\textit{p}-adic lattices are not Kähler groups

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\textbf{Abstract.} We show that any lattice in a simple \textit{p}-adic Lie group is not the fundamental group of a compact Kähler manifold, as well as some variants of this result.

\textbf{Keywords.} Kähler groups; lattices in Lie groups

\textbf{2010 Mathematics Subject Classification.} 57M05; 32Q55

\textbf{[Français]}

\textbf{Titre.} Les réseaux \textit{p}-adiques ne sont pas des groupes kählériens

\textbf{Résumé.} Dans cette note, nous montrons qu'un réseau d'un groupe de Lie \textit{p}-adique simple n'est pas le groupe fondamental d'une variété kählérienne compacte, ainsi que des variantes de ce résultat.
1. Results

A group is said to be a Kähler group if it is isomorphic to the fundamental group of a connected compact Kähler manifold. In particular such a group is finitely presented. As any finite étale cover of a compact Kähler manifold is still a compact Kähler manifold, any finite index subgroup of a Kähler group is a Kähler group. The most elementary necessary condition for a finitely presented group to be Kähler is that its finite index subgroups have even rank abelianizations. A classical question, due to Serre and still largely open, is to characterize Kähler groups among finitely presented groups. A standard reference for Kähler groups is [ABCKT96].

In this note we consider the Kähler problem for lattices in simple groups over local fields. Recall that if $G$ is a locally compact topological group, a subgroup $\Gamma \subset G$ is called a lattice if it is a discrete subgroup of $G$ with finite covolume (for any $G$-invariant measure on the locally compact group $G$).

We work in the following setting. Let $I$ be a finite set of indices. For each $i \in I$ we fix a local field $k_i$ and a simple algebraic group $G_i$ defined and isotropic over $k_i$. Let $G = \prod_{i \in I} G_i(k_i)$. The topology of the local fields $k_i$, $i \in I$, makes $G$ a locally compact topological group. We define $\text{rk} \ G := \sum_{i \in I} \text{rk}_{k_i} G_i$.

We consider $\Gamma \subset G$ an irreducible lattice: there does not exist a disjoint decomposition $I = I_1 \sqcup I_2$ into two non-empty subsets such that, for $j = 1, 2$, the subgroup $\Gamma_j := \Gamma \cap G_{I_j}$ of $G_{I_j} := \prod_{i \in I_j} G_i(k_i)$ is a lattice in $G_{I_j}$.

The reference for a detailed study of such lattices is [Mar91]. In Section 2 we recall a few results for the convenience of the reader.

Most of the lattices $\Gamma$ as in Section 1.B are finitely presented (see Section 2.C). The question whether $\Gamma$ is Kähler or not has been studied by Simpson using his non-abelian Hodge theory when at least one of the $k_i$’s is archimedean. He shows that if $\Gamma$ is Kähler then necessarily for any $i \in I$ such that $k_i$ is archimedean the group $G_i$ has to be of Hodge type (i.e. admits a Cartan involution which is an inner automorphism), see [Si92, Corollary 5.3 and 5.4]. For example $\text{SL}(n, Z)$ is not a Kähler group as $\text{SL}(n, \mathbb{R})$ is not a group of Hodge type. In this note we prove:

**Theorem 1.1.** Let $I$ be a finite set of indices and $G$ be a group of the form $\prod_{i \in I} G_i(k_i)$, where $G_i$ is a simple algebraic group defined and isotropic over a local field $k_i$. Let $\Gamma \subset G$ be an irreducible lattice.

Suppose there exists an $i \in I$ such that $k_i$ is non-archimedean. If $\text{rk} \ G > 1$ and $\text{char}(k_i) = 0$, or if $\text{rk} \ G = 1$ (i.e. $G = G(k)$ with $G$ a simple isotropic algebraic group of rank 1 over a local field $k$) then $\Gamma$ is not a Kähler group.

**Remark 1.2.** Notice that the case $\text{rk} \ G = 1$ is essentially folkloric. As we did not find a reference in this generality let us give the proof in this case.

If $\Gamma$ is not cocompact in $G$ (this is possible only if $k$ has positive characteristic) then $\Gamma$ is not finitely generated by [L91, Corollary 7.3], hence not Kähler.

Hence we can assume that $\Gamma$ is cocompact. In that case it follows from [L91, Theorem 6.1 and 7.1] that $\Gamma$ admits a finite index subgroup $\Gamma'$ which is a (non-trivial) free group. But a non-trivial free group is never Kähler, as it always admits a finite index subgroup with odd Betti number (see [ABCKT96, Example 1.19 p.7]). Hence $\Gamma'$, thus also $\Gamma$, is not Kähler.
On the other hand, to the best of our knowledge not a single case of Theorem 1.1 in the case where \( \text{rk } G > 1 \) and all the \( k_i, i \in I \), are non-archimedean fields of characteristic zero was previously known. The proof in this case is a corollary of Margulis’ superrigidity theorem and the recent integrality result of Esnault and Groechenig [EG17, Theorem 1.3], whose proof was greatly simplified in [EG17-2].

1.D. Let us mention some examples of Theorem 1.1:

- Let \( p \) be a prime number, \( I = \{1\}, k_1 = \mathbb{Q}_p \), \( G = \text{SL}(n) \). A lattice in \( \text{SL}(n, \mathbb{Q}_p), n \geq 2 \), is not a Kähler group. This is new for \( n \geq 3 \).

- \( I = \{1, 2\}, k_1 = \mathbb{R} \) and \( G_1 = \text{SU}(r, s) \) for some \( r \geq s > 0 \), \( k_2 = \mathbb{Q}_p \) and \( G_2 = \text{SL}(r + s) \). Then any irreducible lattice in \( \text{SU}(r, s) \times \text{SL}(r + s, \mathbb{Q}_p) \) is not Kähler. In Section 2 we recall how to construct such lattices (they are \( S \)-arithmetic). The analogous result that any irreducible lattice in \( \text{SL}(n, \mathbb{R}) \times \text{SL}(n, \mathbb{Q}_p) \) (for example \( \text{SL}(n, \mathbb{Z}[1/p]) \)) is not a Kähler group already followed from Simpson’s theorem.

1.E. I don’t know anything about the case not covered by Theorem 1.1: can a (finitely presented) irreducible lattice in \( G = \prod_{i \in I} G_i(k_i) \) with \( \text{rk } G > 1 \) and all \( k_i \) of (necessarily the same, see Theorem 2.1) positive characteristic, be a Kähler group? This question already appeared in [BKT13, Remark 0.2 (5)].

2. Reminder on lattices

2.A. Examples of pairs \((G, \Gamma)\) as in Section 1.B are provided by \( S \)-arithmetic groups: let \( K \) be a global field (i.e. a finite extension of \( \mathbb{Q} \) or \( \mathbb{F}_q(t) \), where \( \mathbb{F}_q \) denotes the finite field with \( q \) elements), \( S \) a non-empty set of places of \( K \), \( S_\infty \) the set of archimedean places of \( K \) (or the empty set if \( K \) has positive characteristic), \( O^{S \cup S_\infty} \) the ring of elements of \( K \) which are integral at all places not belonging to \( S \cup S_\infty \) and \( G \) an absolutely simple \( K \)-algebraic group, anisotropic at all archimedean places not belonging to \( S \). A subgroup \( \Lambda \subset G(K) \) is said \( S \)-arithmetic (or \( S \cup S_\infty \)-arithmetic) if it is commensurable with \( G(O^{S \cup S_\infty}) \) (this last notation depends on the choice of an affine group scheme flat of finite type over \( O^{S \cup S_\infty} \), with generic fiber \( G \); but the commensurability class of the group \( G(O^{S \cup S_\infty}) \) is independent of that choice).

If \( S \) is finite the image \( \Gamma \) in \( \prod_{v \in S} G(K_v) \) of an \( S \)-arithmetic group \( \Lambda \) by the diagonal map is an irreducible lattice (see [B63] in the number field case and [H69] in the function field case). In the situation of Section 1.B, a (necessarily irreducible) lattice \( \Gamma \subset G \) is called \( S \)-arithmetic if there exist \( K, G, S \) as above, a bijection \( i : S \rightarrow I \), isomorphisms \( K_v \rightarrow k_{i(v)} \) and, via these isomorphisms, \( k_i \)-isomorphisms \( \varphi_i : G \rightarrow G_i \) such that \( \Gamma \) is commensurable with the image via \( \prod_{i \in I} \varphi_i \) of an \( S \)-arithmetic subgroup of \( G(K) \).

2.B. Margulis’ and Venkatarana’s arithmeticity theorem states that as soon as \( \text{rk } G \) is at least 2 then every irreducible lattice in \( G \) is of this type:

**Theorem 2.1 (Margulis, Venkatarana).** In the situation of Section 1.B, suppose that \( \Gamma \subset G \) is an irreducible lattice and that \( \text{rk } G \geq 2 \). Suppose moreover for simplicity that \( G_i, i \in I \), is absolutely simple. Then:

(a) All the fields \( k_i \) have the same characteristic.

(b) The group \( \Gamma \) is \( S \)-arithmetic.

**Remark 2.2.** Margulis [Mar84] proved Theorem 2.1 when \( \text{char}(k_i) = 0 \) for all \( i \in I \). Venkatarana [V88] had to overcome many technical difficulties in positive characteristics to extend Margulis’ strategy to the general case.

On the other hand, if \( \text{rk } G = 1 \) (hence \( I = \{1\} \)) and \( k = k_1 \) is non-archimedean, there exist non-arithmetic lattices in \( G \), see [L91, Theorem A].
2.C. With the notations of Section 2.A, an $S$-arithmetic lattice $\Gamma$ is always finitely presented except if $K$ is a function field, and $\text{rk}_K G = \text{rk} G = |S| = 1$ (in which case $\Gamma$ is not even finitely generated) or $\text{rk}_K G > 0$ and $\text{rk} G = 2$ (in which case $\Gamma$ is finitely generated but not finitely presented). In the number field case see the result of Raghunathan [R68] in the classical arithmetic case and of Borel-Serre [BS76] in the general $S$-arithmetic case; in the function field case see the work of Behr, e.g. [Behr98]. For example the lattice $\text{SL}_2(\mathbb{F}_q[t])$ of $\text{SL}_2(\mathbb{F}_q((1/t)))$ is not finitely generated, while the lattice $\text{SL}_3(\mathbb{F}_q[t])$ of $\text{SL}_3(\mathbb{F}_q((1/t)))$ is finitely generated but not finitely presented.

3. Proof of Theorem 1.1

Thanks to Remark 1.2 we can assume that $\text{rk} G > 1$. In this case the main tools for proving Theorem 1.1 are the recent result of Esnault and Groechenig and Margulis’ superrigidity theorem.

3.A. Recall that a linear representation $\rho : \Gamma \to \text{GL}(n, k)$ of a group $\Gamma$ over a field $k$ is cohomologically rigid if $H^1(\Gamma, \text{Ad} \rho) = 0$. A representation $\rho : \Gamma \to \text{GL}(n, \mathbb{C})$ is said to be integral if it factorizes through $\rho : \Gamma \to \text{GL}(n, K)$, $K \to \mathbb{C}$ a number field, and moreover stabilizes an $O_K$-lattice in $\mathbb{C}^n$ (equivalently, see [Ba80, Corollary 2.3 and 2.5]) for any embedding $v : K \hookrightarrow k$ of $K$ in a non-archimedean local field $k$ the composed representation $\rho_v : \Gamma \to \text{GL}(n, K) \hookrightarrow \text{GL}(n, k)$ has bounded image in $\text{GL}(n, k)$. A group will be said complex projective if is isomorphic to the fundamental group of a connected smooth complex projective variety. This is a special case of a Kähler group (the question whether or not any Kähler group is complex projective is open).

In [EG17-2, Theorem 1.1] Esnault and Groechenig prove that if $\Gamma$ is a complex projective group then any irreducible cohomologically rigid representation $\rho : \Gamma \to \text{GL}(n, \mathbb{C})$ is integral. This was conjectured by Simpson.

3.B. A corollary of [EG17-2, Theorem 1.1] is the following:

**Corollary 3.1.** Let $\Gamma$ be a complex projective group. Let $k$ be a non-archimedean local field of characteristic zero and let $\rho : \pi_1(X) \to \text{GL}(n, k)$ be an absolutely irreducible cohomologically rigid representation. Then $\rho$ has bounded image in $\text{GL}(n, k)$.

**Proof.** Let $\bar{k}$ be an algebraic closure of $k$. As $\rho$ is absolutely irreducible and cohomologically rigid there exists $g \in \text{GL}(n, \bar{k})$ and a number field $K \subset \bar{k}$ such that $\rho^\sigma(\Gamma) := g \cdot \rho(\Gamma) \cdot g^{-1}(\Gamma) \subset \text{GL}(n, K)$ lies in $\text{GL}(n, K)$.

Let $k'$ be the finite extension of $k$ generated by $K$ and the matrix coefficients of $g$ and $g^{-1}$. This is still a non-archimedean local field of characteristic zero, and both $\rho(\Gamma)$ and $\rho^\sigma(\Gamma)$ are subgroups of $\text{GL}(n, k')$. As $\rho : \Gamma \to \text{GL}(n, k) \subset \text{GL}(n, k')$ has bounded image in $\text{GL}(n, k)$ if and only if $\rho^\sigma : \Gamma \to \text{GL}(n, k')$ has bounded image in $\text{GL}(n, k')$, we can assume, replacing $\rho$ by $\rho^\sigma$ and $k$ by $k'$ if necessary, that $\rho(\Gamma)$ is contained in $\text{GL}(n, K)$ with $K \subset k$ a number field.

Let $\sigma : K \to \mathbb{C}$ be an infinite place of $K$ and consider $\rho^\sigma : \Gamma \to \text{GL}(n, K) \to \text{GL}(n, \mathbb{C})$ the associated representation. As $\rho$ is absolutely irreducible, the representation $\rho^\sigma$ is irreducible. As $H^1(\Gamma, \text{Ad} \circ \rho^\sigma) \to H^1(\Gamma, \text{Ad} \circ \rho) \otimes_{K, \sigma} \mathbb{C} = 0$ the representation $\rho^\sigma$ is cohomologically rigid.

It follows from [EG17, Theorem 1.3] that $\rho^\sigma$ is integral. In particular, considering the embedding $K \subset k$, it follows that the representation $\rho : \Gamma \to \text{GL}(n, k)$ has bounded image in $\text{GL}(n, k)$.

3.C. Notice that we can upgrade Corollary 3.1 to the Kähler world if we restrict ourselves to faithful representations:

**Corollary 3.2.** The conclusion of Corollary 3.1 also holds for $\Gamma$ a Kähler group and $\rho : \pi_1(X) \to \text{GL}(n, k)$ a faithful representation.
Proof. As the representation ρ is faithful, the group Γ is a linear group in characteristic zero. It then follows that the Kähler group Γ is a complex projective group (see [CCE14, Theorem 0.2] which proves that a finite index subgroup of Γ is complex projective, and its refinement [C17, Corollary I.3] which proves that Γ itself is complex projective). The result now follows from Corollary 3.1.

3.D. Let us apply Corollary 3.1 to the case of Theorem 1.1 where \( \text{rk} G > 1 \). Renaming the indices of \( I \) if necessary, we can assume that \( I = \{1, \cdots, r\} \) and \( k_1 \) is non-archimedean of characteristic zero. Let us choose an absolutely irreducible \( k_1 \)-representation \( \rho_{G_1} : G_1 \longrightarrow \text{GL}(V) \). Let

\[
\rho : \Gamma \longrightarrow G \overset{p_1}{\longrightarrow} G_1(k_1) \longrightarrow \text{GL}(V)
\]

be the representation of Γ deduced from \( \rho_{G_1} \) (where \( p_1 : G \longrightarrow G_1(k_1) \) denotes the projection of G onto its first factor). As \( p_1(\Gamma) \) is Zariski-dense in \( G_1 \) it follows that \( \rho \) is absolutely irreducible.

As \( \text{rk} G > 1 \), Margulis’ superrigidity theorem applies to the lattice \( \Gamma \) of \( G \): it implies in particular that \( H^1(\Gamma, \text{Ad} \circ \rho) = 0 \) (see [Mar91, Theorem (3)(iii) p. 3]). Hence the representation \( \rho : \Gamma \longrightarrow \text{GL}(V) \) is cohomologically rigid.

Suppose by contradiction that \( \Gamma \) is a Kähler group. By Theorem 2.1(a) and the assumption that \( k_1 \) has characteristic zero it follows that \( \Gamma \) is linear in characteristic zero. As in the proof of Corollary 3.2 we deduce that \( \Gamma \) is a complex projective group. It then follows from Corollary 3.1 that \( \rho \) has bounded image in \( \text{GL}(V) \), hence that \( p_1(\Gamma) \) is relatively compact in \( G(k_1) \). This contradicts the fact that \( \Gamma \) is a lattice in \( G = G(k_1) \times \prod_{j \in I \backslash \{1\}} G(k_j) \).

References


