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Pluricomplex Green's functions and Fano manifolds

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Abstract. We show that if a Fano manifold does not admit Kähler-Einstein metrics then the Kähler potentials along the continuity method subconverge to a function with analytic singularities along a subvariety which solves the homogeneous complex Monge-Ampère equation on its complement, confirming an expectation of Tian-Yau.

Keywords. Fano manifold; pluricomplex Green function; algebraic Kähler potentials

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[Français]

Titre. Fonctions de Green pluricomplexes et variétés de Fano

Résumé. Nous montrons que si une variété de Fano n'admet aucune métrique de Kähler-Einstein alors, suivant la méthode de continuité, les potentiels kählériens sous-convergent vers une fonction à singularités analytiques le long d'une sous-variété, sur le complémentaire de laquelle la fonction est solution de l'équation de Monge-Ampère complexe homogène. Cela confirme une attente de Tian-Yau.

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Contents

1. Introduction

Let X^n be a Fano manifold, i.e. a compact complex manifold with $c_1(X) > 0$. A Kähler-Einstein metric on X is a Kähler metric ω which satisfies

$$\mathrm{Ric}(\omega) = \omega.$$

This implies that $[\omega] = c_1(X)$. We assume throughout this paper that X does not admit a Kähler-Einstein metric. This is known to be equivalent to K-unstability by [13] (see also [40]), but we will not use this fact.

We fix a Kähler metric ω with $[\omega] = c_1(X)$, with Ricci potential F defined by $\mathrm{Ric}(\omega) = \omega + \sqrt{-1}\partial\overline{\partial}F$ (normalized by $\int_X (e^F - 1)\omega^n = 0$). We consider Kähler metrics ω_t with $[\omega_t] = c_1(X)$ which satisfy

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega.$$

We can write $\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$ and the functions φ_t solve the complex Monge-Ampère equation [46]

$$\omega_t^n = e^{F - t\varphi_t}\omega^n. \tag{1.1}$$

A solution φ_t exists on $[0, R(X))$ where $R(X) \leqslant 1$ is the greatest lower bound for the Ricci curvature of Kähler metrics in $c_1(X)$ [36]. It is known [35, 38] that since X does not admit Kähler-Einstein metrics, we must have that $\lim_{t\to R(X)} \sup_X \varphi_t = +\infty$. We fix a sequence $t_i \to R(X)$ and write $\varphi_i := \varphi_{t_i}$ and $\omega_i := \omega_{t_i}$. Using this result, together with multiplier ideal sheaves, Nadel [29, Proposition 4.1] proved that (up to passing to a subsequence) the measures ω_i^n converge to zero (as measures) on compact sets of $X \backslash V$ for some proper analytic subvariety $V \subset X$, and in [44] the second-named author improved this to uniform convergence.

By weak compactness of closed positive currents in a fixed cohomology class, up to subsequences we can extract a limit ρ of $\varphi_i - \sup_X \varphi_i$ (which may depend on the subsequence), which is an unbounded ω-psh function, and the convergence happens in the L^1 topology.

In their work [41, p.178], Tian-Yau expressed the expectation that ρ should have logarithmic poles along a proper analytic subvariety $V \subset X$, and that it should satisfy $(\omega + \sqrt{-1}\partial\overline{\partial}\rho)^n = 0$ on $X\backslash V$, so that ρ could be thought of as a kind of pluricomplex Green's function (see also [38, p.238] and [39, p.109]).

In this note we confirm Tian-Yau's expectation:

Theorem 1.1. *Let X be a Fano manifold without a Kähler-Einstein metric, and let $\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$ be the solutions of the continuity method (1.1). Given any sequence $t_i \in [0, R(X))$ with $t_i \to R(X)$, choose a subsequence such that $\varphi_{t_i} - \sup_X \varphi_{t_i}$ converge in $L^1(X)$ to an ω-psh function ρ. Then we can find $m \geqslant 1$ and an ω-psh function ψ on X with analytic singularities*

$$\psi = \frac{1}{m}\log\sum_{j=1}^{p}\lambda_j^2|S_j|^2_{h^m}, \tag{1.2}$$

for some $\lambda_j \in (0,1]$ and some sections $S_j \in H^0(X, K_X^{-m})$, with nonempty common zero locus $V \subset X$ such that $\rho - \psi$ is bounded on X, and on $X \backslash V$ we have

$$(\omega + \sqrt{-1}\partial\overline{\partial}\rho)^n = 0, \tag{1.3}$$

where the Monge-Ampère product is in the sense of Bedford-Taylor [6].

In particular, Theorem 1.1 implies that the non-pluripolar Monge-Ampère operator of ρ (defined in [12]) vanishes identically on X. On the other hand, there is another meaningful Monge-Ampère operator that can be applied to ρ. Indeed, the fact that $\rho - \psi \in L^\infty(X)$ implies that ρ itself has analytic singularities. In [3] Andersson-Błocki-Wulcan defined a Monge-Ampère operator for ω-psh functions with analytic singularities (generalizing earlier work of Andersson-Wulcan [4] in the local setting). In general, applying this Monge-Ampère operator to ρ will produce a Radon measure μ on X (which may be identically zero in some cases), which by Theorem 1.1 is supported on the analytic set V, thus providing geometrically interesting examples of unbounded quasi-psh functions on compact Kähler manifolds with Monge-Ampère operator concentrated on a subvariety (see also [1, 2] for related results in the local setting). In particular, this answers [11, Question 1 (c)], an open problem raised at the AIM workshop "The complex Monge-Ampère equation" in August 2016 (cf. the related [23, Question 12]).

Note that in general a formula for the total mass of μ is proved in [3, Theorem 1.2], and it satisfies

$$\int_X \mu \leqslant \int_X \omega^n,$$

with strict inequality in general (but it is not hard to see that if $\dim X = 2$ and V is a finite set then equality holds). Therefore, the measure μ is in general different from the measures that one obtains as weak limits of $(\omega + \sqrt{-1}\partial\overline{\partial}\varphi_i)^n$ (up to subsequences), whose total mass is always equal to $\int_X \omega^n$.

Remark 1.2. [Remark added in proof] After this work was posted on the arXiv, and partly prompted by it, Błocki [10] modified the definition of the Monge-Ampère operator for ω-psh functions with analytic singularities of [4, 3], and with his definition the total mass is always equal to $\int_X \omega^n$. It is an interesting question to determine whether this Monge-Ampère operator equals the weak limit of $(\omega + \sqrt{-1}\partial\overline{\partial}\varphi_i)^n$.

Remark 1.3. As in the second-named author's previous work [44], Theorem 1.1 has a direct counterpart for solutions of the normalized Kähler-Ricci flow, instead of the continuity method (1.1). The statement is identical to Theorem 1.1, except that now the sequence t_i goes to $+\infty$. The proof is also almost verbatim the same, and the partial C^0 estimate along the flow is proved in [14, 15] (see also [5]). All other ingredients used also have well-known counterparts for the flow (see [44]). We leave the simple details to the interested reader.

Remark 1.4. The behavior of the solutions ω_t of (1.1) as $t \to R(X)$ has been investigated in the past. If the manifold is K-stable, [20] show that ω_t converge smoothly to a Kähler-Einstein metric. If on the other hand no such metric exists, the blowup behavior of ω_t has been investigated in [29, 44] in the setting of this paper, and also in [20, 26, 33] by allowing reparametrizations of the metrics by diffeomorphisms.

The proof of Theorem 1.1 relies on the partial C^0 estimate for solutions of (1.1) which was established by Székelyhidi [37]. We recall this in section 2, together with a well-known reformulation of this estimate (Proposition 2.1). In section 3 we observe that this gives us the singularity model function ψ in (1.2), and it also implies that ρ has the same singularity type as ψ. In section 4 we show the general fact that every ω-psh function on X with the same singularity type as ψ has vanishing Monge-Ampère operator outside V, thus proving Theorem 1.1. This relies on a geometric understanding of the rational map defined by the sections $\{S_j\}$ as in Theorem 1.1. Lastly, in section 5 we discuss the pluricomplex Green's function with the same singularity type as ψ.

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2. The partial C^0 estimate

To start we fix some notation. We choose a Hermitian metric h on K_X^{-1} with curvature $R_h = \omega$ (such h is unique up to scaling), and let h^m be the induced metric on K_X^{-m}, for all $m \geqslant 1$. Let $N_m = \dim H^0(X, K_X^{-m})$, and for any $m \geqslant 1$ define the density of states function

$$\rho_m(\omega) = \sum_{j=1}^{N_m} |S_j|^2_{h^m},$$

where $S_1, \dots, S_{N_m}$ are a basis of $H^0(X, K_X^{-m})$ which is orthonormal with respect to the L^2 inner product $\int_X \langle S_1, S_2\rangle_{h^m}\omega^n$. Clearly $\rho_m(\omega)$ is independent of the choice of basis, and is also unchanged if we scale h by a constant. The integral $\int_X \rho_m(\omega)\omega^n$ equals N_m, and if m is sufficiently large so that K_X^{-m} is very ample, then $\rho_m(\omega)$ is strictly positive on X. If we apply this same construction to the metrics ω_t and Hermitian metrics $h_t = he^{-\varphi_t}$ we get a density of states function $\rho_m(\omega_t)$. Following [39], we say that a "partial C^0 estimate" holds if there exist $m \geqslant 1$ and a constant $C > 0$ such that

$$\inf_X \rho_m(\omega_t) \geqslant C^{-1}, \tag{2.1}$$

holds for all $t \in [0, R(X))$. The reason for this name is explained by the following proposition, which is essentially well-known (see [39, Lemma 2.2] and [43, Proposition 5.1]), but we provide the details for convenience:

Proposition 2.1. *If a partial C^0 estimate holds then there exists $m \geqslant 1$, such that for all $\varepsilon > 0$ we can find a constant $C > 0$ so that for all $t \in [\varepsilon, R(X))$ we can find real numbers $1 = \lambda_1(t) \geqslant \dots \geqslant \lambda_{N_m}(t) > 0$ and a basis $\{S_j(t)\}_{1\leqslant j \leqslant N_m}$ of $H^0(X, K_X^{-m})$, orthonormal with respect to the L^2 inner product of ω, h^m, such that for all $t \in [0, R(X))$ we have*

$$\sup_X \left| \varphi_t - \sup_X \varphi_t - \frac{1}{m} \log \sum_{j=1}^{N_m} \lambda_j(t)^2 |S_j(t)|^2_{h^m} \right| \leqslant C. \tag{2.2}$$

In the rest of the paper we will fix a value of $\varepsilon > 0$ once and for all, for example $\varepsilon = R(X)/2$. The precise choice is irrelevant, since we are only interested in the behavior as $t \to R(X)$.

Proof. First, it is well-known that for all $m \geqslant 1$ and $\varepsilon > 0$ there is a constant C such that for all $t \in [\varepsilon, R(X))$ we have

$$\rho_m(\omega_t) \leqslant C. \tag{2.3}$$

To see this, first observe for every $S \in H^0(X, K_X^{-m})$ we have

$$\Delta_{\omega_t} |S|^2_{h_t^m} = |\nabla S|^2_t - 2m|S|^2_{h_t^m} \geqslant -2m|S|^2_{h_t^m}, \tag{2.4}$$

and that since $\mathrm{Ric}(\omega_t) \geqslant t\omega_t \geqslant \varepsilon\omega_t$, Myers' Theorem gives a uniform upper bound for $\mathrm{diam}(X, \omega_t)$ and then Croke [19] and Li [27] show that the Sobolev constant of (X, ω_t) has a uniform upper bound. We can then apply Moser iteration to (2.4) to get

$$\sup_X |S|^2_{h_t^m} \leqslant C \int_X |S|^2_{h_t^m} \omega_t^n \leqslant C, \tag{2.5}$$

provided we assume that $\int_X |S|^2_{h_t^m}\omega_t^n = 1$. Taking now an orthonormal basis of sections and summing we obtain (2.3).

Thanks to (2.3) we know that for $t \in [\varepsilon, R(X))$ a partial C^0 estimate is equivalent to

$$\sup_X |\log \rho_m(\omega_t)| \leqslant C. \tag{2.6}$$

We now take a basis $\{\tilde{S}_j(t)\}_{1\leqslant j\leqslant N_m}$ of $H^0(X, K_X^{-m})$ orthonormal with respect to the L^2 inner product of ω_t, h_t^m and notice that since $h_t^m = e^{-m\varphi_t}h^m$ we clearly have

$$\varphi_t = \frac{1}{m}\log\frac{\sum_{j=1}^{N_m}|\tilde{S}_j(t)|^2_{h^m}}{\sum_{j=1}^{N_m}|\tilde{S}_j(t)|^2_{h_t^m}},$$

which is equivalent to

$$\varphi_t - \frac{1}{m}\log\sum_{j=1}^{N_m}|\tilde{S}_j(t)|^2_{h^m} = -\frac{1}{m}\log\rho_m(\omega_t). \tag{2.7}$$

It follows from (2.6) and (2.7) that that for $t \in [\varepsilon, R(X))$ a partial C^0 estimate is equivalent to an estimate

$$\sup_X\left|\varphi_t - \frac{1}{m}\log\sum_{j=1}^{N_m}|\tilde{S}_j(t)|^2_{h^m}\right| \leqslant C.$$

We now choose another basis $\{S_j\}_{1\leqslant j\leqslant N_m}$ of $H^0(X, K_X^{-m})$ orthonormal with respect to the L^2 inner product of ω, h^m. After modifying S_j and $\tilde{S}_j(t)$ by t-dependent unitary transformations, we obtain orthonormal bases $\{S_j(t)\}_{1\leqslant j\leqslant N_m}$ with respect to ω, h^m, and $\{\tilde{S}_j(t)\}_{1\leqslant j\leqslant N_m}$ with respect to ω_t, h_t^m such that

$$\tilde{S}_j(t) = \mu_j(t)S_j(t),$$

for some positive real numbers $\mu_j(t)$, with $\mu_1(t) \geqslant \ldots \geqslant \mu_{N_m}(t) > 0$. We then let $\lambda_j(t) = \mu_j(t)/\mu_1(t)$ and we see that a partial C^0 estimate is equivalent to

$$\sup_X\left|\varphi_t - \frac{2}{m}\log\mu_1(t) - \frac{1}{m}\log\sum_{j=1}^{N_m}\lambda_j(t)^2|S_j(t)|^2_{h^m}\right| \leqslant C. \tag{2.8}$$

We now claim that if a partial C^0 estimate holds, then for all $t \in [\varepsilon, R(X))$ we also have

$$\left|\frac{2}{m}\log\mu_1(t) - \sup_X\varphi_t\right| \leqslant C. \tag{2.9}$$

Once this is proved, combining (2.8) and (2.9) we get (2.2). To prove (2.9), first use (2.5) to get

$$C \geqslant \sup_X|\tilde{S}_1(t)|^2_{h_t^m} \geqslant \mu_1(t)^2\sup_X|S_1(t)|^2_{h^m}e^{-m\sup_X\varphi_t},$$

and the fact $\int_X |S_1(t)|^2_{h^m}\omega^n = 1$ implies that $\sup_X |S_1(t)|^2_{h^m} \geqslant 1/\mathrm{Vol}(X,\omega)$, and so

$$\left(\frac{2}{m}\log\mu_1(t) - \sup_X\varphi_t\right) \leqslant C.$$

On the other hand the partial C^0 estimate (2.1) implies that

$$C^{-1} \leqslant \rho_m(\omega_t) = \sum_{j=1}^{N_m}|\tilde{S}_j(t)|^2_{h_t^m} \leqslant \mu_1(t)^2\sum_{j=1}^{N_m}|S_j(t)|^2_{h^m}e^{-m\varphi_t}, \tag{2.10}$$

and we clearly have that

$$\sup_j \sup_X |S_j(t)|^2_{h^m} \leqslant C, \tag{2.11}$$

since the sections $\{S_j(t)\}$ are just varying in a compact unitary group (or one can also repeat the Moser iteration argument of (2.3) for the fixed metric ω). This together with (2.10), evaluated at the point where φ_t achieves its maximum, gives the reverse inequality

$$\left(\sup_X \varphi_t - \frac{2}{m}\log \mu_1(t)\right) \leqslant C,$$

which completes the proof of (2.9). □

3. The singularity model function

The next goal is to use the partial C^0 estimate in Proposition 2.1 to construct a singular ω-psh function ψ which will have the same singularity type of any weak limit of the normalized solutions $\varphi_i - \sup_X \varphi_i$ of the continuity method.

Let the notation be as in Proposition 2.1, and in particular we fix once and for all a value of $m \geqslant 1$ given there. We can find a sequence $t_i \to R(X)$ and an ω-psh function ρ with $\sup_X \rho = 0$ such that $\varphi_i - \sup_X \varphi_i \to \rho$ in $L^1(X)$, and pointwise a.e. Passing to a subsequence, we can find a basis $\{S_j\}_{1\leqslant j\leqslant N_m}$ of $H^0(X, K_X^{-m})$ orthonormal with respect to the L^2 inner product of ω, h^m, such that $S_j(t_i) \to S_j$ smoothly as $i \to \infty$, for all $1 \leqslant j \leqslant N_m$. The change of basis matrix from $\{S_j\}_{1\leqslant j\leqslant N_m}$ to $\{S_j(t)\}_{1\leqslant j\leqslant N_m}$ induces an automorphism $\sigma(t)$ of $\mathbb{CP}^{N_m-1}$, such that $\sigma(t_i) \to \mathrm{Id}$ smoothly as $i \to \infty$.

For ease of notation, write

$$\psi_t = \frac{1}{m}\log\sum_{j=1}^{N_m} \lambda_j(t)^2 |S_j(t)|^2_{h^m}.$$

These functions are Kähler potentials for ω since

$$\omega + \sqrt{-1}\partial\overline{\partial}\psi_t = \frac{\iota^*\sigma(t)^*\tau(t)^*\omega_{FS}}{m} > 0, \tag{3.1}$$

where $\iota : X \hookrightarrow \mathbb{CP}^{N_m-1}$ is the Kodaira embedding map given by the sections $\{S_j\}_{1\leqslant j\leqslant N_m}$, the map $\tau(t)$ is the automorphism of $\mathbb{CP}^{N_m-1}$ induced by the diagonal matrix with entries $\{\lambda_j(t)\}_{1\leqslant j\leqslant N_m}$, and ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^{N_m-1}$. The identity in (3.1) follows directly from the definition of the Fubini-Study metric ω_{FS} on $\mathbb{CP}^{N_m-1}$, which on $\mathbb{C}^{N_m}\backslash\{0\}$ is given explicitly by $\omega_{FS} = \sqrt{-1}\partial\overline{\partial}\log\sum_{j=1}^{N_m}|z_j|^2$, and from the fact that the curvature of h is ω.

Up to passing to a subsequence of t_i, we may assume that $\lambda_j(t_i) \to \lambda_j$ as $i \to \infty$ for all j, and we have

$$1 = \lambda_1 \geqslant \ldots \geqslant \lambda_p > 0 = \lambda_{p+1} = \cdots = \lambda_{N_m},$$

for some $1 \leqslant p < N_m$. The case $p = N_m$ is impossible because by (2.2) it would imply a uniform L^∞ bound for φ_t and so X would admit a Kähler-Einstein metric. For the same reason, the set $V := \{S_1 = \cdots = S_p = 0\}$ must be a nonempty proper analytic subvariety of X.

Note that thanks to (2.2) we can write

$$\omega_t^n = e^{F - t(\varphi_t - \sup_X \varphi_t)} e^{-t\sup_X \varphi_t}\omega^n \leqslant C e^{t\psi_t} e^{-t\sup_X\varphi_t}\omega^n,$$

and since the term $e^{t\psi_t}$ is uniformly bounded on compact sets of $X\backslash V$, we see immediately that

$$\omega_t^n \to 0, \tag{3.2}$$

uniformly on compact sets of $X\backslash V$ (this result was proved in [44] without using the partial C^0 estimate, which was not available at the time, with weaker results established earlier in [29]).

Let then

$$\psi = \frac{1}{m}\log\sum_{j=1}^{p}\lambda_j^2|S_j|^2_{h^m},$$

which is a smooth function on $X\backslash V$ which approaches $-\infty$ uniformly on V. Since $e^{m\psi_t}\to e^{m\psi}$ smoothly on X, and since ψ_t are smooth and ω-psh, it follows that ψ is ω-psh. This will be our singularity model function in the rest of the argument, as we now explain:

Lemma 3.1. *Define the class*

$$\mathcal{C} = \{\eta \in PSH(X,\omega) \mid \eta - \psi \in L^\infty(X)\},$$

of ω-psh functions with the same singularity type as ψ. Then we have that $\rho \in \mathcal{C}$.

Proof. Recall that we have $\varphi_i - \sup_X \varphi_i \to \rho$ a.e. on X. Thanks to (2.2), the function ρ satisfies

$$|\rho - \psi| \leqslant C, \tag{3.3}$$

a.e. on X, which implies the same inequality on all of X by elementary properties of psh functions (cf. [25, Theorem K.15]), thus showing that $\rho\in\mathcal{C}$. □

4. Understanding the class $\mathcal{C}$

We now exploit the geometry of our setting to gain a better understanding of the class of functions $\mathcal{C}$.

The sections $\{\lambda_j S_j\}_{1\leqslant j\leqslant p}$ define a rational map $\Phi : X \dashrightarrow \mathbb{CP}^{p-1}$, with indeterminacy locus $Z\subset V$ (this inclusion is in general strict, since $\mathrm{codim} Z \geqslant 2$ while V may contain divisorial components). Let Y be the image of Φ, i.e. the closure of $\Phi(X\backslash Z)$ in $\mathbb{CP}^{p-1}$, which is an irreducible projective variety. By resolving the indeterminacies of Φ we get a modification $\mu : \tilde{X}\to X$, obtained as a sequence of blowups with smooth centers, and a holomorphic map $\Psi : \tilde{X}\to Y$ such that $\Psi = \Phi\circ\mu$ holds on $\tilde{X}\backslash\mu^{-1}(Z)$. We may also assume without loss of generality that μ principalizes the ideal sheaf generated by $\{S_j\}_{1\leqslant j\leqslant p}$, so that we have

$$\mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\psi) = \theta + [E],$$

where E is an effective $\mathbb{R}$-divisor with $\mu(E)\subset V$, and θ is a smooth closed semipositive $(1,1)$ form on $\tilde{X}$. We will denote by $\omega_{FS,p}$ the Fubini-Study metric on $\mathbb{CP}^{p-1}$. To identify θ, note that on $X\backslash V$ we have by definition $\omega + \sqrt{-1}\partial\overline{\partial}\psi = \frac{\Phi^*\omega_{FS,p}}{m}$, and so on $\tilde{X}\backslash\mu^{-1}(V)$ we have

$$\mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\psi) = \frac{\mu^*\Phi^*\omega_{FS,p}}{m} = \frac{\Psi^*\omega_{FS,p}}{m},$$

and so $\theta = \frac{\Psi^*\omega_{FS,p}}{m}$ on $\tilde{X}\backslash\mu^{-1}(V)$, and hence everywhere since both sides of this equality are smooth forms on all of $\tilde{X}$. This proves the key relation

$$\mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\psi) = \frac{\Psi^*\omega_{FS,p}}{m} + [E]. \tag{4.1}$$

Let $\tilde{X}\xrightarrow{\nu}\tilde{Y}\xrightarrow{q} Y$ be the Stein factorization of Ψ, where $\tilde{Y}$ is an irreducible projective variety, the map ν has connected fibers, and q is a finite morphism.

We have that $q^*\omega_{FS,p}$ is a smooth semipositive $(1,1)$ form on $\tilde{Y}$, in the sense of analytic spaces. Since ν has compact connected fibers, a standard argument shows that the set of $\frac{\Psi^*\omega_{FS,p}}{m}$-psh functions on $\tilde{X}$

can be identified with the set of (weakly) $\frac{q^*\omega_{FS,p}}{m}$-psh functions on $\tilde{Y}$ via ν^* (indeed the restriction of every $\frac{\Psi^*\omega_{FS,p}}{m}$-psh function to any fiber of ν is plurisubharmonic and hence constant on that fiber). We will use this standard argument several other times in the following.

Here and in the following, as in [21], a weakly quasi-psh function on a compact analytic space means a quasi-psh function on its regular part which is locally bounded above near the singular set. As shown in [21, §1], weakly quasi-psh functions are the same as usual quasi-psh functions if the analytic space is normal, and otherwise they can be identified with quasi-psh functions on its normalization.

Proposition 4.1. *Given any function $\eta \in \mathcal{C}$, there is a unique bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh function u on $\tilde{Y}$ such that*

$$\mu^*\eta = \mu^*\psi + \nu^* u. \tag{4.2}$$

Conversely, given any bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh function u on $\tilde{Y}$ there is a unique function $\eta \in \mathcal{C}$ such that (4.2) holds.

The relation in (4.2) thus allows us to identify the class $\mathcal{C}$ with the class of bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh functions on $\tilde{Y}$.

Next, we observe that

Proposition 4.2. *We have that*

$$\dim Y < \dim X.$$

This is a consequence of our assumption that X does not admit a Kähler-Einstein metric.

Lastly, every function $\eta \in \mathcal{C}$ belongs to $L^\infty_{\text{loc}}(X \backslash V)$, and so its Monge-Ampère operator $(\omega + \sqrt{-1}\partial\overline{\partial}\eta)^n$ is well-defined on $X \backslash V$ thanks to Bedford-Taylor [6]. Combining the results in Propositions 4.1 and 4.2 we will obtain:

Theorem 4.3. *For every $\eta \in \mathcal{C}$ we have that*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\eta)^n = 0,$$

on $X \backslash V$.

In particular, this holds for the function ρ, thanks to Lemma 3.1, and Theorem 1.1 thus follows from these.

Proof of Proposition 4.1. If η is an ω-psh function on X with $\eta - \psi \in L^\infty(X)$, i.e. η is an element of $\mathcal{C}$, then using (4.1) we can write

$$\mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\eta) = \frac{\Psi^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}\mu^*(\eta - \psi) + [E],$$

where E is as in (4.1) and $\mu^*(\eta - \psi) \in L^\infty(\tilde{X})$. Applying the Siu decomposition, we see that

$$\frac{\Psi^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}\mu^*(\eta - \psi) \geqslant 0,$$

weakly, and so

$$\mu^*(\eta - \psi) = \nu^* u_\eta,$$

for a bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh functions u_η on $\tilde{Y}$, which is uniquely determined by η (and ψ, which we view as fixed).

Conversely, given a bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh function u on $\tilde{Y}$, we have that ν^*u is $\frac{\Psi^*\omega_{FS,p}}{m}$-psh and bounded on $\tilde{X}$ and so

$$0 \leqslant \frac{\Psi^*\omega_{FS,p}}{m} + [E] + \sqrt{-1}\partial\overline{\partial}\nu^*u = \mu^*\omega + \sqrt{-1}\partial\overline{\partial}(\mu^*\psi + \nu^*u),$$

and so $\mu^*\psi + \nu^*u$ descends to an ω-psh function η_u on X with $\eta_u - \psi \in L^\infty(X)$, i.e. $\eta_u \in \mathcal{C}$.

These two constructions are inverses to each other, and so we obtain the desired bijective correspondence between functions in $\mathcal{C}$ and bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh functions on $\tilde{Y}$. □

Proof of Proposition 4.2. On X we have the estimate

$$\omega_t \geqslant C^{-1}\frac{\iota^*\sigma(t)^*\tau(t)^*\omega_{FS}}{m}, \tag{4.3}$$

which is a direct consequence of the partial C^0 estimate (see e.g. [24, Lemma 4.2]). We can also give a direct proof by calculating

$$\Delta_{\omega_t}\left(\log \mathrm{tr}_{\omega_t}\left(\frac{\iota^*\sigma(t)^*\tau(t)^*\omega_{FS}}{m}\right) - A(\varphi_t - \sup_X \varphi_t - \psi_t)\right) \geqslant \mathrm{tr}_{\omega_t}\left(\frac{\iota^*\sigma(t)^*\tau(t)^*\omega_{FS}}{m}\right) - C,$$

if A is sufficiently large, and applying the maximum principle together with the partial C^0 estimate (2.2) (for this calculation we used that the bisectional curvature of the metrics $\frac{\iota^*\sigma(t)^*\tau(t)^*\omega_{FS}}{m}$ have a uniform upper bound independent of t).

If we had $\dim Y = \dim X$ then the rational map Φ would be generically finite, so there would be a nonempty open subset $U \Subset X\backslash V$ such that $\Phi|_U$ is a biholomorphism with its image. Recall that Φ is the rational map defined by the sections $\{\lambda_j S_j\}_{1\leqslant j\leqslant p}$, while $\iota : X \hookrightarrow \mathbb{CP}^{N_m-1}$ is the embedding defined by the sections $\{S_j\}_{1\leqslant j\leqslant N_m}$, and so $\Phi = \tilde{\tau} \circ P \circ \iota$ where $P : \mathbb{CP}^{N_m-1} \dashrightarrow \mathbb{CP}^{p-1}$ is the linear projection given by $[z_1 : \cdots : z_{N_m}] \mapsto [z_1 : \cdots : z_p]$ and $\tilde{\tau} : \mathbb{CP}^{p-1} \to \mathbb{CP}^{p-1}$ is the automorphism given by

$$[z_1 : \cdots : z_p] \mapsto [\lambda_1 z_1 : \cdots : \lambda_p z_p].$$

In particular, on the embedded open n-fold $\iota(U)$, we have that $P|_{\iota(U)}$ is also a biholomorphism with its image. The automorphisms $\tau(t_i)$ descend to automorphisms $\tilde{\tau}(t_i)$ on $\mathbb{CP}^{p-1}$, and now as $i \to \infty$ these converge smoothly to the automorphism $\tilde{\tau}$. Thus $P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota = \tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota$, which converge smoothly as maps to $\tilde{\tau} \circ P \circ \iota = \Phi$ on U as $i \to \infty$.

Since Φ is an isomorphism on U, smooth convergence gives us that $P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota$ is a local isomorphism. Thus, after possibly shrinking U,

$$P : (\tau(t_i) \circ \sigma(t_i) \circ \iota)(U) \to (P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota)(U) = (\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U)$$

is an isomorphism, and for i large the open sets $(\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U) \subset \mathbb{CP}^{p-1}$ converge to the open set $(\tilde{\tau} \circ P \circ \iota)(U)$ in the Hausdorff sense. Up to shrinking U, there is an open subset $V \subset \mathbb{CP}^{p-1}$ that contains $(\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U)$ for all i large, and still P^{-1} is well-defined on V (and $P : P^{-1}(V) \to V$ is a biholomorphism), so that $P^{-1}(V)$ contains $(\tau(t_i) \circ \sigma(t_i) \circ \iota)(U)$ for all i large, and on $P^{-1}(V)$ we have

$$P^*\omega_{FS,p} \leqslant C\omega_{FS}, \tag{4.4}$$

On U we also have that $\frac{\iota^*\sigma(t_i)^*\tau(t_i)^*P^*\omega_{FS,p}}{m}$ converges smoothly to $\frac{\Phi^*\omega_{FS,p}}{m}$, which is a Kähler metric on U. Thanks to (4.3) and (4.4), on U we have

$$\omega_i \geqslant C^{-1}\frac{\iota^*\sigma(t_i)^*\tau(t_i)^*\omega_{FS}}{m} \geqslant C^{-1}\frac{\iota^*\sigma(t_i)^*\tau(t_i)^*P^*\omega_{FS,p}}{m} \geqslant C^{-1}\frac{\Phi^*\omega_{FS,p}}{m},$$

for all i large, which implies that $\int_U \omega_i^n \geqslant C^{-1}$, which is absurd thanks to (3.2). □

Remark 4.4. In particular we see that if $\dim Y = 0$ (i.e. Y is a point) then we have $\mathcal{C} = \{\psi + s\}_{s\in\mathbb{R}}$. On the other hand as long as $\dim Y > 0$ the class $\mathcal{C}$ is always rather large.

Proof of Theorem 4.3. Thanks to Proposition 4.1, every $\eta \in \mathcal{C}$ satisfies $\mu^*\eta = \mu^*\psi + \nu^*u$ for some bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh function u on $\tilde{Y}$. Then using (4.1) we have

$$\begin{aligned}\mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\eta) &= \frac{\Psi^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}\nu^*u + [E] \\ &= \nu^*\left(\frac{q^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}u\right) + [E],\end{aligned}$$

and so if K is any compact subset of $X\backslash V$, since μ is an isomorphism on $\mu^{-1}(K)$, we get

$$\begin{aligned}\int_K (\omega + \sqrt{-1}\partial\overline{\partial}\eta)^n &= \int_{\mu^{-1}(K)} \mu^*(\omega + \sqrt{-1}\partial\overline{\partial}\eta)^n \\ &= \int_{\mu^{-1}(K)} \nu^*\left(\frac{q^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}u\right)^n = 0,\end{aligned}$$

since $\dim \tilde{Y} = \dim Y < \dim X$ by Proposition 4.2. □

5. The pluricomplex Green's function

We can also consider the pluricomplex Green's function with singularity type determined by ψ, namely

$$G = \sup\{u \mid u \in PSH(X,\omega), u \leqslant 0, u \leqslant \psi + O(1)\}^*, \tag{5.1}$$

which is the compact manifold analog of the construction in [31], and has been studied in detail in [18, 30, 32] and references therein. In particular, since ψ has analytic singularities, it follows from [31, 32] that $G \in \mathcal{C}$.

Thanks to Proposition 4.1 we can write

$$\mu^*G = \mu^*\psi + \nu^*F, \tag{5.2}$$

for a bounded weakly $\frac{q^*\omega_{FS,p}}{m}$-psh function F on $\tilde{Y}$. The function F is itself given by a suitable envelope.

Proposition 5.1. *The pluricomplex Green's function G satisfies (5.2) where F is the envelope on $\tilde{Y}$ given by*

$$F = \sup\{w \mid w \in PSH(\tilde{Y}, q^*\omega_{FS,p}/m), w \leqslant -\nu_*\mu^*\psi\}^*, \tag{5.3}$$

and where we are writing

$$\nu_*(f)(y) = \sup_{x\in\nu^{-1}(y)} f(x),$$

for any function f on $\tilde{X}, y \in \tilde{Y}$.

In other words, F is given by a quasi-psh envelope with obstacle $-\nu_*\mu^*\psi$ on $\tilde{Y}$.

Proof. Write $E = \sum_i \lambda_i E_i$ for E_i prime divisors and $\lambda_i \in \mathbb{R}_{>0}$, and for each i fix a defining section s_i of $\mathcal{O}(E_i)$ and a smooth metric h_i on $\mathcal{O}(E_i)$ with curvature form R_i. For brevity, we will write $|s|_h^2 = \prod_i |s_i|_{h_i}^{2\lambda_i}$ and $R_h = \sum_i \lambda_i R_i$. Then the Poincaré-Lelong formula gives

$$[E] = \sqrt{-1}\partial\overline{\partial}\log|s|_h^2 + R_h,$$

and we obtain that $\mu^*\omega - R_h$ is cohomologous to $\frac{\Psi^*\omega_{FS,p}}{m}$ and

$$\mu^*\omega - R_h = \frac{\Psi^*\omega_{FS,p}}{m} + \sqrt{-1}\partial\overline{\partial}(\log|s|_h^2 - \mu^*\psi),$$

and $\mu^*\psi - \log|s|_h^2$ is smooth on all of $\tilde{X}$. Note that if we denote by

$$\tilde{G} = \sup\{u \mid u \in PSH(\tilde{X}, \mu^*\omega), u \leqslant 0, u \leqslant \log|s|_h^2 + O(1)\}^*,$$

then we have that $\tilde{G} = \mu^* G$ (this is again because every $\mu^*\omega$-psh function on $\tilde{X}$ is in fact the pullback of an ω-psh function on X).

As in [28], we use a trick from [8, Section 4] (see also [31]), to show that

$$\tilde{G} = \log|s|_h^2 + \sup\{v \mid v \in PSH(\tilde{X}, \mu^*\omega - R_h), v \leqslant -\log|s|_h^2\}^*.$$

For the reader's convenience, we supply the simple proof. Denote the right hand side by $\hat{G}$. For one direction, if v is $(\mu^*\omega - R_h)$-psh and satisfies $v \leqslant -\log|s|_h^2$, then $u := v + \log|s|_h^2$ satisfies $u \leqslant 0$ but also since $v \leqslant C$ on $\tilde{X}$, we see that $u \leqslant \log|s|_h^2 + C$, and also

$$\begin{aligned}\mu^*\omega + \sqrt{-1}\partial\overline{\partial}u &= \mu^*\omega + \sqrt{-1}\partial\overline{\partial}\log|s|_h^2 + \sqrt{-1}\partial\overline{\partial}v \\ &= \mu^*\omega - R_h + [E] + \sqrt{-1}\partial\overline{\partial}v \\ &\geqslant \mu^*\omega - R_h + \sqrt{-1}\partial\overline{\partial}v \geqslant 0,\end{aligned}$$

and so $\hat{G} \leqslant \tilde{G}$. Conversely, if u is $\mu^*\omega$-psh and satisfies $u \leqslant 0$ and $u \leqslant \log|s|_h^2 + C$ for some C, then the Siu decomposition of $\mu^*\omega + \sqrt{-1}\partial\overline{\partial}u$ contains $[E]$ and so

$$0 \leqslant \mu^*\omega + \sqrt{-1}\partial\overline{\partial}u - [E] = \mu^*\omega - R_h + \sqrt{-1}\partial\overline{\partial}(u - \log|s|_h^2),$$

and so $v := u - \log|s|_h^2$ is $(\mu^*\omega - R_h)$-psh and satisfies $v \leqslant -\log|s|_h^2$, and it follows that $\tilde{G} \leqslant \hat{G}$, which proves our claim.

But finally note that for all $x \in \tilde{X}$ we have

$$\begin{aligned}&\log|s|_h^2(x) + \sup\{v(x) \mid v \in PSH(\tilde{X}, \mu^*\omega - R_h), v \leqslant -\log|s|_h^2\} \\ &= \mu^*\psi(x) + \sup\{v(x) \mid v \in PSH(\tilde{X}, \Psi^*\omega_{FS,p}/m), v \leqslant -\mu^*\psi\} \\ &= \mu^*\psi(x) + \sup\{w(\nu(x)) \mid w \in PSH(\tilde{Y}, q^*\omega_{FS,p}/m), w \leqslant -\nu_*\mu^*\psi\}\end{aligned}$$

and taking the upper-semicontinuous regularization and using the claim above gives $\mu^*G = \mu^*\psi + \nu^*F$, which completes the proof. □

Using Proposition 5.1 we can see that F is continuous on a Zariski open subset of $\tilde{Y}$, using the following argument. Let $g : Y' \to \tilde{Y}$ be a resolution of the singularities of $\tilde{Y}$. Then we have:

$$g^*F = \sup\{w \mid w \in PSH(Y', g^*q^*\omega_{FS,p}/m), w \leqslant -g^*\nu_*\mu^*\psi\}^*.$$

Note that $g^*q^*\omega_{FS,p}/m$ is semi-positive and big, and that $-g^*\nu_*\mu^*\psi$ is continuous off of $g^{-1}(\nu(\mu^{-1}(\psi^{-1}(-\infty))))$, where it is unbounded. Using the trick in [28], we can replace the obstacle $-g^*\nu_*\mu^*\psi$ with a globally continuous obstacle h without changing g^*F. Now, approximate h uniformly by smooth functions h_j. It is easy to see that the envelopes:

$$F_j := \sup\{w \mid w \in PSH(Y', g^*q^*\omega_{FS,p}/m), w \leqslant h_j\}^*.$$

converge uniformly to g^*F. But then by [8], the F_j are continuous away from the non-Kähler locus of $g^*q^*\omega_{FS,p}/m$ (a proper Zariski closed subset, see e.g. [12]), so we are done.

Remark 5.2. One is naturally led to wonder about what the optimal regularity of G is. The sharp $C^{1,1}$ regularity (on a Zariski open subset) of envelopes of the form (5.3) has been recently obtained in [17, 45] in Kähler classes and in [16] in nef and big classes (see also [7, 8, 9]) when the obstacle is smooth (or at least $C^{1,1}$), but in our case the regularity of $-\nu_*\mu^*\psi$ does not seem to be very good, especially near the points where ν is not a submersion.

On the other hand, the first-named author [28] has very recently obtained $C^{1,1}$ regularity (on a Zariski open subset) of envelopes with prescribed analytic singularities, which include those of the form (5.1), generalizing results in [32] in the case of line bundles. In our situation, the results of [28, 32] do not apply since in (5.1) the functions u and ψ are both ω-psh (while for these results one would need them to be quasi-psh with respect to two different $(1,1)$-forms such that the cohomology class of their difference is big). Moreover, the main result of [28] also allows for u and ψ being both ω-psh, but then needs the condition that the total mass of the non-pluripolar Monge-Ampère operator of ψ be strictly positive. This is obviously not the case in our situation however, by Theorem 4.3.

Remark 5.3. One possibly interesting approach to studying higher regularity of functions $v \in \mathcal{C}$ which are already continuous on $X \setminus V$ is the following. Suppose $\sup_X v = 0$. Fix an $M > 0$ and let Ω be the open set $\Omega := \{v < -M\}$. Then one can easily show using the comparison principle and Theorem 4.3 that we have:

$$\max\{v, -M\} = V_\Omega - M,$$

where here V_Ω is the global (Siciak) extremal function for Ω. In particular, one sees that Ω is regular. There is then a well-developed theory about Hölder continuous regularity for such functions (the so called HCP property), see e.g. [34]. It may be possible to use this theory to study G, if one can first show that it is continuous in at least a neighborhood of V. Another possibility may be to study regularity of the boundary of Ω – see the very end of [28].

Remark 5.4. On can also naturally ask whether the function ρ (and therefore also its singularity type ψ) in Theorem 1.1 is actually independent of the choice of subsequence t_i, and also how regular ρ is on $X\setminus V$. Our guess is that ρ is indeed uniquely determined, and is smooth on $X\setminus V$. These properties would both follow if one could show that the map $\Phi : X \dashrightarrow Y$ is independent of the chosen subsequence, and that the corresponding function u on $\tilde{Y}$ given by Lemma 3.1 and Proposition 4.1 which satisfies

$$\mu^*\rho = \mu^*\psi + \nu^* u,$$

actually solves a suitable complex Monge-Ampère equation on $\tilde{Y}$. In a related setting of Calabi-Yau manifolds fibered over lower-dimensional spaces, such a limiting equation after collapsing the fibers was obtained by the second-named author in [42, Theorem 4.1].

Remark 5.5. Lastly, we can also ask whether the limit ρ (if it is unique) is necessarily equal to the pluricomplex Green's function G up to addition of a constant. By remark 4.4 this is the case if the rational map Φ is constant, so that Y is a point. In general though this seems rather likely false.

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