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# The fundamental group of quotients of products of some topological spaces by a finite group — A generalization of a theorem of Bauer–Catanese–Grunewald–Pignatelli

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**Abstract.** We provide a description of the fundamental group of the quotient of a product of topological spaces $X_i$, each admitting a universal cover, by a finite group $G$, provided that there is only a finite number of path-connected components in $X_i^g$ for every $g \in G$. This generalizes previous work of Bauer–Catanese–Grunewald–Pignatelli and Dedieu–Perroni.

**Keywords.** fundamental group; quotients by finite group; orbifolds

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**[Français]**

**Le groupe fondamental de quotients de produits de certains espaces topologiques par un groupe fini — Généralisation d'un théorème de Bauer–Catanese–Grunewald–Pignatelli**

**Résumé.** Nous fournissons une description du groupe fondamental du quotient d'un produit d'espaces topologiques $X_i$, chacun admettant un revêtement universel, par un groupe fini $G$, pourvu qu'il n'existe qu'un nombre fini de composantes connexes par arcs dans $X_i^g$ pour chaque $g \in G$. Cela généralise des résultats antérieurs de Bauer–Catanese–Grunewald–Pignatelli et de Dedieu–Perroni.

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## Contents



# 1. Introduction

The fundamental group of a quotient of a Haussdorf space $X$ by a finite group $G$ acting freely can be computed noticing that $X \to X/G$ is a covering map, and then using the long exact sequence of homotopy groups associated to a fibration. When $X = X_1 \times \cdots \times X_k$ and $G$ acts on each $X_i$ freely and diagonally on $X$, the fundamental group of $X_1 \times \cdots \times X_k$ sits as a finite-index normal subgroup of $\pi_1(X/G)$.

In the case where each $X_i$ is a projective smooth curve and the action of $G$ is only *faithful*, the following Theorem was shown in [BCGP12].

**Theorem 1.1.** [BCGP12, Theorems 0.10 and 4.1] *Let $C_1, \ldots, C_k$ be smooth projective curves and let $G$ be a finite group acting faithfully by automorphisms on each of them. Consider the diagonal action of $G$ on the product $C_1 \times \cdots \times C_k$, then the fundamental group of $(C_1 \times \cdots \times C_k)/G$ admits a normal finite index subgroup $\mathcal{N}$ isomorphic to a product of fundamental groups of smooth projective curves.*

It was later extended in [DP12] to the case when the action of $G$ is non-necessarily faithful. There, they quotient $G$ to obtain a group acting faithfully, follow the subsequent arguments and then extend again to $G$.

Let us explain briefly the method of proof in [BCGP12]. First, they consider the orbifold surface groups $T_i$ of $C_i/G$, which are an extension of $G$ by $\pi_1(C_i)$ and hence come with a surjective morphism to $G$ (see Subsection 2.2.1). They show that the fundamental group $\pi_1((C_1 \times \cdots \times C_k)/G)$ is isomorphic to the quotient of the fiber product $\mathbb{H} := T_1 \times_G \cdots \times_G T_k$ by the normal subgroup $\mathrm{Tors}(\mathbb{H})$ generated by the elements of torsion.

The second part relies on the following proposition whose proof uses abstract group theoretic arguments.

**Proposition 1.2.** [DP12, Proposition 3.5] *There exists a short exact sequence of groups*

$$1 \to E \to \mathbb{H}/\operatorname{Tors}\mathbb{H} \to T \to 1$$

*where $E$ is finite and $T$ is a group of finite index in a product of orbifold surfaces groups.*

They finally use Proposition 1.2 and properties of the orbifold surface groups such as residually finiteness and cohomological goodness to construct a subgroup of $\mathbb{H}/\operatorname{Tors}\mathbb{H}$ intersecting $E$ trivially and satisfying the required properties.

Here, a more geometric approach is used via fundamental groups of stacks or orbispaces [Noo05, Che01]. This theory permits to see $X \to [X/G]$ as a covering map under some conditions on $X$, where $[X/G]$ denotes the quotient stack, and a long exact sequence of homotopy groups is available. We will denote the fundamental group of the stack $[X/G]$ by $\pi_1([X/G])$.

For $i = 1, \ldots, k$, let $X_i$ be a connected, locally path-connected and semi-locally simply connected topological space with an action of a finite group $G$, consider the diagonal action of $G$ in $X := X_1 \times \cdots \times X_k$

and denote by $I$ the subgroup generated by the elements having a fixed point in every $X_i$ for $i = 1, \dots, k$. We can formulate now our first main Theorem.

**Theorem 1.3**. *Let $X, X_1, \dots, X_k$ and $G$ as above. Suppose that the number of path connected components in the fixed locus set $X_i^g$ of an element $g \in G$ is finite for every $g \in G$ and $i = 1, \dots, k$. Then there exists a homomorphism*

$$\pi_1(X/G) \to \prod_{i=1}^{k} \pi_1\left([(X_i/I)/(G/I)]\right)$$

*which has finite kernel and its image is a finite-index subgroup.*

This can be seen as a generalization of Proposition 1.2 (Bauer–Catanese–Grunewald–Pignatelli) by the remarks preceding the statement of the Proposition.

The action of $G/I$ on $X_i/I$ is induced by the action of $G$ in $X_i$. Note that if $k = 1$ then $G/I$ can be seen to act freely on $X_1/I$ and $\pi_1([(X_1/I)/(G/I)]) = \pi_1((X_1/I)/(G/I))$ but $(X_1/I)/(G/I) \cong X_1/G$. The same argument works if we make the product of the same topological space, which gives the following corollary.

**Corollary 1.4**. *Let $X_i = X_1$ for $i = 2, \dots, k$ and $G$ satisfy the hypothesis of the above theorem. Then the homomorphism $\pi_1(X/G) \to \pi_1(X_1/G)^k$ has finite kernel and its image is a finite-index subgroup.*

An important case of Theorem 1.3 and Corollary 1.4 is when $X_i$ is a smooth complex algebraic variety for $i = 1, \dots, k$. Indeed, the fundamental group of a variety with quotient singularities is the fundamental group of a smooth variety.

Our second main Theorem generalizes Theorem 1.1 (Bauer–Catanese–Grunewald–Pignatelli). It can be stated without using the language of stacks or orbispaces.

**Theorem 1.5**. *Let $X, X_1, \dots X_k$ and $G$ satisfy the hypothesis of Theorem 1.3. Suppose that $\pi_1(X/G)$ is residually finite. Then there exists a normal finite-index subgroup $\mathcal{N} \lhd \pi_1(X/G)$ isomorphic to a product:*

$$\mathcal{N} \cong \prod_{i=1}^{k} H_i.$$

*with $H_i \lhd \pi_1(X_i/I)$ normal finite-index subgroups.*

As a corollary, following closely the arguments used in [BCGP12], we show that for smooth curves $C_1, \dots, C_k$ the group $\pi_1((C_1 \times \cdots \times C_k)/G)$ is residually finite. Hence, we have that Theorem 1.1 (Bauer–Catanese–Grunewald–Pignatelli, Dedieu–Perroni) is valid in the case when the curve is non-necessarily compact.

**Corollary 1.6**. *Let $C_1, \dots, C_k$ be smooth algebraic curves and let $G$ be a finite group acting on each of them. Then there exists a normal subgroup $\mathcal{N} \lhd \pi_1((C_1 \times \cdots \times C_k)/G)$ of finite index, isomorphic to a product $\Pi_1 \times \cdots \times \Pi_k$, where $\Pi_j$ is either the fundamental group of a smooth projective curve or a free group of finite rank.*

The paper is organized as follows: in Section 2 preliminary results are given. Then the first main Theorem is proved in Section 3 and the proof of the second main Theorem together with some applications are given in Section 4.

## 2. Preliminaries

### 2.1. Properties of fundamental group of topological stacks

Let $X$ be a connected, semi-locally simply connected and locally path-connected topological space and $G$ a finite group acting continuously on it.

**2.1.1. Fiber homotopy exact sequence.**— There exists a homotopy theory for stacks and the existence of the long exact sequence of homotopy, see [Noo14], is more general than what follows, however we only need the following case: consider the topological stack $\mathcal{X} = [X/G]$, a point $x \in X$ and denote by $\bar{x} \in \mathcal{X}$ the image of $x$. We have an associated fibration $G \to X \to \mathcal{X}$ and a long exact sequence of homotopy groups,

$$\cdots \to \pi_{n+1}(\mathcal{X}, \bar{x}) \to \pi_n(G, Id) \to \pi_n(X, x) \to \pi_n(\mathcal{X}, \bar{x}) \to \pi_{n-1}(G, \mathrm{Id}) \to \cdots$$

the map $\pi_n(G, Id) \to \pi_n(X, x)$ is induced by the orbit $G \cdot x \hookrightarrow X$.

**2.1.2. Action on the universal cover.**— The hypothesis made on $X$ ensures that there exists an universal cover $\tilde{X}$ and moreover, if we let $\mathcal{X} = [X/G]$ as in §2.1.1, we have an action of $\pi_1(\mathcal{X}, \bar{x})$ on $\tilde{X}$ (see §3.2.1). We will use several times the following lemma in what follows.

**Lemma 2.1.** *Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ in $\tilde{X}$, let $y \in \tilde{X}$ and denote by $I_y$ the isotropy group of the action. Then there exists a monomorphism $I_y \to G$.*

*Proof.* By §2.1.1 we obtain a short exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(\mathcal{X}, \bar{x}) \to G \to 1,$$

as the action of $\pi_1(X, x)$ on $\tilde{X}$ is free, we obtain that the restriction of $\pi_1(\mathcal{X}, \bar{x}) \to G$ to $I_y$ is injective. □

## 2.2. Product of topological spaces

**2.2.1. Fundamental group of the quotient of a product.**— For $i = 1, \ldots, k$ let $X_i$ as in §2.1 be a connected, semi-locally simply connected and locally path-connected topological space and $G$ a finite group acting on each of them.

By §2.1.1 we have $k$ exact sequences

$$1 \to \pi_1(X_i, x_i) \to \pi_1(\mathcal{X}_i, \bar{x}_i) \xrightarrow{\varphi_i} (G, \mathrm{Id}) \to 1 \tag{2.1}$$

where $\mathcal{X}_i = [X_i/G]$, $x_i \in X_i$ and its image in $\mathcal{X}_i$ is denoted by $\bar{x}_i$.

Denote by $\mathbb{H} := \pi_1(\mathcal{X}_1, x_1) \times_G \cdots \times_G \pi_k(\mathcal{X}_k, x_k)$. The exact sequences in (2.1) can be assembled as follows

$$1 \to \pi_1(X_1 \times \cdots \times X_k, x) \to \mathbb{H} \to G \to 1 \tag{2.2}$$

with $x = (x_1, \ldots, x_k)$. The geometric nature of $\mathbb{H}$ is shown in the following lemma.

**Lemma 2.2.** *Let $G$ act diagonally on $X = X_1 \times \cdots \times X_k$. Consider the stack $\mathcal{X} = [X/G]$ then $\pi_1(\mathcal{X}, \bar{x}) \cong \mathbb{H}$.*

*Proof.* We have natural projection maps $\mathcal{X} \to \mathcal{X}_i$ for $i = 1, \ldots, k$, which together with the morphisms $\varphi_i : \pi_1(\mathcal{X}_i, \bar{x}_i) \to G$ and the universal property of the fiber product give us a morphism $\pi_1(\mathcal{X}, \bar{x}) \to \mathbb{H}$. By the exact sequence of a fibration §2.1.1 applied to the action of $G$ to $X_1 \times \cdots \times X_k$ and by (2.2) we obtain

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_1 \times \cdots \times X_k, x) & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \mathrm{id} & & \\
1 & \longrightarrow & \pi_1(X_1 \times \cdots \times X_k, x) & \longrightarrow & \mathbb{H} & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

which implies the result. □

**Lemma 2.3.** *Let $X, X_i$ and $G$ be as above. Then*

$$\pi_1(X/G, [x]) \cong \pi_1(\mathcal{X}, \bar{x})/N \cong \pi_1(\mathcal{X}, \bar{x})/\mathbf{I}$$

*where $N$ is the normal subgroup generated by the image of the inertia groups $I_x$ and $\mathbf{I}$ is the subgroup generated by the elements of $\pi_1(\mathcal{X})$ having fixed points in the universal cover of $X_1 \times \cdots \times X_k$.*

*Proof.* By [Noo08, Theorem 8.3(i)] we have that $\pi_1(X/G,[x]) \cong \pi_1(\mathcal{X},\bar{x})/N$. The group $\pi_1(\mathcal{X},\bar{x})$ acts on $\tilde{X} \cong \tilde{X}_1 \times \cdots \times \tilde{X}_k$ the universal cover of $X_1 \times \cdots \times X_k$ in such a way that $[(\tilde{X}_1 \times \ldots \times \tilde{X}_k)/\pi_1(\mathcal{X},\bar{x})] \cong \mathcal{X}$. As $G$ is finite, by Lemma 2.1 any stabilizer $I_x$ for the action of $\pi_1(\mathcal{X})$ on $\tilde{X}$ is finite, therefore it has the slice property and by [Noo08, Theorem 9.1] we obtain that $\pi_1(X/G,[x]) \cong \pi_1(\mathcal{X},\bar{x})/\mathbf{I}$ .

□

# 3. The fundamental group of the product of topological spaces

## 3.1. Constructing the homomorphism

**3.1.1. Finite index of the group in the product.**— Let $I_y$ denote the isotropy at the point $y$ in $\tilde{X}$ for the action of $\pi_1(\mathcal{X},\bar{x})$. By Lemma 2.1 the map $\pi_1(\mathcal{X},\bar{x}) \to G$ restricted to $I_y$ is injective, therefore we can identify $I_y$ with a subgroup of $G$. When we do such identification we will denote it by $I'_y < G$.

Now as $\pi_1(\mathcal{X},\bar{x}) \cong \pi_1(\mathcal{X}_1,\bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k,\bar{x}_k)$, if $y = (y_1,\ldots,y_k)$ we define $I_i < \pi_1(\mathcal{X}_i,\bar{x}_i)$ as the image of $I_y$ via the morphism $\pi_1(\mathcal{X},\bar{x}) \to \pi_1(\mathcal{X}_i,\bar{x}_i)$.

**Lemma 3.1.** *We have that $I_y \cong I_i$ for all $i = 1,\ldots,k$ and $I_y = I_1 \times_{I'_y} \cdots \times_{I'_y} I_k$.*

*Proof.* For $\gamma = (\gamma_1,\ldots,\gamma_k) \in I_y$ note that $\gamma_i \in \pi_1(\mathcal{X}_i,\bar{x}_i)$ fixes $y_i \in \tilde{X}_i$, otherwise $\gamma$ can not fix a point in $\tilde{X}$. As above, the restriction of $\pi_1(\mathcal{X}_i,\bar{x}_i) \to G$ to $I_{y_i}$ is injective and as $I_i \subset I_{y_i}$ we have that $\gamma_i \neq \beta_i$ for $\gamma,\beta \in I_y \subset \pi_1(\mathcal{X}_1,\bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k,\bar{x}_k)$ with $\gamma \neq \beta$. Therefore we can construct an inverse to the projection. The result follows. □

Note that we obtain that $I_i < I_{y_i}$, but in general $I_{y_i}$ can be bigger. Let us define the homomorphism $I_y \to \prod I_{y_i}$ given by decomposing an element in its components. By Lemma 3.1 it is injective. Denote by $N$ the subgroup in $\pi_1(\mathcal{X},\bar{x})$ generated by all the $I_y$ and by $N'_i$ the subgroup in $\pi_1(\mathcal{X}_i,\bar{x}_i)$ generated by $I_i$.

**Lemma 3.2.** *The subgroup $N'_i$ is normal in $\pi_1(\mathcal{X}_i,\bar{x}_i)$.*

*Proof.* Let $\gamma_i \in N'_i$ and $t_i \in \pi_1(\mathcal{X}_i,\bar{x}_i)$. We can write $\gamma_i = \gamma_{i_1} \cdots \gamma_{i_j}$ with each $\gamma_{i_l} \in I_{i_l}$ coming from $\gamma_l = (\gamma_{1_l},\ldots,\gamma_{i_l},\ldots,\gamma_{k_l}) \in I_{y_l} \subset \pi_1(\mathcal{X},\bar{x})$ and the point $y_l = (y_{1_l},\ldots,y_{i_l},\ldots,y_{k_l}) \in \tilde{X}$ for $l = 1,\ldots,j$. As every $\pi_1(\mathcal{X}_j,\bar{x}_j) \to G$ is surjective, for $j = 1,\ldots,i-1,i+1,\ldots,k$, there exists $t_j \in \pi_1(\mathcal{X}_j,\bar{x}_j)$ such that $t = (t_1,\ldots,t_k) \in \pi_1(\mathcal{X},\bar{x})$.

As $t \cdot \gamma_l \cdot t^{-1} \in I_{ty_l}$ it follows that $t_i \gamma_{i_l} t_i^{-1} \in N'_i$ and therefore

$$t_i \gamma_i t_i^{-1} = (t_i \gamma_{i_1} t_i^{-1}) \cdot t_i \cdots t_i^{-1} \cdot (t_i \gamma_{i_j} t_i^{-1}) \in N'_i.$$

□

**Proposition 3.3.** *There is an homomorphism*

$$\pi_1(X/G,[x]) \to \prod_{i=1}^{k} \pi_1(\mathcal{X}_i,\bar{x}_i)/N'_i$$

*such that the image has finite index.*

*Proof.* By Lemma 2.2 we have that $\pi_1(\mathcal{X},\bar{x}) \cong \pi_1(\mathcal{X}_1,\bar{x}_1) \times_G \cdots \times_G \pi_1(\mathcal{X}_k,\bar{x}_k)$. Therefore there is an injective homomorphism $\pi_1(\mathcal{X},\bar{x}) \to \prod \pi_1(\mathcal{X}_i,\bar{x}_i)$. By Lemma 3.2 we obtain the exact sequence

$$1 \to \prod_{i=1}^{k} N'_i \to \prod_{i=1}^{k} \pi_1(\mathcal{X}_i,\bar{x}_i) \to \prod_{i=1}^{k} \pi_1(\mathcal{X}_i,\bar{x}_i)/N'_i \to 1, \tag{3.1}$$

and together with Lemma 2.3 we obtain a commutative diagram

$$
\begin{array}{ccccc}
 & & 1 & & 1 \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & N & \longrightarrow & \prod_{i=1}^k N_i' \\
 & & \downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1(\mathcal{X},\bar{x}) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i) \\
 & & \downarrow & & \downarrow \\
 & & \pi_1(X/G,[x]) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow \\
 & & 1 & & 1.
\end{array}
\tag{3.2}
$$

This diagram provides a homomorphism

$$\pi_1(X/G,[x]) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)/N_i'$$

and shows that it is well defined. We can not complete (3.2) to a commutative diagram of groups with short exact sequence in the rows because usually $\pi_1(\mathcal{X},\bar{x})$ is not normal in $\prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)$. It will be normal, for example, if $G$ is abelian.

As $G$ is finite we obtain that $\pi_1(\mathcal{X},\bar{x})$ has finite index in $\prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)$. In fact the upper-bound

$$\left[\prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i) : \pi_1(\mathcal{X},\bar{x})\right] \leq |G|^{k-1}$$

can be seen as follows. For each surjection $\varphi_i : \pi_1(\mathcal{X}_i,\bar{x}_i) \to G$ consider a lift $G_i \subset \pi_1(\mathcal{X}_i,\bar{x}_i)$ of $G$ with $|G_i| = |G|$. In $\prod_{i=1}^k G_i$, let us consider the equivalence relation

$$(g_1,\ldots,g_k) \sim (g_1',\ldots,g_k') \Leftrightarrow (\varphi_1(g_1),\ldots,\varphi_k(g_k)) = (g\varphi_1(g_1'),\ldots,g\varphi_k(g_k')) \text{ with } g \in G.$$

It is easily seen that the quotient $\left(\prod_{i=1}^k G_i\right)/\sim \cong (G\times\cdots\times G)/\Delta_G$ is a set of representatives of left cosets $(\prod_{i=1}^k \pi_1(\mathcal{X}_i),\bar{x}_i)/\pi_1(\mathcal{X},\bar{x})$. By considering as coset representatives in $\prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)/N_i'$ the image of $\prod_{i=1}^k G_i$ and using the diagram (3.2) we have that $\pi_1(X/G,[x])$ has finite index in $\prod_{i=1}^k \pi_1(\mathcal{X}_i,\bar{x}_i)/N_i'$. □

## 3.2. The homomorphism has finite kernel

**3.2.1. The subgroup $N_i'$ is finitely normally generated.**— Let $X$ be a connected, semi-locally simply connected and locally path connected topological space. Let $G$ be a discrete finite group acting on $X$, $x \in X$ and denote by $\bar{x} \in \mathcal{X} = [X/G]$ the image of the point $x$ and by $p : X \to [X/G]$ the quotient map.

Let us briefly recall the description of $\pi_1(\mathcal{X},\bar{x})$ as given in [Che01]. It can be defined as $\pi_0(\Omega(\mathcal{X},\bar{x}))$ where $\Omega(\mathcal{X},\bar{x})$ denote the space loop of $\mathcal{X}$ pointed at the constant loop of value $\bar{x}$. Every loop is given locally as a map from an open subset of $S^1$ to a given uniformization of an open subset of $\mathcal{X}_{\text{top}}$ and plus some gluing conditions. In our case of a global quotient, a more explicit description of $\Omega(\mathcal{X},\bar{x})$ can be given as follows.

Let $P(X,x)$ consist of paths in $X$ starting at $x$. As a subspace of $\Lambda(X)$, the free loop space of $X$, it inherits a structure of a topological space. By considering the constant loop $x$ of value $x \in X$, we obtain $(P(X,x),x)$ a pointed topological space. Define $P(X,G,x)$ as the subspace of $P(X,x)\times G$ consisting of the elements $(\gamma,g)$ satisfying $\gamma(1) = g\cdot\gamma(0) = g\cdot x$. As a topological space it is pointed at $(x,1_G)$

**Lemma 3.4.** [Che01, Lemma 3.4.2] *There exists a natural homeomorphism between the pointed topological spaces* $(\Omega(\mathcal{X},\bar{x}),x)$ *and* $(P(X,G,x),(x,1_G))$.

*Remark* 3.5. When $(\mathcal{X},\bar{x})$ is a pointed topological stack there exists $(B[R\rightrightarrows X],x')$ a pointed topological space, where $B[R\rightrightarrows X]$ is the classifying space of the topological groupoïd $[R\rightrightarrows X]$, such that we can take $\pi_1(\mathcal{X},\bar{x}):=\pi_1(B[R\rightrightarrows X],x')$. In the case of a global quotient $\mathcal{X}=[X/G]$ it happens that $B[R\rightrightarrows X]$ equals the Borel construction $X\times_G EG$, see [Noo12].

Now, the construction of Chen also gives a natural isomorphism between $\pi_1(\mathcal{X},\bar{x})$ and $\pi_1(X\times_G EG,x')$ [Che01, Theorem 3.4.1] linking both definitions.

There exists a canonical projection $(P(X,G,x),(x,1_G))\to(G,1_G)$ given by sending $(\gamma,g)$ to $g$. This map can be seen to be a fibration [Che01, Lemma 3.4.3] having as fiber at $1_G$ the space loop $\Omega(X,x)$ via the embedding $\Omega(X,x)\hookrightarrow P(X,G,x)$ where $\gamma$ maps to $(\gamma,1_G)$.

With this description at hand, suppose there is $y\in X$ such that it is fixed by an element $g$, that is, $y\in X^g$. Denote by $\gamma_y$ a path starting at $x$ and finishing at $y$, then $\gamma_y(g\gamma_y^{-1})\in P(X,G,x)$, where $g\gamma_y^{-1}$ denotes the action of $g$ applied to each point of the path.

**Lemma 3.6.** *Let* $I_y<G$ *denote the inertia (stabilizer) of the action of* $G$ *at* $y\in X$*. Every homotopy class of a path* $[\gamma_y]\in\pi_1(X,x,y)$ *induces an injective morphism* $I_y\to\pi_1(\mathcal{X},x)$.

*Proof.* As $G$ is discrete $g\mapsto\gamma_g(g\gamma_g^{-1})$ is continuous, with $g\in I_y$. Then by taking the functor $\pi_0$ we got a morphism of groups $\pi_0(I_y)\to\pi_0(P(X,G,x))=\pi_1(\mathcal{X},\bar{x})$. Finally, by composing with the projection $(\pi_0(P(X,G,x),x))\to\pi_0((G,1_G))$ we obtain that different points under $\pi_0(I_y)\to\pi_1(\mathcal{X},\bar{x})\to\pi_0(G)$ have different images, thus the morphism in injective. ☐

**Lemma 3.7.** *Let* $Y\in\pi_0(X^g)$, $y_1,y_2\in Y$ *and let* $\gamma_{y_1},\gamma_{y_2}$ *be paths starting at* $x\in X$ *and finishing at* $y_1$ *and* $y_2$ *respectively, then* $\gamma_{y_1}(g\gamma_{y_1}^{-1})$ *is a conjugate of* $\gamma_{y_2}(g\gamma_{y_2}^{-1})$ *in* $\pi_1(\mathcal{X},\bar{x})$ *by elements of* $\pi_1(X,x)$.

*Proof.* There exists a path $\beta\subset Y$ connecting $y_1$ and $y_2$, therefore $\gamma_{y_1}\beta(g\beta^{-1}\gamma_{y_1}^{-1})\in P(X,G,x)$ but as $g\beta=\beta$ passing to $\pi_0(P(X,G,x),x)$ it equals $[\gamma_{y_1}(g\gamma_{y_1}^{-1})]$.

Now consider the path $\gamma_{y_2}$. Note that $\theta:=\gamma_{y_1}\beta\gamma_{y_2}^{-1}\in\Omega(X,x)$. There exists a continuous map

$$\#:P(X,G,x)\times P(X,G,x)\to P(X,G,x)$$

which induces the multiplication in the fundamental group (see [Che01, Section 3.1]). The element $\theta\#(\gamma_{y_2}(g\gamma_{y_2}^{-1})\#\theta^{-1})$ can be seen to be $\theta(\gamma_{y_2}\cdot(g(\theta\cdot\gamma_{y_2})^{-1}))\in P(X,G,x)$. By passing to the group $\pi_1(\mathcal{X},\bar{x})=\pi_0(P(X,G,x),x)$ we have that $[\theta][\gamma_{y_2}(g\gamma_{y_2}^{-1})][\theta^{-1}]=[\gamma_{y_1}(g\gamma_{y_1}^{-1})]$. ☐

Recall that given $(X,x)$ as above, we have a pointed universal cover map $p:(\tilde{X},\tilde{x})\to(X,x)$ where $\tilde{x}$ represents the constant loop of value $x$. Every element in $\gamma\in\pi_1(X)$ corresponds to a point in $p^{-1}(x)$. So given a pointed map $p_\gamma:(\tilde{X},\gamma)\to(X,x)$ it induces a deck transformation of $\tilde{X}$ in the following way: given $y\in\tilde{X}$ take a path $\alpha_y\subset\tilde{X}$ starting at $\gamma$ and finishing at $y$. Consider the unique lift $\widetilde{p_\gamma(\alpha_y)}\subset\tilde{X}$ starting at $x$ and assign to $y$ the point $\widetilde{p_\gamma(\alpha_y)}(1)$. It can be seen to be a well-defined map (see [Hat00]).

Now, by the description given above of $\pi_1(\mathcal{X},\bar{x})$, any $\gamma\in\pi_1(\mathcal{X},\bar{x})$ such that $\varphi(\gamma)=g$ (recall that $\varphi:\pi_1(\mathcal{X},\bar{x})\to G$) have as a representative an element in $P(X,G,x)$ which we still denote by $\gamma$. So $\gamma$ starts at $x$ and finishes at $gx$. Denote by $\tilde{\pi}:(\tilde{X},\tilde{x})\to(\mathcal{X},\bar{x})$ the universal cover morphism, note that $\tilde{\pi}_\gamma:(\tilde{X},\gamma)\to(\mathcal{X},\bar{x})$ is also a cover morphism. By [Che01, Theorem 4.1.6] we obtain a deck transformation in the following way: given $y\in\tilde{X}$ take a path $\alpha_y\subset\tilde{X}$ starting at $\gamma$ and ending at $y$. Using the notation of the precedent paragraph, the path $p_\gamma(\alpha_y)$ starts at $gx$. Then the path $g^{-1}p_\gamma(\alpha_y)$ starts at $x$ so we can lift it to $g^{-1}\widetilde{p_\gamma(\alpha_y)}$ in $(\tilde{X},\tilde{x})$, the end point of this lift is then defined as the image of $y$. It is shown that it is a well defined map and does not depend on the path chosen.

**Lemma 3.8**. *Let $y \in X$ be fixed by $g \in G$, consider a path $\gamma_y$ connecting $x$ and $y$. Consider the action of $\pi_1(\mathcal{X}, \bar{x})$ on $\tilde{X}$ given by deck transformations* $\mathrm{Deck}(\tilde{X}, \mathcal{X})$, *then the element $\gamma_y(g\gamma_y^{-1}) \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in $\tilde{X}$. Moreover, any element of $\pi_1(\mathcal{X}, \bar{x})$ fixing a point in $\tilde{X}$ is of this form.*

*Proof.* As the endpoint of $\gamma_y(g\gamma_y^{-1})$ is $gx$ we have a pointed covering morphism

$$\tilde{\pi}_{\gamma_y(g\gamma_y^{-1})} : (\tilde{X}, \gamma_y(g\gamma_y^{-1})) \to (\mathcal{X}, \bar{x})$$

and we can consider $g\gamma_y$ as a path in $\tilde{X}$ connecting $\gamma_y(g\gamma_y^{-1})$ and $\gamma_y$ as follows: let us define $f(t) = \gamma(g\gamma_y^{-1}) \cdot (g\gamma_y|_t)$ where $g\gamma_y|_t(t') := g\gamma_y(t'/t)$ denote the path starting at $gx$ and finishing at $g\gamma_y(t)$ in time $t$ for $t \neq 0$ and being the constant path with value $gx$ if $t = 0$. We project then $f(t)$ to $X$ and obtain $g\gamma_y$ which starts at $gx$ and finishes at $\bar{y}$. By the discussion before the lemma, we obtain that it lifts to $\gamma_y$ in $(\tilde{X}, \tilde{x})$, as $g$ fixes $y$ we obtain that the point $\gamma_y \in \tilde{X}$ is fixed by the induced deck transformation.

Consider the exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(\mathcal{X}, \bar{x}) \xrightarrow{\varphi} G \to 1,$$

let $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ and $z \in \tilde{X}$ such that $\gamma$ fixes $z$. Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be the projection, as it is $\varphi$-invariant we have that $\varphi(\gamma)p(z) = p(z)$. Then by considering the path in $X$ corresponding to $z$, we can construct an element $z\varphi(\gamma)z^{-1}$, which fixes $z \in \tilde{X}$. As in the isotropy $\varphi$ is injective by Lemma 2.1, we have that $z\varphi(\gamma)z^{-1} = \gamma$. □

**Proposition 3.9**. *Suppose that there are only a finite number of elements in $\pi_0(X^g)$ for each $g \in G$, then there exists a finite set $L \subset \pi_1(\mathcal{X}, \bar{x})$ consisting of elements having fixed points in $\tilde{X}$ such that if $\gamma \in \pi_1(\mathcal{X}, \bar{x})$ fixes a point in $\tilde{X}$ then it is conjugate to an element of $L$ by elements in $\pi_1(X, x)$.*

*Proof.* By Lemma 3.7 for every element in $Y \in \pi_0(X^g)$ it suffices to fix an element $\gamma_y(g\gamma_y^{-1})$ with $y \in Y$. For every $g \in G$ and every element in $\pi_0(X^g)$ we pick such an element. We define $L$ as the set consisting of such elements. By Lemma 3.8 every such element fixes a point in $\tilde{X}$ and any other fixing a point will be conjugate of the element in $L$ corresponding to its connected component. □

**3.2.2. Proof that the homomorphism has finite kernel.**— Let us return to the case of $k$-topological spaces $X_1, \ldots, X_k$ and let $G$ be a finite group acting on each one of them on the left as in 2.2.1. Proposition 3.9 gives us $k$ subsets $L(\mathcal{X}_i) \subset \pi_1(\mathcal{X}_i, \bar{x}_i)$ whose elements correspond to the element of $\pi_0(X_i^g)$ with $g \in G$. Now consider the subsets $L_i \subset L(\mathcal{X}_i)$ consisting of elements corresponding to $\pi_0(X_i^g)$ where $g$ fixes a point in $X_i$ for $i = 1, \ldots, k$.

Recall that $N < \pi_1(\mathcal{X}, \bar{x})$ (with $\mathcal{X} = [(X_1 \times \cdots \times X_k)/G]$) is the subgroup generated by the inertia subgroups $I_y$ given by the action of $\pi_1(\mathcal{X}, \bar{x})$ in $\tilde{X}$ and $N_i' < \pi_1(\mathcal{X}_i, \bar{x}_i)$ is the image of the $i$-projection of $N$. The following lemma is immediate from Proposition 3.9

**Lemma 3.10**. *We have that $N_i' = \left\langle \gamma_i l_i \gamma_i^{-1} \mid l_i \in L_i, \gamma_i \in \pi_1(X_i, x_i) \right\rangle$ in $\pi_1(\mathcal{X}_i, \bar{x}_i)$ for $i = 1, \ldots, k$.*

**Definition 3.11**. Let us define

$$C_i = C_i(\pi_1(X_i), L_i) := \left\langle\!\left\langle \gamma_i l_i \gamma_i^{-1} l_i^{-1} \mid \gamma_i \in \pi_1(X_i, x_i), l_i \in L_i \right\rangle\!\right\rangle_{\pi_1(\mathcal{X}_i, \bar{x}_i)},$$

to be the normal subgroup generated by the commutators of elements in $\pi_1(X_i, x_i)$ and in $L_i$. Denote by $\mathbb{T}_i := \pi_1(\mathcal{X}_i, \bar{x}_i)/C_i$ and by $\hat{L}_i$ the image of $L_i$ in $\mathbb{T}_i$.

**Lemma 3.12**. *It happens that $C_i < N_i'$ and moreover we can consider $C_i$ as a subgroup of $N$ via $\{e\} \times \cdots \times C_i \times \ldots \times \{e\}$ and $C_1 \times \cdots \times C_k < N$.*

*Proof.* Let $l_i \in L_i$ and $\gamma_i \in \pi_1(X_i, x_i)$, the elements of $L_i$ were chosen such that there exists $l_j \in L_j$ and $y \in \tilde{X}$ such that $l = (l_1, \dots, l_i, \dots, l_k) \in I_y < N$. We have that $\gamma_i' = (e, \dots, \gamma_i, \dots, e) \in \pi_1(\mathcal{X}, \bar{x})$ and as $N$ is normal in $\pi_1(\mathcal{X}, \bar{x})$ we have that $\gamma_i' l \gamma_i'^{-1} \in N$, so

$$\gamma_i' l \gamma_i'^{-1} l^{-1} = (e, \dots, \gamma_i l_i \gamma_i^{-1} l_i^{-1}, \cdots, e) \in N$$

This element projects to $[\gamma_i, l_i] \in C_i$. Finally given $\beta_i \in \pi_1(\mathcal{X}_i, \bar{x}_i)$, as every $\varphi_j$ is surjective, there exists $\beta_j \in \pi_1(\mathcal{X}_j, \bar{x}_j)$ such that $\varphi_i(\beta_i) = \varphi_j(\beta_j)$, so $\beta = (\beta_1, \dots, \beta_k) \in \pi_1(\mathcal{X}, \bar{x})$ and every conjugate of $[\gamma_i, l_i]$ can be seen as an element of $N$.

Finally, by considering the product of the identification of the elements in $C_j$ we have $C_1 \times \cdots \times C_k < N$. □

Before stating the next lemma recall that $N < N_1' \times_G \cdots \times_G N_k'$.

**Lemma 3.13.** *The subgroup $C_i$ has finite index in $N_i'$, in particular $C_1 \times \cdots \times C_k$ has finite index in $N_1' \times \cdots \times N_k'$ hence also in $N$.*

*Proof.* First note that by Lemma 3.10 and by definition of $\mathbb{T}_i$ we have

$$N_i'/C_i = \langle\langle L_i \rangle\rangle_{\pi_1(X,x)}/C_i \cong \langle\langle \hat{L}_i \rangle\rangle_{R_i} = \langle \hat{L}_i \rangle,$$

with $R_i$ the image of $\pi_1(X_i, x_i)$ in $\mathbb{T}_i$.

Moreover as $\varphi(C_i) = \{e\}$ we have that $C_i < \ker \varphi \cong \pi_1(X_i, x_i)$. As $\pi_1(X_i, x_i)$ has finite index in $\pi_1(\mathcal{X}_i, \bar{x}_i)$, it follows that $R_i$ has finite index in $\mathbb{T}_i$, which implies that $R_i \cap \langle \hat{L}_i \rangle$ has finite index in $\langle \hat{L}_i \rangle$. Note that $\langle \hat{L}_i \rangle$ is generated by a finite number of torsion elements and that by construction $R_i \cap \langle \hat{L}_i \rangle$ is a central group in $\langle \hat{L}_i \rangle$. As any group generated by a finite number of torsion elements and such that the center has finite index is finite (see [BCGP12, Lemma 4.6]) the result follows. □

**Theorem 3.14.** *The homomorphism $\pi_1(X/G, [x]) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ has finite kernel.*

*Proof.* By composing the quotient map $\prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ with the inclusion $\pi_1(\mathcal{X}, \bar{x}) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)$ we obtain $\pi_1(\mathcal{X}, \bar{x}) \to \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i'$ with kernel $(N_1' \times \cdots \times N_k') \cap \pi_1(\mathcal{X}, \bar{x}) = N_1' \times_G \cdots \times_G N_k'$ by the description of $\pi_1(\mathcal{X}, \bar{x})$ as fiber product. We put this as a row in the following commutative diagram together with a vertical column given by Lemma 2.3 and complete to

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N & \xrightarrow{\text{id}} & N & & \\
 & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N_1' \times_G \cdots \times_G N_k' & \longrightarrow & \pi_1(\mathcal{X}, \bar{x}) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} \\
1 & \longrightarrow & N_1' \times_G \cdots \times_G N_k'/N & \longrightarrow & \pi_1(X/G, [x]) & \longrightarrow & \prod_{i=1}^k \pi_1(\mathcal{X}_i, \bar{x}_i)/N_i' \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
\end{array} \tag{3.3}$$

By Lemma 3.13 both $N_1' \times_G \cdots \times_G N_k'/C_1 \times \cdots \times C_k$ and $N/C_1 \times \cdots \times C_k$ are finite hence

$$\frac{N_1' \times_G \cdots \times_G N_k'/C_1 \times \cdots \times C_k}{N/C_1 \times \cdots \times C_k} \cong N_1' \times_G \cdots \times_G N_k'/N$$

is finite.

□

**3.2.3. Geometric interpretation of the groups $\pi_1(\mathcal{X}_i,\bar{x}_i)/N_i'$.**— Let us denote by $I$, the subgroup of $G$ generated by the elements having a fixed point in every $X_i$ for $i=1,\dots,k$. Note that $I$ is a normal subgroup.

Let $x_i'$ denote the class of $x_i$ in $X/I$ and $\bar{x}_i'$ the image of $x_i'$ in $[(X_i/I)/(G/I)]$.

**Proposition 3.15.** *There is an isomorphism*

$$\pi_1(\mathcal{X}_i,\bar{x}_i)/N_i' \xrightarrow{\sim} \pi_1([(X_i/I)/(G/I)],\bar{x}_i').$$

*Proof.* Observe that the action of $G$ on $X_i$ descends to an action of $G/I$ on $X_i/I$ and therefore we can define $[(X_i/I)/(G/I)]$. Recall by the previous subsection 3.2.1 that $\pi_1(\mathcal{X}_i,\bar{x}_i)$ can be identified with the set of path-components of $P(X_i,G,x)$. Therefore an element $[\gamma]\in\pi_1(\mathcal{X}_i,\bar{x}_i)$ can be represented by a path $\gamma$ in $X_i$ starting at $x_i$ and finishing at $gx_i$ for some $g\in G$. Denote by $p_i : X_i \to X_i/I$ the quotient map. By considering $p_i(\gamma)$, we obtain a morphism between $\pi_1(\mathcal{X}_i,\bar{x}_i)$ and $\pi_1([(X_i/I/G/I)],\bar{x}_i')$.

It is immediate to see that the paths coming from the inertia of $I$ in $X_i$, that is, the elements of the form $\gamma_y(g\gamma_y^{-1})$ with $g\in I$ and $y\in X_i^g$, are sent to the trivial element in $\pi_1(X_i/I,x_i')$.

Now consider $\gamma\in\ker\left(\pi(\mathcal{X}_i,\bar{x}_i)\to\pi_1([(X_i/I)/\ (G/I)],\bar{x}_i')\right)$. Then $\gamma$ is represented by a path in $X_i$, which we still denote by $\gamma$, starting at $x_i$ and finishing at $gx_i$ with $g\in G$. Note that moreover $g\in I$, otherwise by the projection $\pi_1([X_i/I/G/I],\bar{x}_i')\to G$ the element would be sent to a non-zero element. Hence the image of $\gamma$ lies in $\pi_1(X_i/I,x_i')$ and it is trivial. By the exact sequence

$$1\longrightarrow N_{[X_i/I]}\longrightarrow\pi_1([X_i/I],\bar{x}_i)\longrightarrow\pi_1(X/I,x_i')\to 1$$

and noticing that $N_{[X_i/I]}=N_i'$ we have that $\gamma\in N_i'$ which proves the result. □

# 4. Applications

## 4.1. Product of the same topological space

Now let us describe a case where $N_i'$ equals the whole subgroup $N_i$ generated by the elements having a fixed point in the universal cover.

**Corollary 4.1.** *Let $X_i=X_1$ for $i=2,\dots,k$ and $G$ finite acting on $X_1$. Then the morphism*

$$\pi_1\left((X_1\times\cdots\times X_1)/G,[x]\right)\longrightarrow\prod_{i=1}^{k}\pi_1(X_1/G,[x_i])$$

*has finite kernel.*

*Proof.* We only have to show that $N_1'=N_1$ and then we obtain the result by applying Theorem 3.14. By construction we have that $N_1'\subset N_1$. Let us show the inverse inclusion. Take $\gamma_1\in N_1$, then we can write $\gamma_1=\gamma_{1_1}\cdots\gamma_{1_l}$ such that there exists $y_{1_j}\in\tilde{X}_1$ satisfying $\gamma_{1_j}\in I_{y_{1_j}}$ for $j=1,\dots,l$. As $\tilde{X}=\tilde{X}_1\times\cdots\times\tilde{X}_k$ by taking $y_j=(y_{1_j},\dots,y_{1_j})\in\tilde{X}$ we have that $\gamma^j=(\gamma_{1_j},\dots,\gamma_{1_j})\in I_{y_j}$ and therefore $\gamma=\gamma^1\cdots\gamma^l\in N$ and the image of $\gamma$ in $N_1$ equals $\gamma_1$. □

Another proof using Proposition 3.15 can be obtained as follows: the action of $G/I$ is free in $X_1/I$ and since $X_1/G\cong(X_1/I)/(G/I)$ we have $\pi_1\left([(X_1/I)/(G/I)]\right)\cong\pi_1(X_1/G)$.

## 4.2. Second Main Theorem

**Theorem 4.2.** *Let $X_1,\dots,X_k$ admit a universal cover and let $G$ be a finite group acting on each of them such that $\left|\pi_0(X_i^g)\right|<+\infty$ for every $g\in G$ and $i=1,\dots,k$. Denote $X=X_1\times\cdots\times X_k$ and consider the diagonal action of $G$ on it. Suppose $\pi_1(X/G,[x])$ is residually finite, then $\pi_1(X/G,[x])$ has a normal finite-index subgroup $\mathcal{N}\cong H_1\times\cdots\times H_k$ isomorphic to a product of normal finite index subgroups subgroups $H_i\lhd\pi_1(X_i/I,[x_i])$.*

*Proof.* By Theorem 1.3 we get a morphism $\Theta : \pi_1(X/G,[x]) \to \prod_{i=1}^k \pi_1([X_i/I/G/I])$ having finite kernel $E$. As $\pi_1(X/G,[x])$ is residually finite we can construct a finite-index normal subgroup $\Gamma \lhd \pi_1(X/G,[x])$ such that $\Gamma \cap E = \{e\}$.

The morphism $\Theta|_\Gamma : \Gamma \to \prod_{i=1}^k \pi_1([(X_i/I)/(G/I)],\bar{x}_i')$ is therefore injective and moreover as the subgroup $\Theta(\pi_1(X/G)) < \prod_{i=1}^k \pi_1([(X_i/I)/(G/I)],\bar{x}_i')$ has finite index it follows that $\Theta(\Gamma) < \prod_{i=1}^k \pi_1([(X_i/I)/(G/I)],\bar{x}_i')$ has finite index.

For every $i = 1,\dots,k$, we have $\pi_1(X_i/I,[x_i]) < \pi_1([(X_i/I)/(G/I)],\bar{x}_i')$ as a normal finite-index subgroup. Define the subgroup

$$\Theta(\Gamma)_i := \Theta(\Gamma) \cap (\{e_1\} \times \cdots \times \pi_1(X_i/I,[x_i]) \times \cdots \times \{e_k\})$$

where $e_k \in \pi_1(X_j/I,[x_j])$ is the identity element. As $\Theta(\Gamma)_i$ has finite index in $\pi_1(X_i/I,[x_i])$, there exists a normal subgroup of finite index $H_i$ of $\pi_1([(X_i/I)/(G/I)])$ contained in $\Theta(\Gamma)_i$. Set $H := H_1 \times \cdots \times H_k$, then $H \lhd \Theta(\Gamma)$ and it is a finite-index normal subgroup of $\prod_{i=1}^k \pi_1([(X_i/I)/(G/I)],\bar{x'}_i)$. The subgroup $\mathcal{N} := \Theta^{-1}(H) \cap \Gamma$ satisfies the stated properties. □

### 4.2.1. Case of smooth curves.—

**Corollary 4.3.** *Let $C_1,\dots,C_k$ be smooth algebraic curves and let $G$ be a finite group acting on each $C_i$. Denote $C = C_1 \times \cdots \times C_k$. Consider $\mathcal{C} = [C/G]$ with $G$ acting diagonally on $C$. Then $\pi_1(C/G)$ has a normal subgroup $\mathcal{N}$ of finite index isomorphic to $\Pi_1 \times \cdots \times \Pi_k$ where $\Pi_i$ is either a surface group or a finitely generated free group for $i = 1,\dots,k$.*

By Theorem 1.3 we have a morphism $\pi_1(C/G) \to \prod_{i=1}^k \pi_1([C_i/I/G/I])$ with finite kernel, however if the action of $G/I$ is not faithful on $C_i/I$ then $\pi_1([(C_i/I)/(G/I)])$ is not necessarily an orbifold surface group. This can be overcome as follows: let $K_i := \ker(G/I \to \operatorname{Aut} C_i/I)$ and $H_i := (G/I)/K_i$. Denote by $\mathcal{C}_i := [(C_i/I)/G/I]$ and by $\mathcal{C}_i' := [(C_i/I)/H_i]$, we have a canonical morphism $\mathcal{C}_i \to \mathcal{C}_i'$.

**Lemma 4.4.** *The induced homomorphism $q_i : \pi_1(\mathcal{C}_i) \to \pi_1(\mathcal{C}_i')$ is surjective and has finite kernel.*

*Proof.* By choosing a point $x_i \in C_i$ and denoting by $\bar{x}_i$ its image in both $\mathcal{C}_i$ and $\mathcal{C}_i'$ we obtain a fibration $[\mathrm{pt}/K,\mathrm{pt}] \hookrightarrow (\mathcal{C}_i,\bar{x}_i) \to (\mathcal{C}_i',\bar{x}_i)$. By taking the long homotopy exact sequence

$$\ldots \to \pi_2(\mathcal{C}_i',\bar{x}_i) \to \pi_1(\mathrm{pt}/K,\mathrm{pt}) \to \pi_1(\mathcal{C}_i,\bar{x}_i) \to \pi_1(\mathcal{C}_i',\bar{x}_i) \to 1,$$

as there is an isomorphism between $\pi_1(\mathrm{pt}/K,\mathrm{pt})$ and $\pi_0(K,1_K)$, the result follows. □

So by composing, we obtain a morphism $\Theta : \pi_1(C/G) \to \prod_{i=1}^k \pi_1(\mathcal{C}_i) \to \prod_{i=1}^k \pi_1(\mathcal{C}_i')$, this allows us to prove the following lemma, which together with Theorem 4.2 will imply Corollary 4.3.

**Lemma 4.5.** *The group $\pi_1(C/G)$ is residually finite.*

*Proof.* First note that, as $\pi_1(\mathcal{C}_i')$ is an orbifold surface group, it is in particular residually finite. Now, it follows that $\Theta(\pi_1(C/G))$ is residually finite as it is a finite-index subgroup of a direct product of residually finite groups.

We need another property of these groups to continue. Let $H$ be a group and let $\hat{H}$ be its profinite completion. A group $H$ is called *good* if for each $k \geq 0$ and for each finite $H$-module $M$ the natural homomorphism

$$H^k(\hat{H},M) \to H^k(H,M)$$

is an isomorphism. In [GJZZ08, Lemmas 3.2 and 3.4, Proposition 3.6] it is shown that a finite-index subgroup of a good group is good, the product of good groups is good and that $\pi_1(\mathcal{C})$ for $\mathcal{C}$ an algebraic orbifold curve is good. We obtain therefore that $\Theta(\pi_1(C/G))$ is good.

Finally, [GJZZ08, Proposition 6.1] asserts that if $T$ is a residually finite good group and $\varphi : H \to T$ is a surjective homomorphism with finite kernel then $H$ is residually finite. Applying this to $\Theta' : \pi_1(C/G) \to \Theta(\pi_1(C/G))$ we obtain the result. □

### 4.3. Partial compactifications of arrangement of lines

The original motivation of this work was to study the partial compactifications of the complement of an arrangement of lines in $\mathbb{P}^2_{\mathbb{C}}$ which is the topic of my Ph.D. thesis. In [Agu19] a general method for computing a presentation of the fundamental group was given and some examples studied. A family of arrangements related to the studied in *op. cit.* is available for any $n \in \mathbb{N}$, however this will require a treatment one by one. The results obtained here can be used to study some of these partial compactifications in family.

**4.3.1. Partial compactification of the complement of an arrangement of lines.**— Consider the projective plane $\mathbb{P}^2_{\mathbb{C}}$ with homogeneous coordinates $(z_1 : z_2 : z_3)$.

Let $\mathscr{A} = \sum_{i=1}^{k} L_i$ be a divisor in $\mathbb{P}^2$ such that the irreducible components $L_i$ are copies of $\mathbb{P}^1$ (lines). Then the singular set $\operatorname{Sing}\mathscr{A}$ of $\mathscr{A}$ consists only of points. Consider $\pi : \operatorname{Bl}_{\operatorname{Sing}\mathscr{A}} \mathbb{P}^2 \to \mathbb{P}^2$ the blow up of the projective plane at the points $\operatorname{Sing}\mathscr{A}$. The divisor $\pi^*\mathscr{A} = \sum_{i=1}^{k+|\operatorname{Sing}\mathscr{A}|} D_i$ has as irreducible components copies of $\mathbb{P}^1$, with $D_1, \dots, D_k$ being the strict transform of $L_1, \dots, L_k$ respectively and $D_{k+1}, \dots, D_{k+|\operatorname{Sing}\mathscr{A}|}$ being the exceptional divisors. Take a subset $J \subset \{1, 2, \dots, k + |\operatorname{Sing}\mathscr{A}|\}$. The surface $\operatorname{Bl}_{\operatorname{Sing}\mathscr{A}} \mathbb{P}^2 \setminus (\cup_{j\in J} D_j)$ is called a ***partial compactification of*** $\mathbb{P}^2 \setminus (\cup L_i)$. We are interested in how the fundamental group changes when we partially compactify the complement of such an arrangement $\mathscr{A} \subset \mathbb{P}^2$.

**4.3.2. Examples.**— The subvariety of $\mathbb{P}^2$

$$\operatorname{Ceva}(n) := \{(z_1 : z_2 : z_3) \mid (z_1^n - z_2^n)(z_1^n - z_3^n)(z_2^n - z_3^n) = 0\}$$

can be seen as the union of the closure of the three singular fibers of the rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $(z_1 : z_2 : z_3) \mapsto ((z_1^n - z_2^n) : (z_2^n - z_3^n))$. The map $f$ is not defined in a subset $S = \{p_1, \dots, p_{n^2}\} \subset \operatorname{Sing}\operatorname{Ceva}(n)$ consisting of $n^2$ points where $\mathscr{A}_1 := \{z_1^n - z_2^n = 0\} = \sum_{i=1}^{n} L_i$ intersects $\mathscr{A}_2 := \{z_1^n - z_3^n = 0\} = \sum_{i=n+1}^{2n} L_i$. It actually happens that $S \subset \mathscr{A}_3 := \{z_2^n - z_3^n = 0\} = \sum_{i=2n+1}^{3n} L_i$ and $S$ consists of points where 3 lines of $\operatorname{Ceva}(n)$ meet. We have another 3 points $p_{n^2+i}$ in $\operatorname{Sing}(\operatorname{Ceva}(n))$ which correspond to each singular point of $\mathscr{A}_i$ for $i = 1, 2, 3$ and hence of multiplicity $n$.

The rational map $f$ extends to a morphism $\tilde{f} : \operatorname{Bl}_{\operatorname{Sing}\operatorname{Ceva}(n)} \mathbb{P}^2 \to \mathbb{P}^1$ having as generic fiber the *Fermat curve of degree* $n$ defined as $F(n) := \{z_1^n + z_2^n + z_3^n = 0\} \subset \mathbb{P}^2$. Therefore $\tilde{f}$ is an isotrivial fibration.

Denote by $\mu(n)$ the group of $n^{\text{th}}$ roots of unity. By taking 3 copies of it we define $H(n) := \mu_1(n) \oplus \mu_2(n) \oplus \mu_3(n)/\langle \mu_1\mu_2\mu_3 = 1\rangle$ where $\mu_i \in \mu_i(n)$. It acts on $F(n)$ via $(z_1 : z_2 : z_3) \mapsto (\mu_1 z_1 : \mu_2 z_2 : \mu_3 z_3)$. The proof of the following theorem will appear elsewhere.

**Theorem 4.6.** *Consider the diagonal action of* $H(n)$ *in* $F(n) \times F(n)$*. Denote by* $S$ *the minimal resolution of* $F(n) \times F(n)/H(n)$.

*(1) The fibration* $S \to (F(n) \times F(n))/H(n) \to F(n)/H(n) \cong \mathbb{P}^1$ *is isomorphic to* $\tilde{f}$.

*(2) Every singular point in* $F(n) \times F(n)/H(n)$ *corresponds to the contraction of the strict transform* $D_i$ *of some line* $L_i \in \operatorname{Ceva}(n)$.

*(3) The contraction of the* $n$ *lines corresponding to* $\mathscr{A}_i$ *lie in the line* $E_i$ *which is the exceptional divisor corresponding to the unique singular point in* $\mathscr{A}_i$.

*(4)* $E_i$ *is mapped to a point via* $F(n) \times F(n)/H(n) \to \mathbb{P}^1$.

The Fermat curve $F(n)$ of degree $n$ can be seen as a branched covering of $\mathbb{P}^1$ of degree $n^2$ via the morphism in $\mathbb{P}^2$ given by $F(n) \ni (z_1 : z_2 : z_3) \to (z_1^n : z_2^n : z_3^n) \in \{w_0 + w_1 + w_2 = 0\}$ which branches at the points $(1 : -1 : 0), (1 : 0 : -1), (0 : 1 : -1)$. Over each branching point there are $n$ points, we denote by $X_1, \dots, X_n$ for those over $(1 : -1 : 0)$, by $Y_1, \dots, Y_n$ over $(1 : 0 : -1)$ and $Z_1, \dots, Z_n$ over $(1 : -1 : 0)$.

Recall that for $S \to S'$ be a resolution of singularities of $S'$, if $S'$ has only quotient singularities, by [Kol93, Theorem 7.8.1] we have that $\pi_1(S) \to \pi_1(S')$ is an isomorphism.

**Example 4.7.** Consider the surface $S_1 := \big(F(n) \times (F(n) \setminus \{X_1,\dots,X_n\})\big)/H(n)$. The subgroup $I$ generated by the elements of $H(n)$ having fixed points both in $F(n)$ and in $F(n) \setminus \{X_1,\dots,X_n\})$ equals $H(n)$. As $F(n)/H(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_1,\dots,X_n\})/H(n) \cong \mathbb{C}$ and by Theorem 1.3 the morphism

$$\pi_1(S_1) \to \pi_1(\mathbb{P}^1) \times \pi_1(\mathbb{C})$$

has finite kernel, it follows that $\pi_1(S_1)$ is finite.

The minimal resolution of singularities $S_1' \to S_1$ can be identified with the following partial compactification of $\mathrm{Ceva}(n)$. Consider

$$J := \{1,\dots,n,3n+n^3+1\} \subset \{1,\dots,3n+n^2+3\}$$

then following the construction given in §4.3.1 we have that

$$\mathrm{Bl}_{\mathrm{Sing\,Ceva}(n)} \mathbb{P}^2 \setminus \{\cup_{j\in J} D_j\} \cong S_1'.$$

That is, from the surface $\mathrm{Bl}_{\mathrm{Sing\,Ceva}(n)} \mathbb{P}^2$ we remove only the strict transform of $\mathscr{A}_1$ and the exceptional divisor coming from the singular point of $\mathscr{A}_1$. This can be identified with a singular fiber or $\tilde{f}$.

**Example 4.8.** Consider now $S_2 := (F(n) \times F(n) \setminus \{X_i, Y_i\})/H(n)$. In this case the subgroup $I$, defined as in the previous paragraph, is isomorphic to $\mu(n)$. As $F(n)/\mu(n) \cong \mathbb{P}^1$, $F(n) \setminus \{X_i, Y_i\})/\mu(n) \cong \mathbb{C}^*$ and by Theorem 1.3 the morphism

$$\pi_1(S_2) \to \pi_1([\mathbb{P}^1/\mu(n)]) \times \pi_1([\mathbb{C}^*/\mu(n)])$$

has finite kernel and the image is a finite-index subgroup.

By Theorem 4.2 and Corollary 4.3, we have that $\mathbb{Z} \triangleleft \pi_1(S_2)$ has finite index. As in Example 4.7 the minimal resolution of singularities $S_2' \to S_2$ can be identified with $\mathrm{Bl}_{\mathrm{Sing\,Ceva}(n)} \mathbb{P}^2$ minus two singular fibers of $\tilde{f}$.

**Example 4.9.** If we consider $S_3 := (F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\})/H(n)$ it can be identified with $\mathrm{Bl}_{\mathrm{Sing}\,\mathscr{A}} \mathbb{P}^2$ minus the three singular fibers of $\tilde{f}$. As $H(n)$ acts freely in $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\}$, the long exact sequence of homotopy associated to the covering map $F(n) \times F(n) \setminus \{X_i, Y_i, Z_i\} \to S_3$ yields

$$1 \longrightarrow \pi_1(F(n)) \times \pi_1(F(n) \setminus \{X_i, Y_i, Z_i\}) \longrightarrow \pi_1(S_3) \longrightarrow H(n) \to 1.$$

*Remark* 4.10. We can remove points also in the first component $F(n)$ of the product. However, we can not get more partial compactifications of $\mathrm{Ceva}(n)$ in this way. This can be shown by drawing the dual graph of the divisor $\pi^*\mathrm{Ceva}(n)$ and noticing that the lines obtained by removing points does not satisfy the intersection pattern of the graph.

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