

*p***-adic lattices are not Kähler groups**

Bruno Klingler

Abstract. We show that any lattice in a simple *p*-adic Lie group is not the fundamental group of a compact Kähler manifold, as well as some variants of this result.

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[Français]

Titre. Les réseaux *p***-adiques ne sont pas des groupes kählériens**

Résumé. Dans cette note, nous montrons qu'un réseau d'un groupe de Lie *p*-adique simple n'est pas le groupe fondamental d'une variété kählérienne compacte, ainsi que des variantes de ce résultat.

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Bruno Klingler Humboldt-Universität zu Berlin, Germany *e-mail*: bruno.klingler@hu-berlin.de

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Contents

1. Results

1.A. A group is said to be a Kähler group if it is isomorphic to the fundamental group of a connected compact Kähler manifold. In particular such a group is finitely presented. As any finite étale cover of a compact Kähler manifold is still a compact Kähler manifold, any finite index subgroup of a Kähler group is a Kähler group. The most elementary necessary condition for a finitely presented group to be Kähler is that its finite index subgroups have even rank abelianizations. A classical question, due to Serre and still largely open, is to characterize Kähler groups among finitely presented groups. A standard reference for Kähler groups is [\[ABCKT96\]](#page-4-0).

1.B. In this note we consider the Kähler problem for lattices in simple groups over local fields. Recall that if *G* is a locally compact topological group, a subgroup $\Gamma \subset G$ is called a *lattice* if it is a discrete subgroup of *G* with finite covolume (for any *G*-invariant measure on the locally compact group *G*).

We work in the following setting. Let *I* be a finite set of indices. For each $i \in I$ we fix a local field k_i and a simple algebraic group G_i defined and isotropic over k_i . Let $G = \prod_{i \in I} G_i(k_i)$. The topology of the local fields k_i , $i \in I$, makes G a locally compact topological group. We define $\mathrm{rk}\,G:=\sum_{i\in I}\mathrm{rk\,}_{k_i}$ \mathbf{G}_i .

We consider $\Gamma \subset G$ an *irreducible* lattice: there does not exist a disjoint decomposition $I = I_1 \coprod I_2$ into two non-empty subsets such that, for $j = 1, 2$, the subgroup $\Gamma_j := \Gamma \cap G_{I_j}$ of $G_{I_j} := \prod_{i \in I_j} \mathbf{G}_i(k_i)$ is a lattice in G_{I_j} .

The reference for a detailed study of such lattices is [\[Mar91\]](#page-5-0). In Section [2](#page-2-0) we recall a few results for the convenience of the reader.

1.C. Most of the lattices Γ as in Section [1.B](#page-1-2) are finitely presented (see Section [2.C\)](#page-3-1). The question whether Γ is Kähler or not has been studied by Simpson using his non-abelian Hodge theory when at least one of the k_i 's is archimedean. He shows that if Γ is Kähler then necessarily for any $i \in I$ such that k_i is archimedean the group G_i has to be of Hodge type (i.e. admits a Cartan involution which is an inner automorphism), see [\[Si92,](#page-5-1) Corollary 5.3 and 5.4]. For example $SL(n, \mathbb{Z})$ is not a Kähler group as $SL(n, \mathbb{R})$ is not a group of Hodge type. In this note we prove:

Theorem 1.1. Let I be a finite set of indices and G be a group of the form $\prod_{j\in I} G_j(k_j)$, where G_j is a simple *algebraic group defined and isotropic over a local field k^j . Let* Γ ⊂ *G be an irreducible lattice.*

Suppose there exists an $i \in I$ *such that* k_i *is non-archimedean.* If $rk G > 1$ *and* $char(k_i) = 0$ *, or if* $rk G = 1$ *(i.e. G* = G(*k*) *with* G *a simple isotropic algebraic group of rank* 1 *over a local field k) then* Γ *is not a Kähler group.*

Remark 1.2. Notice that the case $rkG = 1$ is essentially folkloric. As we did not find a reference in this generality let us give the proof in this case.

If Γ is not cocompact in *G* (this is possible only if *k* has positive characteristic) then Γ is not finitely generated by [\[L91,](#page-5-2) Corollary 7.3], hence not Kähler.

Hence we can assume that Γ is cocompact. In that case it follows from [\[L91,](#page-5-2) Theorem 6.1 and 7.1] that Γ admits a finite index subgroup Γ' which is a (non-trivial) free group. But a non-trivial free group is never Kähler, as it always admits a finite index subgroup with odd Betti number (see [\[ABCKT96,](#page-4-0) Example 1.19 p.7]). Hence Γ', thus also Γ, is not Kähler.

On the other hand, to the best of our knowledge not a single case of Theorem [1.1](#page-1-1) in the case where $rk\ G > 1$ and all the k_i , $i \in I$, are non-archimedean fields of characteristic zero was previously known. The proof in this case is a corollary of Margulis' superrigidity theorem and the recent integrality result of Esnault and Groechenig ([\[EG17,](#page-4-1) Theorem 1.3], whose proof was greatly simplified in [\[EG17-2\]](#page-4-2)).

1.D. Let us mention some examples of Theorem [1.1:](#page-1-1)

– Let *p* be a prime number, $I = \{1\}$, $k_1 = \mathbb{Q}_p$, $G = SL(n)$. A lattice in $SL(n, \mathbb{Q}_p)$, $n \ge 2$, is not a Kähler group. This is new for $n \geq 3$.

 $I = \{1; 2\}$, $k_1 = \mathbb{R}$ and $G_1 = SU(r, s)$ for some $r \ge s > 0$, $k_2 = \mathbb{Q}_p$ and $G_2 = SL(r + s)$. Then any irreducible lattice in $SU(r,s)\times\mathbf{SL}(r+s,\mathbb{Q}_p)$ is not Kähler. In Section $\overset{\cdot}{2}$ $\overset{\cdot}{2}$ $\overset{\cdot}{2}$ we recall how to construct such lattices (they are *S*-arithmetic). The analogous result that any irreducible lattice in $SL(n, \mathbb{R})\times SL(n, \mathbb{Q}_p)$ (for example $SL(n, \mathbb{Z}[1/p]))$ is not a Kähler group already followed from Simpson's theorem.

1.E. I don't know anything about the case not covered by Theorem [1.1:](#page-1-1) can a (finitely presented) irreducible lattice in $G = \prod_{i \in I} \mathbf{G}_i(k_i)$ with $rk G > 1$ and all k_i of (necessarily the same, see Theorem [2.1\)](#page-2-1) *positive characteristic*, be a Kähler group? This question already appeared in [\[BKT13,](#page-4-3) Remark 0.2 (5)].

2. Reminder on lattices

2.A. Examples of pairs (*G,*Γ) as in Section [1.B](#page-1-2) are provided by *S-arithmetic groups*: let *K* be a global field (i.e a finite extension of Q or $\mathbb{F}_q(t)$, where \mathbb{F}_q denotes the finite field with q elements), S a non-empty set of places of *K*, S_{∞} the set of archimedean places of *K* (or the empty set if *K* has positive characteristic), O*S*∪*S*[∞] the ring of elements of *K* which are integral at all places not belonging to *S* ∪ *S*[∞] and G an absolutely simple *K*-algebraic group, anisotropic at all archimedean places not belonging to *S*. A subgroup $\Lambda \subset G(K)$ is said *S*-arithmetic (or $S \cup S_{\infty}$ -arithmetic) if it is commensurable with $G(\mathcal{O}^{S \cup S_{\infty}})$ (this last notation depends on the choice of an affine group scheme flat of finite type over O*S*∪*S*[∞] , with generic fiber G; but the commensurability class of the group $G(\mathcal{O}^{S\cup S_{\infty}})$ is independent of that choice).

If *S* is finite the image Γ in $\prod_{v\in S}\mathbf{G}(K_v)$ of an *S*-arithmetic group Λ by the diagonal map is an irreducible lattice (see [\[B63\]](#page-4-4) in the number field case and [\[H69\]](#page-5-3) in the function field case). In the situation of Section [1.B,](#page-1-2) a (necessarily irreducible) lattice $\Gamma \subset G$ is called *S*-arithmetic if there exist *K*, **G**, *S* as above, a bijection $i: S \longrightarrow I$, isomorphisms $K_v \longrightarrow k_{i(v)}$ and, via these isomorphisms, k_i -isomorphisms $\varphi_i: \mathbf{G} \longrightarrow \mathbf{G}_i$ such that Γ is commensurable with the image via $\prod_{i\in I}\varphi_i$ of an *S*-arithmetic subgroup of $\mathbf{G}(K)$.

2.B. Margulis' and Venkataramana's arithmeticity theorem states that as soon as rk*G* is at least 2 then every irreducible lattice in *G* is of this type:

Theorem 2.1 (Margulis, Venkataramana). *In the situation of Section [1.B,](#page-1-2) suppose that* Γ ⊂ *G is an irreducible lattice and that* $rk G \geq 2$. Suppose moreover for simplicity that \tilde{G}_i , $i \in I$, is absolutely simple. Then.

- (a) *All the fields kⁱ have the same characteristic.*
- (b) *The group* Γ *is S-arithmetic.*

Remark 2.2. Margulis [\[Mar84\]](#page-5-4) proved Theorem [2.1](#page-2-1) when $char(k_i) = 0$ for all $i \in I$. Venkatarama [\[V88\]](#page-5-5) had to overcome many technical difficulties in positive characteristics to extend Margulis' strategy to the general case.

On the other hand, if $rk G = 1$ (hence $I = \{1\}$) and $k = k_1$ is non-archimedean, there exist non-arithmetic lattices in *G*, see [\[L91,](#page-5-2) Theorem A].

2.C. With the notations of Section [2.A,](#page-2-2) an *S*-arithmetic lattice Γ is always finitely presented except if *K* is a function field, and $rk_K\mathbf{G} = rkG = |S| = 1$ (in which case Γ is not even finitely generated) or $rk_K\mathbf{G} > 0$ and $rk G = 2$ (in which case Γ is finitely generated but not finitely presented). In the number field case see the result of Raghunathan [\[R68\]](#page-5-6) in the classical arithmetic case and of Borel-Serre [\[BS76\]](#page-4-5) in the general *S*-arithmetic case; in the function field case see the work of Behr, e.g. [\[Behr98\]](#page-4-6). For example the lattice $SL_2(\mathbb{F}_q[t])$ of $SL_2(\mathbb{F}_q((1/t)))$ is not finitely generated, while the lattice $SL_3(\mathbb{F}_q[t])$ of $SL_3(\mathbb{F}_q((1/t)))$ is finitely generated but not finitely presented.

3. Proof of Theorem [1.1](#page-1-1)

Thanks to Remark [1.2](#page-1-3) we can assume that rk*G >* 1. In this case the main tools for proving Theorem [1.1](#page-1-1) are the recent result of Esnault and Groechenig and Margulis' superrigidity theorem.

3.A. Recall that a linear representation $\rho : \Gamma \longrightarrow GL(n,k)$ of a group Γ over a field k is cohomologically r igid if $H^1(Γ, A dρ) = 0$. A representation $ρ: Γ → GL(n, ℂ)$ is said to be integral if it factorizes through $\rho: \Gamma \longrightarrow GL(n,K)$, $K \hookrightarrow \mathbb{C}$ a number field, and moreover stabilizes an \mathcal{O}_K -lattice in \mathbb{C}^n (equivalently, see [\[Ba80,](#page-4-7) Corollary 2.3 and 2.5]: for any embedding $v : K \hookrightarrow k$ of K in a non-archimedean local field k the composed representation $\rho_v : \Gamma \longrightarrow GL(n,K) \hookrightarrow GL(n,k)$ has bounded image in $GL(n,k)$). A group will be said *complex projective* if is isomorphic to the fundamental group of a connected smooth complex projective variety. This is a special case of a Kähler group (the question whether or not any Kähler group is complex projective is open).

In [\[EG17-2,](#page-4-2) Theorem 1.1] Esnault and Groechenig prove that if Γ is a complex projective group then any irreducible cohomologically rigid representation $\rho : \Gamma \longrightarrow GL(n, \mathbb{C})$ is integral. This was conjectured by Simpson.

3.B. A corollary of [\[EG17-2,](#page-4-2) Theorem 1.1] is the following:

Corollary 3.1. *Let* Γ *be a complex projective group. Let k be a non-archimedean local field of characteristic zero and let* $\rho : \pi_1(X) \longrightarrow \text{GL}(n,k)$ *be an absolutely irreducible cohomologically rigid representation. Then* ρ *has bounded image in* $GL(n, k)$ *.*

Proof. Let *k* be an algebraic closure of *k*. As ρ is absolutely irreducible and cohomologically rigid there exists $g \in GL(n,\overline{k})$ and a number field $K \subset \overline{k}$ such that $\rho^g(\Gamma) := g \cdot \rho \cdot g^{-1}(\Gamma) \subset GL(n,\overline{k})$ lies in $GL(n,K)$.

Let *k'* be the finite extension of *k* generated by *K* and the matrix coefficients of *g* and g^{-1} . This is still a non-archimedean local field of characteristic zero, and both $\rho(\Gamma)$ and $\rho^{g}(\Gamma)$ are subgroups of $GL(n, k')$. As $\rho: \Gamma \longrightarrow GL(n,k) \subset GL(n,k')$ has bounded image in $GL(n,k)$ if and only if $\rho^g: \Gamma \longrightarrow GL(n,k')$ has bounded image in $GL(n, k')$, we can assume, replacing ρ by ρ^g and k by k' if necessary, that $\rho(\Gamma)$ is contained in $GL(n, K)$ with $K \subset k$ a number field.

Let $\sigma: K \hookrightarrow \mathbb{C}$ be an infinite place of K and consider $\rho^{\sigma}:\Gamma \stackrel{\rho}{\longrightarrow} \mathbf{GL}(n,K) \stackrel{\sigma}{\hookrightarrow} \mathbf{GL}(n,\mathbb{C})$ the associated representation. As ρ is absolutely irreducible, the representation ρ^σ is irreducible. As

$$
H^1(\Gamma, \text{Ad}\circ\rho^{\sigma}) = H^1(\Gamma, \text{Ad}\circ\rho) \otimes_{K,\sigma} \mathbb{C} = 0
$$

the representation ρ^{σ} is cohomologically rigid.

It follows from [\[EG17,](#page-4-1) Theorem 1.3] that ρ^{σ} is integral. In particular, considering the embedding $K \subset k$, it follows that the representation $\rho : \Gamma \longrightarrow GL(n,k)$ has bounded image in $GL(n,k)$.

3.C. Notice that we can upgrade Corollary [3.1](#page-3-2) to the Kähler world if we restrict ourselves to faithful representations:

Corollary 3.2. *The conclusion of Corollary [3.1](#page-3-2) also holds for* Γ *a Kähler group and* $\rho : \pi_1(X) \longrightarrow GL(n,k)$ *a* faithful *representation.*

Proof. As the representation ρ is faithful, the group Γ is a linear group in characteristic zero. It then follows that the Kähler group Γ is a complex projective group (see [\[CCE14,](#page-4-8) Theorem 0.2] which proves that a finite index subgroup of Γ is complex projective, and its refinement [\[C17,](#page-4-9) Corollary 1.3] which proves that Γ itself is complex projective). The result now follows from Corollary [3.1.](#page-3-2) \Box

3.D. Let us apply Corollary [3.1](#page-3-2) to the case of Theorem [1.1](#page-1-1) where rk*G >* 1. Renaming the indices of *I* if necessary, we can assume that $I = \{1, \dots, r\}$ and k_1 is non-archimedean of characteristic zero. Let us choose an absolutely irreducible k_1 -representation $\rho_{\mathbf{G}_1} : \mathbf{G}_1 \longrightarrow \mathbf{GL}(V)$. Let

$$
\rho : \Gamma \longrightarrow G \xrightarrow{p_1} \mathbf{G}_1(k_1) \longrightarrow \mathbf{GL}(V)
$$

be the representation of Γ deduced from $\rho_{\mathbf{G}_1}$ (where $p_1:G \longrightarrow \mathbf{G}_1(k_1)$ denotes the projection of G onto its first factor). As $p_1(\Gamma)$ is Zariski-dense in G_1 it follows that ρ is absolutely irreducible.

As rk*G >* 1, Margulis' superrigidity theorem applies to the lattice Γ of *G*: it implies in particular that $H^1(\Gamma, \text{Ad} \circ \rho) = 0$ (see [\[Mar91,](#page-5-0) Theorem (3) (iii) p. 3]). Hence the representation $\rho : \Gamma \longrightarrow \text{GL}(V)$ is cohomologically rigid.

Suppose by contradiction that Γ is a Kähler group. By Theorem [2.1](#page-2-1)(a) and the assumption that k_1 has characteristic zero it follows that Γ is linear in characteristic zero. As in the proof of Corollary [3.2](#page-3-3) we deduce that Γ is a complex projective group. It then follows from Corollary [3.1](#page-3-2) that *ρ* has bounded image in $GL(V)$, hence that $p_1(\Gamma)$ is relatively compact in $G(k_1)$. This contradicts the fact that Γ is a lattice in $G = G(k_1) \times \prod_{j \in I \setminus \{1\}} G(k_j)$). \Box

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