N M)$.*

Proof. For every irreducible compact analytic subset A of M and each $g \in N$, the image $g(A)$ is again an irreducible compact analytic subset of M. Hence, we obtain a holomorphic N-action on $\mathscr{C}_k(M)$ by the obvious extension of this action to k-cycles. Moreover, the inclusion $\iota \colon M \to G \times_N M$, $x \mapsto [e, x]$ induces a proper holomorphic embedding of $\mathscr{C}_k(M)$ into $\mathscr{C}_k(G \times_N M)$ by sending $A \in \mathscr{C}_k(M)$ to $\iota(A)$. From this we obtain a well-defined injective and immersive holomorphic map

$$\Phi_k \colon G \times_N \mathscr{C}_k(M) \hookrightarrow \mathscr{C}_k(G \times_N M), \quad \Phi_k([g, A]) := g \bullet \iota(A).$$

Suppose for a moment that $A \subset G \times_N M$ is an irreducible compact analytic subset. Since G/N is quasi-affine, and hence in particular holomorphically separable, the bundle projection $\pi \colon G \times_N X \to G/N$ maps A to a point $gN \in G/N$. Consequently, $\mathscr{C}_k(G \times_N M)$ coincides with the space $\mathscr{C}_k(G \times_N M)_\pi$ of π-relative cycles. Moreover, the natural G-action on $\mathscr{C}_k(G \times_N M)$ makes the resulting projection $\mathscr{C}_k(G \times_N M) \to G/N$ equivariant.

We hence conclude from the fact that Φ_k induces an isomorphism of the fibres over eN that the image of Φ_k is all of $\mathscr{C}_k(G \times_N M)_\pi = \mathscr{C}_k(G \times_N M)$. The claim follows. □

Proposition 3.15. **("(3) $\Rightarrow$ (2)")** *If $G \times_N X$ is Kähler, then N acts meromorphically on X.*

Proof. Composed with the the natural embedding $\mathrm{Aut}_{\mathrm{aff}}(X) \hookrightarrow \mathscr{C}_n(X \times X)$, the action map $N \to \mathrm{Aut}_{\mathrm{aff}}(X)$ yields the following holomorphic map:

$$\iota \colon N \to \mathscr{C}_n(X \times X)$$
$$g \mapsto \Gamma_{x \mapsto g \bullet x} := \{(x, g \bullet x) \mid x \in X\}.$$

We set $n := \dim X$ and consider the N-action on the product $X \times X$ given by $g \bullet (x, y) = (x, g \bullet y)$. We have $G \times_N (X \times X) \cong (G \times_N X) \times X$, which shows that $G \times_N (X \times X)$ is Kähler. Hence, the cycle space $\mathscr{C}_n(G \times_N (X \times X))$ is Kähler by [BV89, Théorème 2]. Applying Lemma 3.14 to $M = N \times N$ we infer that $G \times_N \mathscr{C}_n(X \times X)$ is likewise Kähler. It therefore follows from [GMO11, Theorem 3.6] that all N-orbits in $\mathscr{C}_n(X \times X)$ are locally closed in the analytic Zariski-topology.

Now, we notice that $\iota(N)$ coincides with the N-orbit of $\Delta_X \in \mathscr{C}_n(X \times X)$, where Δ_X denotes the diagonal in $X \times X$. This implies that $\iota(N)$ is a locally Zariski-closed, hence Zariski-closed subgroup of $\mathrm{Aut}_{\mathrm{aff}}(X) < \mathrm{Aut}^0(X) \subset \mathscr{C}_n(X \times X)$, cf. [Hum75, Section 7.4], as was to be shown. □

For later usage, we note the following, related observation.

Lemma 3.16. *An algebraic subgroup H of G which acts meromorphically on X acts automatically trivially on* $\mathrm{Alb}(X)$.

Proof. This follows from the fact that every meromorphic homomorphism from an affine linear group to a compact complex torus is constant, see [Fuj78, Lemma 3.8]. $\qquad\square$

3.C.   Existence of a Hamiltonian G-extension

In this section, we will show the crucial implication "(2) $\Rightarrow$ (1)" of Theorem 3.7, i.e., the fact that meromorphic actions of unipotent algebraic subgroups of semisimple groups G always admit Hamiltonian G-extensions, thereby completing the proof.

Let us recall the setup: Let N be a unipotent subgroup of the simply-connected complex semisimple Lie group G, and let X be a connected compact Kähler manifold endowed with a meromorphic N-action.

3.C.a.   The case of unipotent radicals of Borel subgroups

Let B be a Borel subgroup of G having Levi decomposition $B = TU = UT$. In a first step, we make the additional assumption that N coincides with the the unipotent radical U of B; i.e., $N = U$.

Let us consider the twisted product $M := B \times_U X$ and the U-equivariant inclusion $\iota_X \colon X \hookrightarrow M$ as the fibre over eU. Since the principal bundle $B \to B/U \cong T$ is holomorphically trivial, the same holds for the associated fibre bundle; i.e., we have $M \cong T \times X$. Explicitly, an isomorphism $B \times_U X \to T \times X$ is given by $[tu, x] \mapsto (t, u \bullet x)$. A direct calculation shows that the induced B-action on $T \times X$ is given by the formula

$$(tu) \bullet (s, x) = (ts, (s^{-1} u s) \bullet x). \tag{3.2}$$

Let $\overline{G}$ be a projective meromorphic structure on G. Since the subgroups T, U and B are algebraic in G, their topological closures in $\overline{G}$ are analytic.

Lemma 3.17. *If U acts meromorphically on X, then the B-action on $T \times X$ defined in Equation* (3.2) *extends to a meromorphic B-action on $\overline{T} \times X$. In particular, there exists a B-equivariant Kähler compactification $(\overline{M}, \omega_{\overline{M}})$ of $M \supset \iota_X(X)$ such that $[\iota_X^*(\omega_{\overline{M}})] = [\omega_Z]$.*

Proof. By the definition of a meromorphic structure the map $c \colon T \times U \to U$, $(s, u) \mapsto s^{-1} u s$, extends to a meromorphic map $\overline{c} \colon \overline{T} \times \overline{U} \to \overline{U}$ which is holomorphic on $(\overline{T} \times U) \cup (T \times \overline{U})$. Analogously, for the same reason the multiplication map $m \colon T \times T \to T$ extend meromorphically to $\overline{T} \times \overline{T} \to \overline{T}$ making $\overline{T}$ a T-bi-equivariant compactification of T. Denote the extended map by $\overline{m}$. With this notation, B acts on $\overline{T} \times X$ by the formula

$$(tu) \bullet (s, x) := (\overline{m}(t, s), \overline{\alpha}(\overline{c}(s, u), x)),$$

where $\overline{\alpha} \colon \overline{U} \times X \to X$ is the meromorphic extension of the U-action on X. It is hence clear that B acts meromorphically on $\overline{T} \times X$. $\qquad\square$

In the next step we consider the holomorphic fibre bundle $Z := G \times_B \overline{M} \xrightarrow{\ \pi\ } G/B$. The natural inclusion $\overline{M} \hookrightarrow G \times_B \overline{M}$ as the fibre over eB is denoted by $\iota_{\overline{M}}$. Both the typical fibre $\overline{M}$ and the base G/B of Z are Kähler. Using Blanchard's theorem [Bla56, Théorème principal II], we will construct a Kähler form on Z such that Z is a Hamiltonian G-extension of the U-action on X.

Since G/B is simply-connected, in order to be able to apply Blanchard's result and hence show that $G \times_B \overline{M}$ is Kähler, we only have to check that the *transgression map* from $H^1(\overline{M}, \mathbb{R})$ to $H^2(G/B, \mathbb{R})$ is identically zero. Recall that this map is defined on the set of transgressive elements in $H^1(\overline{M}, \mathbb{R})$ in the following way. A class $[\alpha] \in H^1(\overline{M}, \mathbb{R})$ is called *transgressive* if there exists a 1-form β on Z such that $[\iota_{\overline{M}}^* \beta] = [\alpha]$ and such that there exists a 2-form τ on G/B such that $d\beta = \pi^* \tau$. Since π^* is injective, we

have $d\tau = 0$, and the transgression map then associates $[\tau] \in H^2(G/B, \mathbb{R})$ to $[\alpha] \in H^1(\overline{M}, \mathbb{R})$, see [BT82, Proposition 18.13]. As a side-remark we note that this transgression map is zero if and only if $b_1(Z) = b_1(G/B) + b_1(\overline{M})$ where b_1 denotes the first Betti number, see [Bla56, p. 192].

Lemma 3.18. *Suppose that B acts trivially on* $\mathrm{Alb}(\overline{M})$. *Then, every* $[\alpha] \in H^1(\overline{M}, \mathbb{R})$ *is transgressive, and the transgression map is identically zero.*

Proof. As $\overline{M}$ is Kähler by Lemma 3.17, we may use the Hodge decomposition to write

$$[\alpha] \in H^1(\overline{M}, \mathbb{R}) \subset H^1(\overline{M}, \mathbb{C}) = H^{1,0}(\overline{M}) \oplus \overline{H^{1,0}(\overline{M})} = H^0(\Omega^1_{\overline{M}}) \oplus \overline{H^0(\Omega^1_{\overline{M}})}$$

as $[\alpha] = \eta + \overline{\eta}$ where η is a holomorphic 1-form on $\overline{M}$. Since by hypothesis and Lemma 3.16 the algebraic subgroup $B < G$ acts trivially on $\mathrm{Alb}(\overline{M})$, the form η is B-invariant, hence extends to a G-invariant holomorphic 1-form $\hat{\eta} \in H^0(\Omega^1_Z)^G$. Since $d\hat{\eta}$ is likewise G-invariant, it is uniquely determined by its restriction $\iota^*_{\overline{M}}(d\hat{\eta}) = d(\iota^*_{\overline{M}}\hat{\eta}) = d\eta = 0$. Consequently, $d\hat{\eta} = 0 = \pi^*\mathbf{0}$. We conclude that $[\alpha]$ is transgressive and is mapped to $\mathbf{0} + \mathbf{0} = \mathbf{0} \in H^2(G/B, \mathbb{R})$ by the transgression map. □

Remark 3.19. The proof works more generally for a *parabolic* subgroup $P < G$ acting on a compact Kähler manifold such that the induced action on the Albanese is trivial. To be more precise, we see that for every such P-manifold X the twisted product $G \times_P X$ is Kähler. As the G-action on this manifold is Hamiltonian, by applying [GMO11, Theorem 3.6] or [Fuj96, Remark after Lemma 2.1] together with [Fuj78, Proposition 6.10] we conclude that every P-orbit in X is Zariski-open in its closure and that the P-action on X is meromorphic. This generalises and gives a new proof for a result of Sommese, cf. [Som75, §3]. For criteria guaranteeing triviality of induced actions on Albanese tori see [Fuj78, §6c)].

Combining Lemmata 3.17, 3.16, and 3.18 with Blanchard's theorem we conclude that the twisted product $Z = G \times_B \overline{M}$ is Kähler. Moreover, the first step of Blanchard's proof [Bla56, p. 187] shows that the cohomology class of the constructed Kähler form on $Z = G \times_B \overline{M}$ pulls back under $\iota_{\overline{M}}$ to $[\omega_{\overline{M}}]$ on $\overline{M}$. Embedding X into $\overline{M}$ by ι_X and further into Z by $\iota_{\overline{M}}$ we obtain a Hamiltonian G-extension of the U-action on X. Notice that Z contains $G \times_U X \cong G \times_B (B \times_U X)$ as a Zariski open subset. In fact, $Z = G \times_B \overline{M} \cong G \times_B (\overline{T} \times X) \to G \times_B \overline{T}$ is an extension of the X-fibre bundle $G \times_U X \to G/N$ to the projective completion $G \times_B \overline{T}$ of G/N.

3.C.b.   The general case

In the second and final step we show that the case of an arbitrary unipotent subgroup N of G can be reduced to the unipotent radical U of a Borel subgroup $B \subset G$.

Lemma 3.20. *Let N be a unipotent subgroup of G. Then there exists a Borel subgroup $B = TU$ of G such that $N \subset U$. In addition, if N acts effectively and meromorphically on X, then $U \times_N X$ admits a U-equivariant Kähler compactification on which U acts effectively and meromorphically and whose Kähler class extends the given Kähler class $[\omega_X]$ on X.*

Proof. The first statement is [Hum75, 30.4, Theorem]. Moreover, from the first paragraph in the proof of [Akh95, Chapter 5, Theorem 4] we conclude that the N-principal bundle $U \to U/N$ admits an algebraic section, whose image we call S. As a consequence, we obtain an N-equivariant isomorphism $N \times S \to U$ with $S \cong U/N$. Let $\overline{S}$ be a U-equivariant smooth projective compactification of S. Following the line of argumentation presented at the beginning of Section 3.C.c we conclude as before that $\overline{S} \times X$ is a compact Kähler manifold endowed with a meromorphic U-action, containing $U \times_N X$ as a Zariski-open set. Again, we can choose the Kähler form on $\overline{S} \times X$ such that its class extends the given Kähler class $[\omega_X]$. □

Finally, by applying the discussion of Section 3.C.c to the compact Kähler manifold $X' = \overline{S} \times X$ with meromorphic U-action obtained in Lemma 3.20 and noticing that

$$Y = G \times_N X \cong G \times_B (B \times_U (U \times_N X)) \cong G \times_B (T \times (U/N) \times X) \tag{3.3}$$

we arrive at

Proposition 3.21. ("(2) ⇒ (1)") *Let N be a unipotent subgroup of G acting effectively and meromorphically on the compact Kähler manifold X. Then, there exists a Hamiltonian G-extension Z of the N-action on X, which can be chosen such that it contains $Y := G \times_N X$ as a Zariski open subset. More precisely, we might choose*

$$Z = \overline{Y} = G \times_B (\overline{T} \times \overline{U/N} \times X),$$

where $B = TU$ is a Borel subgroup of G with $N \subset U$ and where $\overline{T} \times \overline{U/N}$ is the B-equivariant compactification of $T \times (U/N)$ constructed in Lemmata 3.17 and 3.20.

Remark 3.22. (Non-uniqueness) There are many possible ways to choose the compactifications $\overline{T}$ and $\overline{S}$; in the first case, we have all smooth projective toric varieties of dimension $\dim T$ to choose from.

3.C.c.   Additional observations

We note a minimality property of the above construction that is crucial for subsequent arguments on the independence of the set of semistable points from the chosen G-extensions, see Section 4.B below.

Proposition 3.23. *Let X be a compact Hamiltonian N-manifold and let $\overline{Y}$ be the G-equivariant compactification of $Y := G \times_N X$ whose existence is guaranteed by Proposition 3.21. Then, for any Hamiltonian G-extension Z of the N-action on X, there exists a G-equivariant embedding $\overline{Y} \hookrightarrow \overline{G/N} \times Z$ where $\overline{G/N}$ is a certain G-equivariant compactification of G/N.*

Proof. Let Z be any G-extension of the N-action on X. The N-equivariant embedding $X \hookrightarrow Z$ induces a G-equivariant embedding $Y = G \times_N X \hookrightarrow G \times_N Z \cong G/N \times Z$. The identification $G/N \cong G \times_B (T \times U/N)$, cf. Equation (3.3), suggests choosing the compactified fibre bundle

$$\overline{G/N} := G \times_B (\overline{T} \times \overline{U/N}) \tag{3.4}$$

as a well-adapted G-equivariant compactification of G/N. With this definition we obtain the desired embedding $\overline{Y} = G \times_B (\overline{T} \times \overline{U/N} \times X) \hookrightarrow G \times_B (\overline{T} \times \overline{U/N} \times Z) \cong \overline{G/N} \times Z$. □

Remark 3.24. Since N is a connected subgroup of the simply-connected group G, the quasi-affine variety G/N is simply-connected. Consequently, every smooth compactification of G/N is likewise simply-connected, see [FL81, 0.7(B)]; in particular, this observation applies to the compactification constructed in Proposition 3.23.

Example 3.25. Suppose that $N = \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix} \right) \subset G = \mathrm{SL}(2, \mathbb{C})$. Choosing $\overline{T} = \mathbb{P}_1$, we see that $\overline{G/N} = G \times_B \overline{T}$ is the blow-up of $\mathbb{P}_2$ at the origin, i.e., the first Hirzebruch surface.

4.   The set of semistable points with respect to unipotent groups

Let $G = K^{\mathbb{C}}$ be a simply-connected semisimple complex Lie group with maximal compact subgroup K, let N be a unipotent subgroup of G, and let (X, ω_X) be a compact Hamiltonian N-manifold. In this section we explain how one can use the concept of Hamiltonian G-extensions in order to define the set of semistable points $X_N^{ss}[\omega_X]$ for the N-action on X.

4.A.   Defining the set of semistable points

We will slowly approach the goal of defining the correct notion of semistability, see Definition 4.2 below.

4.A.a. The naive approach

Since (X, ω_X) is a compact Hamiltonian N-manifold, we can find a Hamiltonian G-extension (Z, ω_Z) of the N-action on X with N-equivariant embedding $\iota \colon X \hookrightarrow Z$. It is tempting to define the set of N-semistable points in X as the N-invariant open subset $\iota^{-1}(Z_G^{ss}[\omega_Z])$. As the following example shows, this set heavily depends on the choice of the G-extension (Z, ω_Z).

Example 4.1. Let $X = \mathbb{P}_1$ with the Fubini-Study metric ω_{FS} and let $N = \mathbb{C}$ act on $\mathbb{P}_1$ by $t \cdot [x_0 : x_1] = [x_0 + t x_1 : x_1]$. Since the N-action extends to an action of $G = \mathrm{SL}(2, \mathbb{C})$, we have the following three Hamiltonian G-extensions.

(1) $Z_1 = X$ where $\iota_1 = \mathrm{id}_X$,

(2) $Z_2 = X \times X$ endowed with $\omega_{Z_2} = \frac{1}{2}(\omega_{FS} \oplus \omega_{FS})$ and the diagonal G-action, where the embedding is given by $\iota_2(x) = (x, x)$, and

(3) $Z_3 = Z_2$ with $\omega_{Z_3} = 2\omega_{Z_2}$ and with the diagonal G-action where $\iota_3(x) = ([1 : 0], x)$.

It is not hard to show that $\iota_1^{-1}(Z_{1,G}^{ss}) = \iota_2^{-1}(Z_{2,G}^{ss}) = \emptyset$ while $\iota_3^{-1}(Z_{3,G}^{ss}) = \mathbb{P}_1 \setminus \{[1 : 0]\}$.

4.A.b. The case of linear actions on projective manifolds

Let us recall the approach taken by Doran and Kirwan in [DK07]. There, the authors study the situation where X is a subvariety of $\mathbb{P}(W)$ for some finite-dimensional G-module W and they consider the embedding $\hat{\iota} \colon G \times_N X \hookrightarrow G \times_N \mathbb{P}(W) \cong (G/N) \times \mathbb{P}(W)$ that is induced by the embedding $\iota \colon X \hookrightarrow \mathbb{P}(W)$. Then, they equip $G \times_N X$ with the G-linearised ample line bundle $L := \hat{\iota}^*(\mathrm{pr}_1^*(\mathscr{O}_{G/N}) \otimes \mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}(W)}(1)))$. In this way, Doran and Kirwan can consider the set of Mumford-semistable points $(G \times_N X)_G^{ss}(L)$ in $G \times_N X$ and they proceed by defining the set of Mumford-semistable points in X as $X_N^{ss}(L) := \iota^{-1}(G \times_N X)_G^{ss}(L)$, see [DK07, Definition 5.1.5]. As shown in [DK07, Propositions 5.1.8 and 5.1.9], this set X_N^{ss} does not depend on the choice of the group G and can be intrinsically defined knowing only the N-action on X.

4.A.c. The definition of semistability

The algebraic approach suggests that in our situation, we should choose a Hamiltonian G-extension (Z, ω_Z) of the N-action on X and then consider the analogous embedding $Y := G \times_N X \hookrightarrow G \times_N Z \cong G/N \times Z$, such that the Hamiltonian G-extension Z plays the role of the projectivised G-module $\mathbb{P}(W)$.

To implement this idea, first note that the N-equivariant embedding $\iota \colon X \hookrightarrow Z$ induces the G-equivariant proper embedding

$$\hat{\iota} \colon Y = G \times_N X \hookrightarrow G \times_N Z \cong G/N \times Z \tag{4.1}$$

given by $\hat{\iota}([g, x]) = (gN, g \cdot \iota(x))$. Secondly, recalling that G/N is quasi-affine, choose a G-representation space V containing G/N as an orbit as well as a K-invariant hermitian inner product on V, which induces K-invariant Kähler forms ω_V on G/N and

$$\hat{\omega}_Z := \omega_V \oplus \omega_Z \tag{4.2}$$

on $G/N \times Z$. Then, we take

$$\omega_Y := \hat{\iota}^* \hat{\omega}_Z \tag{4.3}$$

as Kähler form on Y. The spaces considered so far fit into following commutative diagram

$$
\begin{array}{ccc}
X & \stackrel{\iota}{\hookrightarrow} & Z \\
\downarrow & & \downarrow \\
Y = G \times_N X & \stackrel{\hat{\iota}}{\hookrightarrow} & (G/N) \times Z,
\end{array}
$$

where the vertical arrows correspond to the maps given by $x \mapsto [e, x]$ and $z \mapsto (eN, z)$, respectively. It follows that $[\omega_Y]$ extends $[\omega_X]$ to Y. Recall that, as K is semisimple and ω_Y is K-invariant, there exists a unique moment map $\mu_Y \colon Y \to \mathfrak{k}^*$ for the K-action on Y, see Remark 2.1; in other words, Y is a (non-compact) Hamiltonian G-manifold. We will use μ_Y to define a notion of semistability for the N-action:

Definition 4.2. Let (X, ω_X) be a Hamiltonian N-manifold. We define the *set of N-semistable points (with respect to $[\omega_X]$)* as

$$\boxed{X_N^{ss}[\omega_X] := X \cap Y_G^{ss}[\omega_Y]\,,}$$

where the Kähler form ω_Y on $Y = G \times_N X$ is given by Equations (4.2) and (4.3) above. Analogously, the *set of N-stable points* in X is defined as $X_N^s[\omega_X] := X \cap Y_G^s[\omega_Y]$, where we set $Y_G^s[\omega_Y] := G \cdot \mu_Y^{-1}(0)$.[4]

Note that $X_N^{ss}[\omega_X]$ a priori depends on the choice of G, ω_V, and Z although we do not convey this information in our notation. We will discuss the choices regarding G and the metric on G/N in Section 4.C below. As we will see, the definition is actually independent of the choice of a Hamiltonian G-extension, once G and the metric on G/N are fixed; see the subsequent section.

Example 4.3. (A non-projective compact Kähler manifold with meromorphic $\mathbb{C}$-action) Let S be a non-projective $K3$-surface with $\mathrm{Pic}(S) \neq \{e\}$, and let $L \to S$ be a non-trivial holomorphic line bundle on S with zero section $Z_L \subset L$. Let $P := L \setminus Z_L \to S$ be the associated $\mathbb{C}^*$-principal bundle, and consider the non-trivial $\mathbb{P}^1$-bundle

$$M := P \times_{\mathbb{C}^*} \mathbb{P}^1 \longrightarrow P/\mathbb{C}^* = S.$$

As the Albanese of $\mathbb{P}^1$ is trivial and since S is a simply-connected compact Kähler manifold, the compact complex manifold M is Kähler by Blanchard's theorem, cf. the discussion in Section 3.C.c. On the other hand, as the non-projective surface S embeds into M as zero section, Y is likewise non-projective. Since the corresponding vector field has zeroes, the effective $\mathbb{C}^*$-action from the left is trivial on the Albanese of M, see [Fuj78, Prop. 6.8] or [Lie78, Proposition 3.14]. Trivially extend the action of $\mathbb{C}^*$ on M to an action of the Borel subgroup $\mathbb{C}^* < B$ of lower-triangular matrices in $\mathrm{SL}(2, \mathbb{C}) =: G$, and define

$$X := G \times_B M.$$

Since the B-action on the Albanese of M is trivial and since $\mathbb{P}^1$ is simply-connected, it again follows from Blanchard's theorem that the compact complex manifold X is Kähler, see also Lemma 3.18. Moreover, the G-action on X induces an action of the unipotent algebraic subgroup $N < G$ of strictly upper-triangular elements of $\mathrm{SL}(2, \mathbb{C})$. Set $K := \mathrm{SU}(2)$. Then, any K-invariant Kähler form ω_X on X produces a moment map $\mu_X \colon X \to \mathfrak{k}^*$ for the K-action; i.e., X is its own Hamiltonian G-extension, so that the N-action on X is meromorphic.

Now, apply the construction described at the beginning of the present section to X. As $G > N$ already acts on X, we can choose $Z = X$ as Hamiltonian G-extension. Let ω_V be the essentially unique flat K-invariant Kähler form on $G/N \cong \mathbb{C}^2 \setminus \{0\} \subset V = \mathbb{C}^2$, and hence consider the Kähler form $\omega_Y = \omega_V \oplus \omega_X$ on $Y := G \times_N X \cong G/N \times X$, together with the resulting moment map $\mu_Y = \mu_{G/N} + \mu_X \colon Y \to \mathfrak{k}^*$. We claim that $X_N^{ss}[\omega_X] \neq \emptyset$, which will conclude our construction. In order to establish the claim, we first note that $\mu_X(X) \neq \{0\}$, as otherwise Equation (2.2) would imply that the K-action on X is trivial. Let $\beta_0 \in \mu_X(X) \setminus \{0\}$ and choose $x_0 \in \mu_X^{-1}(\beta_0)$. We have $\mu_{G/N}(G/N) = \mathfrak{k}^* \setminus \{0\}$, cf. Section 2.D, and hence there exists $g_0 \in G$ such that $\mu_{G/N}(g_0 U) = -\beta_0$. By construction, we then have $\mu_Y(g_0 \cdot (eU, g_0^{-1} \cdot x_0)) = \mu_Y(g_0 U, x_0) = 0$, and hence $(eU, g_0^{-1} \cdot x_0) \in Y_G^{ss}[\omega_Y] \cap (\{eU\} \times X) = X_N^{ss}[\omega_X]$, which is therefore non-empty, as claimed.

---

[4] $\uparrow$ With this definition, a semistable point $x \in X$ is N-stable if and only if $N \cdot x$ is closed in $X_N^{ss}[\omega_X]$. In view of Lemma 5.1 and Theorem 5.3 below this is the correct notion in our situation. The reader should however be aware that there are several notions of (proper) stability in use in the literature.

4.B.   Semistability does not depend on the G-extension

In contrast to the naive definition of semistability discussed in Section 4.A.a above, it turns out the set of G-semistable points in Y with respect to ω_Y as defined in Definition 4.2 does not depend on the choice of the Hamiltonian G-extension (Z, ω_Z).

Theorem 4.4. *Let N be a unipotent subgroup of the simply-connected semisimple complex Lie group G, acting meromorphically on the compact Kähler manifold (X, ω_X). Let (Z_j, ω_j), $j = 1, 2$, be two Hamiltonian G-extensions of the N-action on X. Choose a G-equivariant algebraic embedding $G/N \hookrightarrow V$ into a G-representation space V, and a K-invariant Hermitian inner product on V, inducing Kähler forms ω_V on G/N, $\hat{\omega}_Z$ on $G/N \times Z$, and $\omega_{Y,1}$, $\omega_{Y,2}$ on Y, as described in Section 4.A.a. Then, we have $Y_G^{ss}[\omega_{Y,1}] = Y_G^{ss}[\omega_{Y,2}]$.*

Proof. We will prove that on the smooth Kähler compactification $\overline{Y}$ of Y constructed in Proposition 3.21 there exists a $(1,1)$-form $\alpha \in \mathcal{A}^{1,1}(\overline{Y})$ satisfying the following two properties:

(a) $\alpha|_Y = \omega_{Y,2} - \omega_{Y,1}$, and

(b) $[\alpha] = 0 \in H^2(\overline{Y}, \mathbb{R})$.

It then follows from the $\partial\bar{\partial}$-lemma on the compact Kähler manifold $\overline{Y}$ that there is a smooth function $\varphi \in \mathcal{C}^\infty(\overline{Y})$ such that $\alpha = i\partial\bar{\partial}\varphi$. Restricting everything to Y, we obtain a *bounded* smooth function φ on Y such that $\omega_{Y,2} = \omega_{Y,1} + i\partial\bar{\partial}\varphi$. In this situation, we can repeat the proof of Proposition 2.4 to deduce $Y_G^{ss}[\omega_{Y,1}] = Y_G^{ss}[\omega_{Y,2}]$. Hence, in order to complete the argument, we must show existence of $\overline{Y}$ and α with the above properties.

    To do so, we note first that $(Z, \omega_Z) := (Z_1 \times Z_2, \frac{1}{2}(\omega_1 \oplus \omega_2))$ is another Hamiltonian G-extension of the N-action on X; here, G acts diagonally on $Z_1 \times Z_2$ and $\iota \colon X \hookrightarrow Z$ is given by the direct product $\iota(x) = (\iota_1(x), \iota_2(x))$ of the two inclusions $\iota_j \colon X \hookrightarrow Z_j$, $j = 1, 2$. Our situation can be summarised by the following diagram:

$$
\begin{array}{ccc}
 & G/N \times Z_1 \longrightarrow Z_1 & \\
\nearrow^{\hat{\iota}_1} & \uparrow q_1 & \uparrow \\
Y \xrightarrow{\hat{\iota}} G/N \times Z \xrightarrow{\mathrm{pr}_Z} Z & & \\
\searrow_{\hat{\iota}_2} & \downarrow q_2 & \downarrow \\
 & G/N \times Z_2 \longrightarrow Z_2. &
\end{array}
$$

We denote the Kähler form $\omega_V \oplus \omega_j$ on $G/N \times Z_j$ by $\hat{\omega}_j$, see Equation (4.2), and note that by assumption the same form ω_V appears in both formulas. Then, it follows from the general construction and from the diagram above that

$$\omega_{Y,2} - \omega_{Y,1} = \hat{\iota}^*(q_2^*(\hat{\omega}_2) - q_1^*(\hat{\omega}_1)) = \hat{\iota}^*(\mathrm{pr}_Z^*(\omega_2 - \omega_1)). \tag{4.4}$$

Let $\overline{G/N}$ as defined in Equation (3.4) and let us denote the natural projection $\overline{G/N} \times Z \to Z$ by $\overline{\mathrm{pr}}_Z$. As Z is a Hamiltonian G-extension, by Proposition 3.23 there exists a G-equivariant holomorphic embedding $\psi \colon \overline{Y} \hookrightarrow \overline{G/N} \times Z$. It now follows from Equation (4.4) that

$$\alpha := \psi^*(\overline{\mathrm{pr}}_Z^*(\omega_2 - \omega_1)) \in \mathcal{A}^{1,1}(\overline{Y}) \tag{4.5}$$

fulfils property (a), as desired.

    We still must show that the K-invariant 2-form α is cohomologuous to zero on $\overline{Y}$. For this, we consider the holomorphic fibre bundle $q := \mathrm{pr}_{\overline{G/N}} \circ \psi \colon \overline{Y} \hookrightarrow \overline{G/N} \times Z \to \overline{G/N}$ with typical fibre X and base $\overline{G/N}$. Since $\overline{Y}$ is Kähler, and as by Remark 3.24 the manifold $\overline{G/N}$ is simply-connected, it follows from [Bla56, Théorème II.1.1] that the transgression map $H^1(X, \mathbb{R}) \to H^2(\overline{G/N}, \mathbb{R})$ is zero. Consequently,

the Leray spectral sequence for q degenerates at the E_2-term by [Bla56, Théorème II.1.2]; see also [Voi07, Theorem 4.15 and Remark 4.16]. Again using simple-connectedness of $\overline{G/N}$, we conclude that

$$H^k(\overline{G/N}, \mathbb{R}) \otimes H^l(X, \mathbb{R}) = E_2^{k,l} = E_\infty^{k,l} = \mathrm{Gr}^k H^{k+l}(\overline{Y}, \mathbb{R}) \quad \text{and} \quad E_\infty^{1,1} = \{0\}.$$

Computing the corresponding filtration of $H^2(\overline{Y}, \mathbb{R})$ and comparing it with the Leray spectral sequence for $\mathrm{pr}_{\overline{G/N}}$ then leads to the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(\overline{G/N}, \mathbb{R}) & \xrightarrow{q^*} & H^2(\overline{Y}, \mathbb{R}) & \xrightarrow{j_X^*} & H^2(X, \mathbb{R}) & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle \psi^*} & & \Big\uparrow{\scriptstyle \iota^*} & & \\
0 & \longrightarrow & H^2(\overline{G/N}, \mathbb{R}) & \xrightarrow{\mathrm{pr}_{\overline{G/N}}^*} & H^2(\overline{G/N}, \mathbb{R}) \oplus H^2(Z, \mathbb{R}) & \xrightarrow{j_Z^*} & H^2(Z, \mathbb{R}) & \longrightarrow & 0
\end{array}
$$

where $j_X \colon X \hookrightarrow \overline{Y}$ is the inclusion as the fibre over $eN \in G/N$, and similarly for j_Z.

Now, Equation (4.5) says that $[\alpha] = \psi^*([0] \oplus [\omega_2 - \omega_1])$. Together with $[j_X^*(\alpha)] = [\iota_2^*(\omega_2)] - [\iota_1^*(\omega_1)] = [\omega_X] - [\omega_X] = 0$ this implies that $[\alpha] = 0$, as claimed. $\qquad\square$

Remark 4.5. Note that in contrast to their difference the forms $\omega_{Y,j}$ themselves *do not extend* to the compactification $\overline{Y}$.

4.C. Discussion regarding the choice of Kähler metric on G/N

As this is a subtle issue, let us discuss the choice of Kähler forms on G/N made in Section 4.A.a and the fact that we have to fix such a form in some detail. We will provide examples showing that the independence statement of Theorem 4.4 is optimal from many points of view. The problems occuring are closely related to the ones encountered in the algebraic situation when searching for various kinds of "reductive envelopes", cf. [DK07, Sections 5.2 and 5.3].

4.C.a. The algebraic situation

Let us compare with Doran–Kirwan's approach in the algebraic situation, see Section 4.A.a: The key point that explains the choice of the trivial line bundle on G/N and that eventually makes the proof of [DK07, Proposition 5.1.9] on the independence of semistability from the choice of the embedding into G work is the following. Given any pair G and G' of reductive groups such that $N \subset G \subset G'$, the line bundles L on $G \times_N X$ and L' on $G' \times_N X$ constructed in [DK07] as above verify $\iota^* L' = L$ where $\iota \colon G \times_N X \hookrightarrow G' \times_N X$ is the embedding induced by the inclusion $G \hookrightarrow G'$. In the analytic category, such a canonical choice of Kähler metric on G/N does not exist, even among curvature forms in the trivial line bundle. Indeed, every K-invariant Kähler metric of the form $\omega = i\partial\overline{\partial}\rho$ with $\rho \in \mathcal{C}^\infty(G/N)^K$ is the curvature form of a K-invariant hermitian metric in the trivial line bundle on G/N, cf. [Dem12, Chapter V, (12.6)]. Even if we restrict to metrics of this form, the set of semistable points might change, as the following example shows.

Example 4.6. Let us consider the algebraic and hence meromorphic action of $\mathbb{C} \cong N \subset G = \mathrm{SL}(2, \mathbb{C})$ on $X = \mathbb{P}_1$, endowed with the Fubini-Study form ω_{FS}. As Hamiltonian G-extension of the N-action on X we take $Z = X$. Let $K = \mathrm{SU}(2)$.

According to [Dem12, Lemma 7.10], every K-invariant Kähler form on $G/N \cong \mathbb{C}^2 \setminus \{0\}$ is of the form $i\partial\overline{\partial}\rho(\|z\|)$ where ρ is a smooth function on $\mathbb{R}^{>0}$ such that $\rho \circ \exp$ is strictly increasing and strictly convex. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^{>0})$ be the function defined by $\varphi(t^2) = \rho(t)$. Then the unique moment map for the action of $\mathrm{SU}(2)$ on $\mathbb{C}^2 \setminus \{0\}$ is given by

$$z = (z_1, z_2) \mapsto \varphi'(\|z\|^2) \begin{pmatrix} \frac{|z_1|^2 - |z_2|^2}{2} & \overline{z_1} z_2 \\ z_1 \overline{z_2} & -\frac{|z_1|^2 - |z_2|^2}{2} \end{pmatrix}. \tag{4.6}$$

We now consider the one-parameter family of Kähler forms $\omega_{G/N,c}$ given by $\omega_{G/N,c} := i\partial\overline{\partial}\rho_c(\|z\|)$, where $\rho_c(t) = c\log(1 + t^2)$ with $c > 0$. Following the construction of Section 4.A.a, the induced Kähler form on $Y = G/N \times \mathbb{P}_1$ is $\omega_c = i\partial\overline{\partial}\rho_c(\|z\|) \oplus \omega_{FS}$. Identifying $\mathfrak{su}(2)^*$ with $i\mathfrak{su}(2)$ via the Killing form, the corresponding moment map $\mu\colon (G/N) \times \mathbb{P}_1 \to i\mathfrak{su}(2)$ is given by

$$\mu(z, [x_0 : x_1]) = \frac{c}{1 + \|z\|^2}\begin{pmatrix} \frac{|z_1|^2 - |z_2|^2}{2} & \overline{z_1}z_2 \\ z_1\overline{z_2} & -\frac{|z_1|^2 - |z_2|^2}{2} \end{pmatrix} + \frac{1}{|x_0|^2 + |x_1|^2}\begin{pmatrix} \frac{|x_0|^2 - |x_1|^2}{2} & \overline{x_0}x_1 \\ x_0\overline{x_1} & -\frac{|x_0|^2 - |x_1|^2}{2} \end{pmatrix}.$$

A slice for the SU(2)-action on $(G/N) \times \mathbb{P}_1$ is given by $S = \{((z, r), [0 : 1]) \mid z \in \mathbb{C}, r \geq 0\}$. The point $((z, r), [0 : 1]) \in S$ is mapped under μ to

$$\frac{c}{1 + |z|^2 + r^2}\begin{pmatrix} \frac{|z|^2 - r^2}{2} - \frac{1}{2} & r\overline{z} \\ rz & -\frac{|z|^2 - r^2}{2} + \frac{1}{2} \end{pmatrix}.$$

Consequently, $\mu^{-1}(0)$ is non-empty if and $(c-1)|z|^2 = 1$ for some $z \in \mathbb{C}^*$, which is the case if and only if $c > 1$. In summary, we see that, depending on $c > 0$, the set of semistable points X_N^{ss} can be empty or not.

4.C.b. Proper moment maps

Notice that Example 4.6 shows that for non-projective G-varieties in general the set of GIT-semistable points for the linearisation of the G-action in an ample line bundle L and the set of semistable points with respect to a moment map μ computed using the curvature form of a Hermitian metric in the same line bundle L do not have to coincide. In case the moment map under discussion is *proper*, the two sets coincide by [Sj95, Theorem 2.18]. Hence, in the above example one could look for Kähler forms leading to proper moment maps that would then give a link to the algebraic theory and establish independence of semistabilty from the choice of the metric.

Looking at formula (4.6) one sees that a moment map of the most general form possible in the given situation is proper on $\mathbb{C}^2 \setminus \{0\}$ if and only if

$$\lim_{t\to 0}\varphi'(t)t = \lim_{t\to\infty}\varphi'(t)t = \infty.$$

Since $t \mapsto \rho(e^t) = \varphi(e^{2t})$ is strictly increasing and strictly convex, we see that $t \mapsto 2\varphi'(e^{2t})e^{2t}$ is strictly increasing. Hence, $\lim_{t\to 0}\varphi'(t)t = \infty$ is impossible. This proves that there is no proper SU(2)-equivariant moment map on $\mathbb{C}^2 \setminus \{0\}$. Using compactness of $\mathbb{P}^1$, one can use this to conclude that none of the moment maps for the corresponding Kähler forms on $G/N \times \mathbb{P}^1$ is proper either, so that proper moment maps just do not exist in the situation at hand.

4.C.c. Metrics arising from embedding into representations

As we have seen in Section 2.D, a natural choice in the situation at hand is to consider Kähler metrics on G/N that are obtained by embedding this homogeneous space as an orbit in a G-representation space, and this is also the choice made in the construction presented in Section 4.A.a. We will see in Section 5 below that this leads to a number of desirable properties. However, also a restriction to this class of metrics does not lead to a common notion of semistability, as the following example shows.

Example 4.7. We continue the discussion at the end of Example 2.9 and consider $G = \mathrm{SL}(3, \mathbb{C})$ and $N = G_v$ the unipotent radical of a certain Borel subgroup of G. If we equip G/N with the restriction of the flat Kähler metric for which the image of the moment map μ_V has complement with non-empty interior, and if we take $X = X_\alpha$ to be the G-flag manifold corresponding to the coadjoint orbit $\mathrm{Ad}^*(K)\cdot\alpha$ through a point $\alpha \in \mathfrak{k}_{\mathrm{reg}}^*$ such that $-\alpha$ does not lie in the image of μ_V, then the set of semistable points for the G-action on $G \times_N X$ is empty. On the other hand, as N is a Grosshans subgroup of G, there exists a G-module V' inducing a moment map $\mu_{V'}$ on G/N whose image is a K-invariant dense open subset of $\mathfrak{k}^*$, see Lemma 2.8. Without loss of generality, we may assume that the point $\alpha \in \mathfrak{k}_{\mathrm{reg}}^*$ chosen above fulfils $-\alpha \in \mu_{V'}(G/N)$. Then, for the Kähler metric on $G \times_N X$ induced by the second embedding the set of semistable points is not empty.

4.C.d.   Unipotent radicals of parabolic subgroups

The next example shows that even in the case that we are able to embed N as the unipotent radical of parabolic subgroups of two different semisimple groups G_1 and G_2 and hence, as explained in Remark 2.10, for each of the two embeddings there exists a very natural choice of a Kähler form on G_j/N, we cannot expect X_N^{ss} to be independent of the group G_j.

Example 4.8. Let us consider the action of $N = \mathbb{C}^2$ on $X = \mathbb{P}_2$ given by the embedding

$$N \hookrightarrow G_1 = \mathrm{SL}(3, \mathbb{C}) = \mathrm{SU}(3)^{\mathbb{C}} = K_1^{\mathbb{C}}, \quad (t, s) \mapsto \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the obvious Hamiltonian G_1-extension $Z_1 = X$, we have $G_1 \times_N X = G_1 \times_N Z_1 = (G_1/N) \times \mathbb{P}_2$ with moment map $\mu = \mu_V + \mu_{\mathbb{P}_2} \colon (G_1/N) \times \mathbb{P}_2 \to \mathfrak{k}_1^* = \mathfrak{su}(3)^*$. Since N is embedded as the unipotent radical of a parabolic subgroup P of G_1, we may consider the canonical affine completion $\overline{G_1/N}^{\mathrm{a}}$ and equip it with the canonical Kähler form that is described in the paragraph before Remark 3.4 in [Kir11], cf. Remark 2.10.

The behaviour of the corresponding moment map μ_V on $\overline{G_1/N}^{\mathrm{a}}$ is best understood in terms of its description as the universal $K^{(P)}$-imploded cross-section $(T^*K)_{\mathrm{impl}}^{K, K^{(P)}}$, see [Kir11, Definition 3.11]. According to the discussion following Remark 3.13 in [Kir11], the G_1-orbits in $\overline{G_1/N}^{\mathrm{a}}$ correspond to the strata

$$(K_1 \times \mathrm{Ad}^*(K_1^{(P)}) \bullet \sigma) / \approx_{K_1^{(P)}},$$

where σ runs through the open faces of $(\mathfrak{t}_1^*)_+$. In particular, the open orbit G_1/N is associated with the interior $\mathrm{int}(\mathfrak{t}_1^*)_+$ of $(\mathfrak{t}_1^*)_+$. The description of the moment map μ_V given in [Kir11, Theorem 3.12] now implies that $\mu_V(G_1/N)$ is contained in $(\mathfrak{k}_1)_{\mathrm{reg}}^* = \mathrm{Ad}^*(K_1) \bullet \mathrm{int}(\mathfrak{t}_1^*)_+$. Since $\mu_{\mathbb{P}_2}(\mathbb{P}_2)$ does not intersect the interior of $(\mathfrak{t}_1^*)_+$, the zero fibre of μ is empty, hence $X_N^{ss} = \emptyset$.

Now, let us consider the second embedding

$$N \hookrightarrow G_2 = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) = (\mathrm{SU}(2) \times \mathrm{SU}(2))^{\mathbb{C}} = K_2^{\mathbb{C}}, \quad (t, s) \mapsto \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \right).$$

Here, N is embedded as the unipotent radical of a Borel subgroup of G_2, and thus in particular again a Grosshans subgroup of G_2. As G_2-extension of the N-action on $X = \mathbb{P}_2$ we choose the embedding

$$\iota \colon \mathbb{P}_2 \hookrightarrow \mathbb{P}_3, \quad \iota([z_0 : z_1 : z_2]) = [z_0 : z_2 : z_1 : z_2],$$

which is N-equivariant for the N-action on $\mathbb{P}_3$ given by

$$N \hookrightarrow G_2 \hookrightarrow \mathrm{SL}(4, \mathbb{C}), \quad (t, s) \mapsto \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A moment map $\mu_{\mathbb{P}_3} \colon \mathbb{P}_3 \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ for the K_2-action on $\mathbb{P}_3$ with respect to the Fubini-Study metric is given by the explicit formula

$$\mu_{\mathbb{P}_3}([z_0 : z_1 : z_2 : z_3]) = \frac{1}{|z_0|^2 + \cdots + |z_3|^2} \left[\begin{pmatrix} \frac{|z_0|^2 - |z_1|^2}{2} & \overline{z_0} z_1 \\ z_0 \overline{z_1} & -\frac{|z_0|^2 - |z_1|^2}{2} \end{pmatrix} \oplus \begin{pmatrix} \frac{|z_2|^2 - |z_3|^2}{2} & \overline{z_2} z_3 \\ z_2 \overline{z_3} & -\frac{|z_2|^2 - |z_3|^2}{2} \end{pmatrix} \right],$$

see Example 4.6. In order to determine the semistable locus $Y_{G_2}^{ss}(\mu_Y)$ in $Y = G_2 \times_N X$ we consider the closed embedding

$$Y = G_2 \times_N X \hookrightarrow G_2 \times_N Z \cong G_2/N \times \mathbb{P}_3.$$

and the moment map $\mu_Y = (\mu_V + \mu_{\mathbb{P}_3})|_Y$. The canonical affine closure of

$$G_2/N \cong (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$$

is $V = \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^4$, which we equip with the Hermitian structure $\frac{1}{2}\langle \cdot, \cdot \rangle_{st}$, where $\langle \cdot, \cdot \rangle_{st}$ is the standard Hermitian product of $\mathbb{C}^4$. A direct calculation using the formulae given in Example 4.6 and the explicit expression for $\mu_{\mathbb{P}_3}$ given above yields

$$\mu_Y(eN, [0:0:1]) = \mu_V((1,0),(1,0)) + \mu_{\mathbb{P}_3}([0:1:0:1]) = 0.$$

Hence, we have $X_N^{ss} \neq \emptyset$.

4.D. Semistable points induced by affine completions of G/N

There is a further way to define N-semistable points in X, less directly linked to the intrinsic geometry of G/N and X. Instead of discussing the diagonal G-action on $G/N \times Z$ let us consider an affine completion $\overline{G/N}^a$ and consider the diagonal G-action on $\overline{G/N}^a \times Z$. Let $\bar{\iota}: X \hookrightarrow \overline{G/N}^a \times Z$ be the N-equivariant embedding and define $X_N^{\overline{ss}}[\omega_X] := \bar{\iota}^{-1}((\overline{G/N}^a \times Z)_G^{ss}[\omega_V + \omega_Z])$. Then, $X_N^{\overline{ss}}[\omega_X]$ is an open N-invariant subset which contains but in general is strictly bigger than $X_N^{ss}[\omega_X]$. Analogously, we define $X_N^{\bar{s}}$ as $\bar{\iota}^{-1}((\overline{G/N}^a \times Z)_G^s[\omega_V + \omega_Z])$.

Lemma 4.9. *Let (X, ω_X) be a compact Hamiltonian N-manifold with a Hamiltonian G-extension (Z, ω_Z). If N is a Grosshans subgroup of G, i.e., if $\mathbb{C}[G]^N$ is finitely generated, then for the canonical affine completion $\mathrm{Spec}\,\mathbb{C}[G]^N$ of G/N the set $X_N^{\overline{ss}}$ is non-empty.*

Proof. We already noticed in Section 2.D that under the Grosshans assumption the corresponding moment map $\mu_V: \mathrm{Spec}\,\mathbb{C}[G]^N \to \mathfrak{k}^*$ is surjective. Every moment map $\mu = \mu_V + \mu_Z: \mathrm{Spec}\,\mathbb{C}[G]^N \times Z \to \mathfrak{k}^*$ thus has non-empty zero fibre. □

4.E. Algebraic actions on projective manifolds

In this section, we study the following situation: let X be a projective manifold and N a unipotent group acting *linearly* on X in the sense that there exists a finite-dimensional N-representation W such that the corresponding homomorphism $N \to \mathrm{GL}(W)$ embeds N into a semisimple subgroup G of $\mathrm{SL}(W)$, and an N-equivariant embedding $\iota: X \hookrightarrow \mathbb{P}(W)$. We will compare the moment map approach presented in earlier sections with the Geometric Invariant Theory approach of Doran–Kirwan [DK07].

Consider the (very ample) line bundle $L_X := \iota^*(\mathcal{O}_{\mathbb{P}(W)}(1))$ on X, which is N-linearised by construction. Let $\langle \cdot, \cdot \rangle$ be a Hermitian inner product on W and set $K := \mathrm{SU}(W, \langle \cdot, \cdot \rangle) \cap G$, so that $G = K^{\mathbb{C}}$. Endow $\mathbb{P}(W)$ and hence X with the corresponding Fubini-Study Kähler form ω_{FS} and its restriction $\omega_X := \iota^*(\omega_{FS})$, respectively, so that $[\omega_X] = c_1(L_X) \in H^2(X, \mathbb{R})$. Note that $\mathbb{P}(W)$ is a Hamiltonian G-extension of X. Next, as suggested by the construction of semistable points with respect to ω_X, we look at

$$\bar{\iota}: X \hookrightarrow Y = G \times_N X \hookrightarrow G \times_N \mathbb{P}(W) \cong G/N \times \mathbb{P}(W) \hookrightarrow \overline{G/N}^a \times \mathbb{P}(W),$$

cf. Section 4.D, and additionally at the G-linearised ample line bundle $L := \mathcal{O}_{\overline{G/N}^a} \boxtimes \mathcal{O}_{\mathbb{P}(W)}(1)$. In this situation, we define

$$X_N^{\bar{s}}(L_X) := \bar{\iota}^{-1}((\overline{G/N}^a \times \mathbb{P}(W))_G^s(L))$$

to be the pre-image of the GIT-stable points for the G-action on $\overline{G/N}^a \times \mathbb{P}(W)$ and the given linearisation, which is unique as G is semisimple. The main comparison result regarding moment-map-semistability and GIT-semistability can now be formulated as follows:

Proposition 4.10. *In the above situation, assume additionally that $\overline{G/N}^a$ is normal. Then, we have*

$$X_N^{ss}[\omega_X] = X_N^{\bar{s}}(L_X). \tag{4.7}$$

Proof. The inner product $\langle \cdot, \cdot \rangle$ induces a K-invariant Hermitian metric on $\mathscr{O}_{\mathbb{P}(W)}(1)$ such that $\frac{i}{2\pi} \times$ the curvature is ω_{FS}. Using a K-invariant Hermitian metric on the trivial line bundle over $V \supset \overline{G/N}^{\mathrm{a}}$ with $\frac{i}{2\pi} \times$ curvature equal to ω_V, we get a K-invariant Hermitian metric h on $L \to \overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$ with $\frac{i}{2\pi} \times$ curvature equal to $\omega_V + \omega_{FS}$. We are hence in the general situation of [Sj95, Section 2.2][5], with the exception that $\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$ is normal and not smooth, which does not affect Sjamaar's arguments[6]. In particular, the compact complex space $(\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)}) /\!\!/ G$ is projective algebraic by Grauert's version of the Kodaira Embedding Theorem, see [Sj95, Theorem 2.17]. Moreover, we claim that

$$(\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)}) = (\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(L). \tag{4.8}$$

In order to prove this, as the moment map $\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)}$ is proper we can follow the general line of argumentation presented in [Sj95, proof of Theorem 2.18]: Since the possibly singular variety $\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$ is contained in $V \times \mathbb{P}(W)$, and since all differential geometric and symplectic objects are obtained by restriction, the computations regarding the relation between the norms of sections and (the norm square of) the moment map given in the first paragraph of *loc. cit.* continue to hold, so that for any μ-semistable $p \in \overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$ and any G-invariant section s of L over $(\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)})$, the restriction of the function $h(s,s)$ to the closure of $G \cdot p$ inside $(\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)})$ takes on its maximum at the limit $F_\infty(p)$ under the gradient flow of $-\|\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)}\|^2$, from which we conclude that s is bounded on $(\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W))^{ss}_G(\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)})$. Furthermore, an application of [Gre10a, Proposition 7.6] shows that the set of μ-semistable points is Zariski-open. Since in addition $\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$ is normal, it therefore follows from Riemann's Extension Theorem[7] that s extends to a G-invariant section over the whole of $\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$. The arguments for the two implications "algebraically semistable implies analytically semistable" and "analytically semistable implies algebraically semistable" can now be used without changes, proving (4.8).

The analogous equality for stable points follows from the fact that on both sides, these are the ones for which the corresponding fibre of the quotient map consists of a single (closed) orbit. Intersecting with $\bar{\iota}(X)$ yields $X^{\bar{s}}_N[\omega_X] = X^{\bar{s}}_N(L_X)$, from which we conclude using Corollary 5.2 proven in Section 5.A below. $\qquad\square$

Remark 4.11. (Comparison of semistable points) In the given situation, Doran and Kirwan in [DK07, Definition 5.1.6] define the set of *GIT-semistable points* to be

$$X^{ss}_N(L_X) := X \cap Y^{ss}_G(\hat{\iota}^* \mathscr{O}_{G/N} \boxtimes \mathscr{O}_{\mathbb{P}(W)}(1)),$$

where $\hat{\iota}$ is given by (4.1). In general, this set will not coincide with $X^{ss}_N[\omega_X]$, as the following argument shows. Assume we had $X^{ss}_N[\omega_X] = X^{ss}_N(L_X)$. Since the latter set only depends on the N-action on X and its lift to the N-linearised line bundle L_X, see [DK07, Proposition 5.1.9], the same would be true for $X^{ss}_N[\omega_X]$. In particular, $X^{ss}_N[\omega_X]$ would be independent of the chosen embedding $N \hookrightarrow G$ and of the chosen embedding $G/N \hookrightarrow V$ with (normal) affinisation $\overline{G/N}^{\mathrm{a}}$. This however would stand in contradiction to Example 4.8.

In this direction, Equality (4.7) gives the inclusion $X^{ss}_N[\omega_X] \subset X^{ss}_N(L_X)$, which in general is strict, as the gradient flow of the norm square of the moment map of a GIT-semistable point in $\hat{\iota}(Y)$ might converge to a point (in the zero fibre of the moment map) in the *boundary* of $\hat{\iota}(Y)$ in $\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)$.

5.   Properties of quotients by unipotent groups

We establish the existence of a compactifiable geometric quotient of the set of semistable points by the N-action that extends to a meromorphic map from X to the compactification and carries a natural Kähler

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[5] ↑ As G is semisimple, the moment map computed there has to coincide with $\mu_{\overline{G/N}^{\mathrm{a}} \times \mathbb{P}(W)}$.

[6] ↑ See also [HaH99] for the result attributed to Roberts. Regarding [Sj95, Lemma 2.16], see also [HHL94].

[7] ↑ In Sjamaar's setup the application of Riemann's Extension Theorem is not justified, since at this point the complement of the set of μ-semistable points is not known to be small enough; e.g., it could contain interior points (in the Euclidean topology).

form obtained by symplectic reduction. We will use the notation established in Section 4.A.a.

5.A. Existence of geometric quotients

As in the reductive case, sets of semistable points admit quotients, which in the unipotent case are automatically geometric, since unipotent groups cannot have properly semistable orbits by the following

Lemma 5.1. *Let (X, ω_X) be a Hamiltonian N-manifold. Then every N-orbit in $X_N^{ss}[\omega_X]$ is closed in $X_N^{ss}[\omega_X]$, i.e., we have $X_N^{ss}[\omega_X] = X_N^{s}[\omega_X]$.*

Proof. Consider the analytic Hilbert quotient $\pi_G \colon Y_G^{ss}[\omega_Y] \to Y_G^{ss}[\omega_Y]/\!\!/G$. The fibre $\pi_G^{-1}(\pi_G(x))$ is an affine G-variety, see [HH99, Proposition 3.3.7]. It hence follows from a classical result that every N-orbit is closed in $\pi_G^{-1}(\pi_G(x))$ and hence in $Y_G^{ss}[\omega_X]$. The claim follows. $\qquad\square$

Corollary 5.2. *In the situation of Section 4.D, we have $X_N^{ss}[\omega_X] = X_N^{\bar{s}}[\omega_X]$.*

Proof. It follows from Lemma 5.1 that every orbit in $Y_G^{ss}[\omega_Y]$ is closed in $Y_G^{ss}[\omega_Y]$. If $\Phi \colon Y \hookrightarrow \overline{G/N}^{a} \times Z$ is the natural inclusion, we hence have

$$Y^{ss}[\omega_Y] = \{y \in Y \mid G \cdot y \cap \mu_Y^{-1}(0) \neq \emptyset\} = \{y \in Y \mid G \cdot \Phi(y) \cap \mu_{\overline{G/N}^{a} \times Z}^{-1}(0) \neq \emptyset\}.$$

As Φ restricted to $X \subset Y$ coincides with $\bar{\imath}$, the claim follows. $\qquad\square$

Theorem 5.3. *Let (X, ω_X) be a compact Hamiltonian N-manifold. Then, the set $X_N^{ss}[\omega_X]$ of semistable points admits a geometric quotient $\pi \colon X_N^{ss}[\omega_X] \to X_N^{ss}[\omega_X]/N$ by the N-action. In fact, π is a principal N-fibre bundle and $X_N^{ss}[\omega_X]/N =: Q$ is smooth.*

Proof. By the quotient theory for Hamiltonian actions of reductive groups, see Theorem 2.6, the set of G-semistable points $Y_G^{ss}[\omega_Y] = G \cdot X_N^{ss}[\omega_X]$ admits an analytic Hilbert quotient by the G-action. Moreover, by Lemma 5.1, every G-orbit in $Y_G^{ss}[\omega_Y]$ is closed there, hence the quotient $Y_G^{ss}[\omega_Y] \to Y_G^{ss}[\omega_Y]/\!\!/G$ is in fact geometric. By construction of the twisted product, the restriction to $X_N^{ss}[\omega_X] \subset Y_G^{ss}[\omega_Y]$ yields the desired geometric quotient π.

For every $x \in X_N^{ss}[\omega_X]$, the G-orbit is closed, hence the isotropy subgroup G_x is reductive. On the other hand, as $x \in X$, the isotropy is contained in N, and hence unipotent. It follows that $G_x = N_x = \{e\}$, and hence that π is a principal N-fibre bundle. $\qquad\square$

5.B. Compactifications of the quotient

We will establish the existence of natural compactifications of the quotient $X_N^{ss}[\omega_X]/N$, *which we assume to be non-empty in this section.*

Recall that the fundamental construction of Section 4.A.a involves the choice of an embedding of G/N into a Hermitian K-representation V as a G-orbit $G \cdot v$, see Equation (4.2). This leads to an affine completion $\overline{G/N}^{a} := \overline{G \cdot v}$ of G/N to which both the Kähler form and the moment map extend. Consider the composition $\Phi \colon Y \hookrightarrow \overline{G/N}^{a} \times Z$ of the open embedding $G/N \times Z \hookrightarrow \overline{G/N}^{a} \times Z$ with the embedding (4.1) used in the main construction.

Proposition 5.4. *The inclusion $Y = G \times_N X \overset{\Phi}{\hookrightarrow} \overline{G/N}^{a} \times Z$ induces an open embedding*

$$\phi \colon X_N^{ss}[\omega_X]/N \hookrightarrow \overline{Q}$$

of $X_N^{ss}[\omega_X]/N = Q$ into a compact complex space $\overline{Q}$ such that $\overline{Q} \setminus \phi(Q)$ is analytic and nowhere dense.

Proof. We claim that $\Phi(Y)$ is Zariski-open in its closure. For this, we first look at the compactification $V \times Z \hookrightarrow \mathbb{P}(V \oplus \mathbb{C}) \times Z$; by slight abuse of notation, the composition of Φ with this embedding will also be denoted by Φ. Since the G-action on the compact Kähler manifold Z is Hamiltonian, it is meromorphic, see Remark 2.2. As in addition the G-action on V is algebraic, the G-action on the compact Kähler manifold $\mathbb{P}(V \oplus \mathbb{C}) \times Z$ is meromorphic. Secondly, we notice that $\Phi(Y) = G \bullet (\{eU\} \times \iota(X))$, where $\iota\colon X \hookrightarrow Z$ is the extension map; i.e., $\Phi(Y)$ is the G-sweep of a compact complex submanifold of $\mathbb{P}(V \oplus \mathbb{C}) \times Z$. It therefore follows from [Fuj78, Lemma 2.4(1)] that $\Phi(Y)$ is Zariski-open in its closure inside $\mathbb{P}(V \oplus \mathbb{C}) \times Z$, and hence it is Zariski-open in its closure inside $\overline{G/N}^{\mathrm{a}} \times Z$. We denote this closure by $\overline{Y}$.

By Theorem 2.7 the moment map μ_V is proper on $\overline{G/N}^{\mathrm{a}}$. It follows that the moment map for the action of K on $\overline{G/N}^{\mathrm{a}} \times Z$ and hence the restriction $\mu_{\overline{Y}}\colon \overline{Y} \to \mathfrak{k}^*$ of this moment map to the analytic subset $\overline{Y} \subset \overline{G/N}^{\mathrm{a}} \times Z$ is likewise proper. Recalling the construction of the Kähler form ω_Y and the associated moment map μ_Y, we can summarise the situation in the following commutative diagram:

$$
\begin{array}{ccccccc}
Y & \hookrightarrow & \overline{Y} & \hookrightarrow & \overline{G/N}^{\mathrm{a}} \times Z & \hookrightarrow & V \times Z \\
 & \mu_Y \searrow & \downarrow{\scriptstyle \mu_{\overline{Y}}} & & \downarrow & \swarrow{\scriptstyle \mu_V + \mu_Z} & \\
 & & \mathfrak{k}^* & \xrightarrow{=} & \mathfrak{k}^*. & &
\end{array}
$$

Here, in the first line, the first inclusion is open and the other two inclusions are closed.

Since $\mu_{\overline{Y}}$ is proper, its zero fibre is compact, and hence the associated analytic Hilbert quotient $\overline{Y}_G^{ss}(\mu_{\overline{Y}}) /\!\!/ G \simeq \mu_{\overline{Y}}^{-1}(0)/K$ is a compact complex space, which in fact comes with a natural closed embedding into the (non-compact) analytic Hilbert quotient $(V \times Z)_G^{ss}(\mu_V + \mu_Z) /\!\!/ G$. As the inclusion $Y \hookrightarrow \overline{Y}$ has Zariski-open image, and as every G-orbit in $Y_G^{ss}(\mu_Y) = Y_G^{ss}[\omega_Y]$ is closed by the argument in the proof of Theorem 5.3, the inclusion $Y_G^{ss}[\omega_Y] \hookrightarrow \overline{Y}_G^{ss}(\mu_{\overline{Y}})$ is Zariski-open and saturated with respect to the quotient map $\overline{\pi}\colon \overline{Y}_G^{ss}(\mu_{\overline{Y}}) \to \overline{Y}_G^{ss}(\mu_{\overline{Y}}) /\!\!/ G$. It therefore induces the desired Zariski-open embedding

$$
\phi\colon Q = X_N^{ss}[\omega_X]/N \cong Y_G^{ss}[\omega_Y]/G \hookrightarrow \overline{Y}_G^{ss}(\mu_{\overline{Y}}) /\!\!/ G =: \overline{Q}
$$

into the compact complex space $\overline{Q}$. $\square$

5.C. Zariski-openness of semistable points and meromorphic extension of the quotient map

While there is no general result for analyticity of the complement of the set of semistable points in a Hamiltonian G-manifold with non-proper moment map, in our setup this can be shown by hand.

Theorem 5.5. *Let (X, ω_X) be a compact Hamiltonian N-manifold. Then, the set $X_N^{ss}[\omega_X]$ of semistable points is Zariski-open in X. Moreover, the quotient map $\pi\colon X_N^{ss}[\omega_X] \to X_N^{ss}[\omega_X]/N$ extends to a meromorphic map[8] $\pi\colon X \dashrightarrow \overline{Q}$ to the compact complex space $\overline{Q}$ constructed in the proof of Proposition 5.4.*

Proof. By part (1) of Theorem 2.6, $X_N^{ss}[\omega_X]$ is open in the Euclidean topology. Let $\pi_F\colon X \dashrightarrow Q_F$ be a Fujiki quotient of X by the N-action, whose existence is guaranteed by Proposition 1.4, and let $\Gamma \subset X \times Q_F$ be the graph. In particular, Γ is an N-invariant, irreducible, compact analytic subset of $X \times Q_F$, where N acts only on the first factor. Embedding $X \times Q_F$ into $Y \times Q_F$ and further into $\mathbb{P}(V \oplus \mathbb{C}) \times Z \times Q_F$ as in the proof of Proposition 5.4 we can interpret Γ as an N-invariant subvariety in $\mathbb{P}(V \oplus \mathbb{C}) \times Z \times Q_F$. Using again that the G-action on the latter space is meromorphic, we conclude that $\hat{\Gamma} := \overline{G \bullet \Gamma}$ is Zariski-open in its closure in $\overline{Y} \times Q_F$, and in particular irreducible. On $Y \subset \overline{Y}$ it is the graph of the G-invariant extension of the N-invariant meromorphic map π_F from X to $Y = G \times_N X$. It follows that $\hat{\Gamma}$ is the graph of a G-invariant

---

[8] ↑ The reader is referred to [Whi72, Chapter 6, Section 3] for an in depth discussion of meromorphic mappings between complex spaces.

meromorphic map from $\overline{Y}$ to Q_F, which we will call $\hat{\pi}_F$. The graph of the restriction of $\hat{\pi}_F$ to $\overline{Y}_G^{ss}(\mu_{\overline{Y}})$ is equal to $\hat{\Gamma}^{\circ} := \hat{\Gamma} \cap (\overline{Y}_G^{ss}(\mu_{\overline{Y}}) \times Q_F)$. Now, $\overline{Y}_G^{ss}(\mu_{\overline{Y}}) \times Q_F$ admits an analytic Hilbert quotient by the G-action, namely $\Pi = \overline{\pi} \times \mathrm{id}_{Q_F} \colon \overline{Y}_G^{ss}(\mu_{\overline{Y}}) \times Q_F \to \overline{Q} \times Q_F$. As $\hat{\Gamma}^{\circ}$ is a G-invariant, irreducible analytic subset of $\overline{Y}_G^{ss}(\mu_{\overline{Y}}) \times Q_F$, its image $\hat{\Gamma}_{\mathrm{red}}^{\circ} = \Pi(\hat{\Gamma}^{\circ})$ is an irreducible analytic subset of $\overline{Q} \times Q_F$ by the basic properties of analytic Hilbert quotients listed in Section 2.C.

On the one hand, as orbits through points in $Y_G^{ss}(\mu_Y) \subset \overline{Y}_G^{ss}(\mu_{\overline{Y}})$ are closed, $\hat{\Gamma}_{\mathrm{red}}^{\circ}$ defines a meromorphic map $\pi_{F,\mathrm{red}}$ from $\overline{Q}$ to Q_F, cf. the argument given in the proof of [Gre10a, Proposition 4.5]. On the other hand, consider the open subset $U := U_F \cap X_N^{ss}[\omega_X]$, cf. Proposition 1.4. As this set is N-invariant, and since both $\pi_F|_U$ and $\pi|_U$ are geometric quotients for the N-action on U by Proposition 1.4 and Theorem 5.3, respectively, the respective images $\pi_F(U) \subset Q_F$ and $\pi(U) \subset Q \subset \overline{Q}$ are biholomorphic via $\pi_{F,\mathrm{red}}$. It follows that $\pi_{F,\mathrm{red}} \colon \overline{Q} \dashrightarrow Q_F$ is bimeromorphic. From this, we conclude that $(\pi_{F,\mathrm{red}})^{-1} \circ \pi_F \colon X \dashrightarrow \overline{Q}$ is a meromorphic extension of π and that there are Zariski-open, dense subsets $\overline{\Omega} \subset \overline{Q}$ and $\Omega_F \subset Q_F$ that are biholomorphic via $\pi_{F,\mathrm{red}}$. By shrinking U_F if necessary, we may assume that $\Omega_F = \pi_F(U_F)$ and $\overline{\Omega} \subset Q \subset \overline{Q}$. The situation is hence summarised by the following commutative diagram

$$
\begin{array}{ccc}
U_F & \lhook\joinrel\longrightarrow & X_N^{ss}[\omega_X] \\
{\scriptstyle \pi_F}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\Omega_F & \underset{(\pi_{F,\mathrm{red}})^{-1}}{\lhook\joinrel\longrightarrow} Q \lhook\joinrel\longrightarrow & \overline{Q}.
\end{array}
$$

In particular, the Zariski-open subset U_F is contained in $X_N^{ss}[\omega_X]$. Since X is compact, using a Noetherian induction argument applied to the analytic subset $X' := X \setminus U_F$ we conclude that $X_N^{ss}[\omega_X]$ and hence $X \setminus X_N^{ss}[\omega_X]$ is constructible in the Zariski-topology of X. As we know from the start that $X \setminus X_N^{ss}[\omega_X]$ is closed in the Euclidean topology of X, the claim follows. $\square$

5.D.   Reduced Kähler structure on the quotient

We will show that using a symplectic reduction procedure the quotient $X_N^{ss}[\omega_X]/N$ can be endowed with a Kähler form naturally induced from ω_X. This form extends to the compactification $\overline{Q}$ and its class pulls back under π to the class of $[\omega_X]$ on $X_N^{ss}[\omega_X]$.

Theorem 5.6. *In the setup of Proposition 5.4, there exists a Kähler structure[9] $\omega_{\overline{Q}}$ on the compact complex space $\overline{Q}$ whose restriction $\omega_Q = \omega_{\overline{Q}}|_Q$ to $Q \hookrightarrow \overline{Q}$ is smooth and fulfils*

$$[\pi^* \omega_Q] = [\omega_X|_{X_N^{ss}[\omega_X]}] \in H^2(X_N^{ss}[\omega_X], \mathbb{R}).$$

Proof. Once again, recall our setup in the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & \overset{\psi}{\overbrace{\hspace{8cm}}} & & \\
X_N^{ss}[\omega_X] & \lhook\joinrel\longrightarrow & Y_G^{ss}[\omega_Y] & \lhook\joinrel\longrightarrow & \overline{Y}_G^{ss}[\hat{\omega}_Z] & \lhook\joinrel\longrightarrow & \overline{G/N}^{\mathrm{a}} \times Z & \lhook\joinrel\longrightarrow & V \times Z \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \overline{\pi}} & & & & \big\downarrow{\scriptstyle \mathrm{pr}_Z} \\
Q & \underset{\cong}{\longrightarrow} & Y_G^{ss}[\omega_Y]/\!\!/G & \lhook\joinrel\longrightarrow & \overline{Q} & & & & Z.
\end{array}
\qquad (5.1)
$$

By applying the Kählerian reduction procedure of [HHL94] to $\overline{Y}_G^{ss}[\hat{\omega}_Z]$ and to the quotient $\overline{Q}$, we obtain a Kählerian structure $\omega_{\overline{Q}}$ on $\overline{Q}$ induced by restricting local K-invariant potentials of $\hat{\omega}_Z$ to $\mu_{\overline{Y}}^{-1}(0)$ and by the homeomorphism $\mu_{\overline{Y}}^{-1}(0)/K \simeq \overline{Q}$, cf. Theorem 2.6. We denote the restriction of $\omega_{\overline{Q}}$ to Q by ω_Q.

---

[9] ↑ See [Gre10a, Sections 3.1 and 3.2] for the basic definitions regarding Kähler structures on (singular) complex spaces.

In order to show that ω_Q is smooth, we first note that $Y_G^{ss}[\omega_Y] \subset \overline{Y}_G^{ss}[\hat{\omega}_Z]$ is smooth and $\overline{\pi}$-saturated, and secondly recall the observation made in the proof of Theorem 5.3 above that the G-action on $Y_G^{ss}[\omega_Y]$ is free. Therefore, it follows from the construction of the reduced Kähler form $\omega_{\overline{Q}}$, see [HHL94, Lemma 2 on page 132 and the proof on pages 133/134] and also compare with [Sj95, Theorem 2.10], that in the fundamental commutative diagram

$$
\begin{array}{ccc}
\mu_Y^{-1}(0) & \overset{\tau}{\hookrightarrow} & Y_G^{ss}[\omega_Y] \\
{\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\
\mu_Y^{-1}(0)/K & \underset{\simeq}{\overset{\tau_{\mathrm{red}}}{\longrightarrow}} & Q
\end{array}
\tag{5.2}
$$

the fibre $\mu_Y^{-1}(0)$ is smooth, the K-action on $\mu_Y^{-1}(0)$ is free, and that the Kähler structure ω_Q is smooth and fulfils the "symplectic reduction" equation

$$
\tau^*(\overline{\pi}^*\omega_Q) = \pi_K^*(\tau_{\mathrm{red}}^*\omega_Q) = \omega_Y|_{\mu_Y^{-1}(0)} = \tau^*(\omega_Y|_{Y_G^{ss}[\omega_Y]}).
\tag{5.3}
$$

More is true. Since $Y_G^{ss}[\omega_Y]$ is $\overline{\pi}$-saturated, and since the moment map $\mu_{\overline{Y}} \colon \overline{Y} \to \mathfrak{k}^*$ is proper as observed in the proof of Proposition 5.4, the moment map μ_Y is *admissible* in the sense that the gradient flow F_t of $-\|\mu_Y\|^2$ through any point $p \in Y_G^{ss}[\omega_Y]$ exists for all times, cf. [Kir84, §9], and hence there exists a continuous retraction of $Y_G^{ss}[\omega_Y]$ to $\mu_Y^{-1}(0)$ defined by $z \mapsto \lim_{t\to\infty} F_t(z)$, see [Sj95, page 109] and the references given there, as well as [Ler05]. In particular, the inclusion displayed in the first line of Diagram (5.2) induces an isomorphism between de Rham cohomology groups,

$$
\tau^* \colon H^2(Y_G^{ss}[\omega_Y], \mathbb{R}) \overset{\cong}{\longrightarrow} H^2(\mu_Y^{-1}(0), \mathbb{R}).
$$

Equation (5.3) therefore implies that

$$
[\overline{\pi}^*\omega_Q] = [\omega_Y|_{Y_G^{ss}[\omega_Y]}] \in H^2(Y_G^{ss}[\omega_Y], \mathbb{R}).
\tag{5.4}
$$

In addition, from the right hand part of Diagram (5.1), from Equations (4.2) and (4.3), and from the the fact that the de Rham cohomology class of ω_V is trivial we infer that

$$
[\omega_Y|_{Y_G^{ss}[\omega_Y]}] = [\psi^*(\mathrm{pr}_Z^*(\omega_Z))] \in H^2(Y_G^{ss}[\omega_Y], \mathbb{R}),
$$

so that (5.4) becomes

$$
[\overline{\pi}^*\omega_Q] = [\psi^*(\mathrm{pr}_Z^*(\omega_Z))] \in H^2(Y_G^{ss}[\omega_Y], \mathbb{R}).
$$

Finally, using this, the left hand part of Diagram (5.1), and the fact that Z as a G-extension of X fulfils (3.1) we conclude

$$
[\pi^*\omega_Q] = [\omega_X|_{X_N^{ss}[\omega_X]}] \in H^2(X_N^{ss}[\omega_X], \mathbb{R}). \qquad \square
$$

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