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Smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$

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Abstract. We classify all smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$ and we give a first description of their geometry via the Log Minimal Model Program.

Keywords. Projective varieties with small Picard group; Horospherical varieties; Log Minimal Model Program

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[Français]

Variétés horosphériques projectives lisses de groupe de Picard $\mathbb{Z}^2$

Résumé. Nous classifions toutes les variétés horosphériques projectives lisses de groupe de Picard $\mathbb{Z}^2$ et nous donnons une première description de leur géométrie *via* le programme des modèles minimaux logarithmiques.

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Contents

1. Introduction

In this paper, varieties are irreducible algebraic varieties over $\mathbb{C}$ and groups are linear algebraic groups over $\mathbb{C}$. And we study varieties that belong to the family of horospherical varieties. Let us first introduce this family.

1.1. About horospherical varieties

Horospherical varieties are ones of the most studied normal G-varieties (i.e. varieties endowed with an algebraic action of a group G) including flag varieties (i.e. rational projective homogeneous spaces) and toric varieties.

Recall that toric varieties are normal T-varieties where T is a torus and such that, in particular:

* they have an open T-orbit;
* the ring of regular functions of any T-stable affine open subset is a multiplicity free T-module;
* they are classified in terms of fans.

There is a natural way to generalized toric varieties to normal G-varieties, for any connected reductive algebraic group G, with similar properties. And it gives the family of spherical varieties such that in particular:

* they have an open G-orbit;
* the G-modules associated to the varieties, for example global sections of G-linearized line bundles, are multiplicity free G-modules;
* they are classified in terms of colored fans.

The colored fans of spherical varieties depend on data, called spherical data, defined from the open G-orbit. The spherical data can differ a lot from a spherical homogeneous space to another. This makes it difficult to study the geometry of all spherical varieties. That is why we often focus on remarkable subfamilies, as the family of horospherical varieties where the open G-orbit is a torus fibration over a flag variety. For horospherical varieties, the spherical data is quite simple, so that the combinatorial objects (colored fans,...) are a nice mix of combinatorial objects coming from toric varieties (fans, polytopes,...) and from flag varieties (root systems).

We give a (non-exhaustive) list of recent results about horospherical varieties, related to the results and proofs of the paper.

* There exists a smoothness criterion (really easier to apply than the general one existing for spherical varieties) [Pas06].
* Fano horospherical varieties are classified in terms of some types of polytopes [Pas08]; this result was generalized to spherical varieties [GH15].
* Smooth projective horospherical varieties of Picard group $\mathbb{Z}$ are classified [Pas09] and studied in several works: [PP10], [Li15], [Kim17], [GPPS18],... Note that the only smooth projective toric varieties of Picard group $\mathbb{Z}$ are the projective spaces, and that for horospherical varieties we obtained non-homogeneous varieties: 5 families of two-orbit varieties (including two infinite families).
* The Minimal Model Program (MMP) [Pas15] and the Log MMP [Pas17] for horospherical varieties can be constructively described in terms of one-parameter families of polytopes.

1.2. Results of the paper

We classify and give a first study of the geometry of smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$. For toric varieties, these are only decomposable projective bundles over projective spaces [Kle88]. But for horospherical varieties, there are many other cases.

Indeed, in addition to homogeneous spaces, products of two varieties and decomposable projective bundles over projective spaces, we distinguish several other types of such horospherical varieties. We classify them in this paper, in particular by studying their Log MMP.

To state as nicely as possible the classification of smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$, we extend the notion of simple roots to the groups $\mathbb{C}^*$ and $\{1\}$. We first briefly recall the case of simple groups (in this paper, a simple group has positive semi-simple rank).

If G is a simply connected simple group, we fix a maximal torus contained in a Borel subgroup B of G, then it defines a root system and in particular a set of simple roots. To each simple root α are associated a fundamental weight denoted by ϖ_α and a fundamental G-module denoted by $V(\varpi_\alpha)$. More generally, if χ is a dominant weight (a non-negative sum of fundamental weights) we denote by $V(\chi)$ the G-module associated to χ: it is the unique irreducible G-module that contains a unique B-stable line where B acts with weight χ. A non-zero element of the B-stable line of $V(\chi)$ is called a highest weight vector (of weight χ) and the stabilizer of the B-stable line of $V(\chi)$ is denoted by $P(\chi)$ (it is a parabolic subgroup of G containing B).

In this paper, if $G = \mathbb{C}^*$, we call the identity automorphism of $\mathbb{C}^*$ the *simple root* of G; we denote it by α, and we set $\varpi_\alpha = \alpha$. Then the natural $\mathbb{C}^*$-module $\mathbb{C}$ is denoted by $V(\varpi_\alpha)$ where α is the simple root of $\mathbb{C}^*$. And for any $n \in \mathbb{Z}$, $V(n\varpi_\alpha)$ is the $\mathbb{C}^*$-module $\mathbb{C}$ where $\mathbb{C}^*$ acts with weight $n\varpi_\alpha$; in particular, any character of $\mathbb{C}^*$ is dominant. Moreover, if $G = \{1\}$, we call the trivial morphism from G to $\mathbb{C}^*$ the *simple root* of G; we denote it by α, and we set $\varpi_\alpha = 0$. In these two cases a highest weight vector is any non-zero vector.

Suppose now that G is a product $G_0 \times \cdots \times G_t$ of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$. A simple root of G is a simple root of some G_i and it is said to be *trivial* if G_i is equal to $\mathbb{C}^*$ or $\{1\}$. Moreover if $\chi_0, \dots, \chi_t$ are respectively dominant weights of $G_0, \dots, G_t$, the G-module associated to $\chi = \chi_0 + \cdots + \chi_t$ is the tensor product $V(\chi_0) \otimes \cdots \otimes V(\chi_t)$ and a highest weight vector of this G-module is a decomposable tensor product of highest weight vectors.

In Definition 3.9, we define two types of projective horospherical varieties $\mathbb{X}^1$ and $\mathbb{X}^2$ with Picard group $\mathbb{Z}^2$. We describe them explicitly as the closure of some G-orbit of a sum of highest weight vectors in the projectivization of a G-module, with the convention above. These varieties depend on the group G, on a simple root β, on a tuple $\underline{\alpha}$ of, possibly trivial, simple roots of G and on a tuple $\underline{a}$ of positive integers.

We can now state the two main results of this paper.

Theorem 1.1. *Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Suppose that X is not the product of two varieties. Then X is isomorphic to one of the following horospherical varieties (which we still denote by X).*

In all cases, G is a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$.

Case (0): G is simple and X is a homogeneous variety G/P where P is the intersection of two maximal (proper) parabolic subgroups of G containing the same Borel subgroup.

Case (1): X is one of the variety $\mathbb{X}^1(G,\beta,\underline{\alpha},\underline{a})$ as in Definition 3.9 with one of the restricted conditions (a), (b) or (c) described in Definition 4.4.

Case (2): X is a variety $\mathbb{X}^2(G,\underline{\alpha},\underline{a})$ as in Definition 3.9 with one of the restricted conditions (a), (b) or (c) described in Definition 4.5.

Remark 1.2.

(1) In Theorem 1.1, the decomposable projective bundles over projective spaces are some very particular varieties $\mathbb{X}^1(G,\beta,\underline{\alpha},\underline{a})$ with restricted conditions (b) or (c). (See Remark 4.6 for the complete description.)

(2) The restricted conditions are useful for two reasons: to get X smooth (and not only locally factorial) and to delete isomorphic cases.

In Theorem 1.1, isomorphisms are not G-equivariant isomorphisms. Indeed the acting group is not necessarily the same for both varieties, so we cannot even consider G-equivariant isomorphisms. Note that in the paper, if not precised, isomorphisms are not supposed to be G-equivariant. Nevertheless, all contractions appearing in the (Log) MMP from a given horospherical G-varieties are automatically G-equivariant.

The horospherical varieties given in Theorem 1.1 are all distinct, i.e., pairwise not isomorphic. This is a consequence of the following result.

Theorem 1.3. *Let X be one of the varieties described in Theorem 1.1. Then "the" Log MMP from X gives the following in each case, respectively with the restricted conditions (a), (b) or (c).*

Case (0): There are two Mori fibrations from X, respectively onto Y and Z, with (general) fibers respectively not isomorphic to Z and Y.

Case (1): (a) A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers not isomorphic to a projective space (but isomorphic to another homogeneous variety or to a two-orbit variety) and a "second" one consists of a flip from X followed by a fibration.

(b) A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers isomorphic to a projective space and a "second" one consists of a finite sequence (possibly empty) of flips from X followed by a fibration. Moreover, the fibers of this latter fibration are not all isomorphic.

(c) A "first" Log MMP consists of a Mori fibration from X to $G/P(\varpi_\beta)$ with general fibers isomorphic to a projective space and a "second" one consists of a finite sequence (possibly empty) of flips from X followed by a divisorial contraction.

Case (2): A "first" Log MMP consists of a fibration ψ to a two-orbit variety, the general fiber F_ψ of ψ and a "second" Log MMP are described as follows.

(a) F_ψ is not isomorphic to a projective space (but isomorphic to another homogeneous variety or to a two-orbit variety) and a "second" Log MMP consists of a flip from X followed by a fibration.

(b) F_ψ is isomorphic to a projective space and a "second" Log MMP consists of a finite sequence (not empty) of flips from X followed by a fibration.

(c) F_ψ is isomorphic to a projective space and a "second" Log MMP consists of a finite sequence (may be empty) of flips from X followed by a divisorial contraction.

Moreover, in all cases, up to reordering and up to symmetries of Dynkin diagrams, the data G (as a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$), β, $\underline{\alpha}$ and $\underline{a}$ are invariants of the "two canonical ways" to realize the Log MMP from X (and then invariants of X).

Remark 1.4. In the paper (Proposition 3.4), we prove that for any smooth projective horospherical variety X with Picard group $\mathbb{Z}^2$, the nef cone of X is generated by the two elements of a basis of $\mathrm{Pic}(X)$, then this gives us two canonical ways to choose the log pair to compute Log MMP from X (see Section 5 for more details). Also, in Cases (1) and (2), one of the "two canonical" Log MMP is "naturally" defined (see Remark 3.3) and only consists of a fibration.

Remark 1.5. In Case (1b), if the sequence of flips is empty, we get two fibrations from X. They could be both onto homogeneous varieties. But one and only one of these fibrations has all its fibers isomorphic to each other (by Proposition 5.10, items 3 and 4 with $l = k$). On the contrary, in Case (0), each fibration has all their fibers isomorphic to each other.

The paper is organized as follows. We first recall in Section 2 the results on horospherical varieties that we use in the paper. Then, in Section 3, we easily describe a first (but not optimal) combinatorial classification, containing many repetitions, and we give a first geometric description of all these latter cases that permits to define the two types of varieties $\mathbb{X}^1$ and $\mathbb{X}^2$. In Section 4, we first define the restricted conditions used in the statement of Theorem 1.1, and we prove the theorem. Then, in Section 5, we prove Theorem 1.3, by studying the Log MMP of all varieties of Theorem 1.1.

2. Some known results on horospherical varieties

2.1. First definitions, first properties of divisors, and smoothness criterion

In this section, we present the classification of horospherical varieties in terms of colored fans Then we give the criteria for divisors to be Cartier, globally generated, and ample. And we state the smoothness criterion. All are generalizations of the theory of toric varieties (without colors).

Let G be a connected reductive group. Fix a maximal torus T and a Borel subgroup B containing T. Denote by U the unipotent radical of B, by $\mathcal{S}$ the set of simple roots of (G,B,T), by $\mathfrak{X}(T)$ the lattice of characters of T (or B) and by $\mathfrak{X}(T)^+\subset\mathfrak{X}(T)$ the monoid of dominant characters.

For any lattice L we denote by $L_\mathbb{Q}$ the $\mathbb{Q}$-vector space $L\otimes_\mathbb{Z}\mathbb{Q}$.

Definition 2.1. A *horospherical variety* X is a normal G-variety with an open orbit isomorphic to G/H where H is a subgroup of G containing U.

Then G/H is a torus fibration over the flag variety G/P where P is the parabolic subgroup of G containing B defined as the normalizer of H in G. The dimension of the torus is called the rank of G/H or the *rank* of X and is denoted by n.

We denote by M the sublattice of $\mathfrak{X}(T)$ consisting of characters of P whose restrictions to H are trivial. Its dual lattice is denoted by N. (The lattices M and N are of rank n.)

Let $\mathcal{R}$ be the subset of $\mathcal{S}$ consisting of simple roots that are not simple roots of P (i.e., simple roots associated to fundamental weights some multiples of which are characters of P).

For any simple root $\alpha\in\mathcal{R}$, the restriction of the coroot $\alpha^\vee$ to M is a point of N, which we denote by $\alpha^\vee_M$. We denote by σ the map $\alpha\longmapsto\alpha^\vee_M$ from $\mathcal{R}$ to N.

Definition 2.2.

(1) A *colored cone* of $N_\mathbb{Q}$ is a pair $(\mathcal{C},\mathcal{F})$ where $\mathcal{C}$ is a convex cone of $N_\mathbb{Q}$ and $\mathcal{F}$ is a subset of $\mathcal{R}$ (called the set of colors of the colored cone), such that
   (i) $\mathcal{C}$ is generated by finitely many elements of N and contains $\{\alpha^\vee_M\mid\alpha\in\mathcal{F}\}$,
   (ii) $\mathcal{C}$ does not contain any line and $\mathcal{F}$ does not contain any α such that $\alpha^\vee_M$ is zero.

(2) A *colored face* of a colored cone $(\mathcal{C},\mathcal{F})$ is a pair $(\mathcal{C}',\mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is the set of $\alpha\in\mathcal{F}$ satisfying $\alpha^\vee_M\in\mathcal{C}'$.

(3) A *colored fan* is a finite set $\mathbb{F}$ of colored cones such that
   (i) any colored face of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$, and
   (ii) any element of $N_\mathbb{Q}$ is in the relative interior of at most one colored cone of $\mathbb{F}$.

The main result of Luna-Vust Theory of spherical embeddings is the following classification result (see for example [Kno91]).

Theorem 2.3 (D. Luna-T. Vust). *There is an explicit one-to-one correspondence between G-isomorphism classes of horospherical G-varieties with open orbit G/H and colored fans.*

Complete G/H-embeddings correspond to complete fans, i.e., to fans such that $N_\mathbb{Q}$ is the union of the first components of their colored cones.

If $G=(\mathbb{C}^*)^n$ and $H=\{1\}$, we recover the well-known classification of toric varieties in terms of fans.

If X is a G/H-embedding, we denote by $\mathbb{F}_X$ the colored fan of X in $N_\mathbb{Q}$ and we denote by $\mathcal{F}_X$ the subset $\cup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}_X}\mathcal{F}$ of $\mathcal{R}$, called the *set of colors* of X.

From now on, X is a complete horospherical variety as above. Let us recall now the characterization of Cartier, $\mathbb{Q}$-Cartier, globally generated and ample divisors of horospherical varieties, due to M. Brion in the more general case of spherical varieties ([Bri89]).

First, we describe the B-stable prime divisors of X. We denote by $X_1,\dots,X_m$ the G-stable prime divisors of X. The valuations of $\mathbb{C}(X)$ defined by the order of zeros and poles along these divisors define primitive elements of N, denoted by $x_1,\dots,x_m$ respectively.

And the B-stable but not G-stable prime divisors of X are the closures in X of B-stable prime divisors of G/H, which are the inverse images by the torus fibration $G/H \longrightarrow G/P$ of the Schubert divisors of the flag variety G/P. The Schubert divisors of G/P can be naturally indexed by the subset of simple roots $\mathcal{R}$. Hence, we denote the B-stable but not G-stable prime divisors of X by D_α with $\alpha\in\mathcal{R}$ (note that $\sigma(\alpha)$ is the element of N defined by the valuation of $\mathbb{C}(X)$ defined by the zeros and poles along the divisor D_α).

Theorem 2.4 (cf. Section 3.3 in [Bri89]). *Any divisor of X is linearly equivalent to a linear combination of $X_1,\dots,X_m$ and D_α with $\alpha\in\mathcal{R}$. Now, let $D=\sum_{i=1}^m a_iX_i+\sum_{\alpha\in\mathcal{R}}a_\alpha D_\alpha$ be a $\mathbb{Q}$-divisor of X.*

(1) D is $\mathbb{Q}$-Cartier if and only if there exists a piecewise linear function $h_D : N_\mathbb{Q}\longrightarrow\mathbb{Q}$, linear on each colored cone of $\mathbb{F}_X$, such that for any $i\in\{1,\dots,m\}$, $h_D(x_i)=a_i$ and for any $\alpha\in\mathcal{F}_X$, $h_D(\alpha_M^\vee)=a_\alpha$.

And D is linearly equivalent to 0 if and only if h_D is linear on $N_\mathbb{Q}$.

Moreover, if D is a divisor, D is Cartier if and only if it is $\mathbb{Q}$-Cartier and the linear functions defined as above can be identified with elements of M.

(2) Suppose that D is $\mathbb{Q}$-Cartier. Then D is globally generated (resp. ample) if and only if the piecewise linear function h_D is convex (resp. strictly convex) and for any $\alpha\in\mathcal{R}\backslash\mathcal{F}_X$, we have $h_D(\alpha_M^\vee)\le a_\alpha$ (resp. $h_D(\alpha_M^\vee)<a_\alpha$).

Suppose that D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We define the **pseudo-moment polytope** *of (X,D) to be the polytope $\tilde{Q}_D$ in $M_\mathbb{Q}$ given by the following inequalities, where $\chi\in M_\mathbb{Q}$: $(h_D)+\chi\ge 0$ and for any $\alpha\in\mathcal{R}\backslash\mathcal{F}_X$, $a_\alpha+\chi(\alpha_M^\vee)\ge 0$.*

Let $v^0:=\sum_{\alpha\in\mathcal{R}}a_\alpha\varpi_\alpha$, we define the **moment polytope** *of (X,D) to be the polytope $Q_D:=v^0+\tilde{Q}_D$.*

(3) Suppose that D is a Cartier divisor. Note that the weight of the canonical section of D is v^0. Then the G-module $H^0(X,D)$ is the direct sum (with multiplicities one) of the irreducible G-modules of highest weights $\chi+v^0$ with χ in $\tilde{Q}_D\cap M$.

From now on, a divisor of a horospherical variety is always supposed to be B-stable, i.e., of the form $\sum_{i=1}^m a_iX_i+\sum_{\alpha\in\mathcal{R}}a_\alpha D_\alpha$.

Theorem 2.5 (cf. Theorem 0.3 [Pas06]). *Let X be a projective horospherical variety and let D be an ample Cartier divisor of X. Suppose that X is smooth. Then D is very ample.*

Since $H\supset U$ and the unique U-stable lines of irreducible G-modules are the lines generated by highest weight vectors, we deduce from Theorems 2.4 and 2.5 the following result.

Corollary 2.6. *Let X be a smooth projective horospherical variety and let D be an ample Cartier divisor of X. Then X is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in $\mathbb{P}(\oplus_{\chi\in\tilde{Q}_D\cap M}V(\chi+v^0))$.*

We should have $V(\chi+v^0)^*$ instead of $V(\chi+v^0)$, but the corollary is still true as stated above, see [Pas15, Remark 2.13].

From Theorem 2.4, we can also deduce a locally factoriality criterion.

Corollary 2.7. *A horospherical variety X is locally factorial if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, $\mathcal{C}$ is generated by part of a basis of N and the map $\sigma:\alpha\longmapsto\alpha_M^\vee$ induces an injective map from $\mathcal{F}$ to this basis.*

In particular if X is locally factorial, the Picard number of X is given by the following formula

$$\rho_X=m+|\mathcal{R}|-n=(|\mathbb{F}_X(1)|-n)+|\mathcal{R}\backslash\mathcal{F}_X|,$$

where $\mathbb{F}_X(1)$ is the set of edges (one-dimensional colored cones) of $\mathbb{F}_X$.

Note that the characterizations of Cartier, $\mathbb{Q}$-Cartier, globally generated and ample divisors can be also applied without the completeness assumption. In particular, Corollary 2.7 also does not need the completeness assumption.

To formulate the smoothness criterion we need to give the following definition.

Definition 2.8. ([Pas06, Def. 2.4]) Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint subsets of $\mathcal{S}$. Let $\Gamma_{\mathcal{R}_1\cup\mathcal{R}_2}$ be the maximal subgraph of the Dynkin diagram of G whose vertices are in $\mathcal{R}_1\cup\mathcal{R}_2$.

The *pair* $(\mathcal{R}_1,\mathcal{R}_2)$ is said to be *smooth* if, for any connected component Γ of $\Gamma_{\mathcal{R}_1\cup\mathcal{R}_2}$,

(1) there is at most one vertex of Γ in $\mathcal{R}_2$ and,
(2) if $\alpha\in\mathcal{R}_2$ is a vertex of Γ, then Γ is of type A or C and α is a short extremal simple root of Γ.

Theorem 2.9 (cf. Theorem 2.6 of [Pas06]). *Let X be a locally factorial horospherical variety. Then X is smooth if and only if for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, the pair $(\mathcal{S}\backslash\mathcal{R},\mathcal{F})$ is smooth.*

Corollary 2.10 (cf. Proposition 2.17 of [Pas06]). *Let X be a smooth horospherical variety. Any G-stable subvariety of X is a smooth horospherical variety.*

Remark 2.11. If X is a toric variety, Theorem 2.9 is trivial because locally factorial toric varieties are smooth, or because for any colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}_X$, the pair $(\mathcal{S}\backslash\mathcal{R},\mathcal{F})$ is necessarily $(\emptyset,\emptyset)$ (indeed the root system or the Dynkin diagram of a torus is empty).

2.2. Log MMP via moment polytopes

The MMP [Pas15] and Log MMP [Pas17] of horospherical varieties can be completely computed and described by studying one-parameter families of polytopes. In this subsection, we recall the main results of this theory, as briefly as we can, in order to use them in Section 5.

From the previous section, to any horospherical variety X, are associated a parabolic subgroup P and a sublattice M of $\mathfrak{X}(P)$; and moreover, any ample B-stable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D defines a pseudo-moment polytope $\tilde{Q}$ and a moment polytope Q. In fact, the map $(X,D)\longmapsto(P,M,Q,\tilde{Q})$ classifies polarized projective horospherical varieties in terms of quadruples $(P,M,Q,\tilde{Q})$.

Definition 2.12. A *quadruple* $(P,M,Q,\tilde{Q})$ is called *admissible* if it satisfies the following:

- P is a parabolic subgroup of G containing B, M is a sublattice of $\mathfrak{X}(P)$, Q is a polytope of $\mathfrak{X}(P)_{\mathbb{Q}}$ included in $\mathfrak{X}(P)^+_{\mathbb{Q}}$ and $\tilde{Q}$ is a polytope of $M_{\mathbb{Q}}$;
- there exists (a unique) $v^0\in\mathfrak{X}(P)_{\mathbb{Q}}$ such that $Q=v^0+\tilde{Q}$;
- the polytope $\tilde{Q}$ is of maximal dimension in $M_{\mathbb{Q}}$ (i.e., its interior in $M_{\mathbb{Q}}$ is not empty);
- the polytope Q intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$.

Example 2.13. Suppose that $\mathfrak{X}(P)=\mathbb{Z}\varpi_1\oplus\mathbb{Z}\varpi_2$ and $M=\mathbb{Z}\varpi_2$, then Q and $\tilde{Q}$ are vertical segments of the same length, $\tilde{Q}$ is in $\mathbb{Q}\varpi_2$ and Q is in $\mathbb{Q}_{\geq 0}\varpi_1\oplus\mathbb{Q}_{\geq 0}\varpi_2$ (but not in $\mathbb{Q}\varpi_2$). In Figure 1, we draw three possible pairs $(Q,\tilde{Q})$ to get three admissible quadruples $(P,M,Q,\tilde{Q})$ respectively corresponding to polarized varieties (X,D_1), (X,D_2) and (X',D'), with $D_1\neq D_2$ and $X\not\simeq X'$.

Proposition 2.14 (Corollary 2.10 of [Pas17] together with Propositions 2.10 and 2.11 [Pas15]).

(1) The map $(X,D)\longmapsto(P,M,Q,\tilde{Q})$ is a bijection from the set of isomorphism classes of polarized projective horospherical varieties to the set of admissible quadruples.
(2) It induces a bijection between the set of G-orbits in X and the set of non-empty faces of Q (or $\tilde{Q}$), preserving the natural orders of both sets. Also, the G-orbit in X associated to a non-empty face $F=v^0+\tilde{F}$ of Q is isomorphic to a horospherical homogeneous space corresponding to (P_F,M_F) where P_F is the minimal parabolic subgroup of G containing P and M_F is the maximal sublattice of M such that $(P_F,M_F,F,\tilde{F})$ is an admissible quadruple. Moreover $(P_F,M_F,F,\tilde{F})$ is the quadruple associated to the (horospherical) closure in X of the G-orbit associated to F (polarized by some D_F we do not need to explicit here).

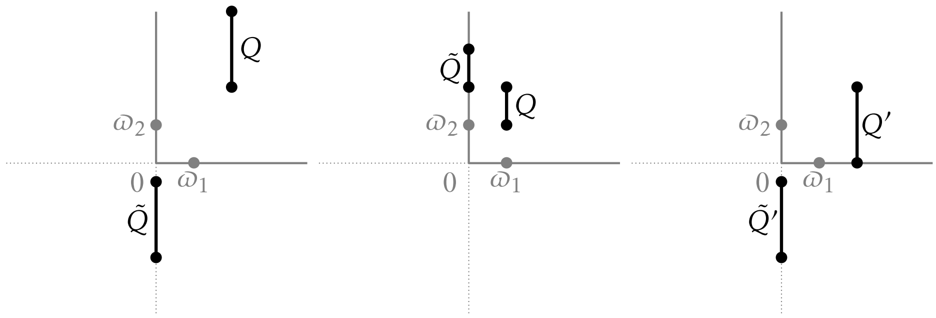

Figure 1. Some (pseudo-)moment polytopes

Example 2.15. Consider the moment polytopes of Example 2.13. And suppose that D_1, D_2 and D' are very ample (otherwise it would be enough to consider multiples of the divisors and of the polytopes).

Then X is the closure of $G\cdot[v_{2\varpi_1+2\varpi_2}+v_{2\varpi_1+3\varpi_2}+v_{2\varpi_1+4\varpi_2}]$ in

$$\mathbb{P}(V(2\varpi_1+2\varpi_2)\oplus V(2\varpi_1+3\varpi_2)\oplus V(2\varpi_1+4\varpi_2))$$

but also the closure of $G\cdot[v_{\varpi_1+\varpi_2}+v_{\varpi_1+1\varpi_2}]$ in $\mathbb{P}(V(\varpi_1+\varpi_2)\oplus V(\varpi_1+2\varpi_2))$. In the first case for example, one can easily check that there are exactly two (closed) G-orbits in addition to the open one in X; moreover, they are $G\cdot[v_{2\varpi_1+2\varpi_2}]\simeq G/(P(\varpi_1)\cap P(\varpi_2))$ and $G\cdot[v_{2\varpi_1+4\varpi_2}]\simeq G/(P(\varpi_1)\cap P(\varpi_2))$, and they correspond to the two vertices of the segment Q. Here, for both closed G-orbits, $P_F=P$ and $M_F=\{0\}$.

Similarly, X' is the closure of $G\cdot[v_{2\varpi_1}+v_{2\varpi_1+\varpi_2}+v_{2\varpi_1+2\varpi_2}]$ in $\mathbb{P}(V(2\varpi_1)\oplus V(2\varpi_1+\varpi_2)\oplus V(2\varpi_1+2\varpi_2))$. There are exactly two (closed) G-orbits in addition to the open one in X', that is $G\cdot[v_{2\varpi_1}]\simeq G/P(\varpi_1)$ and $G\cdot[v_{2\varpi_1+2\varpi_2}]\simeq G/(P(\varpi_1)\cap P(\varpi_2))$. Here, we still have $M_F=\{0\}$ for both closed G-orbits and $P_F=P$ for the second closed G-orbit, but $P_F\neq P$ for the first one ($\mathfrak{X}(P_F)=\mathbb{Z}\varpi_1$).

From Proposition 2.14, we easily get the following result.

Corollary 2.16. *Let (X,D) be a polarized projective horospherical variety and $(P,M,Q,\tilde{Q})$ be the corresponding admissible quadruple. Let F be a non-empty face of Q (or $\tilde{Q}$) and Ω be the corresponding G-orbit in X. Then*

$$\dim(\Omega)=\dim(G/P_F)+\operatorname{rank}(M_F)=\dim(G/P_F)+\dim(F).$$

We can also describe G-equivariant morphisms between horospherical G-varieties, in terms of moment polytopes [Pas15, 2.4]. We summarize, very briefly, this description here.

Without loss of generality, we can reduce to dominant G-equivariant morphisms, i.e. G-equivariant morphisms from a G/H-embbedding to a G/H'-embbedding where $H\subset H'$, i.e., G-equivariant morphisms that extend the projection $G/H\longrightarrow G/H'$. In that case, we have $P\subset P'$ and $M'\subset M$. We keep the same notations as above for the data associated to G/H and we use the same notations with prime for the data associated to G/H'.

Let X be a projective G/H-embedding corresponding to an admissible quadruple $(P,M,Q,\tilde{Q})$ and let X' be a projective G/H'-embedding corresponding to an admissible quadruple $(P',M',Q',\tilde{Q}')$. Then the projection $G/H\longrightarrow G/H'$ extends to a G-equivariant morphism from X to X' if and only if for any non-empty face F of Q, the set of facets (or the corresponding halfspaces in $M_{\mathbb{Q}}$) and the set of walls of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ that contain F define naturally a non-empty face F' of Q'. Moreover in that case the G-orbit of X corresponding to F is sent to the G-orbit of X' corresponding to F'.

Example 2.17. Consider the varieties X and X' of Example 2.13. Each vertex of Q, which is a facet, naturally correspond to a vertex of Q'. But, the vertex $2\varpi_1$ of Q' is contained in a wall of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ and will correspond to the empty face of Q. Then, here, there exists a G-equivariant morphism ϕ from X to X'

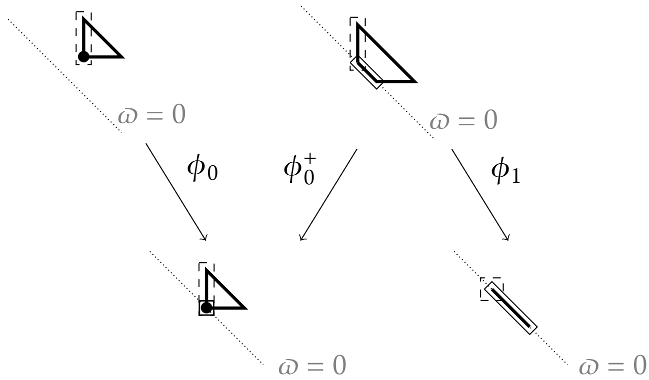

Figure 2. Moment polytopes and G-equivariant morphisms

but there is no such morphism from X' to X. Moreover, ϕ is an isomorphism outside one closed G-orbit where ϕ is the projection $G/(P(\varpi_1)\cap P(\varpi_2)) \longrightarrow G/P(\varpi_2)$.

To complete this example, consider some G/H of rank 2 such that P has a unique fundamental weight ϖ. In Figure 2 we draw 3 moments polytopes of G/H and another moment polytope of a horospherical homogeneous space G/H' of rank 1 with $P' = G$ (in fact $G/H' \simeq \mathbb{C}^*$ and the segment corresponds to the variety $\mathbb{P}^1$). We also draw all G-equivariant morphisms between the corresponding varieties. Note that this picture is similar to Figure 8 with moment polytopes instead of pseudo-moment polytopes.

We also emphasis some vertices and some edges to illustrate images of G-orbits. More precisely, if we focus at the G-orbits distinguished by a $\bullet$, ϕ_0 restricts to the projection $G/P(\varpi) \longrightarrow \mathrm{pt}$. If we focus at the G-orbits distinguished by a non-dashed rectangle, ϕ_0^+ restricts to the fibration $\mathbb{P}^1 \longrightarrow \mathrm{pt}$ and ϕ_1 restricts to the identity morphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. If we focus at the G-orbits distinguished by a dashed rectangle, ϕ_0 and ϕ_0^+ restrict to identity morphisms and ϕ_1 restricts to a fibration to a point.

Now we can state the description of the Log MMP for horospherical varieties in terms of moment polytopes.

First we fix a basis of M (and consider the dual basis for N). Also we choose an order in the set $\{x_1,\dots,x_m\}\cup\{\alpha_M^\vee \mid \alpha\in\mathcal{R}\}$. Then we define a matrix $\mathcal{A}$ of size $(m+|\mathcal{R}|)\times n$ whose rows are the coordinates of the vectors of $\{x_1,\dots,x_m\}\cup\{\alpha_M^\vee \mid \alpha\in\mathcal{R}\}$ in the chosen basis.

Theorem 2.18 (cf. Theorem 1.3 and Section 3 in [Pas17]). *Let X be a $\mathbb{Q}$-factorial projective horospherical variety and let Δ be a B-stable $\mathbb{Q}$-divisor of X. Then for any (general) choice of an ample B-stable $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D of X, a Log MMP from the pair (X,Δ) is described by the following one-parameter families of polytopes*

$$\tilde{Q}^\epsilon := \{x\in M_{\mathbb{Q}} \mid \mathcal{A}x \geq \mathcal{B}+\epsilon\mathcal{C}\} \text{ and } Q^\epsilon := v^\epsilon + \tilde{Q}^\epsilon$$

where $\mathcal{B}$, $\mathcal{C}$ and $v^\epsilon = v^0+\epsilon v^1$ are such that, for any $\epsilon\geq 0$ small enough, $\tilde{Q}^\epsilon$ and Q^ϵ are respectively the pseudo-moment and moment polytope of $(X, D+\epsilon(K_X+\Delta))$.

Note that the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ can be easily computed. Indeed, $\mathcal{A}$ is given by the primitive elements of the rays of the colored fan of X and the images of the colors of G/H; the coefficients of $\mathcal{B}$ are the opposites of the coefficients of D; and the coefficients of $\mathcal{C}$ are the opposites of the coefficients of $K_X+\Delta$. Also, the coefficients of v^0 and v^1 correspond to the coefficients of the D_α's for D and $K_X+\Delta$ respectively.

We can rewrite the conclusion of Theorem 2.18 more precisely as the existence of rational numbers

$$0 := \epsilon_{0,0} < \cdots < \epsilon_{0,k_0} < \epsilon_{0,k_0+1} = \epsilon_{1,0} < \cdots$$
$$\cdots < \epsilon_{1,k_1} < \epsilon_{1,k_1+1} = \epsilon_{2,0} < \cdots < \epsilon_{p,k_p} < \epsilon_{p,k_p+1} = \epsilon_{max}$$

(with $p \geq 1$, and for any $i \in \{0,\dots,p\}$, $k_i \geq 0$) such that, $(P,M,Q^\epsilon,\tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon \in [0,\epsilon_{max}[$, and for ϵ, $\eta \in [0,\epsilon_{max}[$ the following three assertions are equivalent:

- X^ϵ is isomorphic to X^η (where X^ϵ and X^η are the varieties associated to the admissible quadruples $(P,M,Q^\epsilon,\tilde{Q}^\epsilon)$ and $(P,M,Q^\eta,\tilde{Q}^\eta)$ respectively);
- the faces of Q^ϵ (or $\tilde{Q}^\epsilon$) and Q^η (or $\tilde{Q}^\eta$) are "the same", in the following sense: up to deleting inequalities corresponding to some x_j with $j \in \{1,\dots,m\}$ but without changing $\tilde{Q}^\epsilon$ and $\tilde{Q}^\eta$, we have that for any set I of rows, the face of $\tilde{Q}^\epsilon$ corresponding to I (defined by replacing inequalities by equalities for the rows in I) is non empty if and only the face of $\tilde{Q}^\eta$ corresponding to I is non empty;
- there exists $i \in \{0,\dots,p\}$ such that ϵ and η are both in $[\epsilon_{i,0},\epsilon_{i,1}[$, or both in $]\epsilon_{i,k},\epsilon_{i,k+1}[$ with $k \in \{1,\dots,k_i\}$, or both equal to $\epsilon_{i,k}$ with $k \in \{1,\dots,k_i\}$.

Moreover, for any $i \in \{0,\dots,p\}$ and $k \in \{1,\dots,k_i\}$ there are morphisms from X^ϵ to $X^{\epsilon_{i,k}}$ with $\epsilon < \epsilon_{i,k}$ big enough and $\epsilon > \epsilon_{i,k}$ small enough, defining flips. For any $i \in \{1,\dots,p\}$, there are morphisms from X^ϵ to $X^{\epsilon_{i,0}}$ with $\epsilon < \epsilon_{i,0}$ big enough, defining divisorial contractions. Actually, divisorial contractions appear exactly when an inequality corresponding to some x_j with $j \in \{1,\dots,m\}$ becomes superfluous to define $\tilde{Q}^\epsilon$.

Also, there exists P' and M' such that $(P',M',Q^{\epsilon_{max}},\tilde{Q}^{\epsilon_{max}})$ is an admissible quadruple associated to a variety $X^{\epsilon_{max}}$ and such that there is a fibration from X^ϵ to $X^{\epsilon_{max}}$ with $\epsilon < \epsilon_{max}$ big enough. Moreover, the general fiber of this fibration is a horospherical variety and can be described.

In fact all fibers could be described with the following strategy: consider a G-orbit G/H'' of $X^{\epsilon_{max}}$ and list all G-orbits of X^ϵ with $\epsilon < \epsilon_{max}$ big enough that are sent to G/H'' by the fibration, then if there is a unique biggest such G-orbit Ω, the fibers over G/H'' are isomorphic to the closure of $L''\cdot v$ where L'' is a Levi subgroup of H'' and v is the projectivization of a sum of highest weight vectors in Ω. Note that in this paper, there will always be such a biggest G-orbit.

All morphisms above are G-equivariant and the image of any G-orbit can be described as follows. To a face of Q^ϵ (or $\tilde{Q}^\epsilon$) we can associate the maximal set of rows for which equality holds for any element x of the face (in the inequalities $Ax \geq B + \epsilon C$). And similarly to a set of rows we can also naturally associate a face of Q^ϵ (may be empty). For any ϵ and $\epsilon_{i,k}$ as above, for any face F^ϵ of $\tilde{Q}^\epsilon$, we construct a face of $\tilde{Q}^{\epsilon_{i,k}}$ by taking the maximal set of rows associated to F^ϵ and then the face $F^{\epsilon_{i,k}}$ associated to these rows. Then, since there is a morphism ϕ from X^ϵ to $X^{\epsilon_{i,k}}$, the non-empty face $F^{\epsilon_{i,k}}$ corresponds to the G-orbit image by ϕ of the G-orbit corresponding to F^ϵ.

Several examples illustrating Theorem 2.18, in rank 2, are given in Sections 5.2 and 5.4.

3. First combinatorial classification and first geometric description

3.1. Reduction to three cases

In this section, we only use Luna-Vust theory and Corollary 2.7 to reduce to the three main cases of Theorem 1.1.

Lemma 3.1. *Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Then one the three following cases occurs (with notation of Section 2).*

Case (0): $n = 0$, $|\mathcal{R}| = 2$, $\mathcal{F}_X = \emptyset$, *and* $X = G/P$.

Case (1): $n \geq 1$, $\mathcal{R} = \mathcal{F}_X \sqcup \{\beta\}$, *there exist a basis* $(e_1, \ldots, e_n)$ *of* N *and* n *integers* $0 \leq a_1 \leq \cdots \leq a_n$ *such that* σ *induces an injective map* $\tilde{\sigma}$ *from* $\mathcal{F}_X$ *to* $\{e_1, \ldots, e_n, e_0 := -e_1 - \cdots - e_n\}$, $\sigma(\beta) = a_1 e_1 + \cdots + a_n e_n$ *and*

$$\mathbb{F}_X = \{(\mathcal{C}_I, \mathcal{F}_I) \mid I \subsetneq \{0, \ldots, n\}\}$$

where $\mathcal{C}_I$ *is the cone generated by the* e_i*'s with* $i \in I$, *and* $\mathcal{F}_I = \tilde{\sigma}^{-1}(\{e_i \mid i \in I\})$.

Case (2): $n \geq 2$, $\mathcal{R} = \mathcal{F}_X$, *there exist integers* $r \geq 1$, $s \geq 1$, $0 \leq a_1 \leq \cdots \leq a_r$ *and a basis* $(u_1, \ldots, u_r, v_1, \ldots, v_s)$ *of* N *such that* σ *induces an injective map* $\tilde{\sigma}$ *from* $\mathcal{F}_X = \mathcal{R}$ *to* $\{u_0, \ldots, u_r, v_1, \ldots, v_{s+1}\}$, *with* $u_0 := -u_1 - \cdots - u_r$ *and* $v_{s+1} := a_1 u_1 + \cdots + a_r u_r - v_1 - \cdots - v_s$, *and*

$$\mathbb{F}_X = \{(\mathcal{C}_{I,J}, \mathcal{F}_{I,J}) \mid I \subsetneq \{0, \ldots, r\} \text{ and } J \subsetneq \{1, \ldots, s+1\}\}$$

where $\mathcal{F}_{I,J} = \tilde{\sigma}^{-1}(\{u_i \mid i \in I\} \cup \{v_j \mid j \in J\})$ *and where* $\mathcal{C}_{I,J}$ *is the cone generated by the* u_i*'s with* $i \in I$ *and the* v_j*'s with* $j \in J$.

Remark 3.2. If X is a toric variety, $\mathcal{R} = \emptyset$ then we are necessarily in Case (2), and the lemma is already known [Kle88, Theorem 1], and X is the decomposable projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$ over $\mathbb{P}^s$.

Proof. By Corollary 2.7, the map σ induces an injective map from $\mathcal{F}_X$ to $\mathbb{F}_X(1)$ and the Picard number of X is $\rho_X = (|\mathbb{F}_X(1)| - n) + |\mathcal{R} \backslash \mathcal{F}_X|$. But, since X and then $\mathbb{F}_X$ are complete, $|\mathbb{F}_X(1)| - n \geq 0$ with equality if and only if $n = 0$. (And $|\mathcal{R} \backslash \mathcal{F}_X| \geq 0$.) Thus, since $\rho_X = 2$ we distinguish three distinct cases:

Case (0): $n = 0$ and $|\mathcal{R} \backslash \mathcal{F}_X| = 2$;

Case (1): $|\mathbb{F}_X(1)| = n + 1$ and $|\mathcal{R} \backslash \mathcal{F}_X| = 1$;

Case (2): $|\mathbb{F}_X(1)| = n + 2$ and $|\mathcal{R} \backslash \mathcal{F}_X| = 0$.

We now detail each case.

Case (0): In the case where $n = 0$, X is the complete homogeneous variety G/P (and $\mathcal{F}_X = \emptyset$). And then $|\mathcal{R}| = 2$.

Case (1): Consider the fan $\tilde{\mathbb{F}} := \{\mathcal{C} \mid (\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X\}$ associated to the colored fan $\mathbb{F}_X$ (in fact it is the fan of the toric fiber Y of the toroidal variety $\tilde{X} := G \times^P Y$ obtained from X by erasing all colors of X). Since X is locally factorial, the fan $\tilde{\mathbb{F}}$ is the fan of a smooth toric variety of Picard number 1 (because $|\tilde{\mathbb{F}}_X(1)| = n + 1$). Then it is well-known that such a fan is the fan of the projective space $\mathbb{P}^n$. In particular, there exists a basis $(e_1, \ldots, e_n)$ of N such that $\tilde{\mathbb{F}} = \{\mathcal{C}_I \mid I \subsetneq \{0, \ldots, n\}\}$ where $e_0 := -e_1 - \cdots - e_n$ and $\mathcal{C}_I$ is the cone generated by the e_i with $i \in I$.

Denote by β the unique element of $\mathcal{R} \backslash \mathcal{F}_X$. Then, up to reordering the e_i's (for $i \in \{0, \ldots, n\}$), we can suppose that $\sigma(\beta)$ is in $\mathcal{C}_{\{1,\ldots,n\}}$ and equals $a_1 e_1 + \cdots + a_n e_n$ with $0 \leq a_1 \leq \cdots \leq a_n$.

Case (2): As above, consider the fan $\tilde{\mathbb{F}}$. Since X is locally factorial, it is the fan of a smooth toric variety of Picard number 2 (because $|\tilde{\mathbb{F}}_X(1)| = n + 2$). Then, by [Kle88, Theorem 1], there exist integers $r \geq 1$, $s \geq 1$, $0 \leq a_1 \leq \cdots \leq a_r$ and a basis $(u_1, \ldots, u_r, v_1, \ldots, v_s)$ of N such that $\tilde{\mathbb{F}} = \{\mathcal{C}_{I,J} \mid I \subsetneq \{0, \ldots, r\}$ and $J \subsetneq \{1, \ldots, s+1\}\}$, where $u_0 := -u_1 - \cdots - u_r$, $v_{s+1} := a_1 u_1 + \cdots + a_r u_r - v_1 - \cdots - v_s$ and $\mathcal{C}_{I,J}$ is the cone generated by the u_i's with $i \in I$ and the v_j's with $j \in J$.

We conclude by using the following facts: for any $\alpha \in \mathcal{F}_X$ and for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}_X$, we have $\alpha \in \mathcal{F}$ if and only if $\sigma(\alpha) \in \mathcal{C}$; and for any $\alpha \in \mathcal{F}_X$, $\sigma(\alpha)$ is the primitive element of an edge of $\mathbb{F}_X$ (using again Corollary 2.7). □

Remark 3.3. In section 5, we will use the MMP or the Log MMP to study and compare geometrically all these varieties X. We can already describe some Mori fibrations from these varieties, by using the following description of G-equivariant morphisms between horospherical varieties in terms of colored fans ([Kno91]). Let G/H and G/H' be two horospherical homogeneous spaces with $H \subset H'$, and denote by $\pi : G/H \longrightarrow G/H'$ the projection. We keep the same notations as before for the data associated to G/H and we use the same notations with prime for the data associated to G/H'. In particular, we have $M' \subset M$,

$P \subset P'$ and $\mathcal{R}' \subset \mathcal{R}$. By duality, we also have a projection $\pi_* : N_{\mathbb{Q}} \longrightarrow N'_{\mathbb{Q}}$. Let X be a G/H-embedding with colored fan $\mathbb{F}_X$ and let X' be a G/H'-embedding with colored fan $\mathbb{F}_{X'}$. Then the morphism π extends to a G-equivariant morphism from X to X' if and only if for any colored cone $(\mathcal{C},\mathcal{F}) \in \mathbb{F}_X$, there exists a colored cone $(\mathcal{C}',\mathcal{F}') \in \mathbb{F}_{X'}$ such that $\pi_*(\mathcal{C}) \subset \mathcal{C}'$ and $\mathcal{F} \cap \mathcal{R}' \subset \mathcal{F}'$.

Case (0): If X is a complete homogeneous variety G/P of Picard group $\mathbb{Z}^2$, then the MMP gives two Mori fibrations from X to the complete homogeneous varieties G/P_1 and G/P_2 of Picard group $\mathbb{Z}$, where P_1 and P_2 are the maximal proper parabolic subgroups of G containing B such that $P = P_1 \cap P_2$. Note moreover that G/P is a product if and only if $\mathrm{Aut}^0(G/P)$ is not simple.

Case (1): Let $G/H' = X'$ be $G/P(\varpi_\beta)$, i.e. $M' = \{0\}$, $P' = P(\varpi_\beta)$, $\mathbb{F}_{X'} = \{(\{0\},\emptyset)\}$ (and $\mathcal{R}' = \{\beta\}$). Then we can easily check the condition above to prove that there exists a G-equivariant morphism from X to $G/P(\varpi_\beta)$. Note that the general fiber of this fibration is smooth horospherical of Picard group $\mathbb{Z}$ (in particular, it is homogeneous or one of the two-orbit varieties described in [Pas09]).

Case (2): Let P' be the parabolic subgroup containing B (and P) such that $\mathcal{R}' := \tilde{\sigma}^{-1}(\{v_j \mid j \in \{1,\dots,s+1\}\})$. Let M' be the sublattice of M orthogonal to $\mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_r \subset N$. The pair (P',M') corresponds to a horospherical homogeneous space G/H' with H' containing H. Also the dual lattice N' of M' is the image of the projection from N to $\mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_r$. We denote by $v'_1,\dots,v'_{s+1}$ the images of $v_1,\dots,v_{s+1}$ in N', in particular $v'_{s+1} = -v'_1 - \cdots - v'_s$. And finally we denote by $\mathbb{F}_{X'}$ the colored fan $\{\mathcal{C}'_J,\mathcal{F}'_J) \mid J \subsetneq \{1,\dots,s\}\}$ where $\mathcal{C}'_J$ is the cone generated by the v'_j with $j \in J$, and $\mathcal{F}'_J = \tilde{\sigma}^{-1}(\{v_j \mid j \in J\})$. The colored fan $\mathbb{F}_{X'}$ corresponds to a G/H'-embedding X'. Then we can check the condition above to prove that there exists a G-equivariant morphism from X to X', which is a Mori fibration. Note that X' and the general fiber of this fibration are smooth horospherical varieties of Picard group $\mathbb{Z}$ (in particular, they are homogeneous or one of the two-orbit varieties described in [Pas09]).

In the rest of the paper, in cases (1) and (2), we will denote this fibration by $\psi : X \longrightarrow Z$.

3.2. Description via polytopes

We now describe X embedded in the projectivization of a G-module, by choosing the smallest ample Cartier divisor of X and by applying Corollary 2.6. We first study the nef cone of X, which is 2-dimensional.

Recall that any Cartier divisor of X is linearly equivalent to a B-stable divisor, and any prime G-stable divisor corresponds to an edge of $\mathbb{F}_X$ that is not generated by some $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$, and any other B-stable prime divisor is the closure of a color of G/H. Then in Cases (1) and (2), we have $n+2$ prime B-stable divisors that we can denote naturally as follows:

Case (1): $D_{n+1} = D_\beta$; for any $i \in \{0,\dots,n\}$, D_i is the B-stable divisor corresponding to the edge generated by e_i (which equals D_α with $\alpha \in \mathcal{F}_X = \mathcal{R}\backslash\{\beta\}$ if and only if the edge is generated by $\sigma(\alpha)$, and which is G-stable otherwise).

Case (2): for any $i \in \{0,\dots,r\}$, D_i is the B-stable divisor corresponding to the edge generated by u_i; and for any $j \in \{1,\dots,s+1\}$, D_{j+r} is the B-stable divisor corresponding to the edge generated by v_i (which equals D_α with $\alpha \in \mathcal{F}_X = \mathcal{R}$ if and only if the edge is generated by $\sigma(\alpha)$, and which is G-stable otherwise).

Proposition 3.4. *In both cases (1) and (2), the nef cone of X is generated by D_0 and D_{n+1}. In particular, $D_0 + D_{n+1}$ is ample. Moreover (D_0, D_{n+1}) is a basis of* $\mathrm{Pic}(X)$.

Proof. We begin by computing the piecewise linear functions h_{D_0} and $h_{D_{n+1}}$ associated to these two Cartier divisors.

Case (1): Consider the basis $(e_1^*,\dots,e_n^*)$ of M that is dual to the basis $(e_1,\dots,e_n)$ of N. Then h_{D_0} is defined on $N_{\mathbb{Q}}$ by: $(h_{D_0})_{|\mathcal{C}_I} = 0$ if $I = \{1,\dots,n\}$; and for any $i \in \{1,\dots,n\}$, $(h_{D_0})_{|\mathcal{C}_I} = -e_i^*$ where $I = \{0,\dots,i-1,i+1,\dots,n\}$. And $h_{D_{n+1}} = 0$.

Case (2): Consider the basis $(u_1^*,\dots,u_r^*,v_1^*,\dots,v_s^*)$ of M that is dual to the basis $(u_1,\dots,u_r,v_1,\dots,v_s)$ of N. Then h_{D_0} is defined on $N_{\mathbb{Q}}$ by: for any $J\subsetneq\{1,\dots,s+1\}$, $(h_{D_0})_{|\mathcal{C}_{I,J}}=0$ if $I=\{1,\dots,r\}$; for any $i\in\{1,\dots,r\}$, $(h_{D_0})_{|\mathcal{C}_{I,J}}=-u_i^*$ where $I=\{0,\dots,i-1,i+1,\dots,r\}$ and $J=\{1,\dots,s\}$; and, for any $i\in\{1,\dots,r\}$, for any $j\in\{1,\dots,s\}$, $(h_{D_0})_{|\mathcal{C}_{I,J}}=-u_i^*-a_iv_j^*$ where $I=\{0,\dots,i-1,i+1,\dots,r\}$ and $J=\{1,\dots,j-1,j+1,\dots,s+1\}$. And $h_{D_{n+1}}$ is defined on $N_{\mathbb{Q}}$ by: for any $I\subsetneq\{0,\dots,r\}$, for any $j\in\{1,\dots,s\}$, $(h_{D_{n+1}})_{|\mathcal{C}_{I,J}}=-v_j^*$ where $J=\{1,\dots,j-1,j+1,\dots,s+1\}$; and for any $I\subsetneq\{0,\dots,r\}$, $(h_{D_{n+1}})_{|\mathcal{C}_{I,J}}=0$ if $J=\{1,\dots,s\}$.

By Theorem 2.4, one checks that D_0 and D_{n+1} are globally generated but not ample. We also check that for any a and b in $\mathbb{Q}$, aD_0+bD_{n+1} is Cartier if and only if a and b are integers. □

Before applying Corollary 2.6, we reduce to the case where G is the product of simply connected simple groups and a torus, with the following lemma.

Lemma 3.5 (cf. proof of Proposition 3.10 in [Pas06]). *Let $G':=[G,G]$ and let $\mathbb{T}$ be the torus P/H. Then X is also a horospherical $G'\times\mathbb{T}$-variety. Moreover, if $\hat{G}'$ is the universal cover of $\hat{G}'$, X is also a horospherical $\hat{G}'\times\mathbb{T}$-variety.*

Without loss of generality by the lemma, we now assume that G is the product G' of simply connected simple groups and a torus $\mathbb{T}$. In particular, P is the product of a parabolic subgroup of G' with $\mathbb{T}$, and the characters of P are sums of weights of the maximal torus of G' and characters of $\mathbb{T}$. Hence a basis of $M\simeq\mathfrak{X}(\mathbb{T})$ is of the form $(\chi_i+\theta_i)_{i\in\{1,\dots,n\}}$ such that $(\chi_i)_{i\in\{1,\dots,n\}}$ form a basis of $M=\mathfrak{X}(\mathbb{T})$, and the θ_i's are weights of the maximal torus of G'.

With these assumptions, we get the following result.

Lemma 3.6. *The embedding of X given by the ample Cartier divisor D_0+D_{n+1} is:*

Case (1):

$$X\hookrightarrow\mathbb{P}\Big(\bigoplus_{i=0}^{n}V(\chi_i+\varpi_i+(1+a_i)\varpi_\beta)\Big),$$

where $\chi_0=0$, $\chi_1,\dots,\chi_n$ are characters of $\mathbb{T}$, and for any $i\in\{0,\dots,n\}$, ϖ_i is either ϖ_α if $e_i=\sigma(\alpha)$ with $\alpha\in\mathcal{F}_X$ or 0 otherwise.

Case (2):

$$X\hookrightarrow\mathbb{P}\Big(\bigoplus_{i,b_1,\dots,b_{s+1}}V(\chi_i+\varpi_i+\sum_{j=1}^{s+1}b_j(\chi_{r+j}+\varpi_{r+j}))\Big),$$

where $\chi_0=\chi_{n+1}=0$, $\chi_1,\dots,\chi_n$ are characters of $\mathbb{T}$, and for any $i\in\{0,\dots,n+1\}$, ϖ_i is either ϖ_α if u_i or v_{i-r} is $\sigma(\alpha)$ with $\alpha\in\mathcal{F}_X$ or 0 otherwise; and where the sum is taken over all $s+2$-tuples of non-negative integers $(i,b_1,\dots,b_{s+1})$ such that $0\le i\le r$ and $\sum_{j=1}^{s+1}b_j=1+a_i$ (with $a_0:=0$).

Proof. In each case, we describe the pseudo-moment polytope of (X,D_0+D_{n+1}) in a particular basis of M and then the moment polytope of (X,D_0+D_{n+1}). Then we use Corollary 2.6 to conclude.

Case (1): By the previous lemma and the description of the images of colors, for any $i\in\{1,\dots,n\}$, the element e_i^* is of the form $\chi_i+\varpi_i-\varpi_0+a_i\varpi_\beta$, where $\chi_1,\dots,\chi_n$ are characters of $\mathbb{T}$ and for any $i\in\{0,\dots,n\}$, ϖ_i is either ϖ_α if $e_i=\sigma(\alpha)$ with $\alpha\in\mathcal{F}_X$ or 0 otherwise.

The pseudo-moment polytope of (X,D_0+D_{n+1}) is the simplex with vertices 0, $e_1^*,\dots,e_n^*$. The weight of the canonical section of D_0+D_{n+1} is $\varpi_0+\varpi_\beta$, where ϖ_0 is either ϖ_α if $e_0=\sigma(\alpha)$ with $\alpha\in\mathcal{F}_X$ or 0 otherwise. Hence, the moment polytope of (X,D_0+D_{n+1}) is the simplex with vertices $0+\varpi_0+\varpi_\beta=\chi_0+\varpi_0+(1+a_0)\varpi_\beta$ and $(\chi_i+\varpi_i-\varpi_0+a_i\varpi_\beta)+(\varpi_0+\varpi_\beta)=\chi_i+\varpi_i+(1+a_i)\varpi_\beta$ for any $i\in\{1,\dots,n\}$.

Case (2): By the previous lemma and the description of the images of colors, for any $i \in \{1,\dots,r\}$ the element u_i^* is of the form $\chi_i + \varpi_i - \varpi_0 + a_i\varpi_{n+1}$ and for any $j \in \{1,\dots,s\}$ the element v_j^* is of the form $\chi_{r+j} + \varpi_{r+j} - \varpi_{n+1}$, where $\chi_1,\dots,\chi_n$ are characters of $\mathbb{T}$, and for any $i \in \{0,\dots,n+1\}$, ϖ_i is either ϖ_α if u_i (with $0 \leq i \leq r$) or v_{i-r} (with $r+1 \leq i \leq n+1$) is $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 otherwise.

The pseudo-moment polytope of $(X, D_0 + D_{n+1})$ is the polytope with the following vertices: 0, $u_1^*,\dots,u_r^*$, $v_1^*,\dots,v_s^*$ and $u_i^* + (a_i+1)v_j^*$ for any $1 \leq i \leq r$ and for any $1 \leq j \leq s$. Note that the lattice points of this polytope are exactly 0, $v_1^*,\dots,v_s^*$ and for any $1 \leq i \leq r$ all the points of the form $u_i^* + \sum_{j=1}^s b_j v_j^*$ where the b_j's are non-negative integers such that $\sum_{j=1}^s b_j \leq a_i + 1$. Moreover, the weight of the canonical section of $D_0 + D_{n+1}$ is $\varpi_0 + \varpi_{n+1}$, where ϖ_0 (respectively ϖ_{n+1}) is either ϖ_α if u_0 (respectively v_{s+1}) equals $\sigma(\alpha)$ with $\alpha \in \mathcal{F}_X$ or 0 otherwise. Hence, the moment polytope of $(X, D_0 + D_{n+1})$ is the polytope with vertices $0 + \varpi_0 + \varpi_{n+1} = \chi_0 + \varpi_0 + (1+a_0)(\chi_{n+1} + \varpi_{n+1})$; for any $i \in \{1,\dots,r\}$, $\chi_i + \varpi_i + (a_i+1)(\chi_{n+1} + \varpi_{n+1})$; for any $j \in \{1,\dots,s\}$, $\chi_0 + \varpi_0 + \chi_{r+j} + \varpi_{r+j}$; and for any $1 \leq i \leq r$, for any $1 \leq j \leq s$, $\chi_i + \varpi_i - \varpi_0 + a_i\varpi_{n+1} + (a_i+1)(\chi_{r+j} + \varpi_{r+j} - \varpi_{n+1}) + \varpi_0 + \varpi_{n+1} = \chi_i + \varpi_i + (a_i+1)(\chi_{r+j} + \varpi_{r+j})$.

In particular, the lattice points of the pseudo-moment polytope translated by $\varpi_0 + \varpi_{n+1}$ are exactly the $\chi_i + \varpi_i + \sum_{j=1}^{s+1} b_j(\chi_{r+j} + \varpi_{r+j})$ where the sum is taken over all $s+2$-tuples of non-negative integers $(i, b_1,\dots,b_{s+1})$ such that $0 \leq i \leq r$ and $\sum_{j=1}^{s+1} b_j = 1 + a_i$.

□

Recall that, by Lemma 3.5, $(\chi_1,\dots,\chi_n)$ is a basis of $\mathfrak{X}(\mathbb{T})$. Hence, there exists a subtorus $\mathbb{S}$ of $\mathbb{T}$ such that: $(\chi_{i|\mathbb{S}})_{i\in\{1,\dots,n\},\varpi_i=0}$ is a basis of $\mathfrak{X}(\mathbb{S})$, and for any $i \in \{1,\dots,n\}$ such that $\varpi_i \neq 0$, we have $\chi_{i|\mathbb{S}} = 0$.

Lemma 3.7. *In both cases (1) and (2), X is also a horospherical $G' \times \mathbb{S}$-variety.*

Proof. Consider Case (1). For any $i \in \{1,\dots,n\}$ such that $\varpi_i \neq 0$, the G-orbit and the $G' \times \mathbb{S}$-orbit of the highest weight vector $v_{\chi_i+\varpi_i+(1+a_i)\varpi_\beta}$ in

$$V_G(\chi_i + \varpi_i + (1+a_i)\varpi_\beta) \simeq V_{G'\times\mathbb{S}}(\chi_i + \varpi_i + (1+a_i)\varpi_\beta) = V_{G'\times\mathbb{S}}(\varpi_i + (1+a_i)\varpi_\beta)$$

are equal. Case (2) is similar. □

We can replace $\chi_i + \varpi_i$ with ϖ_{α_i} such that

* if $\chi_{i|\mathbb{S}} = 0$ and $\varpi_i \neq 0$, α_i is a simple root of G' (that is supposed to be a product of simply connected simple groups);
* $\mathbb{S}$ is a product of $\mathbb{C}^*$'s whose trivial simple roots are the α_i's with i such that $\chi_{i|\mathbb{S}} \neq 0$ and $\varpi_i = 0$;
* if $i = 0$ or $n+1$, $\chi_{i|\mathbb{S}} = 0$, and $\varpi_i = 0$, we have that α_i is the trivial root of $\{1\}$.

This finally gives the following proposition.

Proposition 3.8. *Let X be a smooth projective horospherical variety of Picard group $\mathbb{Z}^2$ as in Case (1) or (2). Then X is isomorphic to a smooth closure of a G-orbit of a sum of highest weight vectors as follows where G is the product $G_0 \times \cdots \times G_t$ of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$:*

Case (1):

$$\mathbb{P}\Big(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)\Big),$$

where

* *$n \geq 1$ and β is a (non-trivial) simple root of G_0;*
* *$\alpha_0,\dots,\alpha_n$ are distinct simple roots (may be trivial) of G distinct from β;*
* *for any $k \in \{1,\dots,t\}$, $G_k = \{1\}$ if and only if $k = 1$ and α_0 is the trivial root of G_1;*
* *and $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ are integers.*

Case (2):

$$\mathbb{P}(\bigoplus_{i,b_1,\dots,b_{s+1}} V(\varpi_{\alpha_i}+\sum_{j=1}^{s+1} b_j(\varpi_{\alpha_{r+j}})),$$

where

* *the sum is taken over all $s+2$-tuples of non-negative integers $(i,b_1,\dots,b_{s+1})$ such that $0\leq i\leq r$ and $\sum_{j=1}^{s+1} b_j = 1+a_i$ (with $a_0 := 0$);*
* *$r\geq 1$, $s\geq 1$ and $r+s=n$;*
* *$\alpha_0,\dots,\alpha_{n+1}$ are distinct simple roots (may be trivial) of G;*
* *for any $k\in\{0,\dots,t\}$, $G_k=\{1\}$ if and only if, $k=0$ and α_0 is the trivial root of G_0, or $k=t$ and α_{n+1} is the trivial root of G_t;*
* *and $0=a_0\leq a_1\leq\cdots\leq a_r$ are integers.*

These two cases of Proposition 3.8 justify *the definition of two types of varieties.* In Case (2), we already only consider the case where $s=1$ to simplify the definition ; we will prove in Section 4.3 that we can reduce to this case.

Definition 3.9. Let $G=G_0\times\cdots\times G_t$ be a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$ (with $t\geq 0$).

(1) Suppose G_0 to be a simple group. Let β be a simple root of G_0 (non-trivial), let $n\geq\max\{1,t\}$, let $\alpha_0,\dots,\alpha_n$ be distinct, possibly trivial, simple roots of G different from β and let $0=a_0\leq a_1\leq\cdots\leq a_n$ be integers. Suppose also that, for any $k\in\{1,\dots,t\}$, $G_k=\{1\}$ if and only if $k=1$ and α_0 is the trivial root of G_1. Denote $\underline{\alpha}:=(\alpha_0,\dots,\alpha_n)$ and $\underline{a}:=(a_0,\dots,a_n)$. We define $\mathbb{X}^1(G,\beta,\underline{\alpha},\underline{a})$ to be the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i}+(1+a_i)\varpi_\beta)\right).$$

(2) Suppose $t\geq 1$. Let $n\geq 2$, let $0=a_0\leq a_1\leq\cdots\leq a_{n-1}$ be integers, and let $\alpha_0,\dots,\alpha_{n+1}$ be distinct, possibly trivial, simple roots of G. Suppose also that, for any $k\in\{0,\dots,t\}$, $G_k=\{1\}$ if and only if, $k=0$ and α_0 is the trivial root of G_0, or $k=t$ and α_{n+1} is the trivial root of G_t. Denote $\underline{\alpha}:=(\alpha_0,\dots,\alpha_{n+1})$ and $\underline{a}:=(a_0,\dots,a_{n-1})$. We define $\mathbb{X}^2(G,\underline{\alpha},\underline{a})$ to be the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i} V(\varpi_{\alpha_i}+b\varpi_{\alpha_n}+(1+a_i-b)\varpi_{\alpha_{n+1}})\right).$$

Remark 3.10. Up to reordering the G_k's and taking t minimal, we can assume that:

Case (1): the map $\{\alpha_0,\dots,\alpha_n\}\backslash R_0\longrightarrow\{1,\dots,t\}$ is surjective and increasing, where R_0 denotes the set of simple roots of G_0;

Case (2): the map $\{\alpha_0,\dots,\alpha_{n+1}\}\longrightarrow\{0,\dots,t\}$ is surjective and increasing.

Example 3.11. We give here some examples of smooth horospherical varieties of types $\mathbb{X}^1$ and $\mathbb{X}^2$:

$W_1:=\mathbb{X}^1(E_6\times\{0\}\times\mathbb{C}^*\times\mathbb{C}^*,\alpha_4(E_6),(\alpha(\{0\}),\alpha_5(E_6),\alpha(\mathbb{C}^*),\alpha(\mathbb{C}^*),\alpha_1(E_6),\alpha_2(E_6)),(0,0,1,1,2,2))$;

$W_2:=\mathbb{X}^2(\mathrm{SL}_2\times\mathrm{SL}_3\times\mathrm{Sp}_8\times\mathrm{Spin}_7,(\alpha_1(\mathrm{SL}_2),\alpha_1(\mathrm{SL}_3),\alpha_1(\mathrm{Sp}_8),\alpha_1(\mathrm{Spin}_7),\alpha_3(\mathrm{Spin}_7)),(0,0,1))$;

and $W_3:=\mathbb{X}^2(\{0\}\times\mathbb{C}^*\times\mathbb{C}^*\times\mathrm{SL}_2\times\{0\},(\alpha(\{0\},\alpha(\mathbb{C}^*),\alpha(\mathbb{C}^*),\alpha(\mathrm{SL}_2),\alpha(\{0\})),(0,2,3))$.

We will see that W_3 is the toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)\oplus\mathcal{O}_{\mathbb{P}^2}(3))$.

4. Reduction to the cases of Theorem 1.1

This section we define the restricted conditions mentioned in Theorem 1.1, and we prove that we can reduce the cases of Proposition 3.8 to the varieties $\mathbb{X}^1(G,\beta,\underline{\alpha},\underline{a})$ and $\mathbb{X}^2(G,\underline{\alpha},\underline{a})$ with these restricted conditions.

4.1. Smooth horospherical varieties and G-modules

To prove Theorem 1.1 from Proposition 3.8, we replace sums of irreducible G-modules with irreducible $\mathbb{G}$-modules with $G\subset\mathbb{G}$ as soon as we can. Then we enlarge the group G and we reduce to "smaller" cases (for example to horospherical varieties with smaller rank). For this, we first need to apply the smoothness criterion to X (Theorem 2.9), which comes from the fact that horospherical G-modules (i.e. G-modules that are horospherical as varieties) are the $\mathbb{C}^*$-modules $\mathbb{C}$, the SL_d-modules $V(\varpi_1)=\mathbb{C}^d$ and Sp_d-modules (with d even) $V(\varpi_1)=\mathbb{C}^d$. And then we use easy facts as "the $\mathrm{SL}_d\times\mathrm{SL}_e$-module $\mathbb{C}^d\oplus\mathbb{C}^e$ is isomorphic to the SL_{d+e}-module $\mathbb{C}^{d+e}$".

As in [Pas09, Theorem 1.7], the smoothness criterion reveals 8 configurations including the 5 configurations that give the five families of horospherical two-orbit varieties corresponding to non-homogeneous smooth projective horospherical varieties of Picard group $\mathbb{Z}$. We recall these 8 configurations in the following definition.

Definition 4.1. Let K be a simple algebraic group over $\mathbb{C}$ and let γ, δ be two simple roots of K. The ***triple*** (K,γ,δ) is said to be ***smooth*** if (type of K,γ,δ) is one of the following 8 cases, up to exchanging γ and δ (with the notation of Bourbaki [Bou75]).

1. (A_m,α_1,α_m), with $m\geq 2$
2. $(A_m,\alpha_i,\alpha_{i+1})$, with $m\geq 3$ and $i\in\{1,\dots,m-1\}$
3. $(B_m,\alpha_{m-1},\alpha_m)$, with $m\geq 3$
4. (B_3,α_1,α_3)
5. $(C_m,\alpha_i,\alpha_{i+1})$ with $m\geq 2$ and $i\in\{1,\dots,m-1\}$
6. $(D_m,\alpha_{m-1},\alpha_m)$, with $m\geq 4$
7. (F_4,α_2,α_3)
8. (G_2,α_1,α_2)

We say that the triple (type of K,γ,δ) is smooth of two-orbit type if it is one of the cases 3, 4, 5, 7 or 8 above.

Remark 4.2. The smooth triples of two-orbit type correspond bijectively to the isomorphism classes of non-homogeneous projective smooth horospherical varieties with Picard group $\mathbb{Z}$. These varieties have two orbits under the action of their automorphism groups, which are given in [Pas09, Theorem 1.11] and justify that all these varieties are distinct.

Here we also need to introduce another "smooth object" (only used in Case (1)).

Definition 4.3. Let K be a simple algebraic group over $\mathbb{C}$ and let β be a simple root of K and let R be a subset of simple roots of K, all distinct from β. Let n be a non-negative integer. Denote by L the Levi subgroup of the maximal parabolic subgroup $P(\varpi_\beta)$ of K, then the semi-simple part of L is a quotient by a finite central group of a product of simple groups $L^1,\dots,L^q$ (with $q\geq 0$).

The ***quadruple*** (K,β,R,n) is said to be ***smooth*** if

(1) $n=1$, $R=\{\gamma,\delta\}$ such that γ and δ are simple roots of the same L^k such that the triple (L^k,γ,δ) is smooth;

(2) or for any $k\in\{1,\dots,q\}$, at most one simple root of L^k is in R, and if $\gamma\in R$ is a simple root of L_k, then L_k is of type A or C and γ is a short extremal simple root of L_k.

We can list all smooth quadruples (K,β,R,n) (see the appendix). We remark, in particular, that R is at most of cardinality 3.

We can now define the restricted conditions that allow us to state Theorems 1.1 and 1.3.

Definition 4.4. Let $X=\mathbb{X}^1(G,\beta,\underline{\alpha},\underline{a})$ as in Definition 3.9. Recall that R_0 is the maximal subset of $\{\alpha_0,\dots,\alpha_n\}$ consisting of simple roots of G_0. We say that X satisfies the restricted condition (a), (b) or (c) respectively if it satisfies all the following properties including (a), (b) or (c) respectively.

(1) The quadruple (G_0,β,R_0,n) is smooth.
(2) If R_0 is empty, then G_0 is the universal cover of the automorphism group of $G/P(\varpi_\beta)$.
(3) If $i<j$ and $a_i=a_j$ then $\alpha_j\in R_0$. Moreover, if α_i and α_j are in R_0, we suppose them to be ordered with Bourbaki's notation as simple roots of G_0.
(4) One of the three following cases occurs.
   (a) We have $n=t=1$, α_0 and α_1 are both simple roots of G_1 such that the triple (G_1,α_0,α_1) is smooth; in particular, $R_0=\emptyset$ and $a_0<a_1$.

   In the two next cases, the map $\{\alpha_0,\dots,\alpha_n\}\backslash R_0\longrightarrow\{1,\dots,t\}$ is surjective and strictly increasing, and for any $k\in\{1,\dots,t\}$, either G_k is isomorphic to some SL_{d_k} and α_{i_k} is the first simple root of G_k, or G_k is isomorphic to $\mathbb{C}^*$ or $\{1\}$ and α_{i_k} is the trivial simple root of G_k.
   (b) The simple root α_n is not trivial (in particular if $a_{n-1}=a_n$).
   (c) The simple root α_n is trivial (and then $a_{n-1}<a_n$).

Definition 4.5. Let $X=\mathbb{X}^2(G,\underline{\alpha},\underline{a})$ as in Definition 3.9. We say that X satisfies the restricted condition (a), (b) or (c) respectively if it satisfies all the following properties including (a), (b) or (c) respectively.

(1) We have $0=a_0<a_1<\dots<a_n$.
(2) The triple $(G_t,\alpha_n,\alpha_{n+1})$ is smooth of two-orbit type; in particular, α_n and α_{n+1} are both simple roots of G_t and $\alpha_0,\dots,\alpha_{n-1}$ are simple roots of $G_0\times G_1\times\dots\times G_{t-1}$.
(3) One of the three following cases occurs.
   (a) We have $n=2$, $t=1$ and the triple (G_0,α_0,α_1) is smooth.

   In the two next cases: $t=n$, the map $\{\alpha_0,\dots,\alpha_{n-1}\}\longrightarrow\{0,\dots,t-1\}$ is surjective and strictly increasing; and for any $i\in\{1,\dots,t\}$, either G_i is isomorphic to some SL_{d_i} and α_i is the first simple root of G_i, or G_i is isomorphic to $\mathbb{C}^*$ or $\{1\}$ and α_i is the trivial simple root of G_i.
   (b) The simple root α_{n-1} is not trivial.
   (c) The simple root α_{n-1} is trivial.

Remark 4.6. In Theorem 1.1, the decomposable projective bundles over projective spaces are the horospherical varieties X in Case (1) with restricted condition (b) or (c), and such that $R_0=\emptyset$ and ϖ_β is the first simple root of $G_0=\mathrm{SL}_{d_0}$ for some $d_0\geq 2$ (and $0<a_1<\dots<a_n$).

Example 4.7. The three varieties given in Examples 3.11 do not satisfy the restricted condition. Indeed, for W_1 we have $a_2=a_3$ but α_3 is not a simple root of G_0. For W_2, we have $a_0=a_1$ and $G_2=\mathrm{Sp}_8$. And for W_3, $(G_t,\alpha_n,\alpha_{n+1})$ is not smooth of two-orbit type.

But we will prove in the rest of the section that these three varieties are isomorphic to horospherical varieties of type $\mathbb{X}^1$ or $\mathbb{X}^2$ satisfying the restricted condition. More precisely,

$$W_1\simeq\mathbb{X}^1(E_6\times\{0\}\times\mathrm{SL}_2,\alpha_4(E_6),(\alpha(\{0\}),\alpha_5(E_6),\alpha(\mathrm{SL}_2),\alpha_1(E_6),\alpha_2(E_6)),(0,0,1,2,2))$$

which satisfies the restricted condition (b);

$$W_2\simeq\mathbb{X}^2(\mathrm{SL}_5\times\mathrm{SL}_8\times\mathrm{Spin}_7,(\alpha_1(\mathrm{SL}_5),\alpha_1(\mathrm{SL}_8),\alpha_1(\mathrm{Spin}_7),\alpha_3(\mathrm{Spin}_7)),(0,1))$$

and satisfies the restricted condition (b);

$$\text{and } W_3 := \mathbb{X}^1(\mathrm{SL}_3\times\{0\}\times\mathbb{C}^*\times\mathbb{C}^*, \alpha_1(\mathrm{SL}_3), (\alpha(\{0\}), \alpha(\mathbb{C}^*), \alpha(\mathbb{C}^*)), (0,2,3))$$

and satisfies the restricted condition (c).

We can give other examples satisfying the restricted condition in Case (1) (a): for any G_0 and β,

$$\mathbb{X}^1(G_0\times\mathrm{SL}_4, \beta, (\alpha_1(\mathrm{SL}_4), \alpha_3(\mathrm{SL}_4)), (0,1));$$

in Case (2) (a):

$$\mathbb{X}^2(\mathrm{SL}_4\times\mathrm{Sp}_6, (\alpha_1(\mathrm{SL}_4), \alpha_2(\mathrm{SL}_4), \alpha_2(\mathrm{Sp}_6), \alpha_3(\mathrm{Sp}_6)), (0,1));$$

and in Case (2) (c):

$$\mathbb{X}^2(\{0\}\times\mathbb{C}^*\times F_4, (\alpha(\{0\}), \alpha(\mathbb{C}^*), \alpha_2(F_4), \alpha_3(F_4)), (0,1)).$$

We begin by applying the smoothness criterion to get some part of the restricted condition. We suppose that X is as in Proposition 3.8. Recall that the colored fans $\mathbb{F}^1$ and $\mathbb{F}^2$ of the horospherical varieties in Cases (1) and (2) respectively are as follows.

The colored fan $\mathbb{F}^1$ is the complete colored fan whose maximal colored cones are generated by all $u_0,\dots,u_n$ except one and with all possible colors except β, where $(u_1,\dots,u_n)$ is a basis of N and $u_0 = -u_1-\dots-u_n$. Recall also that the map σ is injective from the set $\mathcal{R}\backslash\{\beta\}$ of colors of the horospherical variety to $\{u_0,\dots,u_n\}$ and $\sigma(\beta)=\beta_M^\vee$ is $a_1u_1+\dots+a_nu_n$.

The colored fan $\mathbb{F}^2$ is the complete colored fan whose maximal colored cones are generated by all $u_0,\dots,u_r,v_1,\dots,v_{s+1}$ except one u_i and one v_j, and with all possible colors, where $(u_1,\dots,u_r,v_1,\dots,v_s)$ is a basis of N, $u_0=-u_1-\dots-u_r$ and $v_s = a_1u_1+\dots+a_ru_r-v_1-\dots-v_s$. Recall also that the map σ is injective from the set $\mathcal{R}$ of colors of the horospherical variety to $\{u_0,\dots,u_r,v_1,\dots,v_s\}$.

Lemma 4.8.

Case (1): *The quadruple (G_0,β,R_0,n) is smooth. If there exist $0\leq i<j\leq n$ such that α_i and α_j are simple roots of the same simple group G_k with $k\in\{1,\dots,t\}$ then $n=1$, $i=0$ and $j=1$ (also $t=k=1$). Moreover in that case, the triple (G_k,α_i,α_j) is smooth.*
otherwise, for any $i\in\{0,\dots,n\}$, the simple root α_i is either trivial or in G_0 or the short extremal simple root of some simple group G_k with $k\in\{1,\dots,t\}$ that is of type A or C.

Case (2): *If there exist $0\leq i<j\leq n+1$ such that α_i and α_j are simple roots of the same simple group G_k with $k\in\{0,\dots,t\}$ then either $r=1$, $i=0$ and $j=1$, or $s=1$, $i=n$ and $j=n+1$. Moreover in that case, the triple (G_k,α_i,α_j) is smooth.*
For any $i\in\{0,\dots,n\}$, such that the simple root α_i is the unique α_j of a simple group G_k with $k\in\{0,\dots,t\}$, the root α_i is either trivial or the short extremal simple root of G_k that is of type A or C.

Proof.

Case (1): With notation of Definition 4.3 (with $K=G_0$), suppose γ and δ are two simple roots of the same L^j. If $n>1$, then there exists a maximal colored cone of $\mathbb{F}_X$ that contains $\gamma_M^\vee$ and $\delta_M^\vee$. By applying Theorem 2.9, we get a contradiction. Then $n=1$ and applying Theorem 2.9 to the two one-dimensional colored cones of $\mathbb{F}_X$, we have that the pairs $(R_0\backslash\{\beta,\delta\},\gamma)$ and $(R_0\backslash\{\beta,\gamma\},\delta)$ are smooth, so that (L^j,γ,δ) is smooth (from a case by case study done in [Pas09, Proof of Theorem 1.7]).

Suppose that α is the unique simple root of L^j in R_0. By applying Theorem 2.9 to the colored cone $(\mathbb{Q}_{\geq 0}\alpha_M^\vee,\{\alpha\})$ we get that L^j is of type A or C and α is a short extremal simple root of L^j. This finishes the proof of the smoothness of (G_0,β,R_0,n).

If there exist $0\leq i<j\leq n$ such that α_i and α_j are simple roots of the same simple group G_k with $k\in\{1,\dots,t\}$ then as above Theorem 2.9 implies that $n=1$ and (G_k,α_i,α_j) is smooth. The fact that $i=0$, $j=1$ and $t=k=1$ is obvious.

Now, let $i \in \{0,\dots,n\}$ such that the simple root α_i is the unique α_j of a simple group G_k with $k \in \{1,\dots,t\}$ and suppose that α_i is not trivial. Apply again Theorem 2.9 to the colored cone $(\mathbb{Q}_{\geq 0}\alpha_M^\vee, \{\alpha\})$ to get that α_i is the short extremal simple root G_k with $k \in \{1,\dots,t\}$ that is of type A or C. This finishes the proof of the lemma in Case (1).

Case (2): Suppose there exist $0 \leq i < j \leq n+1$ such that α_i and α_j are simple roots of the same simple group G_k with $k \in \{0,\dots,t\}$. Then Theorem 2.9 implies that $(G_k, \alpha_i, \alpha_j)$ is smooth (still from the case by case study done in [Pas09, Proof of Theorem 1.7]). But this also gives a contradiction if there exists a maximal colored cone of $\mathbb{F}_X$ that contains $\alpha_{i,M}^\vee$ and $\alpha_{j,M}^\vee$. This contradiction occurs if and only if $0 \leq i \leq r$ and $r+1 \leq j \leq n+1$, or $0 \leq i,j \leq r$ and $r \geq 2$, or $r+1 \leq i,j \leq n+1$ and $s \geq 2$.

We conclude the proof of the lemma in Case (2) as in Case (1).

□

Now we list different ways to replace sums of irreducible G-modules with irreducible $\mathbb{G}$-modules with $G \subset \mathbb{G}$.

Lemma 4.9. *Let $\tau \geq 1$. For $i \in \{1,\dots,\tau\}$, let G_i be $\mathbb{C}^*$, SL_{d_i} (with $d_i \geq 2$) or Sp_{d_i} (with $d_i \geq 2$ even). If $G_i = \mathbb{C}^*$ set $d_i = 1$ and ϖ_1^i the identity character of $\mathbb{C}^*$. otherwise, set ϖ_1^i the first fundamental weight of G_i. Let $G = G_1 \times \cdots \times G_\tau$.*

(a) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_1 + \cdots + d_\tau$.
Then $V_{\mathbb{G}}(\varpi_1) = \bigoplus_{i=1}^\tau V_G(\varpi_1^i)$ and $G \cdot \left(\sum_{i=1}^\tau v_{\varpi_1}^i\right) \subset \mathbb{G} \cdot v_{\varpi_1}$.

(b) Let $\mathbb{G} = \mathrm{SL}_d$ where $d = d_1 + \cdots + d_\tau + 1$.
Then $V_{\mathbb{G}}(\varpi_1) = V_G(0) \oplus \bigoplus_{i=1}^\tau V_G(\varpi_1^i)$ and $G \cdot \left(1 + \sum_{i=1}^\tau v_{\varpi_1}^i\right) \subset \mathbb{G} \cdot v_{\varpi_1}$, where 1 denotes the unit in the trivial G-module $V_G(0) = \mathbb{C}$.

With notation of Bourbaki [Bou75] (we put primes to write differently fundamental weights of $\mathbb{G}$ from those of G).

(c) Let $G = \mathrm{SL}_d$ (with $d \geq 3$) and $\mathbb{G} = \mathrm{SO}_{2d}$.
Then $V_{\mathbb{G}}(\varpi_1') = V_G(\varpi_1) \oplus V_G(\varpi_{d-1})$ and $G \cdot \left(v_{\varpi_1} + v_{\varpi_{d-1}}\right) \subset \mathbb{G} \cdot v_{\varpi_1'}$.

(d) Let $G = \mathrm{SL}_d$ (with $d \geq 4$), $\mathbb{G} = \mathrm{SL}_{d+1}$ and $1 \leq i \leq d-2$.
Then $V_{\mathbb{G}}(\varpi_{i+1}') = V_G(\varpi_i) \oplus V_G(\varpi_{i+1})$ and $G \cdot \left(v_{\varpi_i} + v_{\varpi_{i+1}}\right) \subset \mathbb{G} \cdot v_{\varpi_{i+1}'}$.

(e) Let $G = \mathrm{Spin}_{2d}$ (with $d \geq 4$) and $\mathbb{G} = \mathrm{Spin}_{2d+1}$.
Then $V_{\mathbb{G}}(\varpi_d') = V_G(\varpi_{d-1}) \oplus V_G(\varpi_d)$ and $G \cdot \left(v_{\varpi_{d-1}} + v_{\varpi_d}\right) \subset \mathbb{G} \cdot v_{\varpi_d'}$.

Moreover in each case, the projectivizations of the G-orbit and the $\mathbb{G}$-orbit have the same dimension, in particular the two projective varieties defined as the closure of these two orbits in the corresponding projective spaces are the same.

Remark 4.10.

(1) In the first case of Lemma 4.9, with $\tau = 1$ we have in particular that, for d even, $V_{\mathrm{Sp}_d}(\varpi_1) = V_{\mathrm{SL}_d}(\varpi_1)$. Note also that $\mathrm{Sp}_d/P(\varpi_1) = \mathrm{SL}_d/P(\varpi_1)(= \mathbb{P}^{d-1})$.

(2) Cases (c), (d) and (e) correspond to the triples of Definition 4.1 that are not of two-orbit type.

Proof. The first two items are easy and left to the reader. The last three items are given in [Pas09, Propositions 1.8, 1.9 and 1.10]. □

In Case (2), we need the following generalization of Lemma 4.9.

Lemma 4.11. *Let $a \in \mathbb{N}^*$.*

Let $\tau \geq 0$. For $i \in \{0,\dots,\tau\}$, let G_i be $\mathbb{C}^*$, SL_{d_i} (with $d_i \geq 2$) or Sp_{d_i} (with $d_i \geq 2$ even). If $G_i = \mathbb{C}^*$ set $d_i = 1$ and ϖ_1^i the identity character of $\mathbb{C}^*$. Else set ϖ_1^i the first fundamental weight of G_i. Let $G = G_0 \times \cdots \times G_\tau$.

(a) Let $\mathbb{G} = \mathrm{SL}_d$ *where* $d = d_0 + \cdots + d_\tau$*. Then*

$$V_{\mathbb{G}}(a\varpi_1) = \bigoplus_{b_0,\ldots,b_\tau} V_G(\sum_{i=0}^{\tau} b_i\varpi_1^i),$$

where the sum is taken over all $(\tau+1)$*-tuples of non-negative integers* $(b_0,\ldots,b_\tau)$ *such that* $\sum_{i=0}^{\tau} b_i = a$*. And*

$$G\cdot\left(\sum_{b_0,\ldots,b_\tau} v_{\sum_{i=0}^{\tau} b_i\varpi_1^i}\right) \subset \mathbb{G}\cdot v_{a\varpi_1}.$$

(b) Let $\mathbb{G} = \mathrm{SL}_d$ *where* $d = d_0 + \cdots + d_\tau + 1$*. Then*

$$V_{\mathbb{G}}(a\varpi_1) = \bigoplus_{b_0,\ldots,b_\tau} V_G(\sum_{i=0}^{\tau} b_i\varpi_1^i),$$

where the sum is taken over all $(\tau+1)$*-tuples of non-negative integers* $(b_0,\ldots,b_\tau)$ *such that* $\sum_{i=0}^{\tau} b_i \leq a$*. And*

$$G\cdot\left(\sum_{b_0,\ldots,b_\tau} v_{\sum_{i=0}^{\tau} b_i\varpi_1^i}\right) \subset \mathbb{G}\cdot v_{a\varpi_1}.$$

With notation of Bourbaki [Bou75] *(we put primes to write differently fundamental weights of* $\mathbb{G}$ *from those of* G*).*

(c) Let $G = \mathrm{SL}_d$ *(with* $d \geq 3$*) and* $\mathbb{G} = \mathrm{SO}_{2d}$*. Then*

$$V_{\mathbb{G}}(a\varpi_1') = \bigoplus_{b=0}^{a} V_G(b\varpi_1 + (a-b)\varpi_{d-1}) \text{ and } G\cdot\left(\sum_{b=0}^{a} v_{b\varpi_1+(a-b)\varpi_{d-1}}\right) \subset \mathbb{G}\cdot v_{a\varpi_1'}.$$

(d) Let $G = \mathrm{SL}_d$ *(with* $d \geq 4$*),* $\mathbb{G} = \mathrm{SL}_{d+1}$ *and* $1 \leq i \leq d-2$*. Then*

$$V_{\mathbb{G}}(a\varpi_{i+1}') = \bigoplus_{b=0}^{a} V_G(b\varpi_i + (b-a)\varpi_{i+1}) \text{ and } G\cdot\left(\sum_{b=0}^{a} v_{b\varpi_i+(a-b)\varpi_{i+1}}\right) \subset \mathbb{G}\cdot v_{a\varpi_{i+1}'}.$$

(e) Let $G = \mathrm{Spin}_{2d}$ *(with* $d \geq 4$*) and* $\mathbb{G} = \mathrm{Spin}_{2d+1}$*. Then*

$$V_{\mathbb{G}}(a\varpi_d') = \bigoplus_{b=0}^{a} V_G(b\varpi_{d-1} + (b-a)\varpi_d) \text{ and } G\cdot\left(\sum_{b=0}^{a} v_{b\varpi_{d-1}+(b-a)\varpi_d}\right) \subset \mathbb{G}\cdot v_{a\varpi_d'}.$$

Moreover in each case, the projectivizations the G*-orbit and the* $\mathbb{G}$*-orbit have the same dimension, in particular the two projective varieties defined as the closure of these two orbits in the corresponding projective spaces are the same.*

Proof. Remark that for $a = 1$ the lemma is Lemma 4.9. For any $a \geq 1$, we denote by V_a the G-module that we consider in each case.

Consider the horospherical G-variety X defined as the closure of the G-orbit of a sum x_1 of highest weight vectors in $\mathbb{P}(V_1)$: it is a smooth projective variety with Picard group $\mathbb{Z}$ (it is isomorphic to $\mathbb{P}^{d-1}$, $\mathbb{P}^{d-1}$, the quadric Q^{2d-2}, the Grassmannian $\mathrm{Gr}(i+1,d+1)$, $\mathrm{Spin}(2d+1)/P(\varpi_d)$ respectively). Moreover V_1^* is the G-module of global sections of $\mathcal{O}_X(1)$. And, for any $a \geq 1$, the G-module V_a^* is the set of global sections of $\mathcal{O}_X(a)$. But, in each case, X is also a homogeneous projective $\mathbb{G}$-variety $\mathbb{G}/P(\varpi)$ (with $\varpi = \varpi_1$, ϖ_1, ϖ_1', ϖ_{i+1}' and ϖ_d' respectively) by Lemma 4.9, then V_a is also the irreducible $\mathbb{G}$-module $V_{\mathbb{G}}(a\varpi)$.

Also, the image of x_1 in $\mathbb{P}(V_a)$ is the projectivization of a highest weight vector in $V_{\mathbb{G}}(a\varpi)$ for a good choice of a Borel subgroup of $\mathbb{G}$ (because $\mathbb{G}\cdot x_1$ is the homogeneous projective $\mathbb{G}$-variety $\mathbb{G}/P(\varpi)$). □

4.2. Proof of Theorem 1.1 in Case (1)

The first part is already proved by Proposition 3.8 and Lemma 4.8, in particular X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n} V(\varpi_{\alpha_i} + (1+a_i)\varpi_\beta)\right).$$

It remains to prove that we can suppose that

* G_0 is the universal cover of the automorphism group of $G_0/P(\varpi_\beta)$ if R_0 is empty;
* if $i<j$ and $a_i = a_j$ then $\alpha_j \in R_0$;
* and some groups G_k of type C can be replaced by groups of type A.

• If R_0 is empty and G_0 is not the universal cover of the automorphism group of $G_0/P(\varpi_\beta)$, then $G_0/P(\varpi_\beta)$ is isomorphic to $G_0'/P(\varpi_{\beta'})$ where G_0' is the universal cover of $\mathrm{Aut}(G_0/P(\varpi_\beta))$ and (G_0,β,G_0',β') is one of the following: $(\mathrm{Sp}_{2m},\varpi_1,\mathrm{SL}_{2m},\varpi_1)$, $(G_2,\varpi_1,\mathrm{Spin}_7,\varpi_1)$, $(\mathrm{Spin}_{2m+1},\varpi_m,\mathrm{Spin}_{2m+2},\varpi_m)$ or $(\mathrm{Spin}_{2m+1},\varpi_m,\mathrm{Spin}_{2m+2},\varpi_{m+1})$. In any case, $V_{G_0}(\varpi_\beta)\simeq V_{G_0'}(\varpi_{\beta'})$ and $G_0\cdot v_{\varpi_\beta}\simeq G_0'\cdot v_{\varpi_{\beta'}}$. Hence, the fact that R_0 is empty implies that $\bigoplus_{i=0}^{n} V_G(\varpi_{\alpha_i}+(1+a_i)\varpi_\beta)\simeq\bigoplus_{i=0}^{n} V_{\mathbb{G}}(\varpi_{\alpha_i}+(1+a_i)\varpi_{\beta'})$ where $\mathbb{G}=G_0'\times G_1\times\cdots\times G_t$, and X is isomorphic to the closure of the $\mathbb{G}$-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n} V_{\mathbb{G}}(\varpi_{\alpha_i} + (1+a_i)\varpi_{\beta'})\right).$$

• Suppose that there is $0\leq i<j\leq n$ such that α_i and α_j are simple roots of the same simple group among $G_1,\ldots,G_t$. Then by Lemma 4.8, we have $n=1$, $i=0$, $j=1$ (also $t=1$) and the triple (G_1,α_i,α_j) is smooth. In particular, X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\Big(V(\varpi_{\alpha_0}+\varpi_\beta)\oplus V(\varpi_{\alpha_1}+(1+a_1)\varpi_\beta)\Big).$$

If $a_1=0$, the G-module $V(\varpi_{\alpha_0}+\varpi_\beta)\oplus V(\varpi_{\alpha_1}+(1+a_1)\varpi_\beta)$ is isomorphic to the tensor product of the G_0-module $V(\varpi_\beta)$ by the G_1-module $V(\varpi_{\alpha_0})\oplus V(\varpi_{\alpha_1})$, so that X is the product of $G/P(\varpi_\beta)$ by the smooth projective horospherical variety of Picard group $\mathbb{Z}$ defined as the closure of the G_1-orbit of a sum of highest weight vectors in $\mathbb{P}(V(\varpi_{\alpha_0})\oplus V(\varpi_{\alpha_1}))$.

We conclude that if X is not a product, X is as in Case (1a) (with $a_1>0$).

From now on, we suppose that there is no $0\leq i<j\leq n$ such that α_i and α_j are simple roots of the same simple group among $G_1,\ldots,G_t$.

• Suppose that there exists $0\leq i<j\leq n$ such that $a_i=a_j$ and both α_i and α_j are not simple roots of G_0.

Up to reordering, assume that α_i and α_j are simple roots of G_1 and G_2 ($t\geq 2$). Note that if $i=0$ and α_0 is trivial, $G_1=\{1\}$. By Lemma 4.8, G_1 and G_2 are $\{1\}$, $\mathbb{C}^*$ ($d_k=1$ in these two cases), SL_{d_k} (with $d_k\geq 2$) or Sp_{d_k} (with $d_k\geq 2$ even) and α_i, respectively α_j, is either a trivial root or a short extremal root of G_1, respectively G_2.

Let $\mathbb{G}=G_0\times G_3\times\cdots\times G_t\times\mathrm{SL}_{d_1+d_2}$. By Lemma 4.9 ((a) if $i>0$ or α_0 is not trivial and (b) otherwise), the G-module $V(\varpi_{\alpha_i}+(1+a_i)\varpi_\beta)\oplus V(\varpi_{\alpha_j}+(1+a_j)\varpi_\beta)$ is isomorphic to the $\mathbb{G}$-module $V((1+a_i)\varpi_\beta)\otimes\mathbb{C}^{d_1+d_2}$. And X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit of a sum of highest weight vectors in $\mathbb{P}$ under the action of $\mathbb{G}$.

We can now compare the dimension of the open $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of $\mathbb{X}$ with the dimension of the open G-orbit of X. Indeed $\Omega_{\mathbb{X}}$ is isomorphic to a horospherical homogeneous space of rank $n-1$ over $((G_0\times G_3\times\cdots\times G_t)/(P\cap G_0\times G_3\times\cdots\times G_t))\times(\mathrm{SL}_{d_1+d_2}/P(\varpi_1))$, while G/H is of rank n over $((G_0\times G_3\times\cdots\times$

$G_t)/P\cap(G_0\times G_3\times\cdots\times G_t))\times((G_1\times G_2)/P\cap(G_1\times G_2))$. But the dimension of $\mathrm{SL}_{d_1+d_2}/P(\varpi_1)$ is d_1+d_2-1 while the dimension of $(G_1\times G_2)/P\cap(G_1\times G_2)$ is $(d_1-1)+(d_2-1)$. Hence $\Omega_{\mathbb{X}}$ and G/H have the same dimension, so that $X=\mathbb{X}$.

Then we can replace, without changing X, the product of the two simple groups corresponding to two simple roots α_i and α_j with $a_i=a_j$, with a unique simple group of type A. Note that n decreases by this change. (Also note that, if $i=0$ and α_0 is trivial then the new α_0 is not trivial any more.)

With similar arguments, we can also replace any group $G_1,\dots,G_t$, of type C and that contains a unique simple root α_i, by a group of type A.

• What we did just above also works in the cases where $n=1$, $a_1=0$, α_0 and α_1 are simple roots of G_1 and G_2 (and $t=2$). In that case, this proves that X is the closure of the $\mathrm{SL}_d\times G_0$-orbit of a highest weight vector in $\mathbb{P}\left(\mathbb{C}^d\bigotimes V(\varpi_\beta)\right)$. Hence, in that case, X is isomorphic to $\mathbb{P}^{d-1}\times G_0/P(\varpi_\beta)$.

Hence, we conclude the proof by iteration.

4.3. Proof of Theorem 1.1 in Case (2)

The first part is already proved by Proposition 3.8 and Lemma 4.8, in particular X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}:=\mathbb{P}\left(\bigoplus_{i,b_1,\dots,b_{s+1}} V(\varpi_{\alpha_i}+\sum_{j=1}^{s+1}b_j\varpi_{\alpha_{r+j}})\right),$$

where the sum is taken over all $s+2$-tuples of non-negative integers $(i,b_1,\dots,b_{s+1})$ such that $0\leq i\leq r$ and $\sum_{j=1}^{s+1}b_j=1+a_i$.

It remains to prove that we can suppose that

* $s=1$, α_n, α_{n+1} are both simple roots of G_t and $(G_t,\alpha_n,\alpha_{n+1})$ is smooth of two-orbit type;
* $0<a_1<\cdots<a_r$;
* and some groups G_k of type C can be replaced by groups of type A.

• Suppose first that $s>1$, or $s=1$ and α_n, α_{n+1} are not simple roots of the same simple group G_k. Up to reordering and applying Lemma 4.8, for any $j\in\{1,\dots,s\}$, α_{r+j} is either a trivial root of G_{t-s+j} that is $\mathbb{C}^*$ or $\{1\}$, or a short extremal simple root of G_{t-s+j} that is of type A or C. Moreover, the simple groups $G_{t-s+1},\dots,G_t$ contain no other α_i with $i\in\{0,\dots,r\}$. Also, $G_{t-s+j}=\{1\}$ if and only if $j=s$ and α_{r+s} is trivial.

We now apply Lemma 4.11 ((a) if α_{r+s} is not trivial and (b) otherwise). Hence, there exists $d\leq 2$ such that, with $G\subset\mathbb{G}:=G_0\times\cdots\times G_{t-s}\times\mathrm{SL}_d$, we have

$$\mathbb{P}=\mathbb{P}\left(\bigoplus_{i,b_1,\dots,b_{s+1}} V(\varpi_{\alpha_i})\otimes V(\sum_{j=1}^{s+1}b_j\varpi_{\alpha_{r+j}})\right)=\mathbb{P}\left(\bigoplus_{i=0}^{r}V_{\mathbb{G}}(\varpi_{\alpha_i})\otimes V_{\mathbb{G}}((1+a_i)\varpi_1)\right),$$

X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim((G_{t+1-s}\times\cdots\times G_t)/P\cap(G_{t+1-s}\times\cdots\times G_t)=d-s-1$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank r) equals the dimension of G/H. Hence, $X=\mathbb{X}$. Now remark that $\mathbb{X}$ is a horospherical variety as in Case (1).

• From now on, we suppose that $s=1$ (and $n=r+1$), and that α_n, α_{n+1} are both simple roots of G_t (up to reordering). In particular, X is of type $\mathbb{X}^2(G,\underline{\alpha},\underline{a})$ and then is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i}V(\varpi_{\alpha_i}+b\varpi_{\alpha_{r+1}}+(1+a_i-b)\varpi_{\alpha_{r+2}})\right).$$

Note now that for any $k \in \{0,\dots,t\}$, $G_k = \{1\}$ if and only if $k = 0$ and α_0 is trivial.

Recall that, by Lemma 4.8, $\alpha_0,\dots,\alpha_r$ are not simple roots of G_t and the triple $(G_t,\alpha_n,\alpha_{n+1})$ is smooth. Then X is embedded as the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P} := \mathbb{P}\left(\bigoplus_{i=0}^{n-1}\bigoplus_{b=0}^{1+a_i} V(\varpi_{\alpha_i})\otimes V(b\varpi_{\alpha_{r+1}} + (1+a_i-b)\varpi_{\alpha_{r+2}})\right).$$

If $(G_t,\alpha_n,\alpha_{n+1})$ is not of two-orbit type, we can apply Lemma 4.11 ((c), (d) or (e)): we get $G \subset \mathbb{G}$ with $\mathbb{G} := G_0\times\cdots\times G_{t-1}\times\mathbb{G}_t$ such that $\mathbb{P} = \mathbb{P}\left(\bigoplus_{i=0}^r V_{\mathbb{G}}(\varpi_{\alpha_i})\otimes V_{\mathbb{G}}((1+a_i)\varpi)\right)$, X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim(G_t/P\cap G_t)+1 = \dim(\mathbb{G}_t/P(\varpi))$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank r) equals the dimension of G/H. Hence, $X = \mathbb{X}$. And remark that $\mathbb{X}$ is a horospherical variety as in Case (1).

• Now suppose that $r > 1$, or $r = 1$ and α_0, α_1 are not simple roots of the same simple group.

Let $i \neq i'$ in $\{0,\dots,r\}$ such that $a_i = a_{i'}$. Up to reordering and applying Lemma. 4.8, α_i and $\alpha_{i'}$ are, trivial or short extremal, simple roots respectively of G_0 and G_1 that are $\mathbb{C}^*$, $\{1\}$ or simple groups of type A or C. Moreover G_0 and G_1 contain no other α_k's.

We can apply Lemma 4.9 ((a) if $i > 0$ or α_0 is trivial and (b) otherwise) to get $G \subset \mathbb{G} := \mathrm{SL}_d \times G_2 \cdots \times G_t$ such that

$$\mathbb{P} = \mathbb{P}\left(\left(\bigoplus_{k\neq i,i'}\bigoplus_{b=0}^{1+a_k} V_{\mathbb{G}}(\varpi_{\alpha_k})\otimes V_{\mathbb{G}}(b\varpi_{\alpha_{r+1}} + (1+a_k-b)\varpi_{\alpha_{r+2}})\right)\right.$$
$$\left.\oplus\left(\bigoplus_{b=0}^{1+a_i} V_{\mathbb{G}}(\varpi)\otimes V_{\mathbb{G}}(b\varpi_{\alpha_{r+1}} + (1+a_i-b)\varpi_{\alpha_{r+2}})\right)\right),$$

X is a subvariety of the closure $\mathbb{X}$ of the $\mathbb{G}$-orbit $\Omega_{\mathbb{X}}$ of a sum of highest weight vectors in $\mathbb{P}$, and $\dim((G_0\times G_1)/P\cap(G_0\times G_1))+1 = d-1$. In particular the dimension of $\Omega_{\mathbb{X}}$ (which is horospherical of rank $(r-1)+1$) equals the dimension of G/H. Hence, $X = \mathbb{X}$. Now remark that $\mathbb{X}$ is either a horospherical variety as in Case (2) of rank one less than X, or a horospherical variety as in Case (1) if $r = 1$.

With similar arguments, we can also replace any group $G_0,\dots,G_{t-1}$, of type C and that contains a unique simple root α_i, by a group of type A.

• By iteration of the above process, we can now assume that $0 < a_1 < \cdots < a_r$, or that $r = 1$ (and $t = 1$) and α_0, α_1 are two simple roots of G_0. In the second case, note that by Lemma. 4.8, the triple (G_0,α_0,α_1) is smooth.

Suppose $r = 1$, α_0, α_1 are two simple roots of G_0 and that $a_1 = a_0 = 0$. Then, X is the closure of the $G_0\times G_1$-orbit of a sum of highest weight vectors in

$$\mathbb{P} = \mathbb{P}\Big((V_{G_0}(\varpi_{\alpha_0})\oplus V_{G_0}(\varpi_{\alpha_1}))\otimes(V_{G_1}(\varpi_{\alpha_2})\oplus V_{G_1}(\varpi_{\alpha_3}))\Big).$$

Hence in that case, X is the product of two varieties: the closure of the G_0-orbit of a sum of highest weight vectors in $\mathbb{P}\Big((V_{G_0}(\varpi_{\alpha_0})\oplus V_{G_0}(\varpi_{\alpha_1}))\Big)$ and the closure of the G_1-orbit of a sum of highest weight vectors in $\mathbb{P}\Big((V_{G_1}(\varpi_{\alpha_2})\oplus V_{G_1}(\varpi_{\alpha_3}))\Big)$.

Hence, in any case we can assume that $0 < a_1 < \cdots < a_r$. This finishes the proof of Theorem 1.1. □

5. The MMP and Log MMP for smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$

The main goal of this section is to prove Theorem 1.3. For this we apply the Log MMP from the horospherical varieties $\mathbb{X}^1$ and $\mathbb{X}^2$.

The principle of the Log MMP is the following. We begin with a pair (X,Δ) where X is a not too singular projective variety and Δ is a $\mathbb{Q}$-divisor such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We want to contract curves having negative intersection with $K_X+\Delta$ in order to get a new variety with smaller Picard number. In general, we can do this by choosing an extremal ray (whose curves have negative intersection with $K_X+\Delta$) in the cone of effective curves up to numerical equivalence.

In our context, note that this cone is two dimensional and then has two extremal rays; this explains why we have two ways to do the Log MMP.

After contracting a curve it may happen that the new variety is too singular, so that we have to partially desingularize it in a natural and unique way; we call this a flip.

To continue the program, we have to choose again an extremal ray in the cone of effective curves of the new variety, until we finish with a minimal model (when there is no curve with negative intersection with $K_X+\Delta$) or a fibration (when the dimension decreases).

For horospherical varieties, we can compute a Log MMP to the end just by choosing an ample divisor at the beginning (and not an extremal ray at each step), and by considering a one-parameter family of polytopes (Theorem 2.18).

5.1. Generalities

Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Here, we suppose that X is as in Case (1) or (2) of Lemma 3.1 (or Theorem 1.1).

By Proposition 3.4, up to linear equivalence, the ample Cartier divisors of X are of the form $D=d_0D_0+d_{n+1}D_{n+1}$ with positive integers d_0 and d_{n+1}.

We can apply [Pas15] to the polarized variety (X,D) and obtain a description of the MMP from X, via moment polytopes (if X is Fano, we obtain two different paths of the program depending on the choice of d_0 and d_{n+1}; if X is not Fano, we obtain a unique path of the program).

Moreover, we can also choose a B-stable $\mathbb{Q}$-divisor Δ of X and apply [Pas17] to the polarized pair $((X,D),\Delta)$ and obtain a description of the Log MMP from (X,Δ), via moment polytopes as described in Section 2.2. To get a uniform Log MMP for any smooth projective horospherical variety with Picard group $\mathbb{Z}^2$, we choose $D=D_0+D_{n+1}$ and $\Delta=-D_i-K_X$ for $i\in\{0,n+1\}$.

Remark 5.1. In Case (1), an anticanonical divisor of X is (see for example [Pas08, Proposition 3.1])

$$-K_X=\sum_{i=0}^n b_iD_i+b_\beta D_\beta\sim(\sum_{i=0}^n b_i)D_0+(b_\beta-\sum_{i=0}^n a_ib_i)D_{n+1},$$

where $b_i=1$ if D_i is G-stable, $b_i=b_{\alpha_i}\geq 2$ if D_i is the color D_{α_i} and $b_\beta\geq 2$ (recall that $D_\beta=D_{n+1}$). In particular, X is Fano (i.e., $-K_X$ ample) if and only if $b_\beta>\sum_{i=1}^n a_ib_i$.

To describe the MMP from X we could choose the ample divisor $D=(\sum_{i=0}^n b_i)D_1+(b_\beta+1)D_\beta$, so that $D+\epsilon K_X$ is ample for any $\epsilon\in[0,1[$ and $D+K_X\sim(\sum_{i=0}^n a_ib_i+1)D_\beta$ is not ample but globally generated. Then, for that choice of D, the MMP from X consists of the Mori fibration to $G/P(\varpi_\beta)$ described in Remark 3.3.

Moreover, this Mori fibration is also the unique contraction of the Log MMP obtained with the choices $D=D_0+D_{n+1}$ and $\Delta=-D_0-K_X$ in Theorem 2.18 (in that case, Q^1 is a multiple of ϖ_β).

In Case (2), an anticanonical divisor of X is

$$-K_X=\sum_{i=0}^{r}b_iD_i+\sum_{j=1}^{s+1}b_{r+j}D_{r+j}\sim(\sum_{i=0}^{r}b_i)D_0+(\sum_{j=1}^{s+1}b_{r+j}-\sum_{i=0}^{r}a_ib_i)D_{n+1},$$

where $b_i=1$ (respectively b_{r+j}) if D_i (resp. D_{r+j}) is G-stable and $b_i=b_{\alpha_i}\geq 2$ (resp. $b_{r+j}=b_{\alpha_{r+j}}\geq 2$) if D_i is the color D_{α_i} (respectively D_{r+j} is the color $D_{\alpha_{r+j}}$). In particular, X is Fano if and only if the inequality $\sum_{j=1}^{s+1}b_{r+j}>\sum_{i=0}^{r}a_ib_i$ is s.

To describe the MMP from X we could choose the ample divisor $D=(\sum_{i=0}^{r}b_i)D_0+(1+\sum_{j=1}^{s+1}b_{r+j})D_{n+1}$, so that $D+\epsilon K_X$ is ample for any $\epsilon\in[0,1[$ and $D+K_X\sim(1+\sum_{i=0}^{r}a_ib_i)D_{n+1}$ is not ample but globally generated. Then, for that choice of D, the MMP from X consists of the Mori fibration ψ from X to Z described in Remark 3.3. Moreover, this Mori fibration is also the unique contraction of the Log MMP obtained with the choices $D=D_0+D_{n+1}$ and $\Delta=-D_0-K_X$ in Theorem 2.18 (in that case, Q^1 is a simplex of dimension s).

Hence, in both cases, we will describe the Log MMP obtained with the choices $D=D_0+D_{n+1}$ and $\Delta=-D_{n+1}-K_X$.

In the next four subsections, X is one the varieties of Theorem 1.1 in Case (1) or (2). We begin by constructing the families of polytopes for the log pairs $(X,\Delta=-D_{n+1}-K_X)$ with the choice of ample divisor $D=D_0+D_{n+1}$, and then we describe in detail the Log MMP's obtained with these families.

5.2. Case (1): the "second" Log MMP via moment polytopes

To describe the one-parameter family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ defined in Theorem 2.18, we consider the basis $(e_i^*)_{i\in\{1,\dots,n\}}$ of M, where for any $i\in\{1,\dots,n\}$, $e_i^*=\varpi_{\alpha_i}-\varpi_{\alpha_0}+a_i\varpi_\beta$, and we define the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as follows

$$\mathcal{A}=\begin{pmatrix}-1&\cdots&\cdots&-1\\1&0&\cdots&0\\0&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&0&1\\a_1&\cdots&\cdots&a_n\end{pmatrix},\ \mathcal{B}=\begin{pmatrix}-1\\0\\\vdots\\\vdots\\0\\-1\end{pmatrix}\text{ and }\mathcal{C}=\begin{pmatrix}0\\\vdots\\\vdots\\0\\0\\1\end{pmatrix}.$$

Then $\tilde{Q}^\epsilon=\{x\in M_{\mathbb{Q}}\mid\mathcal{A}x\geq\mathcal{B}+\epsilon\mathcal{C}\}$ is the set of $x=(x_1,\dots,x_n)$ such that $x_1,\dots,x_n$ are non-negative, $x_1+\cdots+x_n\leq 1$ and $a_1x_1+\cdots+a_nx_n\geq\epsilon-1$.

Example 5.2. If $n=2$ we are in one of the following situations:

(1) $a_2>a_1>0$ and α_2 is not trivial;
(2) $a_2>a_1>0$ and α_2 is trivial;
(3) $a_2>a_1=0$ and α_2 is not trivial;
(4) $a_2>a_1=0$ and α_2 is trivial;
(5) $a_2=a_1>0$;
(6) $a_2=a_1=0$.

We draw, in Figure 3, these polytopes for $\epsilon=0$ in different cases with the hyperplane $H^0:=\{x\in M_{\mathbb{Q}}\mid a_1x_1+a_2x_2=-1\}$. Note that there is no such hyperplane if $a_2=a_1=0$.

• If $a_n=0$, $\tilde{Q}^\epsilon=\tilde{Q}^0$ for any $\epsilon\in[0,1]$ and it is empty if $\epsilon>1$. Moreover, for any $\epsilon\in[0,1]$, Q^ϵ intersects the interior of $\mathfrak{X}(P)^+_{\mathbb{Q}}$ if and only if $\epsilon<1$. In that case, the Log MMP described by the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of a fibration $\phi_0:X\longrightarrow Y^0$.

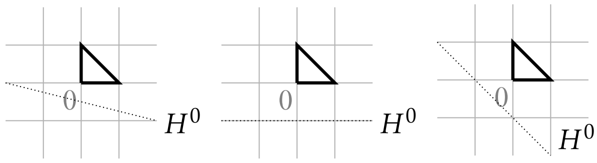

Figure 3. The polytopes $\tilde{Q}^0$ in the cases where $a_1 = 1$ and $a_2 = 2$, $a_1 = 0$ and $a_2 = 1$ and $a_1 = a_2 = 1$ respectively

The fibers of this fibration can be easily computed (by the strategy given in Section 2.2) because the faces of Q^0 are "the same" as the faces of Q^1 and then the fibration induces a bijection between the sets of G-orbits of X and Y^0. Then the fibers of ϕ_0 are isomorphic to the homogeneous projective spaces $(\bigcap_{i\in I} P(\varpi_{\alpha_i}))/(P(\varpi_\beta)\cap\bigcap_{i\in I} P(\varpi_{\alpha_i}))$ (of Picard group $\mathbb{Z}$), with $\emptyset \neq I \subset \{0,\dots,n\}$. Here, we use the following notation: if α_i is trivial, $P(\varpi_{\alpha_i}) = G$ (and otherwise, it is the proper maximal parabolic subgroup of G associated to α_i).

In particular, the general fiber of the fibration is $(\bigcap_{i=0}^n P(\varpi_{\alpha_i}))/(P(\varpi_\beta)\cap\bigcap_{i=0}^n P(\varpi_{\alpha_i}))$ and the smallest fibers are the $P(\varpi_{\alpha_i})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_i}))$ with $i \in \{0,\dots,n\}$. Then we deduce that $\alpha_0 \notin R_0$ if and only if there exists a fiber isomorphic to $G/P(\varpi_\beta)$.

• Suppose now that $a_n \neq 0$, then $\tilde{Q}^\epsilon$ is the intersection of the simplex $\tilde{Q} = \mathrm{Conv}(e_0^*, e_1^*, \dots, e_n^*)$ with the closed half-space $H_+^\epsilon := \{x \in M_\mathbb{Q} \mid a_1x_1 + \cdots + a_nx_n \geq \epsilon - 1\}$, where $e_0^* := 0$. We denote by H_{++}^ϵ the interior of H_+^ϵ and by H^ϵ the hyperplane $H_+^\epsilon \backslash H_{++}^\epsilon$.

In the next proposition, we give a description of the non-empty faces of $\tilde{Q}^\epsilon$ by distinguishing whether a face is in the hyperplane H^ϵ or not.

Note first that the non-empty faces of the simplex $\tilde{Q}$ are the $F_I := \mathrm{Conv}(e_i^* \mid i \in \{0,\dots,n\}\backslash I)$, with $I \subsetneq \{0,\dots,n\}$. In particular, the facets of $\tilde{Q}$ are the $F_i := F_{\{i\}}$ and for any $I \subsetneq \{0,\dots,n\}$, $F_I = \bigcap_{i\in I} F_i$.

Then, for any $I \subsetneq \{0,\dots,n\}$, we define $F_I^\epsilon := F_I \cap H_+^\epsilon$ and $F_{I,\beta}^\epsilon := F_I \cap H^\epsilon$. They are faces (may be empty and not distinct) of $\tilde{Q}^\epsilon$.

Proposition 5.3 (recall that $a_0 = 0$ and that $a_n \neq 0$ here).

The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\epsilon < \max_{i=0}^n(1 + a_i) = 1 + a_n$.

Suppose now that $\epsilon < 1 + a_n$. The non-empty faces of $\tilde{Q}^\epsilon$ are the distinct F_I^ϵ and $F_{I,\beta}^\epsilon$ (with $I \subsetneq \{0,\dots,n\}$) defined as follows:

* *F_I^ϵ (of codimension $|I|$) if $\epsilon < \max_{i\notin I}(1 + a_i)$;*
* *$F_{I,\beta}^\epsilon$ (of codimension $|I|+1$ or $|I|$ respectively) if $\min_{i\notin I}(1+a_i) < \epsilon < \max_{i\notin I}(1+a_i)$ or $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1 + a_i)$.*

In particular, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i \in \{0,\dots,n-1\}$ (for any $\epsilon < 1 + a_n$), F_n^ϵ if $\epsilon < 1 + a_{n-1}$, $F_{\emptyset,\beta}^\epsilon$ if $\epsilon > 1$, and $F_{n,\beta}^\epsilon$ if $\epsilon = 1$ and $a_{n-1} = 0$.

Moreover, we can write any face of $\tilde{Q}^\epsilon$ as the intersection of all the facets that contain it, as follows.
For any $I \subsetneq \{0,\dots,n\}$ such that $\epsilon < \max_{i\notin I}(1 + a_i)$, $F_I^\epsilon = \bigcap_{i\in I} F_i^\epsilon$.
For any $I \subsetneq \{0,\dots,n\}$ such that $\min_{i\notin I}(1 + a_i) < \epsilon < \max_{i\notin I}(1 + a_i)$, $F_{I,\beta}^\epsilon = F_{\emptyset,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$.
For any $I \subsetneq \{0,\dots,n\}$ such that $\epsilon = \min_{i\notin I}(1 + a_i) = \max_{i\notin I}(1 + a_i)$, $F_{I,\beta}^\epsilon = F_{n,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ if $\epsilon = 1$, $n \in I$ and $a_{n-1} = 0$ or $F_{I,\beta}^\epsilon = \bigcap_{i\in I} F_i^\epsilon$ if $\epsilon \neq 1$, $n \notin I$ or $a_{n-1} \neq 0$.

Proof. The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\tilde{Q}$ intersects H_{++}^ϵ if and only if there exists $i \in \{0,\dots,n\}$ such that $e_i^* \in H_{++}^\epsilon$ if and only if there exists $i \in \{0,\dots,n\}$ such that $a_i > \epsilon - 1$ if and only if $a_n > \epsilon - 1$ (because $0 = a_0 \leq \cdots \leq a_n$). This proves the first statement of the proposition.

Suppose now that $\epsilon < 1 + a_n$. For any non-empty face F of $\tilde{Q}^\epsilon$, either $F \not\subset H^\epsilon$ and F is the intersection of a non-empty face of $\tilde{Q}$ with H_+^ϵ, or $F \subset H^\epsilon$ and F is the intersection of a non-empty face of $\tilde{Q}$ with H^ϵ.

Let $I \subsetneq \{0,\ldots,n\}$. The set F_I^ϵ is not empty if and only if there exists $i \notin I$ such that $e_i^* \in H_+^\epsilon$ if and only if there exists $i \notin I$ such that $a_i \geq \epsilon - 1$ if and only if $\epsilon \leq \max_{i\notin I}(1+a_i)$. Moreover, F_I^ϵ is not empty and not included in H^ϵ if and only if it intersects H_{++}^ϵ if and only if there exists $i \notin I$ such that $e_i^* \in H_{++}^\epsilon$ if and only if there exists $i \notin I$ such that $a_i > \epsilon - 1$ if and only if $\epsilon < \max_{i\notin I}(1+a_i)$. Also, in that latter case, the dimension of F_I^ϵ is the same as the dimension of F_I; in particular the non-empty F_I^ϵ that are not included in H^ϵ are all distinct.

Similarly, $F_{I,\beta}^\epsilon$ is not empty if and only if there exist i and j not in I (may be equal) such that $e_i^* \in H_+^\epsilon$ and $e_j^* \notin H_{++}^\epsilon$ (i.e., $a_i \geq \epsilon - 1$ and $a_j \leq \epsilon - 1$). Then $F_{I,\beta}^\epsilon$ is not empty if and only $\min_{i\notin I}(1+a_i) \leq \epsilon \leq \max_{i\notin I}(1+a_i)$. Moreover, $F_{I,\beta}^\epsilon$ is not empty and included in no proper face of F_I (i.e., H^ϵ intersects the relative interior of F_I) if and only if there exist $i \neq j$ not in I such that $e_i^* \in H_{++}^\epsilon$ and $e_j^* \notin H_+^\epsilon$ (i.e., $a_i > \epsilon - 1$ and $a_j < \epsilon - 1$) or for any $i \notin I$ we have $e_i^* \in H^\epsilon$ (i.e., $a_i = \epsilon - 1$). Then $F_{I,\beta}^\epsilon$ is not empty and included in no proper face of F_I if and only $\min_{i\notin I}(1+a_i) < \epsilon < \max_{i\notin I}(1+a_i)$ or $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1+a_i)$. Note also that the non-empty $F_{I,\beta}^\epsilon$ that are not included in a proper face of F_I are all distinct and yield all non-empty faces of $\tilde{Q}^\epsilon$ included in H^ϵ. This finishes the proof of the second statement of the proposition.

To describe the facets, it is sufficient to find the F_i^ϵ with $\epsilon < \max_{j\neq i}(1+a_j)$, the $F_{i,\beta}^\epsilon$ with ϵ equal to both $\min_{j\neq i}(1+a_j)$ and $\max_{j\neq i}(1+a_j)$, and $F_{\emptyset,\beta}^\epsilon$ with $1 = \min_{i=0}^n(1+a_i) < \epsilon < \max_{i=0}^n(1+a_i) = 1+a_n$. We easily find the F_i^ϵ with $i \in \{0,\ldots,n-1\}$ for any $\epsilon < 1+a_n$, and F_n^ϵ for any $\epsilon < 1+a_{n-1}$. We conclude by noticing that, for any $i \in \{0,\ldots,n\}$, we have $\epsilon = \min_{j\neq i}(1+a_j) = \max_{j\neq i}(1+a_j) < 1+a_n$ if and only if $i = n$ and $0 = a_0 = \cdots = a_{n-1}$ (and in particular, $\epsilon = 1$).

To get the last statement, apply the fact that any face of a polytope is the intersection of the facets containing it. □

From Proposition 5.3, we deduce the following result with the following notation. Let us denote by $i_0 := 0, i_1, \ldots, i_k, i_{k+1} := n+1$ some increasing positive integers so that

$$0 = a_{i_0} = \cdots = a_{i_1-1} < a_{i_1} = \cdots = a_{i_1-1} < \cdots < a_{i_l} = \cdots = a_{i_l-1} < \cdots < a_{i_k} = \cdots = a_n.$$

Corollary 5.4. *The isomorphism classes of the horospherical varieties X^ϵ associated to the polytopes in the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ are given by the following subsets of $\mathbb{Q}_{\geq 0}$:*

* $[0,1[$;
* $]1+a_{i_l}, 1+a_{i_{l+1}}[$ *for any* $l \in \{0,\ldots,k-2\}$;
* $\{1+a_{i_l}\}$ *for any* $l \in \{0,\ldots,k-2\}$;
* $]1+a_{i_{k-1}}, 1+a_{i_k}[$ *and* $\{1+a_{i_{k-1}}\}$ *if* $i_k \neq n$ *(i.e., if* $a_{n-1} = a_n$*) or the simple root* α_n *is not trivial (i.e., when* X *is as in Case (1b) of Theorem 1.1);*
* $[1+a_{i_{k-1}}, 1+a_{i_k}[$ *if* $i_k = n$ *(i.e., if* $a_{n-1} < a_n$*) and the simple root* α_n *is trivial (i.e., when* X *is as in Case (1c) of Theorem 1.1).*

Proof. We apply the theory described in Section 2.2, in particular the fact that the isomorphism classes of the varieties X^ϵ are obtained by looking at the ϵ's for which "the faces of Q^ϵ change".

Note first that, by Proposition 5.3, $(P, M, Q^\epsilon, \tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon < 1+a_n$.

Also, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i \in \{0,\ldots,n-1\}$, F_n^ϵ if $\epsilon < 1+a_{n-1}$, $F_{\emptyset,\beta}^\epsilon$ if $\epsilon > 1$, and $F_{n,\beta}^\epsilon$ (orthogonal to $\alpha_{n,M}^\vee$) if $\epsilon = 1$ and $a_{n-1} = 0$. In particular, for any $\epsilon, \eta \in [0, 1+a_n[$, if $a_{n-1} \neq 0$, the facets of Q^ϵ and Q^η are "the same" if and only if ϵ and η are both in $[0,1]$ or $]1, 1+a_{n-1}[$ or $[1+a_{n-1}, 1+a_n[$ (which may be empty). And if $a_{n-1} = 0$, the facets of Q^ϵ and Q^η are "the same" for any $\epsilon, \eta \in [0, 1+a_n[$ (indeed, in that case, the facets F_n^ϵ if $\epsilon < 1$, $F_{\emptyset,\beta}^\epsilon$ if $\epsilon > 1$, and $F_{n,\beta}^\epsilon$ if $\epsilon = 1$ are "the same", in particular all orthogonal to $\beta_M^\vee = a_n\alpha_{n,M}^\vee$).

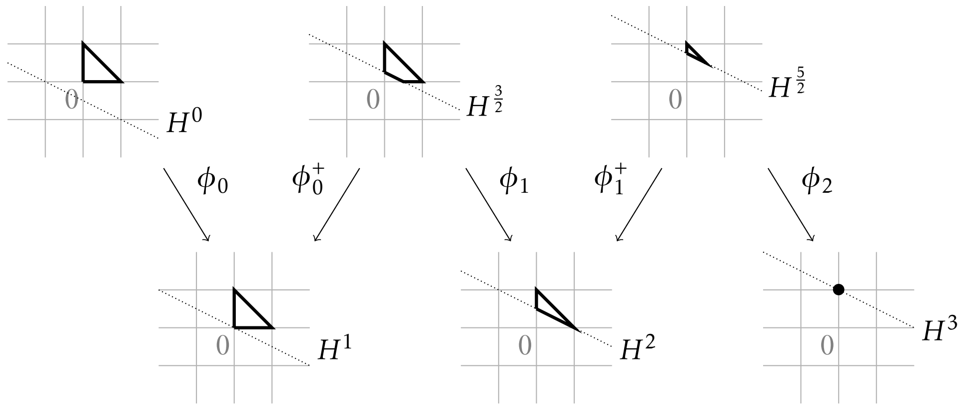

Figure 4. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 1$, $a_2 = 2$ and α_2 is not trivial.

We now use a consequence of the proof of Proposition 5.3: for any $I \subsetneq \{0,\dots,n\}$, $\bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\epsilon \leq \max_{i\notin I}(1+a_i)$, $F_{\emptyset,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\min_{i\notin I}(1+a_i) \leq \epsilon \leq \max_{i\notin I}(1+a_i)$ and $F_{n,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty if and only if $\min_{i\notin I}(1+a_i) = \epsilon = \max_{i\notin I}(1+a_i)$. In particular for any $l \in \{0,\dots,k-2\}$, suppose that for $I = \{i_{l+1},\dots,n\}$ and that $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0,\dots,i_l-1\}$ and that $F_{\emptyset,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon = 1 + a_{i_l}$. Similarly for any $l \in \{0,\dots,k-2\}$, suppose that for $I = \{i_{l+1}-1,\dots,n\}$ and $\bigcap_{i\in I} F_i^\epsilon$ is not empty; suppose also that for $I = \{0,\dots,i_l-1\}$ and that $F_{\emptyset,\beta}^\epsilon \cap \bigcap_{i\in I} F_i^\epsilon$ is not empty; then $\epsilon \in [1+a_{i_l}, 1+a_{i_{l+1}}]$. If $i_k \neq n$, F_n^ϵ is still a facet of Q^ϵ and what we did above with $l \in \{0,\dots,k-2\}$ can be done as well with $l = k-1$.

Hence, this proves that if the two varieties X^ϵ and X^η are isomorphic then ϵ and η are in one of the subsets described in the corollary.

To conclude, we have to prove that the two varieties X^ϵ and X^η are isomorphic when ϵ and η are in one of these subsets. It is obvious from Proposition 5.3 except in the case where $i_k = n$ and the simple root α_n is trivial. But in that case, all polytopes Q^ϵ with $\epsilon \in [1+a_{n-1}, 1+a_n[= [1+a_{i_{k-1}}, 1+a_{i_k}[$ are simplexes with facets F_i^ϵ for $i \in \{0,\dots,n-1\}$ and $F_{\emptyset,\beta}^\epsilon$ or $F_{n,\beta}^\epsilon$ if $\epsilon = 1+a_{n-1} = 1$, i.e., they could be defined even deleting the row corresponding to the simple root α_n that is trivial, so that their faces are "the same". □

We can reformulate this corollary as follows, and get the first statement of Theorem 1.3 in Case (1). We denote $X^0 = X$ and for any $l \in \{1,\cdots,k\}$, $X^l := X^\epsilon$ with $\epsilon \in]1+a_{i_{l-1}}, 1+a_{i_l}[$, and for any $l \in \{0,\cdots,k\}$, $Y^l := X^{1+a_{i_l}}$.

Corollary 5.5. *The family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ describes a Log MMP from X as follows:*

- *k flips $\phi_l : X^l \longrightarrow Y^l \longleftarrow X^{l+1} : \phi_l^+$ for any $l \in \{0,\cdots,k-1\}$ and a fibration $\phi_k : X^k \longrightarrow Y^k$, if $i_k \neq n$ or the simple root α_n is not trivial;*
- *$k-1$ flips $\phi_l : X^l \longrightarrow Y^l \longleftarrow X^{l+1} : \phi_l^+$ for any $l \in \{0,\cdots,k-2\}$, followed by a divisorial contraction $\phi_{k-1} : X^{k-1} \longrightarrow Y^{k-1} \simeq X^k$ and a fibration $X^k \longrightarrow Y^k \simeq \mathrm{pt}$, if $i_k = n$ and the simple root α_n is trivial.*

Example 5.6. In the five different cases with $n = 2$ and $a_2 \neq 0$, we illustrate this corollary in terms of polytopes in Figures 4, 5, 6, 7 and 8.

5.3. Proof of the last statement of Theorem 1.3 in Case (1)

The previous section proves that $a_{i_1},\dots,a_{i_k}$ are invariants of X. To finish the proof of Theorem 1.3 in Case (1), we have to prove that $G_0,\dots,G_t$, $\alpha_0,\dots,\alpha_n,\beta$ and $i_1,\dots,i_k$ are also invariants of X. For this, we have to describe some exceptional loci and some fibers of the different morphisms of the Log MMP.

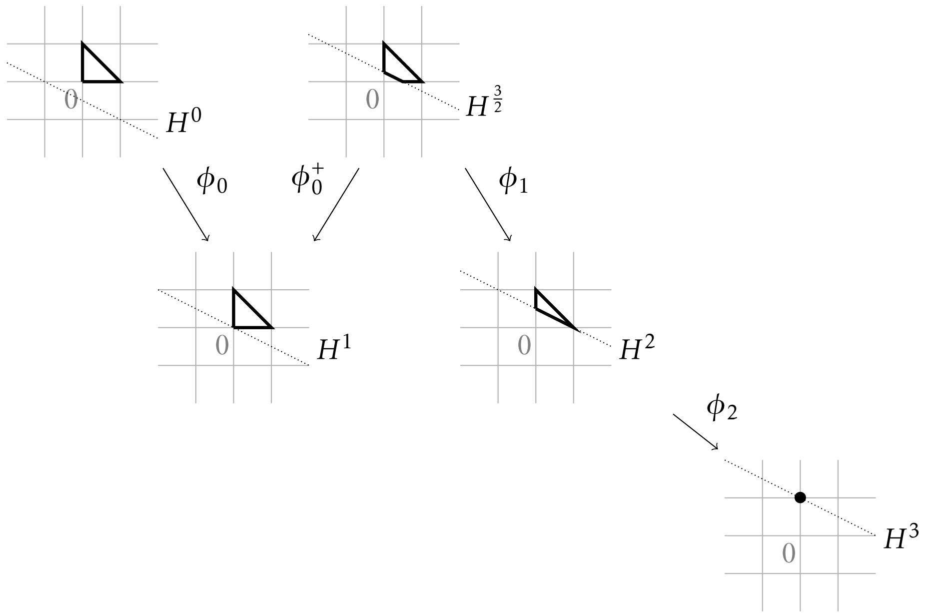

Figure 5. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 1$, $a_2 = 2$ and α_2 is trivial.

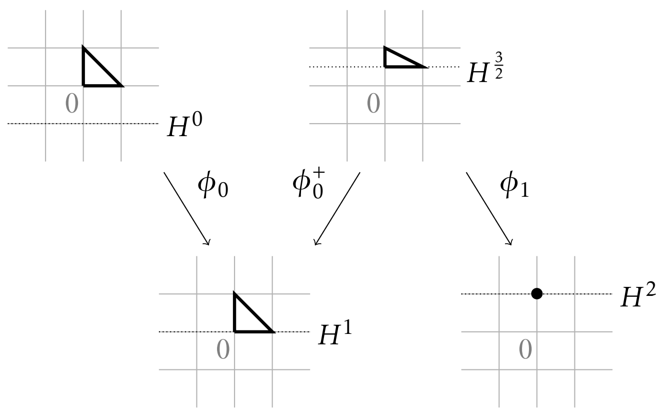

Figure 6. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 0$, $a_2 = 1$ and α_2 is not trivial.

We first distinguish two cases by the following result.

Proposition 5.7. *Define the simple subgroups of $P(\varpi_\beta)$ as in Definition 4.3.*

- *Suppose that $n = 1$ and that α_0 and α_1 are two simple roots of the same simple subgroup of $P(\varpi_\beta)$. Then, the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is either a homogeneous variety different from a projective space (a quadric Q^{2m} with $m \geq 2$, a Grassmannian $\mathrm{Gr}(i,m)$ with $p \geq 5$ and $2 \leq i \leq m-2$, or a spinor variety $\mathrm{Spin}(2m+1)/P(\varpi_m)$ with $m \geq 4$), or a two-orbit variety as in [Pas09].*
- *Suppose that $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$. Then, the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is a projective space.*

Proof. The fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ is the smooth projective $P(\varpi_\beta)$-variety of Picard group $\mathbb{Z}$ isomorphic to the closure of the $P(\varpi_\beta)$-orbit of a sum of highest weight vectors in $\mathbb{P} := \mathbb{P}(V(\varpi_{\alpha_0}) \oplus \cdots \oplus V(\varpi_{\alpha_n}))$. Hence, the proposition is a consequence of [Pas09, Section 1]. □

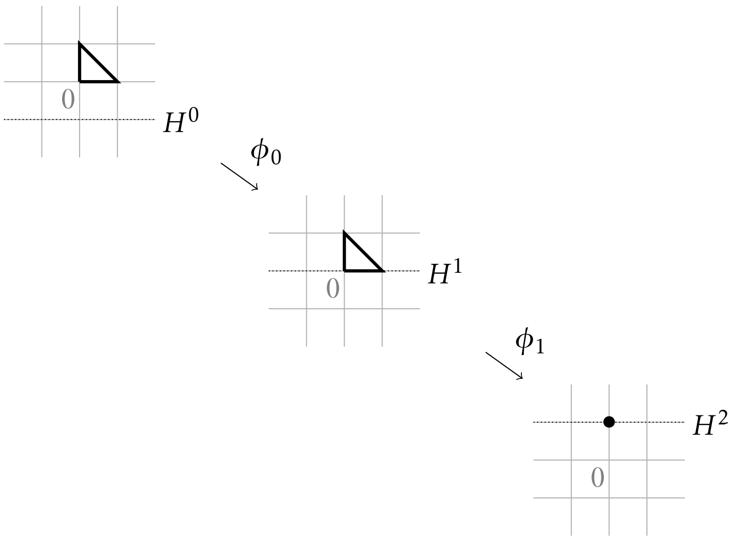

Figure 7. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 0$, $a_2 = 1$ and α_2 is trivial.

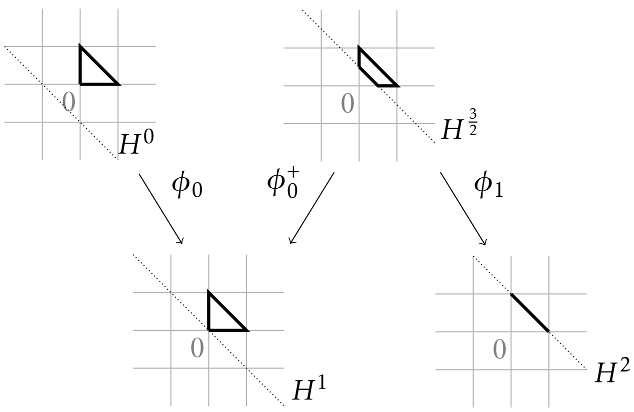

Figure 8. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$ and $a_1 = a_2 = 1$.

• In the case where $n = 1$ and that α_0 and α_1 are two simple roots of the same simple subgroup of $P(\varpi_\beta)$, $G = G_0$, the Log MMP described by Corollary 5.5 consists of a fibration if $a_1 = 0$, or a flip and a fibration if $a_1 > 0$.

– Suppose first that $a_1 = 0$. There are two cases to deal with.

If α_1 is between α_0 and β in the Dynkin diagram of G_0 (and similarly, up to exchanging α_0 and α_1, α_0 is between α_1 and β), since $X \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + \varpi_\beta))$ and $Y^0 \subset \mathbb{P}(V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1}))$, we easily compute that the fibration $\phi_0 : X \longrightarrow Y^0$ has two different types of fibers: one isomorphic to $P(\varpi_{\alpha_0})/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$ over a G-orbit isomorphic to $G/P(\varpi_{\alpha_0})$ and another one of smaller dimension isomorphic to $P(\varpi_{\alpha_1})/(P(\varpi_{\alpha_1}) \cap P(\varpi_\beta))$.

In particular, the pair $(G/P(\varpi_{\alpha_0}), G/P(\varpi_\beta))$ is an invariant of X. Then if G_0 is not the universal cover of the automorphism group of $G/P(\varpi_\beta)$ it must be the universal cover of the automorphism group of $G/P(\varpi_{\alpha_0})$, so that G_0 is an invariant of X. Also, $\phi_0^{-1}(G/P(\varpi_{\alpha_0})) = G/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$, then the pair (α_0, β) is an invariant of X up to symmetries of the Dynkin diagram of G_0.

Moreover, if β is fixed, the possible symmetries are the ones (which fixed β) in type A_m with $m \geq 5$ odd, $\varpi_\beta = \varpi_{\frac{m+1}{2}}$ and any α_0, type E_6 with $\varpi_\beta = \varpi_4$ and $\varpi_{\alpha_0} = \varpi_1, \varpi_3, \varpi_5$ or ϖ_6 , and type D_m with $m \geq 4$, $\varpi_\beta = \varpi_i$ for any $i \in \{1, \ldots, m-2\}$ and $\varpi_{\alpha_0} = \varpi_{m-1}$ or ϖ_m.

The description of the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$, together with Remark 4.2, implies that α_0 and α_1 are also invariants of X up to symmetries of the Dynkin diagram of G_0.

Otherwise (this occurs only in types D and E), G_0 is the universal cover of the automorphism group of $G/P(\varpi_\beta)$, and then G_0 and β are invariants of X up to symmetries of the Dynkin diagram of G_0.

We also easily compute that the fibration $\phi_0 : X \longrightarrow Y^0$ has at least two different types of fibers: one smaller isomorphic to $(P(\varpi_{\alpha_0}) \cap P(\varpi_{\alpha_1}))/(P(\varpi_{\alpha_0}) \cap P(\varpi_{\alpha_1}) \cap P(\varpi_\beta))$ over the open G-orbit of Y^0, and two others fibers isomorphic to $P(\varpi_{\alpha_0})/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$ over $G/P(\varpi_{\alpha_0})$, respectively isomorphic to $P(\varpi_{\alpha_1})/(P(\varpi_{\alpha_1}) \cap P(\varpi_\beta))$ over $G/P(\varpi_{\alpha_1})$ (note that the latter are possibly isomorphic).

In particular, the pair $(G/P(\varpi_{\alpha_0}), G/P(\varpi_{\alpha_1}))$ is an invariant of X and then the pair (α_0, α_1) is also an invariant of X up to symmetries of the Dynkin diagram of G_0.

– Suppose now that $a_1 > 0$. We have then the following inclusions

$$X \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_\beta) \oplus V(\varpi_{\alpha_1} + (1 + a_1)\varpi_\beta)),\ Y^0 \subset \mathbb{P}(V(\varpi_{\alpha_0}) \oplus V(\varpi_{\alpha_1} + a_1\varpi_\beta)),$$
$$X^1 \subset \mathbb{P}(V(\varpi_{\alpha_0} + \varpi_{\alpha_1}) \oplus V(2\varpi_{\alpha_1} + a_1\varpi_\beta)) \text{ and } Y^1 \simeq G/P(\varpi_{\alpha_1}) \subset \mathbb{P}(V(\varpi_{\alpha_1})).$$

In particular X, Y^0 and X^1 have two closed G-orbits and one open G-orbit so that we easily compute exceptional locus and fibers as follows. For example, the exceptional locus of $\phi_0 : X \longrightarrow Y^0$ is the G-orbit of X isomorphic to $G/(P(\varpi_{\alpha_0}) \cap P(\varpi_\beta))$. Then the universal cover of its automorphism group G_0 is an invariant of X. And then β is also an invariant of X up to symmetries of the Dynkin diagram of G_0.

Note now that the exceptional locus of ϕ_0 is sent to the G-orbit of Y^0 isomorphic to $G/P(\varpi_{\alpha_0})$ so that the triple $(G/P(\varpi_{\alpha_0}), G/P(\varpi_{\alpha_1}), G/P(\varpi_\beta))$ is an invariant of G. Also the (same) description of the fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$ implies that the subgroup or $P(\varpi_\beta)$ and the pair (α_0, α_1) are invariants of X (up to symmetries in type A, D and E as in the case where $a_1 = 0$). Hence, the triple $(\beta, \alpha_0, \alpha_1)$ is an invariant of X up to symmetries of the Dynkin diagram of G_0.

• Now we suppose that $n > 1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.

We define various exceptional loci in X as follows. Let $l \in \{0, \ldots, k-1\}$, define E_l to be the closure in X of the set of points $x \in X$ such that x is in the open isomorphism set of the first l contractions and x is in the exceptional locus of ϕ_l.

Proposition 5.8. *For any $l \in \{0, \ldots, k\}$ the exceptional locus E_l is the closure in X of the G-orbit associated to the non-empty face F_{I_l} of Q with $I_l := \{i_{l+1}, \ldots, n\}$. In particular E_l is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in*

$$\mathbb{P} := \mathbb{P}(\bigoplus_{i=0}^{i_{l+1}-1} V(\varpi_{\alpha_i} + (1 + a_i)\varpi_\beta)),$$

and E_l is a smooth projective horospherical of Picard group $\mathbb{Z}^2$ as in Case (1), unless $l = 0$, $i_1 = 1$ so that E_l is homogeneous (projective of Picard group $\mathbb{Z}$ or $\mathbb{Z}^2$).

Note that for $l = k$, $I_k = \emptyset$ and $E_k = X$.

Proof. Let $l \in \{0, \ldots, k\}$ and $\epsilon_l \in \mathbb{Q}_{\geq 0}$ such that $X^l = X^{\epsilon_l}$.

We denote by Ω_I^l and $\Omega_{I,\beta}^l$ the G-orbits of X^l associated to the non-empty faces $F_I^{\epsilon_l}$ and $F_{I,\beta}^{\epsilon_l}$ of the polytope $\tilde{Q}^{\epsilon_l}$. We denote by ω_I^l and $\omega_{I,\beta}^l$ the G-orbits of $Y^l=X^{1+a_{i_l}}$ associated to the non-empty faces $F_I^{1+a_{i_l}}$ and $F_{I,\beta}^{1+a_{i_l}}$ of the polytope $\tilde{Q}^{1+a_{i_l}}$. Recall that, for any $\epsilon\in\mathbb{Q}_{\geq 0}$, we have an order on the G-orbits of X^ϵ compatible with the order on the non-empty faces of $\tilde{Q}^\epsilon$: in particular $\Omega_I^l\subset\overline{\Omega_{I'}^l}$ and $\Omega_{I,\beta}^l\subset\overline{\Omega_{I',\beta}^l}$ respectively if and only if $I'\subset I$, and $\Omega_{I,\beta}^l\subset\overline{\Omega_I^l}$ (as soon as these orbits are defined, i.e., as soon as the corresponding faces are non-empty).

For any $I\subsetneq\{0,\dots,n\}$ such that there exists $i\geq i_l$ not in I (i.e., such that Ω_I^l is defined), $\phi_l(\Omega_I^l)=\omega_I^l$ if there exists $i\geq i_{l+1}$ not in I, and $\phi_l(\Omega_I^l)=\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ if for any $i\geq i_{l+1}$, $i\in I$. Indeed $I\cup\{0,\dots i_l-1\}$ is the minimal subset of $\{0,\dots,n\}$ containing I such that $\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ is defined and there is no I' containing I such that $\omega_{I'}^l$ is defined. And for any $I\subsetneq\{0,\dots,n\}$ such that there exist $i\geq i_l$ and $i'<i_l$ not in I (i.e., such that $\Omega_{I,\beta}^l$ is defined), $\phi_l(\Omega_{I,\beta}^l)=\omega_{I,\beta}^l$ if there exists $i\geq i_{l+1}$ not in I, and $\phi_l(\Omega_{I,\beta}^l)=\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ if for any $i\geq i_{l+1}$, $i\in I$. Indeed $I\cup\{0,\dots i_l-1\}$ is the minimal subset of $\{0,\dots,n\}$ containing I such that $\omega^l_{I\cup\{0,\dots i_l-1\},\beta}$ is defined.

In particular, we have $\phi_l(\Omega_{I_l}^l)=\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}$ (which is also $\phi_l(\Omega^l_{I_l,\beta})$ if $l\geq 1$). But $\Omega_{I_l}^l$ and $\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}$ are non isomorphic horospherical homogeneous spaces by Proposition 2.14, so that $\Omega_{I_l}^l$ is in the exceptional locus of ϕ_l. Moreover, if Ω is a G-orbit of X^{ϵ_l} not contained in $\overline{\Omega_{I_l}^l}$, it is of the form Ω_I^l or $\Omega_{I,\beta}^l$ where $I_l\not\subset I$. Hence, in that case $\phi_l(\Omega)=\Omega$. And then the exceptional locus of ϕ_l is $\overline{\Omega_{I_l}^l}$. Note that $\Omega_{I_l}^0,\dots,\Omega_{I_l}^{l-1}$ are not in the exceptional locus of $\phi_0,\dots,\phi_{l-1}$ respectively, to conclude that $E_l=\overline{\Omega_{I_l}^0}$.

We use again Proposition 2.14 to see that $E_l=\overline{\Omega_{I_l}^0}$ corresponds to the admissible quadruple $(P_F,M_F,F,\tilde{F})$ with $F=F_{I_l}^0$ (and with some ample divisor of E_l). Then we conclude by Corollaries 2.6 and 2.10. ☐

The Log MMP now defines, by restriction, fibrations $\tilde{\phi}_l: E_l\backslash E_{l-1}\longrightarrow E_l':=\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$, for any $l\in\{0,\dots,k\}$.

Definition 5.9. We say that the fibers of $\tilde{\phi}_l$ are locally maximal over $\omega\subset E_l'$ if the dimensions of the fibers of $\tilde{\phi}_l$ over any point of ω are the same and bigger than the dimension of the fibers of $\tilde{\phi}_l$ over any point of a neighborhood of ω that is not in ω.

We say that the fibers of $\tilde{\phi}_l$ are locally almost maximal over $\omega\subset E_l'$ if there exists $\omega'\subsetneq\omega$ such that the fibers of $\tilde{\phi}_l$ are locally maximal over ω' and the fibers of $\tilde{\phi}_{l|\tilde{\phi}_l^{-1}(E_l'\backslash\omega')}$ are locally maximal over $\omega\backslash\omega'\subset E_l'\backslash\omega'$.

We now prove the following result, which implies in particular that $i_1,\dots,i_k$ are invariant of X.

Proposition 5.10. *Suppose that $n>1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$. Let $l\in\{0,\dots,k\}$.*

The map $\tilde{\phi}_l$ is surjective and we distinguish four distinct cases.

1. *we have $i_{l+1}-i_l=1$ and α_{i_l} is not a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over E_l' and $\dim E_l-\dim E_{l-1}=1+\dim E_l'$ (here we set $\dim E_{-1}:=\dim G/P(\varpi_\beta)-1$ so that it still holds for $l=0$). Moreover, E_l' is homogeneous and isomorphic to $G/P(\varpi_{\alpha_{i_l}})$ (which is a point if α_{i_l} is trivial).*
2. *we have $i_{l+1}-i_l=1$ and α_{i_l} is a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over E_l' and $\dim E_l-\dim E_{l-1}\neq 1+\dim E_l'$ (also here $\dim E_{-1}:=\dim G/P(\varpi_\beta)-1$ so that it still holds for $l=0$). Moreover, E_l' is homogeneous and isomorphic to $G/P(\varpi_{\alpha_{i_l}})$.*
3. *we have $i_{l+1}-i_l>1$ and α_{i_l} is not a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over a unique proper subset of E_l', which is a closed G-orbit ω' of E_l' isomorphic to $G/P(\varpi_{\alpha_{i_l}})$. Also the fibers of*

$\tilde{\phi}_l$ are locally almost maximal over exactly $i_{l+1}-i_l-1(>0)$ subvarieties of E'_l containing ω', respectively of dimensions $\dim G/P(\varpi_{\alpha_{i_l}})+\dim G/P(\varpi_{\alpha_j})+1$ with $j\in\{i_l+1,\dots,i_{l+1}-1\}$.

(4) we have $i_{l+1}-i_l>1$ and α_{i_l} is a simple root of G_0. The fibers of $\tilde{\phi}_l$ are locally maximal over $i_{l+1}-i_l$ closed G-orbits, which are respectively isomorphic to $G/P(\varpi_{\alpha_j})$ with $j\in\{i_l,\dots,i_{l+1}-1\}$.

Moreover, in the four cases, the dimension of the fibers over all pointed subsets of E'_l are as follows.

(1) The dimension of the fibers of $\tilde{\phi}_l$ is $1+\dim E_{l-1}$ (in particular $\dim G/P(\varpi_\beta)$ if $l=0$).

(2) The dimension of the fibers of $\tilde{\phi}_l$ is

$$d_{i_l}:=i_l+\dim\left(P(\varpi_{\alpha_{i_l}})/(P(\varpi_\beta)\cap\bigcap_{i=0}^{i_l}P(\varpi_{\alpha_i}))\right).$$

(3) The dimension of the locally maximal fibers of $\tilde{\phi}_l$ is $1+\dim E_{l-1}$ (in particular $\dim G/P(\varpi_\beta)$ if $l=0$). And for any $j\in\{i_l+1,\dots,i_{l+1}-1\}$, the dimension of locally almost maximal fibers of $\tilde{\phi}_l$ over the subset of E'_l of dimension $\dim G/P(\varpi_{\alpha_{i_l}})+\dim G/P(\varpi_{\alpha_j})+1$ is

$$d_j:=i_l+\dim\left(P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap\bigcap_{i=0}^{i_l-1}P(\varpi_{\alpha_i})\cap P(\varpi_{\alpha_j}))\right).$$

(4) For any $j\in\{i_l,\dots,i_{l+1}-1\}$, the dimension of locally maximal fibers of $\tilde{\phi}_l$ over the closed G-orbit isomorphic to $G/P(\varpi_{\alpha_j})$ is

$$d_j:=i_l+\dim\left(P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap\bigcap_{i=0}^{i_l-1}P(\varpi_{\alpha_i})\cap P(\varpi_{\alpha_j}))\right).$$

Proof. We keep the notation of the proof of Proposition 5.8. And we use Corollary 2.16 to compute the dimension of the fibers. Let ω be a G-orbit of Y^l in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$. Then there exists $I\subsetneq\{0,\dots,n\}$ containing $I_l\cup\{0,\dots i_l-1\}$ such that $\omega=\omega^l_{I,\beta}$. Then $\tilde{\phi}_l^{-1}(\omega)=\bigsqcup_J\Omega^l_J$ where the union is taken over all J such that $J\cap I_{l-1}=I\cap I_{l-1}$. In particular, $\tilde{\phi}_l$ is surjective and $\overline{\tilde{\phi}_l^{-1}(\omega)}=\overline{\Omega^l_{I\cap I_{l-1}}}$. We then compute

$$\dim(\omega)=\dim(F^l_{I,\beta})+\dim(G/\bigcap_{i\notin I}P(\varpi_{\alpha_i})),$$
$$\text{and }\dim(\Omega^l_{I\cap I_{l-1}})=\dim(F_{I\cap I_{l-1}})+\dim(G/P(\varpi_\beta)\cap\bigcap_{i\notin I\cap I_{l-1}}P(\varpi_{\alpha_i})),$$

so that the dimension $\delta_{l,\omega}$ of a fiber of $\tilde{\phi}_l$ over ω is

$$\begin{aligned}\delta_{l,\omega}&=\dim(F_{I\cap I_{l-1}})-\dim(F^l_{I,\beta}))+(\dim(G/(P(\varpi_\beta)\cap\bigcap_{i\notin I\cap I_{l-1}}P(\varpi_{\alpha_i}))-\dim(G/\bigcap_{i\notin I}P(\varpi_{\alpha_i})))\\&=i_l+\dim(\bigcap_{i\notin I}P(\varpi_{\alpha_i})/(P(\varpi_\beta)\cap\bigcap_{i\notin I\cap I_{l-1}}P(\varpi_{\alpha_i}))\\&=i_l+\dim(\bigcap_{i\notin I}P(\varpi_{\alpha_i})/(\bigcap_{i\notin I}P(\varpi_{\alpha_i})\cap\bigcap_{i=0}^{i_l-1}P(\varpi_{\alpha_i})\cap P(\varpi_\beta)).\end{aligned}$$

The dimension $\delta_{l,\omega}$ is the biggest when I is as big as possible (it would be $I=\{0,\dots,n\}$ if it was allowed to define ω). Moreover, if we remove from I some i, the dimension changes if and only if j is such that α_i is in G_0 (i.e., α_i is not trivial and not the only simple root α_j in a simple group of G different from G_0, by hypothesis). From this, we will deduce the different following cases.

If α_{i_l} is not a simple root of G_0, then the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fibers of $\tilde{\phi}_l$ are maximal is the unique closed G-orbit $\omega' := \omega^l_{\{0,\dots n\}\backslash\{i_l\},\beta}$ isomorphic to $G/P(\varpi_{\alpha_{i_l}})$. This gives the first case of the proposition if $i_{l+1}-i_l=1$. And if $i_{l+1}-i_l>1$ the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fiber of $\tilde{\phi}_l$ is almost maximal is the union of the subsets $\omega^l_{\{0,\dots n\}\backslash\{i_l,j\},\beta}\cup\omega'$ with $j\in\{i_l+1,\dots,i_{l+1}-1\}$, which are affine cones over $G/P(\varpi_{\alpha_i})$. This gives the third case of the proposition.

Now, if α_{i_l} is a simple root of G_0 (i.e., for any $j\in\{i_l,\dots,i_{l+1}-1\}$, α_j is a simple root of G_0), then the locus in $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$ where the fiber of $\tilde{\phi}_l$ is maximal is the (disjoint) union of the $i_{l+1}-i_l$ closed G-orbits $\omega^l_{\{0,\dots n\}\backslash\{j\},\beta}$ of $\overline{\omega^l_{I_l\cup\{0,\dots i_l-1\},\beta}}$, which are respectively isomorphic to $G/P(\varpi_{\alpha_j})$ for any $j\in\{i_l,\dots,i_{l+1}-1\}$. This gives the second case of the proposition if $i_{l+1}-i_l=1$ and the fourth case if $i_{l+1}-i_l>1$.

□

We easily deduce the following.

Corollary 5.11. *With the notation of Proposition 5.10: for any $j\in\{0,\dots,n\}$, we have*

$$\dim G/P(\varpi_\beta)+d_j-\dim E_{l-1}-1=\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))$$
$$\textit{and}\quad \dim G/P(\varpi_{\alpha_j})+d_j-\dim E_{l-1}-1=\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})).$$

In particular, for any $l\in\{0,\dots,k\}$, the sets

$$\{(\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})),\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})))\mid j\in\{i_l,\dots,i_{l+1}-1\}\}$$

are invariants of X.

And then we conclude the proof of Case (1) of Theorem 1.3 (i.e., that G_0, β, $\alpha_0,\dots,\alpha_n$ are invariants of X) by the following lemma (still in the case where $n>1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$).

Lemma 5.12. *Let G, G' be two products of simply connected simple groups and $\mathbb{C}^*$'s. Let β, β' be two simple roots of two of the simple factors G_0 and G'_0 of G and G' respectively. And let $\alpha_0,\dots,\alpha_n$, respectively $\alpha'_0,\dots,\alpha'_n$ be simple roots of G, G' both as in Case (1) of Theroem 1.1 (with the same integers k and $i_1,\dots,i_k$).*

Suppose that $G/P(\varpi_\beta)$ is isomorphic to $G'/P(\varpi_{\beta'})$ and that for any $l\in\{0,\dots,k\}$,

$$\{(\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})),\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j})))\mid j\in\{i_l,\dots,i_{l+1}-1\}\}=$$
$$\{(\dim P(\varpi_{\alpha'_j})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha'_j})),\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha'_j})))\mid j\in\{i_l,\dots,i_{l+1}-1\}\}.$$

Then $G=G'$, $\beta=\beta'$ and for any $i\in\{0,\cdots,n\}$, $\alpha_i=\alpha'_i$ up to reordering the α_i's and α'_i's inside the sets $\{i_l,\dots,i_{l+1}-1\}$.

Proof. We proceed in several steps.

Step 1. For any $l\in\{0,\dots,k\}$, $\alpha_{i_l}\notin R_0$ if and only if $\alpha_{i'_l}\notin R'_0$, and in that case, α_{i_l} and $\alpha_{i'_l}$ are both extremal simple roots of SL_{m+1} with $m=\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha'_j}))$. Indeed, we have that $\alpha_{i_l}\notin R_0$ if and only if

$$\dim P(\varpi_{\alpha_{i_l}})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_{i_l}}))=\dim G/P(\varpi_\beta)=\dim G/P(\varpi_{\beta'})=\dim P(\varpi_{\alpha'_{i_l}})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha'_{i_l}}))$$

and this is equivalent to saying that $\alpha_{i'_l}\notin R'_0$. The second statement is obvious from the hypothesis on the α_i's and α'_i's. Note that $\alpha_{i_l+1},\dots,\alpha_{i_{l+1}-1}$ are in R_0 by hypothesis.

Step 2. $G_0=G'_0$ and $\beta=\beta'$ up to symmetries of the Dynkin diagram. otherwise, R_0 and R'_0 are not empty and $\{(G_0,\varpi_\beta),(G'_0,\varpi_{\beta'})\}$ is one of the three following sets up to symmetries of the Dynkin diagram (by [Akh95, Section 3.3]): $\{(\mathrm{Sp}_{2m},\varpi_1),(\mathrm{SL}_{2m},\varpi_1)\}$, $\{(\mathrm{Spin}_{2m+1},\varpi_m),(\mathrm{Spin}_{2m+2},\varpi_{m+1})\}$ or

$\{(G_2,\varpi_1),(\mathrm{Spin}_7,\varpi_1)\}$. Let $\alpha_j\in R_0$, there exists $l\in\{0,\dots,k\}$ such that $j\in\{i_l,\dots,i_{l+1}-1\}$. By Step 1, $\alpha_j'\in R_0'$ and up to reordering α_i's and α_i''s in $\{i_l,\dots,i_{l+1}-1\}$ we can suppose that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=\dim P(\varpi_{\alpha_j'})/(P(\varpi_\beta')\cap P(\varpi_{\alpha_j'}))$ and $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha_j'}))$. We have to check that this is not possible in the three cases.

If $((G_0,\varpi_\beta),(G_0',\varpi_{\beta'}))$ is $((\mathrm{Sp}_{2m},\varpi_1),(\mathrm{SL}_{2m},\varpi_1))$ then ϖ_{α_j} is the fundamental weight ϖ_2 of Sp_{2m} (by the smooth condition) so that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=1$ and $\varpi_{\alpha_j'}$ has to be the fundamental weight ϖ_2 (by the smooth condition and because $\dim P(\varpi_{\alpha_j'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha_j'}))=1$). But then

$$\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=2m-3<2m-2=\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha_j'})).$$

If $((G_0,\varpi_\beta),(G_0',\varpi_{\beta'}))$ is $((\mathrm{Spin}_{2m+1},\varpi_m),(\mathrm{Spin}_{2m+2},\varpi_{m+1}))$ then ϖ_{α_j} is the fundamental weight ϖ_1 or ϖ_{m-1} of Spin_{2m+1}. In both cases, $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=m-1$. But $\varpi_{\alpha_j'}$ is the fundamental weight ϖ_1 or ϖ_m of Spin_{2m+2} so that $\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha_j'}))=m$.

If $((G_0,\varpi_\beta),(G_0',\varpi_{\beta'}))$ is $((G_2,\varpi_1),(\mathrm{Spin}_7,\varpi_1))$, then ϖ_{α_j} is the fundamental weight ϖ_2 of G_2 and $\varpi_{\alpha_j'}$ is the fundamental weight ϖ_3 of Spin_7. But then

$$\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=1<3=\dim P(\varpi_{\beta'})/(P(\varpi_{\beta'})\cap P(\varpi_{\alpha_j'})).$$

We can now assume that $G_0=G_0'$ and $\beta=\beta'$. There are at most three simple subgroups of $P(\varpi_\beta)$ (their Dynkin diagram can be obtained from the Dynkin diagram of G_0 by removing β).

Step 3. Let $\alpha_j\in R_0$ and $\alpha_j'\in R_0'$ such that $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j'}))$. By the smooth condition, α_j and α_j' are extremal short simple roots of a simple subgroup of $P(\varpi_\beta)$ of type A or C. We have then $\dim P(\varpi_\beta)/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=p$ (resp. $2p-1$) if the type is A_p (resp. C_p). Hence, we have two cases: they are extremal short simple roots of simple subgroups of $P(\varpi_\beta)$ both of type A_p, or they are extremal short simple roots of simple subgroups of $P(\varpi_\beta)$ of types A_{2p-1} and C_p.

Step 4. Suppose moreover that $\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))=\dim P(\varpi_{\alpha_j'})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j'}))$, then one checks that $\alpha_j=\alpha_j'$ up to symmetries, by studying all cases up to symmetries, where $P(\varpi_\beta)$ has at least two simple subgroups of types A_p and A_p with $p\geq 1$, or A_{2p-1} and C_p with $p\geq 2$.

Type of G_0	ϖ_β	ϖ_{α_j}	$\dim P(\varpi_{\alpha_j})/(P(\varpi_\beta)\cap P(\varpi_{\alpha_j}))$
A_m, $m\geq 5$ m odd	$\varpi_{\frac{m+1}{2}}$	ϖ_1 or $\varpi_{\frac{m+3}{2}}$	$\frac{(m+1)(m-1)}{4}$ or $\frac{m+1}{2}$, and $\frac{(m+1)(m-1)}{4}=\frac{m+1}{2}\frac{m-1}{2}\geq 2\frac{m+1}{2}$
B_3	ϖ_2	ϖ_1 or ϖ_3	2 or 3
B_6	ϖ_4	ϖ_1 or ϖ_6	18 or 8
B_6	ϖ_4	ϖ_3 or ϖ_6	5 or 8
C_m, $m\geq 3$ m multiple of 3	ϖ_i $i=\frac{2}{3}m$	ϖ_1 or ϖ_{i+1}	$\frac{(4m-3i)(i-1)}{2}=\frac{3i(i-1)}{2}$ or i, and $\frac{3i(i-1)}{2}>i$ because $i\geq 2$
C_m, $m\geq 3$ m multiple of 3	ϖ_i $i=\frac{2}{3}m$	ϖ_{i-1} or ϖ_{i+1}	$2m-2i-1=i-1$ or i,
D_7	ϖ_4	ϖ_1 or ϖ_7	21 or 12
D_7	ϖ_4	ϖ_3 or ϖ_7	6 or 12
E_6	ϖ_4	ϖ_1 or ϖ_5	15 or 6

□

5.4. Case (2): the "second" Log MMP via moment polytopes

To describe the one-parameter family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ defined in Theorem 2.18, we consider the basis $(u_i^*)_{i\in\{1,\dots,r\}}\cup(v_1^*)$ of M, where for any $i\in\{1,\dots,r\}$, $u_i^*=\varpi_{\alpha_i}-\varpi_{\alpha_0}+a_i\varpi_{\alpha_{r+2}}$ and $v_1^*=\varpi_{\alpha_{r+1}}-\varpi_{\alpha_{r+2}}$ and we define the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as follows

$$\mathcal{A}=\begin{pmatrix} -1 & \cdots & -1 & 0\\ 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\\ a_1 & \cdots & a_r & -1 \end{pmatrix},\ \mathcal{B}=\begin{pmatrix} -1\\ 0\\ \vdots\\ \vdots\\ 0\\ -1 \end{pmatrix} \text{ and } \mathcal{C}=\begin{pmatrix} 0\\ \vdots\\ \vdots\\ \vdots\\ 0\\ 1 \end{pmatrix}.$$

Then $\tilde{Q}^\epsilon=\{x\in M_{\mathbb{Q}}\mid \mathcal{A}x\geq\mathcal{B}+\epsilon\mathcal{C}\}$ is the set of $x=(x_1,\dots,x_n)$ such that $x_1,\dots,x_n$ are non-negative, $x_1+\cdots+x_r\leq 1$ and $a_1x_1+\cdots+a_rx_r-x_{r+1}-\cdots-x_n\geq\epsilon-1$.

In particular, $\tilde{Q}^\epsilon$ is the intersection of the closed half-space $H_+^\epsilon:=\{x\in M_{\mathbb{Q}}\mid a_1x_1+\cdots+a_rx_r-x_{r+1}\geq\epsilon-1\}$ with $\tilde{Q}^0$. We denote by H_{++}^ϵ the interior of H_+^ϵ and by H^ϵ the hyperplane $H_+^\epsilon\backslash H_{++}^\epsilon$.

Example 5.13. If $n=2$ (i.e., $r=s=1$) we have $a_1>0$, and either α_1 is trivial or not. We draw, in Figure 9, such a polytope for $\epsilon=0$ with the hyperplane $H^0:=\{x\in M_{\mathbb{Q}}\mid a_1x_1-x_2=-1\}$.

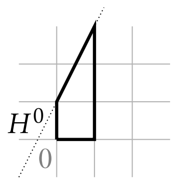

Figure 9. The polytope $\tilde{Q}^0$ in the case where $a_1=2$

Note that $\tilde{Q}^0$ is a polytope with vertices $u_0^*:=0$, $u_1^*,\dots,u_r^*$, $u_0^*+(1+a_0)v_0^*,\dots,u_r^*+(1+a_r)v_1^*$ (recall that $a_0=0$) and facets $F_I:=\operatorname{Conv}((u_i^*\mid i\notin I)\cup(u_i^*+(1+a_i)v_1^*\mid i\notin I))$, $F_{I,1}:=\operatorname{Conv}(u_i^*\mid i\notin I)$ and $F_{I,2}:=\operatorname{Conv}(u_i^*+(1+a_i)v_1^*\mid i\notin I)$ with $I\subsetneq\{0,\dots,r\}$. In particular, the facets of $\tilde{Q}^0$ are the $F_i:=F_{\{i\}}$ with $i\in\{0,\dots,r\}$, $F_{\emptyset,1}$ and $F_{\emptyset,2}$. Moreover for any $I\subsetneq\{0,\dots,n\}$, $F_I=\bigcap_{i\in I}F_i$, $F_{I,1}=\bigcap_{i\in I}F_i\cap F_{\emptyset,1}$ and $F_{I,2}=\bigcap_{i\in I}F_i\cap F_{\emptyset,2}$.

Then, for any $I\subsetneq\{0,\dots,r\}$, we define $F_I^\epsilon:=F_I\cap H_+^\epsilon$, $F_{I,1}^\epsilon:=F_{I,1}\cap H_+^\epsilon$, $F_{I,2}^\epsilon:=F_I\cap H^\epsilon$ and finally $F_{I,1,2}^\epsilon:=F_{I,1}\cap H^\epsilon$. They are faces (possibly empty and not distinct) of $\tilde{Q}^\epsilon$. (Recall $0=a_0<a_1<\cdots<a_r$ and $n=r+1$.)

Proposition 5.14. *The polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\epsilon<1+a_r$.*

Suppose now that $\epsilon<1+a_r$. The non-empty faces of $\tilde{Q}^\epsilon$ are the distinct following F_I^ϵ, $F_{I,1}^\epsilon$, $F_{I,2}^\epsilon$ and $F_{I,1,2}^\epsilon$ with $I\subsetneq\{0,\dots,r\}$:

* F_I^ϵ *(of codimension $|I|$) if $\epsilon<\max_{i\notin I}(1+a_i)$;*
* $F_{I,1}^\epsilon$ *(of codimension $|I|+1$) if $\epsilon<\max_{i\notin I}(1+a_i)$;*
* $F_{I,2}^\epsilon$ *(of codimension $|I|+1$) if $\epsilon<\max_{i\notin I}(1+a_i)$;*
* $F_{I,1,2}^\epsilon$ *(of codimension $|I|+2$, respectively $|I|+1$) if $\min_{i\notin I}(1+a_i)<\epsilon<\max_{i\notin I}(1+a_i)$, respectively if $\epsilon=\min_{i\notin I}(1+a_i)=\max_{i\notin I}(1+a_i)$.*

In particular, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ with $i\in\{0,\dots,r-1\}$, F_r^ϵ if $\epsilon<1+a_{r-1}$, $F_{\emptyset,1}^\epsilon$ and $F_{\emptyset,2}^\epsilon$. Moreover, we can write any face of $\tilde{Q}^\epsilon$ as the intersection of all the facets that contain it, as follows.

For any $I\subsetneq\{0,\dots,r\}$ such that $\epsilon<\max_{i\notin I}(1+a_i)$, $F_I^\epsilon=\bigcap_{i\in I}F_i^\epsilon$.

For any $I \subsetneq \{0,\dots,r\}$ *such that* $\epsilon < \max_{i\notin I}(1+a_i)$, $F^\epsilon_{I,1} = \bigcap_{i\in I} F^\epsilon_i \cap F^\epsilon_{\emptyset,1}$.
For any $I \subsetneq \{0,\dots,r\}$ *such that* $\epsilon < \max_{i\notin I}(1+a_i)$, $F^\epsilon_{I,2} = \bigcap_{i\in I} F^\epsilon_i \cap F^\epsilon_{\emptyset,2}$.
For any $I \subsetneq \{0,\dots,r\}$ *such that* $\min_{i\notin I}(1+a_i) \le \epsilon \le \max_{i\notin I}(1+a_i)$, $F^\epsilon_{I,1,2} = \bigcap_{i\in I} F^\epsilon_i \cap F^\epsilon_{\emptyset,1} \cap F^\epsilon_{\emptyset,2}$.

Remark that, if $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1+a_i)$, then $I = \{0,\dots,r\}\backslash\{i\}$ where i is such that $\epsilon = 1+a_i$. Note also that $\tilde{Q}^{1+a_r}$ is the point u^*_r so that Q^{1+a_r} is the point ϖ_{α_r}.

Proof. For any $\epsilon \ge 0$, the polytope $\tilde{Q}^\epsilon$ is of dimension n if and only if $\tilde{Q}^0$ intersects H^ϵ_{++} if and only if there exists $i \in \{0,\dots,r\}$ such that u^*_i (or $u^*_i + (1+a_i)v^*_1$) is in H^ϵ_{++} if and only if there exists $i \in \{0,\dots,r\}$ such that $a_i > \epsilon - 1$ (or $-1 > \epsilon - 1$) if and only if $a_r > \epsilon - 1$. This proves the first statement of the proposition.

Suppose now that $\epsilon < 1 + a_r$. A non-empty face of $\tilde{Q}^\epsilon$ is either the intersection with H^ϵ_+ of a non-empty face of $\tilde{Q}^0$ that intersects H^ϵ_{++}, or the intersection of a non-empty face of $\tilde{Q}^0$ with H^ϵ.

Let $I \subsetneq \{0,\dots,r\}$. The set F^ϵ_I is not empty if and only if there exists $i \notin I$ such that u^*_i (or $u^*_i + (1+a_i)v^*_1$) is in H^ϵ_+ if and only if there exists $i \notin I$ such that $a_i \ge \epsilon - 1$ (or $-1 \ge \epsilon - 1$) if and only if $\epsilon \le \max_{i\notin I}(1+a_i)$. Moreover with the same argument, F^ϵ_I is not empty and intersects H^ϵ_{++} if and only if $\epsilon < \max_{i\notin I}(1+a_i)$. Also, in that case, the dimension of F^ϵ_I is the same as the dimension of F_I; in particular the non-empty F^ϵ_I that intersect H^ϵ_{++} are all distinct.

Similarly, $F^\epsilon_{I,1}$ is not empty if and only if there exists $i \notin I$ such that $u^*_i \in H^\epsilon_+$ if and only if there exists $i \notin I$ such that $a_i \ge \epsilon - 1$ if and only if $\epsilon \le \max_{i\notin I}(1+a_i)$. Also, $F^\epsilon_{I,1}$ is not empty and intersects H^ϵ_{++} if and only if $\epsilon < \max_{i\notin I}(1+a_i)$. In that case, the dimension of $F^\epsilon_{I,1}$ is the same as the dimension of $F_{I,1}$; in particular the non-empty $F^\epsilon_{I,1}$ that intersect H^ϵ_{++} are all distinct and also distinct from the non-empty F^ϵ_I.

Let $I \subsetneq \{0,\dots,r\}$. Note that for any $\epsilon \ge 0$ (respectively $\epsilon > 0$) and for any $i \in \{0,\dots,r\}$, $u^*_i + (1+a_i)v^*_1 \notin H^\epsilon_{++}$ (respectively $u^*_i + (1+a_i)v^*_1 \notin H^\epsilon_+$). Then the set $F^\epsilon_{I,2}$ is not empty if and only if there exists $i \notin I$ such that $u^*_i \in H^\epsilon_+$ if and only if there exists $i \notin I$ such that $a_i \ge \epsilon - 1$ if and only if $\epsilon \le \max_{i\notin I}(1+a_i)$. Moreover, $F^\epsilon_{I,2}$ is not empty and H^ϵ intersects F_I in its relative interior if and only if there exists $i \notin I$ such that $a_i > \epsilon - 1$ if and only if $\epsilon < \max_{i\notin I}(1+a_i)$. Hence, the dimension of $F^\epsilon_{I,2}$ is the dimension of F_I minus 1 if $\epsilon < \max_{i\notin I}(1+a_i)$ and it equals the dimension of F_I if $\epsilon = \max_{i\notin I}(1+a_i)$. In the first case, the $F^\epsilon_{I,2}$ are all distinct and yield all non-empty faces of $\tilde{Q}^\epsilon$ included in H^ϵ but not in $F_{\emptyset,1}$. In the second case, $F^\epsilon_{I,2} = F^\epsilon_{I,1,2}$.

Now, the set $F^\epsilon_{I,1,2}$ is not empty if and only if there exist i and j not in I (may be equal) such that $u^*_i \in H^\epsilon_+$ and $u^*_j \notin H^\epsilon_{++}$ if and only if there exist i and j not in I such that $a_i \ge \epsilon - 1$ and $a_j \le \epsilon - 1$ if and only if $\min_{i\notin I}(1+a_i) \le \epsilon \le \max_{i\notin I}(1+a_i)$. Moreover, $F^\epsilon_{I,1,2}$ is not empty and included in no proper face of $F_{I,1}$ if and only if there exist i and j not in I such that $u^*_i \in H^\epsilon_{++}$ and $u^*_j \notin H^\epsilon_+$ if and only if there exist i and j not in I such that $a_i > \epsilon - 1$ and $a_j < \epsilon - 1$ (i.e., $a_i < \epsilon - 1$ and $a_j > \epsilon - 1$) or for any $i \notin I$ we have $u^*_i \in H^\epsilon$ (i.e., $a_i = \epsilon - 1$). Then $F^\epsilon_{I,1,2}$ is not empty and included in no proper face of $F_{I,1}$ if and only if $\min_{i\notin I}(1+a_i) < \epsilon < \max_{i\notin I}(1+a_i)$ or $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1+a_i)$. In particular, the dimension of $F^\epsilon_{I,1,2}$ is the dimension of $F_{I,1}$ minus 1 if $\min_{i\notin I}(1+a_i) < \epsilon < \max_{i\notin I}(1+a_i)$ and it equals the dimension of $F_{I,1}$ if $\epsilon = \min_{i\notin I}(1+a_i) = \max_{i\notin I}(1+a_i)$. Note also that the non-empty $F^\epsilon_{I,1,2}$ that are not included in a proper face of $F_{I,1}$ are all distinct and yield all non-empty faces of $\tilde{Q}^\epsilon$ included in $H^\epsilon \cap F_{\emptyset,1}$. This finishes the proof of the second statement of the proposition.

To get the last statements, use again the fact that a facet is a face of codimension 1 and that any face of a polytope is the intersection of the facets containing it. □

From Proposition 5.14, we deduce the following result.

Corollary 5.15. *The isomorphism classes of the horospherical varieties* X^ϵ *associated to the polytopes in the family* $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\ge 0}}$ *are given by the following subsets of* $\mathbb{Q}_{\ge 0}$*:*

- $[0,1[$*;*
- $]1+a_i, 1+a_{i+1}[$ *for any* $i \in \{0,\dots,r-2\}$*;*

* $\{1+a_i\}$ *for any* $i\in\{0,\dots,r-2\}$;
* $]1+a_{r-1},1+a_r[$ *and* $\{1+a_{r-1}\}$ *if the simple root* α_r *is not trivial (i.e., when* X *is as in Case (2b) of Theorem 1.1);*
* $[1+a_{r-1},1+a_r[$ *if the simple root* α_r *is trivial (i.e., when* X *is as in Case (2c) of Theorem 1.1).*

Proof. We apply the theory described in Section 2.2, in particular the fact that the isomorphism classes of the varieties X^ϵ are obtained by looking at the ϵ's for which "the faces of Q^ϵ change". Note first that, by Proposition 5.14, $(P,M,Q^\epsilon,\tilde{Q}^\epsilon)$ is an admissible quadruple if and only if $\epsilon<1+a_r$. Also, the facets of $\tilde{Q}^\epsilon$ are: F_i^ϵ (orthogonal to $\alpha_{i,M}^\vee$) with $i\in\{0,\dots,r-1\}$, F_r^ϵ (orthogonal to $\alpha_{r,M}^\vee$) if $\epsilon<1+a_{r-1}$, $F_{\emptyset,1}^\epsilon$ (orthogonal to $\alpha_{r+1,M}^\vee$) and $F_{\emptyset,2}^\epsilon$ (orthogonal to $\alpha_{r+2,M}^\vee$). In particular, for any $\epsilon,\eta\in[0,1+a_r[$, the facets of Q^ϵ and Q^η are "the same" if and only if ϵ and η are both in $[0,1+a_{r-1}[$ or $[1+a_{r-1},1+a_r[$.

We now use a consequence of the proof of Proposition 5.14: for any $I\subsetneq\{0,\dots,r\}$, $\bigcap_{i\in I}F_i^\epsilon$ is not empty if and only if $\epsilon\leq\max_{i\notin I}(1+a_i)$, $F_{\emptyset,1}^\epsilon\cap\bigcap_{i\in I}F_i^\epsilon$ is not empty if and only if $\epsilon\leq\max_{i\notin I}(1+a_i)$, $F_{\emptyset,2}^\epsilon\cap\bigcap_{i\in I}F_i^\epsilon$ is not empty if and only if $\epsilon\leq\max_{i\notin I}(1+a_i)$, and $F_{\emptyset,1,2}^\epsilon\cap\bigcap_{i\in I}F_i^\epsilon$ is not empty if and only if we have $\min_{i\notin I}(1+a_i)\leq\epsilon\leq\max_{i\notin I}(1+a_i)$. In particular, for any $i\in\{0,\dots,r-2\}$, suppose that for $I=\{i+1,\dots,r\}$ and that $\bigcap_{i\in I}F_i^\epsilon$ is not empty; suppose also that for $I=\{0,\dots,i-1\}$ and that $F_{\emptyset,1,2}^\epsilon\cap\bigcap_{i\in I}F_i^\epsilon$ is not empty; then $\epsilon=1+a_i$. Similarly for any $i\in\{0,\dots,r-2\}$, suppose that for $I=\{i+2,\dots,n\}$ and that $\bigcap_{i\in I}F_i^\epsilon$ is not empty; suppose also that for $I=\{0,\dots,i-1\}$ and that $F_{\emptyset,1,2}^\epsilon\cap\bigcap_{i\in I}F_i^\epsilon$ is not empty; then $\epsilon\in[1+a_i,1+a_{i+1}]$.

Hence, this proves that if two varieties X^ϵ and X^η are isomorphic then ϵ and η are a one of the subsets described in the corollary.

To conclude, we have to prove that the two varieties X^ϵ and X^η are isomorphic when ϵ and η are in one of these subsets. It is obvious from Proposition 5.14 except in the case where the simple root α_n is trivial. But in that case, all polytopes Q^ϵ with $\epsilon\in[1+a_{r-1},1+a_r[$ could be defined even deleting the row corresponding to the simple root α_r that is trivial, so that their faces are "the same" (they are simplexes with facets F_i^ϵ for $i\in\{0,\dots,r-1\}$, $F_{\emptyset,1}^\epsilon$ and $F_{\emptyset,2}^\epsilon$). □

We can reformulate this corollary as follows, and get the first statement of Theorem 1.3 in Case (2). We denote $X_0=X$ and for any $i\in\{1,\cdots,r\}$, $X^i:=X^\epsilon$ with $\epsilon\in]1+a_{i-1},1+a_i[$ and for any $i\in\{0,\cdots,r\}$, $Y^i:=X^{1+a_i}$.

Corollary 5.16. *The family* $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq0}}$ *describes a Log MMP from* X *as follows:*

* r *flips* $\phi_i:X^i\longrightarrow Y^i\longleftarrow X^{i+1}:\phi_i^+$ *for any* $i\in\{0,\cdots,r-1\}$ *and a fibration* $\phi_r:X^r\longrightarrow Y^r$, *if the simple root* α_r *is not trivial;*
* $r-1$ *flips* $\phi_i:X^i\longrightarrow Y^i\longleftarrow X^{i+1}:\phi_i^+$ *for any* $i\in\{0,\cdots,r-2\}$, *followed by a divisorial contraction* $\phi_{r-1}:X^{r-1}\longrightarrow Y^{r-1}\simeq X^r$ *and a fibration* $X^r\longrightarrow Y^r\simeq\mathrm{pt}$, *if the simple root* α_r *is trivial.*

Example 5.17. In the two different cases with $n=2$ and $a_1=2$, we illustrate this corollary in terms of polytopes in Figures 10 and 11.

5.5. Proof of the last statement of Theorem 1.3 in Case (2)

The previous section proves that $a_1,\dots,a_r$ are invariants of X. To finish the proof of Theorem 1.3 in Case (2), we have to prove that $G_0,\dots,G_t$ and $\alpha_0,\dots,\alpha_{r+2}$ are also invariants. Since the "first" Log MMP consists of a fibration $\psi:X\longrightarrow Z$ where Z is a two-orbit variety embedded in $\mathbb{P}(V(\varpi_{\alpha_{r+1}})\oplus V(\varpi_{\alpha_{r+2}}))$ as in [Pas09], G_t, α_{r+1} and α_{r+2} are invariants of X. As in Case (1), we will describe some exceptional loci and some fibers of different morphisms of the Log MMP, but we first distinguish two cases by the following result.

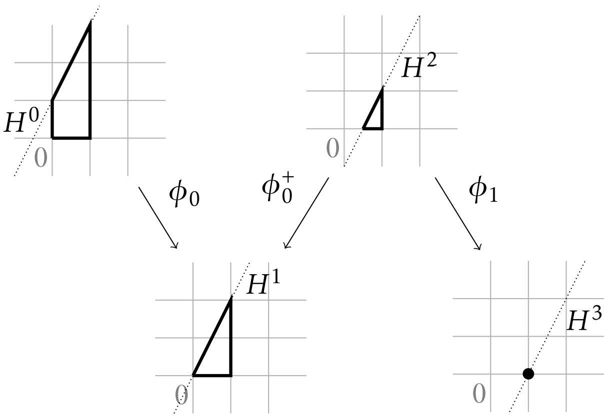

Figure 10. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 2$ and α_1 is not trivial.

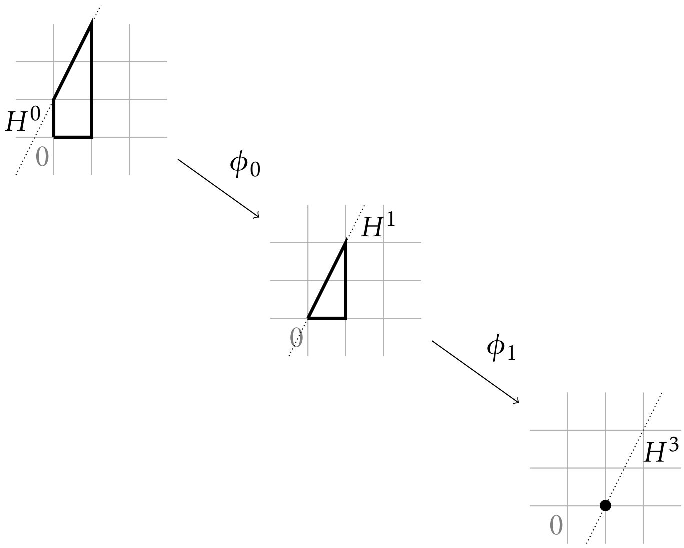

Figure 11. The Log MMP described by the polytopes $\tilde{Q}^\epsilon$ in the case where $n = 2$, $a_1 = 2$ and α_1 is trivial.

Proposition 5.18. *Suppose that $r = 1$ and that α_0 and α_1 are two simple roots of G_0 (and then $t = 1$). Then, the general fiber of $\psi : X \longrightarrow Z$ is either a homogeneous variety different from a projective space (a quadric Q^{2m} with $m \geq 2$, a Grassmannian $\mathrm{Gr}(i, m)$ with $m \geq 5$ and $2 \leq i \leq m-2$, or a spinor variety $\mathrm{Spin}(2m+1)/P(\varpi_m)$ with $m \geq 4$), or a two-orbit variety as in* [Pas09].

Suppose that $r > 1$ or that α_0 and α_1 are simple roots of G_0 and G_1 respectively. Then, the general fiber of $\psi : X \longrightarrow Z$ is a projective space.

Proof. The general fiber of $\psi : X \longrightarrow Z$ is the smooth projective horospherical $G_0 \times \cdots \times G_{t-1}$-variety of Picard group $\mathbb{Z}$ isomorphic to the closure of the $G_0 \times \cdots \times G_{t-1}$-orbit of a sum of highest weight vectors in $\mathbb{P} := \mathbb{P}(V(\varpi_{\alpha_0}) \oplus \cdots \oplus V(\varpi_{\alpha_r}))$. Hence, the proposition is a consequence of [Pas09, Section 1]. □

• In the case where $r = 1$ and that α_0 and α_1 are two simple roots of G_0, $G = G_0 \times G_1$ and the description of the general fiber of $\psi : X \longrightarrow G/P(\varpi_\beta)$, with Remark 4.2, implies that G_0, α_0 and α_1 are invariants of X.

• Now we suppose that $r>1$ or that α_0 and α_1 are not two simple roots of the same simple subgroup of $P(\varpi_\beta)$.

We define various exceptional loci in X as follows. Let $i\in\{0,\dots,r\}$, define E_i to be the closure in X of the set of points $x\in X$ such that x is in the open isomorphism set of the first i contractions and x is in the exceptional locus of ϕ_i.

Proposition 5.19. *For any $i\in\{0,\dots,r\}$ the exceptional locus E_i is the closure in X of the G-orbit associated to the non-empty face F_{I_i} with $I_i:=\{i+1,\dots,r\}$. In particular E_i is isomorphic to the closure of the G-orbit of a sum of highest weight vectors in*

$$\mathbb{P}:=\mathbb{P}\left(\bigoplus_{j=0}^{i}\bigoplus_{b=0}^{1+a_j}V(\varpi_{\alpha_j}+b\varpi_{\alpha_{r+1}}+(1+a_j-b)\varpi_{\alpha_{r+2}})\right),$$

hence for $i\in\{1,\dots,r\}$, E_i is a smooth projective horospherical variety of Picard group $\mathbb{Z}^2$ as in Case (2), and E_0 is the product a two-orbit variety with a homogeneous variety (projective of Picard group $\mathbb{Z}$).

Note that $E_r=X$ and that in any case, the rank of the horospherical G-variety E_i is $i+1$.

Proof. Let $i\in\{0,\dots,r\}$ and $\epsilon_i\in\mathbb{Q}_{\geq 0}$ such that $X^i=X^{\epsilon_i}$.

We denote by Ω^i_I, $\Omega^i_{I,1}$, $\Omega^i_{I,2}$ and $\Omega^i_{I,1,2}$ the G-orbits of X^i associated to the non empty faces $F_I^{\epsilon_i}$, $F_{I,1}^{\epsilon_i}$, $F_{I,2}^{\epsilon_i}$ and $F_{I,1,2}^{\epsilon_i}$ of the polytope $\tilde{Q}^{\epsilon_i}$. We denote by ω^i_I, $\omega^i_{I,1}$, $\omega^i_{I,2}$ and $\omega^i_{I,1,2}$ the G-orbits of $Y^i=X^{1+a_i}$ associated to the non-empty faces $F_I^{1+a_i}$, $F_{I,1}^{1+a_i}$, $F_{I,2}^{1+a_i}$ and $F_{I,1,2}^{1+a_i}$ of the polytope $\tilde{Q}^{1+a_i}$. Recall that, for any $\epsilon\in\mathbb{Q}_{\geq 0}$, we have an order on the G-orbits of X^ϵ compatible with the order on the non-empty faces of $\tilde{Q}^\epsilon$: in particular $\Omega^i_I\subset\overline{\Omega^i_{I'}}$, $\Omega^i_{I,1}\subset\overline{\Omega^i_{I',1}}$, $\Omega^i_{I,2}\subset\overline{\Omega^i_{I',2}}$ and $\Omega^i_{I,1,2}\subset\overline{\Omega^i_{I',1,2}}$ respectively if and only if $I'\subset I$, and $\Omega^i_{I,1}\subset\overline{\Omega^i_I}$, $\Omega^i_{I,2}\subset\overline{\Omega^i_I}$, $\Omega^i_{I,1,2}\subset\overline{\Omega^i_{I,1}}$ and $\Omega^i_{I,1,2}\subset\overline{\Omega^i_{I,2}}$ (as soon as these orbits are defined, i.e., as soon as the corresponding faces are non-empty).

For any $I\subsetneq\{0,\dots,r\}$ such that there exists $j\geq i$ not in I (i.e., such that Ω^i_I is defined), $\phi_i(\Omega^i_I)=\omega^i_I$ if there exists $j\geq i+1$ not in I, and $\phi_i(\Omega^i_I)=\omega^i_{I\backslash\{i\},1,2}$ if for any $j\geq i+1$, $j\in I$. Indeed $I\cup\{0,\dots i-1\}=I\backslash\{i\}$ is the minimal subset of $\{0,\dots,r\}$ containing I such that $\omega^i_{I\backslash\{i\},1,2}$ is defined and there is no I' containing I such that $\omega^i_{I'}$, $\omega^i_{I',1}$ or $\omega^i_{I',2}$ is defined. Similarly, with $k=1$ or 2, for any $I\subsetneq\{0,\dots,r\}$ such that there exists $j\geq i$ not in I (i.e., such that $\Omega^i_{I,k}$ is defined), $\phi_i(\Omega^i_{I,k})=\omega^i_{I,k}$ if there exists $j\geq i+1$ not in I, and $\phi_i(\Omega^i_{I,k})=\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}$ if for any $j\geq i+1$, $j\in I$. Indeed $I\cup\{0,\dots i-1\}=\{0,\dots,r\}\backslash\{i\}$ is the minimal subset of $\{0,\dots,r\}$ containing I such that $\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}$ is defined and there is no I' containing I such that $\omega^i_{I',k}$ is defined.

And for any $I\subsetneq\{0,\dots,r\}$ such that there exist $j\geq i$ and $j'<i$ not in I (i.e., such that $\Omega^i_{I,1,2}$ is defined), $\phi_i(\Omega^l_{I,1,2})=\omega^i_{I,1,2}$ if there exists $i\geq i+1$ not in I, and $\phi_i(\Omega^i_{I,1,2})=\omega^i_{\{0,\dots,r\}\backslash\{i\},\beta}$ if for any $j\geq i+1$, $j\in I$. Indeed $\{0,\dots,r\}\backslash\{i\}=I\cup\{0,\dots i_l-1\}$ is the minimal subset of $\{0,\dots,n\}$ containing I such that $\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}$ is defined.

In particular, we have $\phi_i(\Omega^i_{I_i})=\omega^i_{I\backslash\{i\},1,2}$. But $\Omega^i_{I_i}$ and $\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}$ are non-isomorphic horospherical homogeneous spaces by Proposition 2.14, so that $\Omega^i_{I_i}$ is in the exceptional locus of ϕ_l. Moreover, if Ω is a G-orbit of X^i not contained in $\overline{\Omega^i_{I_i}}$, it is of the form Ω^i_I, $\Omega^i_{I,1}$, $\Omega^i_{I,2}$ or $\Omega^i_{I,1,2}$ where $I_i\not\subset I$. Hence, in that case $\phi_i(\Omega)=\Omega$. And then the exceptional locus of ϕ_i is $\overline{\Omega^i_{I_i}}$. Note that $\Omega^0_{I_i},\dots,\Omega^{l-1}_{I_i}$ are not in the exceptional locus of $\phi_0,\dots,\phi_{i-1}$ respectively, to conclude that $E_i=\overline{\Omega^0_{I_i}}$.

We use again Proposition 2.14 to see that $E_i=\overline{\Omega^0_{I_i}}$ corresponds to the admissible quadruple $(P_F,M_F,F,\tilde{F})$ with $F=F^0_{I_i}$ (and with some ample divisor of E_i). Then we conclude by Corollaries 2.6 and 2.10. ☐

The Log MMP now defines, by restriction, fibrations $\tilde{\phi_i} : E_i\backslash E_{i-1} \longrightarrow E_i' := \overline{\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}}$, for any $i \in \{0,\dots,i\}$.

Proposition 5.20. *For any $i \in \{0,\dots,r\}$, E_i' is a closed G-orbit of Y^i isomorphic to $G/P(\varpi_{\alpha_i})$ (which is a point if α_i is trivial). In particular, the map $\tilde{\phi_i}$ is surjective. Moreover, the dimension of fibers of $\tilde{\phi_i}$ is*

$$i+1+\dim P(\varpi_{\alpha_i})/(P(\varpi_{\alpha_{r+1}})\cap P(\varpi_{\alpha_{r+2}})\cap\bigcap_{j=0}^{i}P(\varpi_{\alpha_j})).$$

Proof. Let $i \in \{0,\dots,r\}$. The face $F^{1+a_i}_{\{0,\dots,r\}\backslash\{i\},1,2}$ of $\tilde{Q}^{1+a_i}$ is the vertex u_i^* and then the corresponding face of Q^{1+a_i} is the vertex ϖ_{α_i}. In particular, the G-orbit $\omega^i_{\{0,\dots,r\}\backslash\{i\},1,2}$ is closed and isomorphic to $G/P(\varpi_{\alpha_i})$.

Now, since $\tilde{\phi_i}$ is G-equivariant, it must be surjective. Moreover, the dimension of the fibers of $\tilde{\phi_i}$ is

$$\dim E_i - \dim E_i' = (i+1+\dim G/(P(\varpi_{\alpha_{r+1}})\cap P(\varpi_{\alpha_{r+2}})\cap\bigcap_{j=0}^{i}P(\varpi_{\alpha_j}))) - \dim G/P(\varpi_{\alpha_i})$$

that is $i+1+\dim P(\varpi_{\alpha_i})/(P(\varpi_{\alpha_{r+1}})\cap P(\varpi_{\alpha_{r+2}})\cap\bigcap_{j=0}^{i}P(\varpi_{\alpha_j}))$. □

Corollary 5.21. *The dimension of the fibers of $\tilde{\phi_i}$ is*

$$i+1+\dim G/(P(\varpi_{\alpha_{r+1}})\cap P(\varpi_{\alpha_{r+2}}))+\sum_{j=0}^{i-1}\dim G/P(\varpi_{\alpha_j}).$$

In particular the dimensions d_j of the $G/P(\varpi_{\alpha_j})$'s, which are projective space under $G_i = \mathrm{SL}_{d_j+1}$, are invariants of X.

Proof. Since $r > 1$, or $r = 1$ and α_0, α_1 are not two simple roots of the same simple subgroup of G, the simple roots $\alpha_0,\dots,\alpha_r$ are respectively the first simple roots of $G_0,\dots,G_r$ that are of type A. (And α_{r+1}, α_{r+2} are simple roots of G_{r+1}.) Then the corollary can be easily deduced from the proposition. □

6. Appendix

Proposition 6.1. *Let (K,β,R,n) be a smooth quadruple. Then we are in one of the following cases, up to symmetries.*

(1) *$n = 1$ and one of the following cases occurs.*
   (a) *K is of type A_m ($m \geq 3$). Then, $\beta = \alpha_k$ with $3 \leq k \leq m$ and $R = \{\alpha_1,\alpha_{k-1}\}$; or $\beta = \alpha_k$ with $4 \leq k \leq m$ and $R = \{\alpha_i,\alpha_{i+1}\}$ with $1 \leq i \leq k-2$.*
   (b) *K is of type B_m ($m \geq 3$). Then, $\beta = \alpha_k$ with $3 \leq k \leq m$ and $R = \{\alpha_1,\alpha_{k-1}\}$ or $R = \{\alpha_i,\alpha_{i+1}\}$ with $1 \leq i \leq k-2$; or $\beta = \alpha_k$ with $1 \leq k \leq m-2$ and $R = \{\alpha_{m-1},\alpha_m\}$; or $\beta = \alpha_{m-3}$ and $R = \{\alpha_{m-2},\alpha_m\}$.*
   (c) *K is of type C_m ($m \geq 3$). Then, $\beta = \alpha_k$ with $3 \leq k \leq m$ and $R = \{\alpha_1,\alpha_{k-1}\}$; or $\beta = \alpha_k$ with $4 \leq k \leq m$ and $R = \{\alpha_i,\alpha_{i+1}\}$ with $1 \leq i \leq k-2$; $\beta = \alpha_k$ with $1 \leq k \leq m-2$ and $R = \{\alpha_i,\alpha_{i+1}\}$ with $1 \leq i \leq k-2$.*
   (d) *K is of type D_m ($m \geq 4$). Then, $\beta = \alpha_k$ with $3 \leq k \leq m-2$ or $k = m$ and $R = \{\alpha_1,\alpha_{k-1}\}$; or $\beta = \alpha_k$ with $4 \leq k \leq m-2$ or $k = m$ and $R = \{\alpha_i,\alpha_{i+1}\}$ with $1 \leq i \leq k-2$; $\beta = \alpha_k$ with $1 \leq k \leq m-4$ and $R = \{\alpha_{m-1},\alpha_m\}$; or $m \geq 5$, $\beta = \alpha_{m-3}$ and R is any subset of cardinality 2 of $\{\alpha_{m-2},\alpha_{m-1},\alpha_m\}$; or $m \geq 5$, $\beta = \alpha_{m-2}$ and $R = \{\alpha_{m-1},\alpha_m\}$; all modulo symmetries.*
   (e) *K is of type E_6. Then $\beta = \alpha_1$ and $R = \{\alpha_2,\alpha_3\}$; or $\beta = \alpha_2$ and $R = \{\alpha_1,\alpha_6\}$, $\{\alpha_1,\alpha_3\}$ or $\{\alpha_3,\alpha_4\}$; or $\beta = \alpha_3$ and $R = \{\alpha_2,\alpha_6\}$, $\{\alpha_2,\alpha_4\}$, $\{\alpha_4,\alpha_5\}$ or $\{\alpha_5,\alpha_6\}$; or $\beta = \alpha_4$ and $R = \{\alpha_1,\alpha_3\}$.*

(f) *K is of type E_7. Then $\beta = \alpha_1$ and $R = \{\alpha_2, \alpha_3\}$; or $\beta = \alpha_2$ and $R = \{\alpha_1, \alpha_7\}$, $\{\alpha_1, \alpha_3\}$, $R = \{\alpha_3, \alpha_4\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_5, \alpha_6\}$ or $\{\alpha_6, \alpha_7\}$; or $\beta = \alpha_3$ and $R = \{\alpha_2, \alpha_7\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_5, \alpha_6\}$ or $\{\alpha_6, \alpha_7\}$; or $\beta = \alpha_4$ and $R = \{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_7\}$, $\{\alpha_5, \alpha_6\}$ or $\{\alpha_6, \alpha_7\}$; or $\beta = \alpha_5$ and $R = \{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_2, \alpha_4\}$ or $\{\alpha_6, \alpha_7\}$; or $\beta = \alpha_6$ and $R = \{\alpha_2, \alpha_5\}$.*

(g) *K is of type E_8. Then $\beta = \alpha_1$ and $R = \{\alpha_2, \alpha_3\}$; or $\beta = \alpha_2$ and $R = \{\alpha_1, \alpha_8\}$, $\{\alpha_1, \alpha_3\}$, $R = \{\alpha_3, \alpha_4\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_5, \alpha_6\}$, $\{\alpha_6, \alpha_7\}$ or $\{\alpha_7, \alpha_8\}$; or $\beta = \alpha_3$ and $R = \{\alpha_2, \alpha_8\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_5, \alpha_6\}$, $\{\alpha_6, \alpha_7\}$ or $\{\alpha_7, \alpha_8\}$; or $\beta = \alpha_4$ and $R = \{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_8\}$, $\{\alpha_5, \alpha_6\}$, $\{\alpha_6, \alpha_7\}$ or $\{\alpha_7, \alpha_8\}$; or $\beta = \alpha_5$ and $R = \{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_2, \alpha_4\}$, $\{\alpha_6, \alpha_8\}$, $\{\alpha_6, \alpha_7\}$ or $\{\alpha_7, \alpha_8\}$; or $\beta = \alpha_6$ and $R = \{\alpha_2, \alpha_5\}$ or $\{\alpha_7, \alpha_8\}$.*

(h) *K is of type F_4. Then $\beta = \alpha_1$ and $R = \{\alpha_3, \alpha_4\}$ or $\{\alpha_2, \alpha_3\}$; $\beta = \alpha_2$ and $R = \{\alpha_3, \alpha_4\}$; $\beta = \alpha_3$ and $R = \{\alpha_1, \alpha_2\}$; $\beta = \alpha_4$ and $R = \{\alpha_2, \alpha_3\}$ or $\{\alpha_1, \alpha_3\}$.*

(2) *R is empty or one of the following cases occurs.*

(a) *K is of type A_m ($m \geq 2$). Then, $\beta = \alpha_1$ and R is $\{\alpha_2\}$ or $\{\alpha_m\}$ (if $m \geq 3$); $\beta = \alpha_k$ with $2 \leq k \leq \frac{m}{2}$ and R is a subset of $\{\alpha_1, \alpha_{k+1}\}$, $\{\alpha_1, \alpha_m\}$, $\alpha_{k-1}, \alpha_{k+1}\}$ (if $k \geq 3$) or $\alpha_{k-1}, \alpha_m\}$ (if $k \geq 3$); or $\beta = \alpha_{\frac{m+1}{2}}$ (if m is odd) and R is a subset of $\{\alpha_1, \alpha_m\}$ or $R = \{\alpha_{k-1}\}$, $\{alpha_1, \alpha_{k+1}\}$ or $\alpha_{k-1}, \alpha_{k+1}\}$ (if $m \geq 5$).*

(b) *K is of type B_m ($m \geq 3$). Then, $m = 3$, $\beta = \alpha_1$ and R is $\{\alpha_3\}$; $\beta = \alpha_k$ with $2 \leq k \leq m-3$ and R is $\{\alpha_1\}$ or $\{\alpha_{k-1}\}$ (if $k \geq 3$); or $\beta = \alpha_{m-2}$ ($m \geq 4$) and R is a subset of $\{\alpha_1, \alpha_m\}$ or $\{\alpha_{m-3}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m - 1$ and R is a subset of $\{\alpha_1, \alpha_m\}$ or R is $\{\alpha_{m-2}\}$ (if $m \geq 4$) or $\{\alpha_{m-2}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m$ and R is $\{\alpha_1\}$ or $\{\alpha_{m-1}\}$.*

(c) *K is of type C_m ($m \geq 2$). Then, $\beta = \alpha_1$ and R is $\{\alpha_2\}$; or $\beta = \alpha_k$ with $2 \leq k \leq m-1$ ($m \geq 3$) and R is a subset of $\{\alpha_1, \alpha_{k+1}\}$ or $\{\alpha_{k-1}, \alpha_{k+1}\}$ (if $k \geq 3$ and $m \geq 4$); or $\beta = \alpha_m$ and $R = \{\alpha_1\}$ or $\{\alpha_{m-1}\}$ (if $m \geq 3$).*

(d) *K is of type D_m ($m \geq 4$). Then, $\beta = \alpha_k$ with $2 \leq k \leq m-4$ ($m \geq 6$) and R is $\{\alpha_1\}$ or $\{\alpha_{k-1}\}$ (if $k \geq 3$ and $m \geq 7$); or $\beta = \alpha_{m-3}$ and R is $\{\alpha_{m-1}\}$, or a subset of $\{\alpha_1, \alpha_{m-1}\}$ (if $m \geq 5$) or $\{\alpha_{m-4}, \alpha_{m-1}\}$ (if $m \geq 6$); or $\beta = \alpha_{m-2}$ and R is $\{\alpha_1\}$, $\{\alpha_1, \alpha_{m-1}\}$ or $\{\alpha_1, \alpha_{m-1}, \alpha_m\}$, or R is a subset of $\{\alpha_{m-3}, \alpha_{m-1}\}$ (if $m \geq 5$), R is $\{\alpha_{m-3}, \alpha_{m-1}, \alpha_m\}$ (if $m \geq 5$); or $\beta = \alpha_m$ and R is $\{\alpha_1\}$ or $\{\alpha_{m-1}\}$.*

(e) *K is of type E_6. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_6\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 6 modulo symmetries.*

(f) *K is of type E_7. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$ or $\{\alpha_7\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_7\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 7; or $\beta = \alpha_5$ and R is a subset of $\{\alpha_i, \alpha_j\}$ with $i = 1$ or 2 and $j = 6$ or 7; or $\beta = \alpha_6$ and $R = \{\alpha_7\}$.*

(g) *K is of type E_8. Then $\beta = \alpha_2$ and $R = \{\alpha_1\}$ or $\{\alpha_8\}$; or $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_2\}$ or $\{\alpha_1, \alpha_8\}$; or $\beta = \alpha_4$ and R is subset of $\{\alpha_2, \alpha_i, \alpha_j\}$ with $i = 1$ or 3 and $j = 5$ or 8; or $\beta = \alpha_5$ and R is a subset of $\{\alpha_i, \alpha_j\}$ with $i = 1$ or 2 and $j = 6$ or 8; or $\beta = \alpha_6$ and R is α_7 or α_8; or $\beta = \alpha_7$ and $R = \{\alpha_8\}$.*

(h) *K is of type F_4. Then $\beta = \alpha_1$ and $R = \{\alpha_4\}$; $\beta = \alpha_2$ and R is a subset of $\{\alpha_1, \alpha_3\}$ or $\{\alpha_1, \alpha_4\}$; $\beta = \alpha_3$ and R is a subset of $\{\alpha_1, \alpha_4\}$ or $\{\alpha_2, \alpha_4\}$.*

(i) *K is of type G_2. Then $\beta = \alpha_1$ and $R = \{\alpha_2\}$; or $\beta = \alpha_2$ and $R = \{\alpha_1\}$*

The proof, which is a long but not difficult case by case verification, is left to the reader.

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