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Coincidence of two Swan conductors of abelian characters

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Abstract. There are two ways to define the Swan conductor of an abelian character of the absolute Galois group of a complete discrete valuation field. We prove that these two Swan conductors coincide.

Keywords. Local field, ramification groups, Swan conductor

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[Français]

Titre. **Coïncidence de deux conducteurs de Swan des caractères abéliens**

Résumé. Il y a deux façons de définir le conducteur de Swan d'un caractère abélien du groupe de Galois absolu d'un corps de valuation discrète complet. Nous montrons que ces deux conducteurs de Swan coïncident.

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Contents

1.   Introduction

1.1.   Let K be a complete discrete valuation field, let $\bar{K}$ be a separable closure of K, and let us consider $\chi \colon \mathrm{Gal}(\bar{K}/K) \to \mathbb{C}^\times$ a homomorphism which factors through $\mathrm{Gal}(L/K)$ for a finite cyclic extension $L \subset \bar{K}$ of K. There are two definitions of the Swan conductor of χ, one is defined by using the logarithmic upper ramification filtration on $\mathrm{Gal}(\bar{K}/K)$ defined geometrically [1], and the other is defined by using the filtrations on the unit groups of complete discrete valuation fields and cup products in Galois cohomology [6].

We prove that the two Swan conductors coincide.

1.2.   We briefly review the two Swan conductors, which we denote in this paper by $\mathrm{Sw}(\chi)$ and $\mathrm{Sw}^{\mathrm{ab}}(\chi)$, respectively.

$\mathrm{Sw}(\chi)$ is defined as follows.   There is a decreasing filtration $\mathrm{Gal}(\bar{K}/K)^t_{\log}$ indexed by $t \in \mathbb{Q}_{\geq 0}$ on $\mathrm{Gal}(\bar{K}/K)$ by closed normal subgroups called the logarithmic upper ramification groups. $\mathrm{Sw}(\chi)$ is defined to be the smallest $t \in \mathbb{Q}_{\geq 0}$ such that $\chi(\mathrm{Gal}(\bar{K}/K)^s_{\log}) = \{1\}$ for all $s > t$ (such t exists). See [1, Theorem 3.16] and Section 4.

$\mathrm{Sw}^{\mathrm{ab}}(\chi)$ is defined as follows.   Fix an injection $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$, say $r \mapsto \exp(2\pi\sqrt{-1}r)$, and identify χ with an element of $H^1(K, \mathbb{Q}/\mathbb{Z}) = H^2(K, \mathbb{Z})$.   Then the cup-product with χ defines a homomorphism $K^\times \to \mathrm{Br}(K)$, denoted by $a \mapsto \{\chi, a\}$, where $\mathrm{Br}(K) = H^2(K, \mathbb{G}_m)$ is the Brauer group of K [13, Chapitre X.4]. $\mathrm{Sw}^{\mathrm{ab}}(\chi)$ is defined to be the smallest integer $n \geq 0$ such that $\{\chi, 1 + \mathfrak{m}_K^n \mathfrak{m}_{K'}\} = 0$ in $\mathrm{Br}(K')$ for any extension $K \to K'$ of complete discrete valuation fields (such n exists; here $\mathfrak{m}_*$ denote the maximal ideals). See [7].

Theorem 1.3. $\mathrm{Sw}(\chi) = \mathrm{Sw}^{\mathrm{ab}}(\chi)$.

1.4.   We will also prove the coincidence of refined Swan conductors. Let $r = \mathrm{Sw}(\chi) = \mathrm{Sw}^{\mathrm{ab}}(\chi)$ and assume $r > 0$. Then we have non-zero elements

$$\mathrm{rsw}(\chi) \in \bar{F} \otimes_F \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log),$$

$$\mathrm{rsw}^{\mathrm{ab}}(\chi) \in \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log)$$

called the (logarithmic) refined Swan conductors. $\mathrm{rsw}(\chi)$ is defined in [10] as a refined version of $\mathrm{Sw}(\chi)$ and $\mathrm{rsw}^{\mathrm{ab}}(\chi)$ is defined in [7] as a refined version of $\mathrm{Sw}^{\mathrm{ab}}(\chi)$. The definition of $\mathrm{rsw}(\chi)$ is recalled in (4.16).

Theorem 1.5. $\mathrm{rsw}(\chi) = \mathrm{rsw}^{\mathrm{ab}}(\chi)$.

1.6. Theorem 1.3 was known, for example, in the following cases. (The cases (i) and (ii) follow from Section 6.1 of [1] and from [6]. The case (iii) is shown in [2, Corollary 9.12].) The case (ii) is proved by comparing Propositions 4.11 and 5.2. In the cases (i) and (ii), the ring $\mathcal{O}_L$ is generated by a single element over $\mathcal{O}_K$ and such an extension is also studied in [14].

(i) (The classical case.) The case where the residue field of K is perfect.

(ii) The case where the residue field F of K is of characteristic p such that $[F : F^p] = p$ and χ factors through $\mathrm{Gal}(L/K)$ for a finite cyclic extension L/K whose ramification index $e(L/K)$ is one.

(iii) The case where K is of positive characteristic.

In the positive characteristic case, the results corresponding to Theorems 1.3 and 1.5 in the non-logarithmic case are proved in [2] and [15].

1.7. Our method to prove Theorem 1.3 is to reduce it to the above case (ii) (not to the classical case (i)). In Theorem 3.1, we prove that for a finite cyclic extension L/K such that χ factors through $\mathrm{Gal}(L/K)$, we can find an extension of complete discrete valuation fields $K \to K'$ such that $\mathrm{Sw}(\chi_{K'}) = e(K'/K)\mathrm{Sw}(\chi)$, $\mathrm{Sw}^{\mathrm{ab}}(\chi_{K'}) = e(K'/K)\mathrm{Sw}^{\mathrm{ab}}(\chi)$, the residue field F' of K' satisfies $[F' : (F')^p] = p$, and $e(LK'/K') = 1$. The refined Swan conductors play important roles to find the field K' above.

Theorem 1.5 is proved also by the reduction to the case (ii). The authors would like to thank an anonymous referee for pointing out that almost the same result as the key step Proposition 4.11 is proved in [5, Theorem 5.9].

2. On the theorem of Epp

The following theorem is not explicitly written in the paper [4] of Epp, but the arguments there (with a correction in [8] of an error in [4]) actually prove this.

Theorem 2.1. *Let K be a complete discrete valuation field whose residue field F is of characteristic $p > 0$, and let L be a finite Galois extension of K. Then there exists a finite extension K' of K satisfying the following conditions* (i) *and* (ii).

(i) $e(LK'/K') = 1$.

(ii) *The residue field of K' is a separable extension of that of K.*

In Theorem 2.1, we may take K' separable over K, although we will not use this fact. To see this, it suffices to modify the construction of $K' = K(\pi')$ in the proof of the case where K is of characteristic $p > 0$ and T is not empty in 2.6.

We use the following lemmas 2.2, 2.3 and 2.4 for the proof of Theorem 2.1.

For a discrete valuation field K, let ord_K be the normalized additive valuation of K. In the case the residue field of K is of characteristic $p > 0$, let $e_K = \mathrm{ord}_K(p)$. (So, $e_K = \infty$ if K is of characteristic p.)

Lemma 2.2. *Let K be a complete discrete valuation field whose residue field F is of characteristic $p > 0$. Let $k = \bigcap_{r \geq 0} F^{p^r}$ be the largest perfect subfield of F and let $W(k) \to \mathcal{O}_K$ be the canonical morphism from the ring of Witt vectors. Then the subring $\bigcap_{r \geq 0}(\mathcal{O}_K/p\mathcal{O}_K)^{p^r} \subset \mathcal{O}_K/p\mathcal{O}_K$ equals the image of $k \to \mathcal{O}_K/p\mathcal{O}_K$ (in the case K is of characteristic p, this means that $\bigcap_{r \geq 0}(\mathcal{O}_K)^{p^r} = k$).*

Proof. Let $A = \bigcap_{r \geq 0}(\mathcal{O}_K/p\mathcal{O}_K)^{p^r}$ denote the subring. Then, $A \subset \mathcal{O}_K/p\mathcal{O}_K$ contains the image of k and the image of A by $\mathcal{O}_K/p\mathcal{O}_K \to F$ is a subring of k. Hence, the assertion follows from $A \cap (\mathfrak{m}_K/p\mathcal{O}_K) = 0$. $\square$

We do not give proofs of the following lemmas 2.3 and 2.4 which are straightforward.

Lemma 2.3. *Let K be a complete discrete valuation field of characteristic $p > 0$ and let F be its residue field. Consider the Artin-Schreier extension $L = K(\alpha)$, $\alpha^p - \alpha = f \in K$. Let π be a prime element of K. Let E be the residue field of L.*

(1) *If $f \in \mathcal{O}_K$, the extension L/K is unramified, possibly trivial.*

(2) *Assume that $-\mathrm{ord}_K f = n \geq 1$ is not divisible by p. Then $e(L/K) = p$ and $E = F$.*

(3) *Assume that $f \in u\pi^{-mp} + \pi^{-mp+1}\mathcal{O}_K$ for some integer $m \geq 1$ and for some $u \in \mathcal{O}_K$ whose residue class $\bar{u}$ does not belong to F^p. Then $E = F(\bar{u}^{1/p})$ and $e(L/K) = 1$.*

Lemma 2.4. *Let K be a complete discrete valuation field of mixed characteristic $(0, p)$. Let F be its residue field. Assume that K contains a primitive p-th root ζ_p of 1. Consider the Kummer extension $L = K(\alpha)$, $\alpha^p = a \in K^\times$. Let π be a prime element of K. Let E be the residue field of L.*

(1) *If $a \in 1 + (\zeta_p - 1)^p\mathcal{O}_K$, the extension L/K is unramified, possibly trivial.*

(2) *Assume that $\mathrm{ord}_K(a)$ is not divisible by p. Then $e(L/K) = p$ and $E = F$.*

(3) *Assume that $a \in (\mathcal{O}_K)^\times$ and that the residue class $\bar{a}$ of a is not contained in F^p. Then $E = F(\bar{a}^{1/p})$ and $e(L/K) = 1$.*

(4) *Assume that $a \in (1 + \pi^n u)(1 + \pi^{n+1}\mathcal{O}_K)$ for some integer n not divisible by p such that $1 \leq n < e_K p/(p-1)$ and for some $u \in (\mathcal{O}_K)^\times$. Then $e(L/K) = p$ and $E = F$.*

(5) *Assume that $a \in (1 + \pi^{mp}u)(1 + \pi^{n+1}\mathcal{O}_K)$ for some integer m such that $1 \leq n = mp < e_K p/(p-1)$ and for some $u \in \mathcal{O}_K$ whose residue class $\bar{u}$ does not belong to F^p. Then $E = F(\bar{u}^{1/p})$ and $e(L/K) = 1$.*

2.5. We start the proof of Theorem 2.1.

First, we reduce the theorem to the case (*) below. Let $K_1 \subset L$ be the maximum tamely ramified extension of K. Then, since $LK_1 = L$ and the residue field of K_1 is a separable extension of that of K, we may assume that $\mathrm{Gal}(L/K)$ equals the inertia subgroup I and is a p-group.

We prove the reduction by induction on the order of I. We may assume that $L \neq K$. Then, since $\mathrm{Gal}(L/K)$ is nilpotent, there exists a subextension $L' \subset L$ such that L' is a Galois extension over K and that L is a cyclic extension of L' of degree p. By induction hypothesis, there exists a finite extension K_1' of K such that $e(L'K_1'/K_1') = 1$ and satisfying (ii). If $e(LK_1'/L'K_1') = 1$, there is nothing to prove. Otherwise, for the maximum unramified extension K_2' of K_1' inside $M_1' = L'K_1'$, the extensions $K_2' \subset M_1' \subset LK_1'$ satisfies the condition (*).

(*) There exists a field M such that $K \subset M \subset L$, $e(M/K) = 1$ and that L is a cyclic extension of M of degree p and $e(L/M) = p$. The residue field E of M is a purely inseparable extension of the residue field F of K.

2.6. We prove Theorem 2.1 in the case K is of characteristic p. Let M be as in (*) in 2.5. We may assume $M = E((\pi))$ with π a prime element of K. We can write $L = M(\alpha)$ where $\alpha^p - \alpha = f = f_I + f_U$ with $f_I, f_U \in M$ such that: $f_I = \sum_{n \in I} a_n \pi^{-n}$ where I is a finite subset of $\mathbb{Z}_{>0}$ and $a_n \in E^\times$, and $f_U \in \mathcal{O}_M$. By Lemma 2.3 (1) applied to the extension L/M, I is not empty because L/M is not unramified. .

In the following, we use the fact that for $u, v \in M$ such that $u \equiv v \mod \{w^p - w \mid w \in M\}$, the extension $M(\beta)$, $\beta^p - \beta = u$, of M is the same as that given by v. If $n \in I$ is divisible by p and $a_n \in E^p$, we have $a_n\pi^{-n} \equiv a_n^{1/p}\pi^{-n/p} \mod \{w^p - w \mid w \in M\}$ and hence we can replace $a_n\pi^{-n}$ by $a_n^{1/p}\pi^{-n/p}$. Hence we may (and do) assume that if $n \in I$ is divisible by p, then $a_n \notin E^p$.

Let S be the subset of I consisting of all $n \in I$ such that $a_n \in \bigcap_{r \geq 0} E^{p^r} = \bigcap_{r \geq 0} F^{p^r} = k$, and let $T = I \setminus S$. Note that if $n \in S$, then n is not divisible by p. By Lemma 2.2, we have $a_n \in k \subset \mathcal{O}_K$ for $n \in S$. Hence $f_S \in K$.

Assume first T is empty. Then $f_I = f_S \in K$. For $K' = K(\alpha_S)$ with $\alpha_S^p - \alpha_S = f_S$, the residue field of K' coincides with F by Lemma 2.3 (2) applied to K'/K, and the extension LK'/MK' is unramified by Lemma 2.3 (1) applied to LK'/MK'.

Assume that T is not empty. For $n \in T$, write $a_n = b_n^{p^{r(n)}}$ where $b_n \in E$, $r(n) \geq 0$, and b_n is not a p-th power in E. Take an integer m such that $m > r(n)$ for any $n \in T$. For $n \in S$, write $a_n = b_n^{p^m}$ ($b_n \in k \subset \mathcal{O}_K$).

Let $K' = K(\pi')$ where π' is a p^m-th root of π and let $M' = MK'$, $L' = LK'$. Then

$$f_I \equiv f_S + f_T \mod \{w^p - w \mid w \in M'\},$$

$$f_S = \sum_{n \in S} b_n (\pi')^{-n}, \quad f_T = \sum_{n \in T} b_n (\pi')^{-np^{m-r(n)}}.$$

Note that $f_S \in k((\pi')) \subset K'$ by Lemma 2.2. Let $n_S := \max(S)$ and $n_T := \max\{np^{m-r(n)} \mid n \in T\}$. If S is empty, we set $n_S = 1$ so that we have $n_S < n_T$. Since n_S is not divisible by p and n_T is divisible by p, we have $n_S \neq n_T$. For the proof of Theorem 2.1, it is sufficient to prove the following Claim 1 and Claim 2.

Claim 1. If $n_S < n_T$, then $e(L'/M') = 1$.

Claim 2. If $n_S > n_T$, let $K'' = K'(\beta)$ where $\beta^p - \beta = f_S$ and let $M'' = MK''$, $L'' = LK''$. Then the residue field of K'' coincides with that of K and $e(L''/M'') = 1$.

We first prove

Claim 3. There is a unique $n \in T$ such that $np^{m-r(n)} = n_T$.

We prove Claim 3. If $n, n' \in T$, $n > n'$ and $np^{m-r(n)} = n'p^{m-r(n')}$, then by $n = n'p^{r(n)-r(n')} > n'$, we have $p \mid n$. Hence $a_n \notin E^p$ and $r(n) = 0$. This contradicts to $r(n) > r(n')$.

Claim 1 follows from Claim 3 and Lemma 2.3 (3) applied to the extension L'/M'.

We prove Claim 2. We have $e(K''/K') = p$ by Lemma 2.3 (2) applied to K''/K'. If τ denotes a prime element of K'', the residue class of the unit $\tau^p(\pi')^{-1}$ is a p-th power. Claim 2 follows from this and Claim 3, and from Lemma 2.3 (3) applied to the extension L''/M''.

2.7. We prove Theorem 2.1 in the case K is of mixed characteristic $(0, p)$. We may assume that K contains a primitive p-th root ζ_p of 1. Note that $\mathrm{ord}_K(\zeta_p - 1) = e_K/(p - 1)$. Let M be as in (*) in 2.5. We have $L = M(\alpha)$, $\alpha^p = a$ for $a \in M^\times$.

The proof consists of two steps. In Step 1, we show that we may assume $a \in 1 + p\mathcal{O}_M$. In Step 2, we give the proof assuming $a \in 1 + p\mathcal{O}_M$.

Let E be the residue field of M and take a ring homomorphism $E \to \mathcal{O}_M/p\mathcal{O}_M$ such that the induced map $E \to \mathcal{O}_M/\mathfrak{m}_M = E$ is the identity map, and its lifting $\iota \colon E \to \mathcal{O}_M$. Let π be a prime element of K.

Step 1. Write $a \equiv c \prod_{n \in T} c_n \mod 1 + p\mathcal{O}_M$ where T is a subset of $\{0, \ldots, e_M - 1\}$ and c_n ($n \in T$) and c are elements of $M^\times$ of the following form. If $0 \in T$, $c_0 = \iota(b)$ for some $b \in E$ such that $b \notin E^p$. If $n \in T$ and $n \geq 1$, $c_n = 1 + \pi^n \iota(b)$ for some $b \in E$ such that $b \notin E^p$. The first term c is a product of a power of π and elements of the form $1 + \pi^m \iota(b)$ with $b \in E^p$ for some integer $m \geq 0$.

Let $K' = K(\pi^{1/p})$, $M' = MK'$, $L' = LK'$. Then we have $c \in ((M')^\times)^p(1 + p\mathcal{O}_{M'})$ since the map $x \mapsto x^p$ on $\mathcal{O}_{M'}/p\mathcal{O}_{M'}$ is a ring homomorphism. Hence if $T \neq \emptyset$, we have $e(L'/M') = 1$ by Lemma 2.4 (3) and (5) applied to the extension L'/M'. If $T = \emptyset$, we also have $L' = M'(\beta)$ with $\beta^p \in 1 + p\mathcal{O}_{M'}$. Thus, the assertion is reduced to the case where $L = M(\alpha)$, $\alpha^p = a$ for $a \in 1 + p\mathcal{O}_M$.

Step 2. Assume $L = M(\alpha)$, $\alpha^p = a \in 1 + p\mathcal{O}_M$. We have an isomorphism

$$(\zeta_p - 1)^{-1}\mathcal{O}_M/\mathcal{O}_M \to (1 + p\mathcal{O}_M)/((1 + (\zeta_p - 1)^p\mathcal{O}_M) ; \ x \mapsto 1 + (\zeta_p - 1)^p x$$

(from the additive group to the multiplicative group). This isomorphism maps $x^p - x$ for $x \in (\zeta_p - 1)^{-1}\mathcal{O}_M$ such that $x^p \in (\zeta_p - 1)^{-1}\mathcal{O}_M$ to a p-th power because

$$(1 + (\zeta_p - 1)x)^p \equiv 1 + (\zeta_p - 1)^p(x^p - x) \bmod 1 + (\zeta_p - 1)^p\mathcal{O}_M.$$

Hence we have a situation similar to the theory of Artin-Schreier extension, and the rest of the proof, which is given below, is similar to the proof of the case where K is of characteristic p in 2.6.

We have $a = a_I a_U$ with $a_I = 1 + \sum_{n \in I}((\zeta_p - 1)^p \pi^{-n})\iota(a_n)$ where I is a subset of $\{n \in \mathbb{Z} \mid 1 \le n \le e_M/(p-1)\}$ and $a_n \in E^\times$ and $a_U \in 1 + (\zeta_p - 1)^p\mathcal{O}_M$. Note that we have $(\zeta_p - 1)^p \pi^{-n} \in p\mathcal{O}_M$ for $n \in I$. By Lemma 2.4 (1) applied to the extension L/M, I is not empty. We may assume that if $n \in I$ and n is divisible by p, then a_n is not a p-th power in E. Let $k = \bigcap_{r \ge 0} F^{p^r} = \bigcap_{r \ge 0} E^{p^r}$, $S = \{n \in I \mid a_n \in k\}$ and let $T = I \smallsetminus S$.

If T is empty, by Lemma 2.2, we have $a_I \equiv c \bmod (M^\times)^p$ where $c = 1 + \sum_{n \in S}((\zeta_p - 1)^p \pi^{-n})[a_n]$ for the Teichmüller lifting $[a_n] \in W(k)^\times \subset \mathcal{O}_K^\times$. Let $K' = K(c^{1/p})$. Then the residue field of K' is F by Lemma 2.4 (4) applied to the extension K'/K and the residue field of K' is the same as that of K, and the extension LK'/MK' is unramified by Lemma 2.4 (1) applied to LK'/MK'.

Assume now that T is not empty. For $n \in T$, define $b_n \in E \smallsetminus E^p$ and $r(n) \ge 0$ as in 2.6. Further take an integer m such that $m > r(n)$ for any $n \in T$ and $b_n \in k$ for $n \in S$ as in 2.6.

Let $K' = K(\pi')$ where π' is a p^m-th root of π and let $M' = MK'$, $L' = LK'$ Then by Lemma 2.2,

$$a_I \equiv a_S a_T \bmod (1 + (\zeta_p - 1)^p\mathcal{O}_{M'}),$$

$$K' \ni a_S = 1 + \sum_{n \in S}((\zeta_p - 1)^p(\pi')^{-n})[b_n], \quad a_T = 1 + \sum_{n \in T}((\zeta_p - 1)^p(\pi')^{-np^{m-r(n)}})\iota(b_n)$$

where $[b_n] \in W(k)^\times \subset \mathcal{O}_K^\times$ for $n \in S$ is the Teichmüller lifting of b_n. Let $n_T := \max\{np^{m-r(n)} \mid n \in T\}$ and $n_S := \max(S)$. Since n_S is not divisible by p and n_T is divisible by p, we have $n_S \ne n_T$. For the proof of Theorem 2.1, it is sufficient to prove the following Claim 1 and Claim 2.

Claim 1. If $n_S < n_T$, then $e(L'/M') = 1$.

Claim 2. If $n_S > n_T$, let $K'' = K'(a_S^{1/p})$. Then the residue field of K'' coincides with that of K and $e(L''/M'') = 1$ where $L'' = LK''$ and $M'' = MK''$.

We first prove

Claim 3. There is a unique $n \in T$ such that $np^{m-r(n)} = n_T$.

The proof of Claim 3 is similar to that of Claim 3 in 2.6. Claim 1 follows from Claim 3 and Lemma 2.4 (5) applied to the extension L'/M'. We prove Claim 2. The residue field of K'' is F by Lemma 2.4 (4) applied to the extension K''/K' and we have $e(K''/K') = p$. If τ denotes a prime element of K'', the residue class of the unit $\tau^p(\pi')^{-1}$ is a p-th power. Claim 2 follows from this and Claim 3 and from Lemma 2.4 (5) applied to the extension L''/M''.

3. Some extensions of complete discrete valuation fields

Theorem 3.1. *Let K be a complete discrete valuation field whose residue field F is of characteristic $p > 0$. Let L/K be a finite Galois extension. Then there is an extension K'/K of complete discrete valuation fields satisfying the following conditions* (i)–(iii). *Let F' be the residue field of K'.*

(i) $e(LK'/K') = 1$.

(ii) $[F' : (F')^p] = p$.

(iii) *The map $\Omega^1_F(\log) \to \Omega^1_{F'}(\log)$ is injective* (*here and in the following $\Omega^1_F(\log) = F \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log)$*).

If F is finitely generated over a perfect subfield k, we can replace (ii) *by the following stronger condition* (ii)'.

(ii)' *There is a perfect subfield k' of F' such that F' is finitely generated and of transcendence degree 1 over k'.*

We will deduce Theorem 3.1 from Theorem 2.1 and the following Propositions 3.2 and 3.3.

Proposition 3.2. *Let K be a complete discrete valuation field whose residue field F is of characteristic $p > 0$. Let π be a prime element of K, let $\mathcal{O}_{K'}$ be the completion of the discrete valuation ring which is the local ring of $\mathcal{O}_K[T, U^{\pm 1}]/(UT^p - \pi)$ at the prime ideal generated by T, let K' be the field of fractions of $\mathcal{O}_{K'}$, and let F' be the residue field of K'. Then we have:*

(i) *The map $\Omega^1_F(\log) \to \Omega^1_{F'}(\log)$ is injective.*

(ii) *The image of this map is contained in $\Omega^1_{F'}$.*

Proof. Straightforward.   □

Proposition 3.3. *Let K be a complete discrete valuation field whose residue field F is of characteristic $p > 0$. Then there is an extension $K \to K'$ of complete discrete valuation fields satisfying the following conditions* (i)–(iii). *Let F' be the residue field of K'.*

(i) $e(K'/K) = 1$.

(ii) $[F' : (F')^p] = p$.

(iii) *The map $\Omega^1_F \to \Omega^1_{F'}$ is injective.*

If F is finitely generated over a perfect field k, we can replace (ii) *by the following stronger condition* (ii)'.

(ii)' *There is a perfect subfield k' of F' such that F' is finitely generated and of transcendence degree 1 over k'.*

Proof. Let $(b_i)_{i \in I}$ be a lifting of a p-base of F to $\mathcal{O}_K$. Let $A_0 = \mathcal{O}_K[T_i, U ; i \in I]$ be the polynomial ring and set $S_i = b_i - UT_i \in A_0$. The residue field F_0 of A_0 at the prime ideal $\mathfrak{p}_0$ generated by $\mathfrak{m}_K$ is $F_0 = F(T_i, U ; i \in I)$ and $(S_i, T_i, U ; i \in I)$ is a p-base. For integers $n \geq 0$, writing $T_i = T_{i,0}$ and $S_i = S_{i,0}$, define $A_{n+1} = A_n[T_{i,n+1}, S_{i,n+1} ; i \in I]/(T^p_{i,n+1} - T_{i,n}, S^p_{i,n+1} - S_{i,n} ; i \in I)$ inductively and $A = \varinjlim_n A_n$. Then A is an integral domain, the ideal $\mathfrak{p}$ of A generated by $\mathfrak{m}_K$ is a prime ideal and the local ring $A_{\mathfrak{p}}$ is a discrete valuation ring. Hence, the residue field F' of A at $\mathfrak{p}$ is the extension of F_0 obtained by adding T^{1/p^n}_i and S^{1/p^n}_i for all i, n. Let $\mathcal{O}_{K'}$ be the completion of the discrete valuation ring $A_{\mathfrak{p}}$ and let K' be the field of fractions of $\mathcal{O}_{K'}$.

Then K' satisfies the conditions (i)–(iii). For (i) and (ii), this is clear. We prove that (iii) is satisfied. The F'-vector space $\Omega^1_{F'}$ is one-dimensional with base dU. The F-vector space Ω^1_F is with base db_i ($i \in I$). In $\Omega^1_{F'}$, we have $db_i = T_i dU$. Since T_i ($i \in I$) are linearly independent over F, we have the injectivity.

Assume that F is finitely generated over a perfect field k. Then I is finite and F is a finite extension of $k(b_i ; i \in I)$. Let $k' \subset F'$ be the extension of the rational function field $k(S_i, T_i ; i \in I) \subset F_0 = F(T_i, U ; i \in I)$ given by $k' := \bigcup_{n \geq 0} k(S^{1/p^n}_i, T^{1/p^n}_i)$. Then k' is perfect and F' is a finite extension of $k'(U)$.   □

3.4.   We prove Theorem 3.1.

Let K_1/K be the extension in Proposition 3.2. By taking K_1 as K in Proposition 3.3, let K_2/K_1 be the extension K'/K of Proposition 3.3. Let K_3/K_2 be the maximal unramified subextension of LK_2/K_2. Then the extension K_3/K satisfies (ii) and (iii) of 3.1 and (ii) of Proposition 3.2.

By applying Theorem 2.1 to LK_3/K_3, we obtain a finite extension K'/K_3 such that $e(LK'/K') = 1$ and the residue field of K' is a separable extension of that of K_3. The extension K'/K has the desired properties. If F is finitely generated over a prime field k, the condition (ii)' is satisfied.

4.   Review and complements on ramification groups

We briefly recall the definition and basic properties of ramification groups. For more detail, we refer to [1, 2, 9, 10, 12]. We introduce the refined logarithmic conductor for a finite Galois extension of a henselian valuation field in (4.15), as a generalization of the refined Swan conductor of an abelian character in the case where the extension is cyclic. We recall the definition of the Swan conductor of an abelian character at the end of Section 4.3. In the case where the residue field is a function field of one variable over a perfect field and the ramification index of the extension is one, we compute explicitly the refined logarithmic conductor in Proposition 4.11 using Lemma 4.6.

4.1.   Let K be a henselian discrete valuation field and F be the residue field of the valuation ring $\mathcal{O}_K$. Let $\bar{K}$ be a separable closure of K and $G_K = \mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group. The residue field $\bar{F}$ of $\bar{K}$ is an algebraic closure of F.

Let L be a finite étale K-algebra and $r > 0$ be a rational number. Let $\mathrm{Spec}\,\mathcal{O}_L \to Q$ be a closed immersion to a smooth scheme Q over $\mathcal{O}_K$. Let $K' \subset \bar{K}$ be a finite extension of K of ramification index e such that er is an integer. Then, we define a dilatation $Q_{\mathcal{O}_{K'}}^{[er]} \to Q_{\mathcal{O}_{K'}} = Q \times_{\mathcal{O}_K} \mathcal{O}_{K'}$ by blowing up the closed subscheme $\mathrm{Spec}\,\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}/\mathfrak{m}_{K'}^{er}$ and by removing the proper transform of the closed fiber. After replacing K' by a separable extension if necessary, the normalization $Q_{\mathcal{O}_{K'}}^{(er)}$ of $Q_{\mathcal{O}_{K'}}^{[er]}$ has geometrically reduced closed fiber and the geometric closed fiber $Q_{\bar{F}}^{(r)} = Q_{\mathcal{O}_{K'}}^{(er)} \times_{\mathcal{O}_{K'}} \bar{F}$ is independent of such K'.

The finite set $F^r(L) = \pi_0(Q_{\bar{F}}^{(r)})$ of connected components is independent of Q. If $F(L) = \mathrm{Mor}_K(L, \bar{K})$, the mapping $F(L) \to F^r(L)$ induced by the canonical morphism $\mathrm{Spec}\,(\mathcal{O}_L \times_{\mathcal{O}_K} \mathcal{O}_{K'})^- \to Q_{\mathcal{O}_{K'}}^{(er)}$ from the normalization is also independent of the choice and is a surjection. We say that the ramification of L over K is bounded by r if the surjection $F(L) \to F^r(L)$ is a bijection. The ramification group $G_K^r \subset G_K = \mathrm{Gal}(\bar{K}/K)$ is defined to be the unique closed normal subgroup such that the surjection $F(L) \to F^r(L)$ induces a bijection $F(L)/G_K^r \to F^r(L)$.

4.2.   A logarithmic variant is defined as follows. Let L be a finite separable extension of K. Let m be an integer divisible by the ramification index $e_{L/K}$ and π be a prime element of K. We define an extension K_m to be the tamely ramified extension $K[t]/(t^m - \pi)$ if m is invertible in F and to be the fraction field of the henselization of $\mathcal{O}_K[u^{\pm 1}, t]/(ut^m - \pi)$ at the prime ideal (t). Then, the finite set $F^{mr}(L \otimes_K K_m)$ is independent of such m and we define $F_{\log}^r(L)$ to be $F^{mr}(L \otimes_K K_m)$. We say that the log ramification of L over K is bounded by r if the surjection $F(L) \to F_{\log}^r(L)$ is a bijection. The ramification group $G_{\log,K}^r \subset G_K$ is defined to be the unique closed normal subgroup such that the surjection $F(L) \to F_{\log}^r(L)$ induces a bijection $F(L)/G_{\log,K}^r \to F_{\log}^r(L)$

Define closed normal subgroups $G_K^{r+} \subset G_K^r$ and $G_{\log,K}^{r+} \subset G_{\log,K}^r$ to be the closures of the unions $\bigcup_{s>r} G_K^s$ and $\bigcup_{s>r} G_{\log,K}^s$ and set $F^{r+}(L) = F(L)/G_K^{r+}$ and $F_{\log}^{r+}(L) = F(L)/G_{K,\log}^{r+}$. We say that the ramification (resp. the log ramification) of L over K is bounded by $r+$ if the surjection $F(L) \to F^{r+}(L)$ (resp. $F(L) \to F_{\log}^{r+}(L)$) is a bijection.

4.3.   We call the largest rational number r such that the ramification (resp. the log ramification) of L over K is not bounded by r the conductor (resp. the logarithmic conductor) of L over K. The conductor (resp. the logarithmic conductor) of L over K is the smallest rational number r such that the ramification (resp. the log ramification) of L over K is bounded by $r+$. The conductor c and the logarithmic conductor $c_{\log}$ satisfies the inequality $c_{\log} \leqq c$. For an extension K' of a henselian discrete valuation field K of ramification index e, the conductor c' and the logarithmic conductor $c'_{\log}$ of a composition field $L' = LK'$ over K' satisfy $c'_{\log} \leqq e \cdot c_{\log}$ and $c' \leqq e \cdot c$. If L is the cyclic extension defined by an abelian character χ of G_K, the Swan conductor $\mathrm{Sw}(\chi)$ is defined as the logarithmic conductor of L over K.

Lemma 4.4. *Assume that the ramification index $e_{L/K}$ is 1. Then, for every rational number $r > 0$, the canonical surjections $F^r_{\log}(L) \to F^r(L)$ and $F^{r+}_{\log}(L) \to F^{r+}(L)$ are bijections.*

Proof. Since we may take $m = 1$, the assertion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.5. Let $Q \to P$ be a quasi-finite and flat morphism of smooth schemes over $\mathcal{O}_K$ and let

$$
\begin{array}{ccc}
Q & \longleftarrow & \operatorname{Spec}\mathcal{O}_L \\
\downarrow & \square & \downarrow \\
P & \longleftarrow & \operatorname{Spec}\mathcal{O}_K
\end{array}
\qquad (4.1)
$$

be a cartesian diagram. Then, by the functoriality of the construction of dilatations, we obtain a finite morphism $Q^{(r)}_{\bar{F}} \to P^{(r)}_{\bar{F}}$ of geometric closed fibers. Define ideals $\mathfrak{m}^{r+}_{\bar{K}} \subset \mathfrak{m}^r_{\bar{K}} \subset \mathcal{O}_{\bar{K}}$ by

$$
\mathfrak{m}^{r+}_{\bar{K}} = \{x \in \bar{K} \mid v(x) > r\} \subset \mathfrak{m}^r_{\bar{K}} = \{x \in \bar{K} \mid v(x) \geqq r\}.
$$

For a k-vector space V, let $\mathbf{V}(V)$ denote the associated covariant scheme $\operatorname{Spec} S^\bullet_k V^\vee$. Then, since $\operatorname{Spec}\mathcal{O}_K \to P$ is a section of a smooth morphism, the conormal sheaf $N_{\operatorname{Spec}\mathcal{O}_K/P}$ is canonically isomorphic to the restriction of $\Omega^1_{P/\mathcal{O}_K}$ and hence the geometric closed fiber $P^{(r)}_{\bar{F}}$ is canonically identified with $\mathbf{V}(\operatorname{Hom}_F(\Omega^1_{P/\mathcal{O}_K} \otimes_{\mathcal{O}_P} F, \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}))$.

The fiber $Q^{(r)}_{\bar{F}} \times_{P^{(r)}_{\bar{F}}} 0$ of the origin is canonically identified with the quotient $F^{r+}(L) = F(L)/G^{r+}_K$. The ramification of L over K is bounded by $r+$ if and only if the finite morphism $Q^{(r)}_{\bar{F}} \to P^{(r)}_{\bar{F}}$ is étale. Assume that L is a Galois extension of K of Galois group G and fix a morphism $L \to \bar{K}$. Let $r = c$ be the conductor of L over K. Then, the connected component $Q^{(r)\circ}_{\bar{F}}$ of $Q^{(r)}_{\bar{F}}$ containing the point corresponding to $L \to \bar{K}$ is a G^r-torsor over $P^{(r)}_{\bar{F}}$. The conductor $r = c$ of L over K is characterized by the condition that the morphism $Q^{(r)\circ}_{\bar{F}} \to P^{(r)}_{\bar{F}}$ is finite étale but is not an isomorphism.

Lemma 4.6. *Assume that $\mathcal{O}_L$ is generated by one element $v \in \mathcal{O}_L$ over $\mathcal{O}_K$, set $d = \operatorname{length}_{\mathcal{O}_L} \Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ and let ord_L be the normalized valuation. Let $v' \neq v$ be a conjugate of v such that the valuation $s = \operatorname{ord}_L(v' - v)$ is the largest.*

1. *The rational number $r = d/[L:K] + s/e_{L/K}$ equals the conductor of L over K.*

2. *Let $\varphi \in \mathcal{O}_K[X]$ be the minimal polynomial of v and define the left vertical arrow of the cartesian diagram*

$$
\begin{array}{ccc}
\mathbf{A}^1_{\mathcal{O}_K} = \operatorname{Spec}\mathcal{O}_K[V] = Q & \longleftarrow & \operatorname{Spec}\mathcal{O}_L \\
{\scriptstyle U \mapsto \varphi(V)}\downarrow & & \downarrow \\
\mathbf{A}^1_{\mathcal{O}_K} = \operatorname{Spec}\mathcal{O}_K[U] = P & \longleftarrow & \operatorname{Spec}\mathcal{O}_K
\end{array}
\qquad (4.2)
$$

   by φ and the bottom horizontal arrow by $U \mapsto 0$. Assume that L is a Galois extension of Galois group G and that $G^r = \langle\sigma\rangle$ is cyclic of order p. Define isomorphisms $\mathbf{F}_p \to G^r$ by σ and $\mathbf{A}^1_{\bar{F}} \to P^{(r)}_{\bar{F}} = \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}$ by $\varphi'(v)(v - \sigma(v))$. Then, there is an isomorphism

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{F}_p & \longrightarrow & \mathbf{A}^1_{\bar{F}} & \xrightarrow{T \mapsto T^p - T} & \mathbf{A}^1_{\bar{F}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^r & \longrightarrow & Q^{(r)\circ}_{\bar{F}} & \longrightarrow & P^{(r)}_{\bar{F}} & \longrightarrow & 0
\end{array}
\qquad (4.3)
$$

   of extensions of smooth group schemes by étale group schemes.

The proof is similar to the computation in [12, Example 3.3.3].

Proof. The left vertical arrow $Q \to P$ in (4.2) is finite flat. Let $v_1, \ldots, v_n \in L$ be the conjugates of v. We fix a numbering so that $v_n = v, v_{n-1} = \sigma(v)$ and $\mathrm{ord}_L(v_i - v_n)$ is increasing. Setting $X - v_n = (v_{n-1} - v_n)T$, we have $\varphi(X) = \prod_{i=1}^n (X - v_i) = \prod_{i=1}^n (v_n - v_i + (v_{n-1} - v_n)T)$. By the assumption that G^r is cyclic of order p, we have $\mathrm{ord}_L(v_i - v_n) < \mathrm{ord}_L(v_{n-1} - v_n)$ for $i \leqq n - p$ and we may assume that $(v_{n-i} - v_n)/(v_{n-1} - v_n) \equiv i \bmod \mathfrak{m}_L$ for $i = 0, \ldots, p - 1$. Hence, we have

$$\varphi(X) \equiv \prod_{i=1}^{n-1} (v_n - v_i) \cdot \prod_{i=1}^{p-1} (1 + iT) \cdot (v_{n-1} - v_n)T = \varphi'(v)(v - \sigma(v))(T^p - T) \tag{4.4}$$

mod $\mathfrak{m}_L^{r+1}$. Thus the assertion 1 follows from the characterization of the conductor at the end of 4.5. The assertion 2 also follows from (4.4). □

Lemma 4.7. *Let P_1, P_2, Q_1, Q_2 be smooth schemes over $\mathcal{O}_K$ and*

$$
\begin{array}{ccc}
Q_i & \longleftarrow & \mathrm{Spec}\, \mathcal{O}_L \\
{\scriptstyle f_i} \downarrow & \square & \downarrow \\
P_i & \longleftarrow & \mathrm{Spec}\, \mathcal{O}_K
\end{array}
\tag{4.5}
$$

for $i = 1, 2$ be cartesian diagrams of schemes over $\mathcal{O}_K$ such that the vertical arrows are quasi-finite and flat. Let

$$
\begin{array}{ccccc}
Q_1 \times_{\mathcal{O}_K} F & \longleftarrow & Q_2 \times_{\mathcal{O}_K} F & \longleftarrow & \mathrm{Spec}\, \mathcal{O}_L \otimes_{\mathcal{O}_K} F \\
{\scriptstyle \bar{f}_1} \downarrow & & {\scriptstyle \bar{f}_2} \downarrow & & \downarrow \\
P_1 \times_{\mathcal{O}_K} F & \longleftarrow & P_2 \times_{\mathcal{O}_K} F & \longleftarrow & \mathrm{Spec}\, F
\end{array}
\tag{4.6}
$$

be a commutative diagram where the right square is induced by (4.5). Then for a rational number $r > 0$, the diagram (4.6) induces a commutative diagram

$$
\begin{array}{ccc}
Q_{1,\bar{F}}^{(r)} & \longleftarrow & Q_{2,\bar{F}}^{(r)} \\
{\scriptstyle \bar{f}_1^{(r)}} \downarrow & & \downarrow {\scriptstyle \bar{f}_2^{(r)}} \\
P_{1,\bar{F}}^{(r)} & \longleftarrow & P_{2,\bar{F}}^{(r)}.
\end{array}
\tag{4.7}
$$

Proof. For $i = 1, 2$, we consider the unions $Q_i' = (Q_i \times_{\mathcal{O}_K} F) \cup \mathrm{Spec}\, \mathcal{O}_L \subset Q_i$ as reduced closed subschemes. Then by the commutative diagram (4.6), the morphism $Q_1 \times_{\mathcal{O}_K} F \to Q_2 \times_{\mathcal{O}_K} F$ and the identity of $\mathrm{Spec}\, \mathcal{O}_L$ define a morphism $Q_1' \leftarrow Q_2'$. Since Q_1 is smooth over $\mathcal{O}_K$, after replacing Q_2 by an étale neighborhood of $\mathrm{Spec}\, \mathcal{O}_L$ if necessary, we may lift $Q_1' \leftarrow Q_2'$ to a morphism $Q_1 \leftarrow Q_2$ over $\mathcal{O}_K$.

The morphism $Q_1 \leftarrow Q_2$ induces a morphism of conormal modules $N_{\mathrm{Spec}\, \mathcal{O}_L/Q_1} \to N_{\mathrm{Spec}\, \mathcal{O}_L/Q_2}$ and defines a commutative diagram

$$
\begin{array}{ccc}
Q_{1,\bar{F}}^{(r)} & \longleftarrow & Q_{2,\bar{F}}^{(r)} \\
\downarrow & & \downarrow \\
\mathbf{V}(\mathrm{Hom}_F(N_{\mathrm{Spec}\, \mathcal{O}_L/Q_1} \otimes_{\mathcal{O}_L} \bar{F}, \mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+})) & \longleftarrow & \mathbf{V}(\mathrm{Hom}_F(N_{\mathrm{Spec}\, \mathcal{O}_L/Q_2} \otimes_{\mathcal{O}_L} \bar{F}, \mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+})).
\end{array}
\tag{4.8}
$$

By the cartesian diagram (4.5), the conormal modules $N_{\mathrm{Spec}\, \mathcal{O}_L/Q_i}$ are the tensor products $N_{\mathrm{Spec}\, \mathcal{O}_K/P_i} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ for $i = 1, 2$. Hence by the commutative diagram (4.6), we may replace $\mathbf{V}(\mathrm{Hom}_F(N_{\mathrm{Spec}\, \mathcal{O}_L/Q_i} \otimes_{\mathcal{O}_L} \bar{F}, \mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}))$ by $P_{i,\bar{F}}^{(r)}$ to get (4.7). □

By slightly enlarging the terminology, we say that the henselization of a local ring of a scheme of finite type is essentially of finite type.

Lemma 4.8 (cf. [2, Lemma 4.4, 4.5]). *Let $\mathcal{O}_K$ be a henselian discrete valuation ring essentially of finite type and flat over $W = W(k)$ for a perfect field k of characteristic $p > 0$.*

1. *There exist a smooth scheme P_0 over W, a divisor $D_0 \subset P_0$ smooth over W, a divisor $X_0 \subset P_0$ flat over over W meeting D_0 transversely and an isomorphism $\mathcal{O}^h_{X_0,\xi} \to \mathcal{O}_K$ over W from the henselization of the local ring at a generic point ξ of the intersection $X_0 \cap D_0$.*

2. *Let L be a finite separable extension of K of ramification index e. Let P_0, D_0, X_0 be as in 1. Further let Q_0 be a smooth scheme over W, $E_0 \subset Q_0$ a smooth divisor over W, $Y_0 \subset Q_0$ a divisor flat over W meeting E_0 transversely and $\mathcal{O}^h_{Y_0,\eta} \to \mathcal{O}_L$ an isomorphism over W. Then, after replacing X_0 and Y_0 by étale neighborhoods of $\operatorname{Spec} \mathcal{O}_K$ and $\operatorname{Spec} \mathcal{O}_L$, there exists a cartesian diagram*

$$
\begin{array}{ccccccc}
e \cdot E_0 & \longrightarrow & Q_0 & \longleftarrow & Y_0 & \longleftarrow & \operatorname{Spec} \mathcal{O}_L \\
\downarrow & & f\downarrow & \square & \downarrow & \square & \downarrow \\
D_0 & \longrightarrow & P_0 & \longleftarrow & X_0 & \longleftarrow & \operatorname{Spec} \mathcal{O}_K
\end{array}
\tag{4.9}
$$

such that the vertical arrows are finite flat.

Proof. 1. Let $u_1, \ldots, u_n \in \mathcal{O}_K$ be liftings of a transcendental basis $\bar{u}_1, \ldots, \bar{u}_n \in F$ over k such that F is a finite separable extension of $k(\bar{u}_1, \ldots, \bar{u}_n)$ and π be a prime element of K. If we set $\mathbf{A}^{n+1}_W = \operatorname{Spec} W[u_1, \ldots, u_n, t]$, the morphism $\mathcal{O}_K \to \mathbf{A}^{n+1}_W$ defined by $u_1, \ldots, u_n, \pi \in \mathcal{O}_K$ is formally unramified. Hence, there exists an étale neighborhood $P_0 \to \mathbf{A}^{n+1}_W$ of the image ξ of the closed point of $\operatorname{Spec} \mathcal{O}_K$, a regular divisor $X_0 \subset P_0$ and an isomorphism $\mathcal{O}^h_{X_0,\xi} \to \mathcal{O}_K$. It suffices to define D_0 by t.

2. Take a function on P_0 defining D_0 and take an étale morphism $P_0 \to \mathbf{A}^{n+1}_W = \operatorname{Spec} W[u_1, \ldots, u_n, t]$ such that $D_0 \subset P_0$ is defined by t. Let $\pi \in \mathcal{O}_K$ be the image of t.

Let s be a function on Q_0 defining E_0 and let $\pi' \in \mathcal{O}_L$ be the image of s. Define $v \in \mathcal{O}^\times_L$ by $\pi = v\pi'^e$ and lift it to a unit $\tilde{v}$ on Q_0. We define a morphism $Q_0 \to \mathbf{A}^{n+1}_W$ satisfying $t \mapsto \tilde{v}s^e$ and lifting the composition $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K \to \mathbf{A}^{n+1}_W$. By replacing Q_0 by an étale neighborhood, we may lift $Q_0 \to \mathbf{A}^{n+1}_W$ to $f \colon Q_0 \to P_0$ satisfying $f^*D_0 = e \cdot E_0$.

We show that the middle and the right squares are cartesian after replacing Q_0 and P_0 by étale neighborhoods. Since the residue fields F and E of K and L are the function field of the closed fibers $D_{0,k}$ and $E_{0,k}$, we may assume $E_0 \to D_0$ and hence $Q_0 \to P_0$ are quasi-finite and hence flat. Further, we may assume that $Q_0 \to P_0$ is finite flat and the right square is cartesian. Then, the morphism $Q_0 \to P_0$ is of degree $[L:K]$ and hence the middle square is cartesian. □

4.9. Assume that $\mathcal{O}_K$ is essentially of finite type and flat over $W = W(k)$ for a perfect field k of characteristic $p > 0$ and let the notation be as in Lemma 4.8.2. We define a dilatation

$$
P^\sim = (P_0 \times_W \mathcal{O}_K)^\sim \to P = P_0 \times_W \mathcal{O}_K
$$

by blowing-up $D_0 \times_W F = (D_0 \times_W \mathcal{O}_K) \cap (P_0 \times_W F) \subset P_0 \times_W \mathcal{O}_K$ and by removing the proper transforms of $D_0 \times_W \mathcal{O}_K$ and of $P_0 \times_W F$. We consider a cartesian diagram

$$
\begin{array}{ccccccc}
Q_0 & \longleftarrow & Q & \longleftarrow & Q^\sim & \longleftarrow & \operatorname{Spec} \mathcal{O}_L \\
\downarrow & \square & \downarrow & \square & \downarrow & \square & \downarrow \\
P_0 & \longleftarrow & P & \longleftarrow & P^\sim & \longleftarrow & \operatorname{Spec} \mathcal{O}_K.
\end{array}
\tag{4.10}
$$

We consider $\mathcal{O}_K$ as a log scheme with the log structure defined by the closed point. With respect to the log structure of $Q^\sim$ defined by the pull-back of E_0, the log scheme $Q^\sim$ is log smooth over $\mathcal{O}_K$. Let K' be

a finite separable extension such that the ramification index $e_{K'/K}$ is divisible by $e = e_{L/K}$. Then, the log product $Q^{\sim}_{\mathcal{O}_{K'}} = Q^{\sim} \times^{\log}_{\mathcal{O}_K} \mathcal{O}_{K'}$ is classically smooth over $\mathcal{O}_{K'}$ and we have a cartesian diagram

$$
\begin{array}{ccc}
Q^{\sim}_{\mathcal{O}_{K'}} & \longleftarrow & \mathrm{Spec}(\mathcal{O}_L \times^{\log}_{\mathcal{O}_K} \mathcal{O}_{K'}) \\
\downarrow & \square & \downarrow \\
P^{\sim}_{\mathcal{O}_{K'}} & \longleftarrow & \mathrm{Spec}\, \mathcal{O}_{K'}.
\end{array}
\tag{4.11}
$$

By the cartesian diagram (4.10), the conormal modules $N_{\mathrm{Spec}\,\mathcal{O}_L/Q}$ and $N_{\mathrm{Spec}\,\mathcal{O}_L/Q^{\sim}}$ are the pull-backs of $N_{\mathrm{Spec}\,\mathcal{O}_K/P} = \Omega^1_{P_0/W} \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_K$ and $N_{\mathrm{Spec}\,\mathcal{O}_K/P^{\sim}} = \Omega^1_{P_0/W}(\log D_0) \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_K$. We have an exact sequence $0 \to N_{D_0/P_0} \to \Omega^1_{P_0/W} \otimes_{\mathcal{O}_{P_0}} \mathcal{O}_{D_0} \to \Omega^1_{D_0/W} \to 0$ and a commutative diagram

$$
\begin{array}{ccc}
\Omega^1_{D_0/W} \otimes_{\mathcal{O}_{D_0}} F & \longrightarrow & \Omega^1_F \\
\downarrow & & \downarrow \\
\Omega^1_{P_0/W}(\log D_0) \otimes_{\mathcal{O}_{D_0}} F & \longrightarrow & \Omega^1_F(\log)
\end{array}
$$

where the horizontal arrows are canonical isomorphisms. Hence the diagram (4.10) and (4.11) define a commutative diagram

$$
\begin{array}{ccc}
Q^{(r)}_{\bar{F}} & \longleftarrow & Q^{\sim(r)}_{\bar{F}} \\
\downarrow & & \downarrow \\
P^{(r)}_{\bar{F}} & \longleftarrow & P^{\sim(r)}_{\bar{F}}
\end{array}
\tag{4.12}
$$

of the reduced closed fibers of dilatations. The bottom arrow is the linear mapping

$$
P^{(r)}_{\bar{F}} = \mathbf{V}(\mathrm{Hom}_F(\Omega^1_{P_0/W} \otimes_{\mathcal{O}_{D_0}} F, \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}))
\tag{4.13}
$$
$$
\leftarrow P^{\sim(r)}_{\bar{F}} = \mathbf{V}(\mathrm{Hom}_F(\Omega^1_F(\log), \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}))
$$

of $\bar{F}$-vector spaces and its image is $\mathbf{V}(\mathrm{Hom}_F(\Omega^1_F, \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}}))$.

Assume that L is a Galois extension of K of Galois group G and let $r = c_{\log} > 0$ be the logarithmic conductor of L over K. We fix a morphism $L \to \bar{K}$ over K. By [10, Theorem 2], the right vertical arrow of (4.12) restricted to the connected component $Q^{\sim(r)\circ}_{\bar{F}} \subset Q^{\sim(r)}_{\bar{F}}$ containing the point corresponding to $L \to \bar{K}$ defines an extension

$$
0 \longrightarrow G^r_{\log} \longrightarrow Q^{\sim(r)\circ}_{\bar{F}} \longrightarrow P^{\sim(r)}_{\bar{F}} = \mathbf{V}(\mathrm{Hom}_F(\Omega^1_F(\log), \mathfrak{m}^r_{\bar{K}}/\mathfrak{m}^{r+}_{\bar{K}})) \longrightarrow 0
\tag{4.14}
$$

of an $\bar{F}$-vector space by an $\mathbf{F}_p$-vector space. By [11, Proposition 1.20], the class of the extension (4.14) defines an element

$$
\omega \in G^r_{\log} \otimes_{\mathbf{F}_p} \Omega^1_F(\log) \otimes_F \mathfrak{m}^{-r}_{\bar{K}}/\mathfrak{m}^{-r+}_{\bar{K}}.
\tag{4.15}
$$

This is independent of the choice of the diagram (4.10) and is called the refined logarithmic conductor of L over K. If G is cyclic and $\chi \colon G \to \mathbb{C}^\times$ is an injective abelian character of G and an injection $\mathbb{Z}/p \to \mathbb{C}^\times$ is fixed, the image of ω in $\Omega^1_F(\log) \otimes_F \mathfrak{m}^{-r}_{\bar{K}}/\mathfrak{m}^{-r+}_{\bar{K}}$ is the refined Swan conductor

$$
\mathrm{rsw}(\chi) \in \Omega^1_F(\log) \otimes_F \mathfrak{m}^{-r}_{\bar{K}}/\mathfrak{m}^{-r+}_{\bar{K}}.
\tag{4.16}
$$

Lemma 4.10. *Let L be a finite Galois extension of K of Galois group G. Let r be the logarithmic conductor of L over K and $\omega \in G^r_{\log} \otimes_{\mathbf{F}_p} \Omega^1_F(\log) \otimes_F \mathfrak{m}^{-r}_{\bar{K}}/\mathfrak{m}^{-r+}_{\bar{K}}$ be the refined logarithmic conductor.*

1. *If the conductor of L over K is the same as the logarithmic conductor r of L over K, then the refined logarithmic conductor ω is in the image of $G_{\log}^r \otimes_{\mathbf{F}_p} \Omega_F^1 \otimes_F \mathfrak{m}_{\bar{K}}^{-r}/\mathfrak{m}_{\bar{K}}^{-r+}$.*

2. *Let K' be an extension of henselian valuation fields of K of ramification index e and of residue field F'. Assume that the image $\omega' \in G_{\log}^r \otimes_{\mathbf{F}_p} \Omega_{F'}^1(\log) \otimes_{F'} \mathfrak{m}_{\bar{K}'}^{-r}/\mathfrak{m}_{\bar{K}'}^{-r+}$ of the refined logarithmic conductor ω of L over K is non-trivial.*

   Then, the logarithmic conductor r' of a composition field L' over K' equals er and ω' is the image of the refined logarithmic conductor of L' over K' by the morphism induced by the injection $\mathrm{Gal}(L'/K')_{\log}^{r'} \to G_{\log}^r$.

Proof. 1. We may assume that the residue field F of K is of finite type over a perfect subfield k. Then, the assertion follows from the commutative diagram (4.12).

2. By the functoriality of construction, the logarithmic ramification of L' over K' is bounded by $er+$. We may assume that the residue fields F and F' of K and K' are of finite type over perfect subfields $k \subset k'$. Then, further by the functoriality of construction, we obtain a morphism $Q_{\bar{F}'}^{\prime(er)} \to Q_{\bar{F}}^{(r)}$ compatible with the injection $G' = \mathrm{Gal}(L'/K') \to G$ and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_{\log}^{\prime er} & \longrightarrow & Q_{\bar{F}'}^{\prime(er)\circ} & \longrightarrow & \mathbf{V}(\mathrm{Hom}_{F'}(\Omega_{F'}^1(\log), \mathfrak{m}_{\bar{K}'}^{er}/\mathfrak{m}_{\bar{K}'}^{er+})) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_{\log}^r & \longrightarrow & Q_{\bar{F}}^{(r)\circ} & \longrightarrow & \mathbf{V}(\mathrm{Hom}_F(\Omega_F^1(\log), \mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+})) & \longrightarrow & 0
\end{array}
\tag{4.17}
$$

of extensions. Since ω' is the extension class of the pull-back of the lower line by the right vertical arrow, the assumption $\omega' \neq 0$ means that the pull-back is non-trivial and $G_{\log}^{\prime er} \neq 0$. Hence er is the logarithmic conductor of L' over K'. The last assertion also follows from the diagram (4.17). $\square$

Proposition 4.11. *Assume that the residue field F of K is a function field of one variable over a perfect subfield k of characteristic $p > 0$ and that the characteristic of K is 0. Let $u \in \mathcal{O}_K$ be a lifting of an element $\bar{u} \in F$ such that F is a finite separable extension of $k(\bar{u})$.*

Let L be a finite Galois extension of K of Galois group G. Assume that the ramification index is 1 and that the residue field E is a purely inseparable extension of F. Let $v \in \mathcal{O}_L$ be a lifting of a generator $\bar{v} \in E = F(\bar{v})$ and let $\varphi \in \mathcal{O}_K[T]$ be the minimal polynomial of v. Assume that $\varphi \equiv T^q - \bar{u} \bmod \mathfrak{m}_K$.

Let r be the logarithmic conductor of L over K. Assume that G^r is cyclic of order p and identify $G^r = \langle \sigma \rangle$ with $\mathbf{F}_p$ by fixing a generater σ. Then, $r = \mathrm{ord}_L \varphi'(v)(v - \sigma(v))$ and the refined logarithmic conductor of L over K is

$$
\frac{d\bar{u}}{\varphi'(v)(v - \sigma(v))} \in \Omega_F^1 \otimes_F \mathfrak{m}_{\bar{K}}^{-r}/\mathfrak{m}_{\bar{K}}^{-r+}.
$$

Almost the same result as Proposition 4.11 is proved in [5, Theorem 5.9] in a similar way. Although we assume that K is of mixed characteristic in Proposition 4.11, the same assertion is proved more easily in the equal characteristic case.

Proof. By Lemma 4.4, the logarithmic conductor equals the conductor. The equality $r = \mathrm{ord}_L \varphi'(v)(v - \sigma(v))$ follows then from Lemma 4.6. We use the notation in Lemma 4.8.2. Since $D_0 \subset P_0$ is smooth over W, there exists a smooth morphism $P_0 \to \mathbf{A}_W^1$ such that D_0 is the pull-back of the 0-section $\mathrm{Spec}\, W \to \mathbf{A}_W^1$. By the assumption that $e = 1$, the divisor $E_0 \subset Q_0$ is also the pull-back of the 0-section $\mathrm{Spec}\, W \to \mathbf{A}_W^1$.

Let $P_1 = P_0 \times_{\mathbf{A}_W^1} \mathcal{O}_K$ and $Q_1 = Q_0 \times_{\mathbf{A}_W^1} \mathcal{O}_K$ be the fiber products with respect to the composition $\mathcal{O}_K \to P_0 \to \mathbf{A}_W^1$. Then P_1 and Q_1 are also smooth over $\mathcal{O}_K$ and we have a cartesian diagram

$$
\begin{array}{ccccc}
Q & \longleftarrow & Q_1 & \longleftarrow & \mathrm{Spec}\,\mathcal{O}_L \\
\downarrow & & \downarrow & & \downarrow \\
P & \longleftarrow & P_1 & \longleftarrow & \mathrm{Spec}\,\mathcal{O}_K.
\end{array}
$$

By this and Lemma 4.10, the refined logarithmic Swan conductor is the image of the class of the extension

$$0 \to G^r \to Q_{1,\bar{F}}^{(r)\circ} \to P_{1,\bar{F}}^{(r)} = \mathbf{V}(\Omega_F^1 \otimes_F \mathfrak{m}_{\bar{K}}^{-r}/\mathfrak{m}_{\bar{K}}^{-r+}) \to 0 \qquad (4.18)$$

defined as the restriction of the G^r-torsor $Q_{\bar{F}}^{(r)\circ}$ over $P_{\bar{F}}^{(r)}$.

We compute the extension (4.18) by comparing it using Lemma 4.7 with that defined by the cartesian diagram (4.2) in Lemma 4.6. Since the diagram

$$\begin{array}{ccc}
Q_0 \times_{\mathbf{A}_W^1} k & \longleftarrow & \operatorname{Spec} E \\
\downarrow & & \downarrow \\
P_0 \times_{\mathbf{A}_W^1} k & \longleftarrow & \operatorname{Spec} F
\end{array}$$

is commutative and the horizontal arrows are the canonical morphisms of the generic points, we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Spec} k[V] = \mathbf{A}_k^1 & \longleftarrow & Q_0 \times_{\mathbf{A}_W^1} k \\
{\scriptstyle U \mapsto V^q} \downarrow & & \downarrow \\
\operatorname{Spec} k[U] = \mathbf{A}_k^1 & \longleftarrow & P_0 \times_{\mathbf{A}_W^1} k
\end{array}$$

where the horizontal arrows are étale. This induces a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Spec} F[V] = \mathbf{A}_F^1 & \longleftarrow & Q_1 \times_{\mathcal{O}_K} F & \longleftarrow & \operatorname{Spec} E \\
{\scriptstyle U \mapsto V^q} \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} F[U] = \mathbf{A}_F^1 & \longleftarrow & P_1 \times_{\mathcal{O}_K} F & \longleftarrow & \operatorname{Spec} F.
\end{array}$$

By comparing this with the cartesian diagram

$$\begin{array}{ccccc}
\operatorname{Spec} F[V] = \mathbf{A}_F^1 & \longrightarrow & \mathbf{A}_{\mathcal{O}_K}^1 & \longleftarrow & \operatorname{Spec} \mathcal{O}_L \\
{\scriptstyle U \mapsto V^q} \downarrow & {\scriptstyle U \mapsto \varphi(V)} \downarrow & & & \downarrow \\
\operatorname{Spec} F[U] = \mathbf{A}_F^1 & \longrightarrow & \mathbf{A}_{\mathcal{O}_K}^1 & \longleftarrow & \operatorname{Spec} \mathcal{O}_K
\end{array}$$

obtained by (4.2), we obtain a commutative diagram

$$\begin{array}{ccc}
\mathbf{A}_{\bar{F}}^1 & \longrightarrow & Q_{1,\bar{F}}^{(r)} \\
\downarrow & & \downarrow \\
\mathbf{A}_{\bar{F}}^1 & \longrightarrow & P_{1,\bar{F}}^{(r)}
\end{array}$$

by Lemma 4.7. Since the left vertical arrow is as in Lemma 4.6.2 and the bottom isomorphism is defined by $U - u$, the assertion follows. $\qquad\square$

5. Coincidence of Swan conductors and of refined Swan conductors

We prove properties of $\operatorname{Sw}^{\mathrm{ab}}$ and $\operatorname{rsw}^{\mathrm{ab}}$ similar to Lemma 4.10.2 and Proposition 4.11.

Proposition 5.1. *Let K be a complete discrete valuation field and let $\chi \colon \operatorname{Gal}(L/K) \to \mathbb{C}^\times$ be a character for a finite abelian extension L of K. Let K' over K be an extension of complete discrete valuation fields of ramification index $e = e(K'/K)$ and let $\chi' \colon \operatorname{Gal}(LK'/K') \to \mathbb{C}^\times$ be the composition of χ with the canonical morphism $\operatorname{Gal}(LK'/K') \to \operatorname{Gal}(L/K)$.*

1. *We have $\mathrm{Sw}^{\mathrm{ab}}_{K'}\chi' \le e \cdot \mathrm{Sw}^{\mathrm{ab}}_K \chi$.*

2. *Assume that $r = \mathrm{Sw}^{\mathrm{ab}}_K \chi \ge 1$. Then, the following conditions are equivalent:*

   (1) *We have $\mathrm{Sw}^{\mathrm{ab}}_{K'}\chi' = e \cdot \mathrm{Sw}^{\mathrm{ab}}_K \chi$.*

   (2) *The image of $\mathrm{rsw}^{\mathrm{ab}}\chi$ by the canonical morphism*

$$\mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log) \to \mathfrak{m}_{K'}^{-er}/\mathfrak{m}_{K'}^{-er+1} \otimes_{\mathcal{O}_{K'}} \Omega^1_{\mathcal{O}_{K'}}(\log)$$

   is non-zero.

   If the equivalent conditions hold, $\mathrm{rsw}^{\mathrm{ab}}\chi'$ equals the image of $\mathrm{rsw}^{\mathrm{ab}}\chi$.

Proof. 1. Let $r = \mathrm{Sw}^{\mathrm{ab}}_K \chi$ and π be a prime element of K. By [7, Proposition (6.3)], we have $\{\chi', 1 + \pi^r \mathfrak{m}_{K'}\} = 0$ and the assertion follows.

2. The condition (2) is equivalent to that $\{\chi', 1 + \pi^r T\} \ne 0$ in $\mathrm{Br}(L')$ where L' is the field of fractions of the henselization of $\mathcal{O}_{K'}[T]_{(\pi')}$ for a prime element π' of K'. Hence, this is equivalent to (1). Further, since the equality $\{\chi, 1 + \pi^r T\} = \lambda_\pi(T\alpha, T\beta)$ is compatible with base change, $\mathrm{rsw}^{\mathrm{ab}}\chi'$ equals the image of $\mathrm{rsw}^{\mathrm{ab}}\chi$. $\qquad\square$

Proposition 5.2. *Let K be a complete discrete valuation field such that the residue field F is of characteristic $p > 0$ and $[F : F^p] = p$. Let $\chi \colon \mathrm{Gal}(L/K) \to \mathbb{C}^\times$ be a faithful character for a cyclic extension L of K of degree $q = p^e$ such that $e(L/K) = 1$ and that the residue field E of L is a purely inseparable extension of F.*

Let $v \in \mathcal{O}_L$ be a lifting of a generator $\bar{v} \in E = F(\bar{v})$ and let $\varphi \in \mathcal{O}_K[T]$ be the minimal polynomial of v. Let $\sigma \in \mathrm{Gal}(L/K)$ be an element of order p and set $\bar{u} = \bar{v}^q \in F$ and $r = \mathrm{ord}_L \varphi'(v)(v - \sigma(v))$.

Then, we have $\mathrm{Sw}^{\mathrm{ab}}\chi = r$ and

$$\mathrm{rsw}^{\mathrm{ab}}\chi = \frac{d\bar{u}}{\varphi'(v)(v - \sigma(v))} \in \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1} \otimes_F \Omega^1_F \subset \mathfrak{m}_K^{-r}/\mathfrak{m}_K^{-r+1} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log).$$

Proof. The assertion follows from [7, Proposition (6.3)] and [6, Theorem (3.6)]. $\qquad\square$

5.3. We prove Theorems 1.3 and 1.5.

Let K be a complete discrete valuation field with residue field of characteristic $p > 0$. We may assume that K is of characteristic 0. Let L be a finite cyclic extension of K and $\chi \colon \mathrm{Gal}(L/K) \to \mathbb{C}^\times$ be a faithful character. We may assume that L is not tamely ramified. We may further assume that the residue field F of K is of finite type over a perfect subfield k, by a standard limit argument.

By Theorem 3.1, there exists an extension K' over K of complete discrete valuation fields satisfying the conditions (i), (ii)' and (iii) in Theorem 3.1. Let $e = e(K'/K)$ be the ramification index and $\chi' \colon \mathrm{Gal}(LK'/K') \to \mathbb{C}^\times$ be the character induced by χ. Then, by the condition (iii), the images of $\mathrm{rsw}(\chi)$ and $\mathrm{rsw}^{\mathrm{ab}}(\chi)$ are non-zero.

Hence by Proposition 5.1 and Lemma 4.10.2, we have $\mathrm{Sw}(\chi') = e \cdot \mathrm{Sw}(\chi)$ and $\mathrm{Sw}^{\mathrm{ab}}(\chi') = e \cdot \mathrm{Sw}^{\mathrm{ab}}(\chi)$. Further $\mathrm{rsw}(\chi')$ and $\mathrm{rsw}^{\mathrm{ab}}(\chi')$ are the images of $\mathrm{rsw}(\chi)$ and $\mathrm{rsw}^{\mathrm{ab}}(\chi)$ respectively. Thus the equality $\mathrm{Sw}(\chi) = \mathrm{Sw}^{\mathrm{ab}}(\chi)$ is equivalent to $\mathrm{Sw}(\chi') = \mathrm{Sw}^{\mathrm{ab}}(\chi')$. Further by the condition (iii) in Theorem 3.1, the equality $\mathrm{rsw}(\chi') = \mathrm{rsw}^{\mathrm{ab}}(\chi')$ is equivalent to $\mathrm{rsw}(\chi) = \mathrm{rsw}^{\mathrm{ab}}(\chi)$. Thus, we may assume that the conditions (i) and (ii)' in Theorem 3.1 are satisfied. In this case, the assertion follows from Propositions 5.2 and 4.11.

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