

p-adic lattices are not Kähler groups

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Abstract. We show that any lattice in a simple *p*-adic Lie group is not the fundamental group of a compact Kähler manifold, as well as some variants of this result.

Keywords. Kähler groups; lattices in Lie groups

2010 Mathematics Subject Classification. 57M05; 32Q55

[Français]

Titre. Les réseaux p-adiques ne sont pas des groupes kählériens

Résumé. Dans cette note, nous montrons qu'un réseau d'un groupe de Lie p-adique simple n'est pas le groupe fondamental d'une variété kählérienne compacte, ainsi que des variantes de ce résultat.

Received by the Editors on September 21, 2018, and in final form on January 22, 2019. Accepted on March 21, 2019.

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B.K.'s research is supported by an Einstein Foundation's professorship

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1. Results

1.A. A group is said to be a Kähler group if it is isomorphic to the fundamental group of a connected compact Kähler manifold. In particular such a group is finitely presented. As any finite étale cover of a compact Kähler manifold is still a compact Kähler manifold, any finite index subgroup of a Kähler group is a Kähler group. The most elementary necessary condition for a finitely presented group to be Kähler is that its finite index subgroups have even rank abelianizations. A classical question, due to Serre and still largely open, is to characterize Kähler groups among finitely presented groups. A standard reference for Kähler groups is [ABCKT96].

1.B. In this note we consider the Kähler problem for lattices in simple groups over local fields. Recall that if *G* is a locally compact topological group, a subgroup $\Gamma \subset G$ is called a *lattice* if it is a discrete subgroup of *G* with finite covolume (for any *G*-invariant measure on the locally compact group *G*).

We work in the following setting. Let *I* be a finite set of indices. For each $i \in I$ we fix a local field k_i and a simple algebraic group \mathbf{G}_i defined and isotropic over k_i . Let $G = \prod_{i \in I} \mathbf{G}_i(k_i)$. The topology of the local fields k_i , $i \in I$, makes *G* a locally compact topological group. We define $\operatorname{rk} G := \sum_{i \in I} \operatorname{rk}_{k_i} \mathbf{G}_i$.

We consider $\Gamma \subset G$ an *irreducible* lattice: there does not exist a disjoint decomposition $I = I_1 \coprod I_2$ into two non-empty subsets such that, for j = 1, 2, the subgroup $\Gamma_j := \Gamma \cap G_{I_j}$ of $G_{I_j} := \prod_{i \in I_j} \mathbf{G}_i(k_i)$ is a lattice in G_{I_i} .

The reference for a detailed study of such lattices is [Mar91]. In Section 2 we recall a few results for the convenience of the reader.

1.C. Most of the lattices Γ as in Section 1.B are finitely presented (see Section 2.C). The question whether Γ is Kähler or not has been studied by Simpson using his non-abelian Hodge theory when at least one of the k_i 's is archimedean. He shows that if Γ is Kähler then necessarily for any $i \in I$ such that k_i is archimedean the group G_i has to be of Hodge type (i.e. admits a Cartan involution which is an inner automorphism), see [Si92, Corollary 5.3 and 5.4]. For example $SL(n, \mathbb{Z})$ is not a Kähler group as $SL(n, \mathbb{R})$ is not a group of Hodge type. In this note we prove:

Theorem 1.1. Let I be a finite set of indices and G be a group of the form $\prod_{j \in I} G_j(k_j)$, where G_j is a simple algebraic group defined and isotropic over a local field k_j . Let $\Gamma \subset G$ be an irreducible lattice.

Suppose there exists an $i \in I$ such that k_i is non-archimedean. If $\operatorname{rk} G > 1$ and $\operatorname{char}(k_i) = 0$, or if $\operatorname{rk} G = 1$ (i.e. $G = \mathbf{G}(k)$ with \mathbf{G} a simple isotropic algebraic group of rank 1 over a local field k) then Γ is not a Kähler group.

Remark 1.2. Notice that the case rk G = 1 is essentially folkloric. As we did not find a reference in this generality let us give the proof in this case.

If Γ is not cocompact in G (this is possible only if k has positive characteristic) then Γ is not finitely generated by [L91, Corollary 7.3], hence not Kähler.

Hence we can assume that Γ is cocompact. In that case it follows from [L91, Theorem 6.1 and 7.1] that Γ admits a finite index subgroup Γ' which is a (non-trivial) free group. But a non-trivial free group is never Kähler, as it always admits a finite index subgroup with odd Betti number (see [ABCKT96, Example 1.19 p.7]). Hence Γ' , thus also Γ , is not Kähler.

On the other hand, to the best of our knowledge not a single case of Theorem 1.1 in the case where $\operatorname{rk} G > 1$ and all the k_i , $i \in I$, are non-archimedean fields of characteristic zero was previously known. The proof in this case is a corollary of Margulis' superrigidity theorem and the recent integrality result of Esnault and Groechenig ([EG17, Theorem 1.3], whose proof was greatly simplified in [EG17-2]).

1.D. Let us mention some examples of Theorem 1.1:

- Let *p* be a prime number, $I = \{1\}$, $k_1 = \mathbb{Q}_p$, $\mathbf{G} = \mathbf{SL}(n)$. A lattice in $\mathbf{SL}(n, \mathbb{Q}_p)$, $n \ge 2$, is not a Kähler group. This is new for $n \ge 3$.

- $I = \{1; 2\}, k_1 = \mathbb{R}$ and $\mathbf{G}_1 = \mathbf{SU}(r, s)$ for some $r \ge s > 0, k_2 = \mathbb{Q}_p$ and $\mathbf{G}_2 = \mathbf{SL}(r + s)$. Then any irreducible lattice in $SU(r, s) \times \mathbf{SL}(r + s, \mathbb{Q}_p)$ is not Kähler. In Section 2 we recall how to construct such lattices (they are *S*-arithmetic). The analogous result that any irreducible lattice in $\mathbf{SL}(n, \mathbb{R}) \times \mathbf{SL}(n, \mathbb{Q}_p)$ (for example $\mathbf{SL}(n, \mathbb{Z}[1/p])$) is not a Kähler group already followed from Simpson's theorem.

1.E. I don't know anything about the case not covered by Theorem 1.1: can a (finitely presented) irreducible lattice in $G = \prod_{i \in I} \mathbf{G}_i(k_i)$ with $\operatorname{rk} G > 1$ and all k_i of (necessarily the same, see Theorem 2.1) *positive characteristic*, be a Kähler group? This question already appeared in [BKT13, Remark 0.2 (5)].

2. Reminder on lattices

2.A. Examples of pairs (G, Γ) as in Section 1.B are provided by *S*-arithmetic groups: let *K* be a global field (i.e a finite extension of \mathbb{Q} or $\mathbb{F}_q(t)$, where \mathbb{F}_q denotes the finite field with *q* elements), *S* a non-empty set of places of *K*, S_{∞} the set of archimedean places of *K* (or the empty set if *K* has positive characteristic), $\mathcal{O}^{S \cup S_{\infty}}$ the ring of elements of *K* which are integral at all places not belonging to $S \cup S_{\infty}$ and **G** an absolutely simple *K*-algebraic group, anisotropic at all archimedean places not belonging to *S*. A subgroup $\Lambda \subset \mathbf{G}(K)$ is said *S*-arithmetic (or $S \cup S_{\infty}$ -arithmetic) if it is commensurable with $\mathbf{G}(\mathcal{O}^{S \cup S_{\infty}})$ (this last notation depends on the choice of an affine group scheme flat of finite type over $\mathcal{O}^{S \cup S_{\infty}}$, with generic fiber **G**; but the commensurability class of the group $\mathbf{G}(\mathcal{O}^{S \cup S_{\infty}})$ is independent of that choice).

If S is finite the image Γ in $\prod_{v \in S} \mathbf{G}(K_v)$ of an S-arithmetic group Λ by the diagonal map is an irreducible lattice (see [B63] in the number field case and [H69] in the function field case). In the situation of Section 1.B, a (necessarily irreducible) lattice $\Gamma \subset G$ is called S-arithmetic if there exist K, **G**, S as above, a bijection $i: S \longrightarrow I$, isomorphisms $K_v \longrightarrow k_{i(v)}$ and, via these isomorphisms, k_i -isomorphisms $\varphi_i: \mathbf{G} \longrightarrow \mathbf{G}_i$ such that Γ is commensurable with the image via $\prod_{i \in I} \varphi_i$ of an S-arithmetic subgroup of $\mathbf{G}(K)$.

2.B. Margulis' and Venkataramana's arithmeticity theorem states that as soon as rkG is at least 2 then every irreducible lattice in G is of this type:

Theorem 2.1 (Margulis, Venkataramana). In the situation of Section 1.B, suppose that $\Gamma \subset G$ is an irreducible lattice and that $\operatorname{rk} G \geq 2$. Suppose moreover for simplicity that G_i , $i \in I$, is absolutely simple. Then:

- (a) All the fields k_i have the same characteristic.
- (b) The group Γ is S-arithmetic.

Remark 2.2. Margulis [Mar84] proved Theorem 2.1 when $char(k_i) = 0$ for all $i \in I$. Venkatarama [V88] had to overcome many technical difficulties in positive characteristics to extend Margulis' strategy to the general case.

On the other hand, if rk G = 1 (hence $I = \{1\}$) and $k = k_1$ is non-archimedean, there exist non-arithmetic lattices in G, see [L91, Theorem A].

2.C. With the notations of Section 2.A, an *S*-arithmetic lattice Γ is always finitely presented except if *K* is a function field, and $\operatorname{rk}_{K}\mathbf{G} = \operatorname{rk} G = |S| = 1$ (in which case Γ is not even finitely generated) or $\operatorname{rk}_{K}\mathbf{G} > 0$ and $\operatorname{rk} G = 2$ (in which case Γ is finitely generated but not finitely presented). In the number field case see the result of Raghunathan [R68] in the classical arithmetic case and of Borel-Serre [BS76] in the general *S*-arithmetic case; in the function field case see the work of Behr, e.g. [Behr98]. For example the lattice $\operatorname{SL}_2(\mathbb{F}_q[t])$ of $\operatorname{SL}_2(\mathbb{F}_q((1/t)))$ is not finitely generated, while the lattice $\operatorname{SL}_3(\mathbb{F}_q[t])$ of $\operatorname{SL}_3(\mathbb{F}_q((1/t)))$ is finitely presented.

3. **Proof of Theorem 1.1**

Thanks to Remark 1.2 we can assume that $\operatorname{rk} G > 1$. In this case the main tools for proving Theorem 1.1 are the recent result of Esnault and Groechenig and Margulis' superrigidity theorem.

3.A. Recall that a linear representation $\rho: \Gamma \longrightarrow \mathbf{GL}(n,k)$ of a group Γ over a field k is cohomologically rigid if $H^1(\Gamma, \operatorname{Ad} \rho) = 0$. A representation $\rho: \Gamma \longrightarrow \mathbf{GL}(n, \mathbb{C})$ is said to be integral if it factorizes through $\rho: \Gamma \longrightarrow \mathbf{GL}(n, K), K \hookrightarrow \mathbb{C}$ a number field, and moreover stabilizes an \mathcal{O}_K -lattice in \mathbb{C}^n (equivalently, see [Ba80, Corollary 2.3 and 2.5]: for any embedding $v: K \hookrightarrow k$ of K in a non-archimedean local field k the composed representation $\rho_v: \Gamma \longrightarrow \mathbf{GL}(n, K) \hookrightarrow \mathbf{GL}(n, k)$ has bounded image in $\mathbf{GL}(n, k)$). A group will be said *complex projective* if is isomorphic to the fundamental group of a connected smooth complex projective variety. This is a special case of a Kähler group (the question whether or not any Kähler group is complex projective is open).

In [EG17-2, Theorem 1.1] Esnault and Groechenig prove that if Γ is a complex projective group then any irreducible cohomologically rigid representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n, \mathbb{C})$ is integral. This was conjectured by Simpson.

3.B. A corollary of [EG17-2, Theorem 1.1] is the following:

Corollary 3.1. Let Γ be a complex projective group. Let k be a non-archimedean local field of characteristic zero and let $\rho : \pi_1(X) \longrightarrow \mathbf{GL}(n,k)$ be an absolutely irreducible cohomologically rigid representation. Then ρ has bounded image in $\mathbf{GL}(n,k)$.

Proof. Let \overline{k} be an algebraic closure of k. As ρ is absolutely irreducible and cohomologically rigid there exists $g \in \mathbf{GL}(n, \overline{k})$ and a number field $K \subset \overline{k}$ such that $\rho^g(\Gamma) := g \cdot \rho \cdot g^{-1}(\Gamma) \subset \mathbf{GL}(n, \overline{k})$ lies in $\mathbf{GL}(n, K)$.

Let k' be the finite extension of k generated by K and the matrix coefficients of g and g^{-1} . This is still a non-archimedean local field of characteristic zero, and both $\rho(\Gamma)$ and $\rho^g(\Gamma)$ are subgroups of $\mathbf{GL}(n,k')$. As $\rho: \Gamma \longrightarrow \mathbf{GL}(n,k) \subset \mathbf{GL}(n,k')$ has bounded image in $\mathbf{GL}(n,k)$ if and only if $\rho^g: \Gamma \longrightarrow \mathbf{GL}(n,k')$ has bounded image in $\mathbf{GL}(n,k')$, we can assume, replacing ρ by ρ^g and k by k' if necessary, that $\rho(\Gamma)$ is contained in $\mathbf{GL}(n,K)$ with $K \subset k$ a number field.

Let $\sigma: K \hookrightarrow \mathbb{C}$ be an infinite place of K and consider $\rho^{\sigma}: \Gamma \xrightarrow{\rho} \mathbf{GL}(n, K) \xrightarrow{\sigma} \mathbf{GL}(n, \mathbb{C})$ the associated representation. As ρ is absolutely irreducible, the representation ρ^{σ} is irreducible. As

$$H^1(\Gamma, \operatorname{Ad} \circ \rho^{\sigma}) = H^1(\Gamma, \operatorname{Ad} \circ \rho) \otimes_{K, \sigma} \mathbb{C} = 0$$

the representation ρ^{σ} is cohomologically rigid.

It follows from [EG17, Theorem 1.3] that ρ^{σ} is integral. In particular, considering the embedding $K \subset k$, it follows that the representation $\rho : \Gamma \longrightarrow \mathbf{GL}(n,k)$ has bounded image in $\mathbf{GL}(n,k)$.

3.C. Notice that we can upgrade Corollary 3.1 to the Kähler world if we restrict ourselves to faithful representations:

Corollary 3.2. The conclusion of Corollary 3.1 also holds for Γ a Kähler group and $\rho : \pi_1(X) \longrightarrow \mathbf{GL}(n,k)$ a faithful representation.

Proof. As the representation ρ is faithful, the group Γ is a linear group in characteristic zero. It then follows that the Kähler group Γ is a complex projective group (see [CCE14, Theorem 0.2] which proves that a finite index subgroup of Γ is complex projective, and its refinement [C17, Corollary 1.3] which proves that Γ itself is complex projective). The result now follows from Corollary 3.1.

3.D. Let us apply Corollary 3.1 to the case of Theorem 1.1 where $\operatorname{rk} G > 1$. Renaming the indices of I if necessary, we can assume that $I = \{1, \dots, r\}$ and k_1 is non-archimedean of characteristic zero. Let us choose an absolutely irreducible k_1 -representation $\rho_{\mathbf{G}_1} : \mathbf{G}_1 \longrightarrow \mathbf{GL}(V)$. Let

$$\rho: \Gamma \longrightarrow G \xrightarrow{p_1} \mathbf{G}_1(k_1) \longrightarrow \mathbf{GL}(V)$$

be the representation of Γ deduced from $\rho_{\mathbf{G}_1}$ (where $p_1 : G \longrightarrow \mathbf{G}_1(k_1)$ denotes the projection of G onto its first factor). As $p_1(\Gamma)$ is Zariski-dense in \mathbf{G}_1 it follows that ρ is absolutely irreducible.

As $\operatorname{rk} G > 1$, Margulis' superrigidity theorem applies to the lattice Γ of G: it implies in particular that $H^1(\Gamma, \operatorname{Ad} \circ \rho) = 0$ (see [Mar91, Theorem (3)(iii) p.3]). Hence the representation $\rho : \Gamma \longrightarrow \operatorname{GL}(V)$ is cohomologically rigid.

Suppose by contradiction that Γ is a Kähler group. By Theorem 2.1(a) and the assumption that k_1 has characteristic zero it follows that Γ is linear in characteristic zero. As in the proof of Corollary 3.2 we deduce that Γ is a complex projective group. It then follows from Corollary 3.1 that ρ has bounded image in **GL**(*V*), hence that $p_1(\Gamma)$ is relatively compact in **G**(k_1). This contradicts the fact that Γ is a lattice in $G = \mathbf{G}(k_1) \times \prod_{j \in I \setminus \{1\}} \mathbf{G}(k_j)$.

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