EPIGA

Density of arithmetic representations of function fields

Hélène Esnault and Moritz Kerz

Abstract. We propose a conjecture on the density of arithmetic points in the deformation space of representations of the étale fundamental group in positive characteristic. This conjecture has applications to étale cohomology theory, for example it implies a Hard Lefschetz conjecture. We prove the density conjecture in tame degree two for the curve $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Hélène Esnault
Freie Universität Berlin, Arnimallee 3, 14195, Berlin, Germany; and
The Institute for Advanced Study, Mathematics, 1 Einstein Dr., Princeton, NJ 08540, USA
e-mail: esnault@math.fu-berlin.de and esnault@ias.edu
Moritz Kerz
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
e-mail: moritz.kerz@mathematik.uni-regensburg.de

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1. Introduction

Let X_0 be a smooth geometrically connected variety defined over a finite field $k = \mathbb{F}_q$ of characteristic p. We fix an algebraic closure $k \subset \bar{k}$ and a geometric point $x \in X_0(\bar{k})$. In this note we study representations of the geometric étale fundamental group $G = \pi_1^{\text{ét}}(X, x)$, where $X = X_0 \otimes_k \bar{k}$, and the action of the Frobenius on the set of representations.

For a given prime $\ell \neq p$, we fix a finite field $\mathbb{F}$ of characteristic ℓ, and a continuous semi-simple representation $\bar{\rho} : G \to \text{GL}_r(\mathbb{F})$. We define the set $\mathcal{S}_{\bar{\rho}}$ of isomorphism classes of continuous semi-simple representations $\rho : \pi_1^{\text{ét}}(X, x) \to \text{GL}_r(\bar{\mathbb{Q}}_\ell)$ with the property that the associated semi-simple residual representation is isomorphic to $\bar{\rho}$. We endow $\mathcal{S}_{\bar{\rho}}$ with a Noetherian Zariski topology in Section 2.

There is a canonical Frobenius action $\Phi : \mathcal{S}_{\bar{\rho}} \xrightarrow{\sim} \mathcal{S}_{\bar{\rho}}$. A point $[\rho] \in \mathcal{S}_{\bar{\rho}}$ is fixed by Φ^n for some integer $n > 0$ if and only if the representation ρ extends to a continuous representation $\pi_1^{\text{ét}}(X_0 \otimes_k k', x) \to \text{GL}_r(\bar{\mathbb{Q}}_\ell)$, for some finite extension $k \subset k'$. We call such a point in $\mathcal{S}_{\bar{\rho}}$ arithmetic and we let $\mathcal{A}_{\bar{\rho}} \subset \mathcal{S}_{\bar{\rho}}$ be the subset of arithmetic points.

The aim of our note is to propose and to study (two variants of) a conjecture about the density of arithmetic points, see Section 3.

Weak Conjecture. *The arithmetic points $\mathcal{A}_{\bar{\rho}}$ are dense in $\mathcal{S}_{\bar{\rho}}$.*

Strong Conjecture. *For a Zariski closed subset $Z \subset \mathcal{S}_{\bar{\rho}}$ with $\Phi^n(Z) = Z$ for some integer $n > 0$ the subset of arithmetic points $Z \cap \mathcal{A}_{\bar{\rho}}$ is dense in Z.*

One application of the Strong Conjecture is that it implies a Hard Lefschetz isomorphism for semi-simple perverse $\bar{\mathbb{Q}}_\ell$-sheaves in characteristic p, see Section 9. This application is motivated by the corresponding work of Drinfeld for complex varieties [Dri01].

For degree $r = 1$ and X either proper or a torus the Strong Conjecture is shown in [EK21, Theorem 1.7 and Lemma 3.1].

In Section 6 we prove the following reductions for the Strong Conjecture, see also Proposition 3.6. Here the algebraically closed field $\bar{k}$ is fixed.

- If the Strong Conjecture holds for given degree r for all smooth curves X over $\bar{k}$ then it holds in degree r for all smooth varieties X over $\bar{k}$.
- If the Strong Conjecture holds in any degree r for $X = \mathbb{P}^1_{\bar{k}} \setminus \{0, 1, \infty\}$ and $\bar{\rho}$ tame then it holds in general over $\bar{k}$.

These reductions motivate our two main theorems, see Section 3.

Theorem A. *The Weak Conjecture holds when X is a curve, $\ell > 2$ and $\bar{\rho}$ is absolutely irreducible.*

Theorem B. *The Strong Conjecture holds for $X = \mathbb{P}^1_{\bar{k}} \setminus \{0, 1, \infty\}$ when $\bar{\rho}$ is tame of degree two.*

We now explain the ideas of our proofs. The main ingredient in the proof of Theorem A is de Jong's conjecture [deJ01] proven in [Gai07, § 1.4] under the assumption $\ell > 2$, using the geometric Langlands program. Indeed, if $\bar{\rho}$ is absolutely irreducible, then $\mathcal{S}_{\bar{\rho}}$ is the set of $\bar{\mathbb{Q}}_\ell$-points of Mazur's deformation space which is smooth if X is a curve, and on which we can apply de Jong's technique [deJ01, § 3.14].

The proof of Theorem B is very different. We embed $\mathcal{S}_{\bar{\rho}}$ in the completion of the affine space of dimension 6 at the closed point which corresponds to the characteristic polynomials of three well chosen elements of the geometric fundamental group G on which Φ acts by raising to the qth power. We can then apply our main density theorem in [EK21] on the cover which separates the roots of those polynomials. In particular, this also shows that the arithmetic points are precisely those which have quasi-unipotent monodromy at infinity. We remark in Section 9.2 that our method yields a proof de Jong's conjecture in this particular case, which does not use automorphic forms.

Acknowledgements

2. The Zariski topology on the set of semi-simple representations

Let ℓ be a prime number, $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ with residue field $\mathbb{F}$, $\mathcal{O} \hookrightarrow \bar{\mathbb{Q}}_\ell$ be an embedding of $\mathcal{O}$ into an algebraic closure of $\bar{\mathbb{Q}}_\ell$ defining an embedding of $\mathbb{F}$ into an algebraic closure $\bar{\mathbb{F}}$. Let G be a pro-finite group which satisfies Mazur's ℓ-finiteness property, *i.e.* for any open subgroup $U \subset G$ the set $\text{Hom}_{\text{cont}}(U, \mathbb{Z}/\ell\mathbb{Z})$ is finite. Let

$$\bar{\rho} : G \to \text{GL}_r(\mathbb{F})$$

be a continuous representation. We define $\mathcal{S}_{\bar{\rho}}$ to be the *set of isomorphism classes of continuous semi-simple representations* $\rho : G \to \text{GL}_r(\bar{\mathbb{Q}}_\ell)$ *with semi-simple reduction isomorphic to* $\bar{\rho}^{\text{ss}} : G \to \text{GL}_r(\mathbb{F}) \subset \text{GL}_r(\bar{\mathbb{F}})$. In this section we define a Zariski topology on $\mathcal{S}_{\bar{\rho}}$. In Section 4 we relate $\mathcal{S}_{\bar{\rho}}$ to the deformation space of pseudorepresentations.

For a finite family $\underline{g} = (g_1, \ldots, g_m) \in G^m$, let p_i be the characteristic polynomial $\text{char}(\bar{\rho}(g_i))$ of $\bar{\rho}(g_i)$. Then $\underline{p} = (p_1, \ldots, p_m)$ is an $\mathbb{F}$-point of the affine space $\mathbb{A}_{\mathcal{O}}^{rm}$ over $\mathcal{O}$. Let $R_{\underline{p}} = R_{\bar{\rho}(\underline{g})}$ be the complete local ring of $\mathbb{A}_{\mathcal{O}}^{rm}$ at the closed point $\underline{p}$. The ring $R_{\bar{\rho}(\underline{g})} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is Noetherian Jacobson. Its maximal ideals correspond to the m-tuples of polynomials over $\bar{\bar{\mathbb{Z}}}_\ell$ with reduction $\underline{p}$, see [GL96, Propositions A.2.2.2

and A.2.2.3]. Sending a representation $\rho\colon G \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ in $\mathcal{S}_{\bar{\rho}}$ to the family of characteristic polynomials $(\mathrm{char}(\rho(g_1)),\ldots,\mathrm{char}(\rho(g_m)))$ therefore induces a map

$$\mathrm{char}_{\underline{g}}\colon \mathcal{S}_{\bar{\rho}} \to \mathrm{Spm}(R_{\bar{\rho}(\underline{g})} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell).$$

We endow the maximal spectrum with the usual Zariski topology, which is thus Noetherian.

Proposition 2.1. *There exists an integer $\tilde{m} > 0$ and a family $\underline{\tilde{g}} \in G^{\tilde{m}}$ such that for any finite family $\underline{g} \in G^m$ which contains $\underline{\tilde{g}}$ we have:*

(1) $\mathrm{char}_{\underline{g}}$ *is injective with Zariski closed image.*
(2) *The induced topologies on $\mathcal{S}_{\bar{\rho}}$ via the embeddings $\mathrm{char}_{\underline{g}}$ and $\mathrm{char}_{\underline{\tilde{g}}}$ are the same.*

Proposition 2.1 is an immediate consequence of Lemma 5.1. *From now on we endow $\mathcal{S}_{\bar{\rho}}$ with the induced Zariski topology from Proposition* 2.1.

Remark 2.2. By the same procedure we can define the ℓ-adic topology on $\mathcal{S}_{\bar{\rho}}$, which we do not consider in this note, compare [Che14, Theorem D], and [Lit21] where it is used in an essential way.

3. The density conjectures

In this section we formulate a strong conjecture and a weak one on the density of arithmetic representations in the Zariski space of all semi-simple representations $\mathcal{S}_{\bar{\rho}}$ defined in Section 2. Then we formulate our main results concerning them.

Let X_0 be a smooth geometrically connected variety defined over a finite field $k = \mathbb{F}_q$ of characteristic $p \neq \ell$. Set $X = X_0 \otimes_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Fix a geometric point $x \in X_0(\bar{k})$ and let G be the geometric fundamental group $\pi_1^{\text{ét}}(X, x)$. Fix a lift $\Phi \in \pi_1^{\text{ét}}(X_0, x)$ of the *arithmetic Frobenius*. Then Φ acts by conjugation on G. This action depends on the lift up to an inner automorphism, so it canonically acts on isomorphism classes of representations of G. We assume that

$$\Phi(\bar{\rho}) \ \text{ is isomorphic to } \ \bar{\rho}$$

which is always fulfilled after replacing Φ by a power, or equivalently X_0 by $X_0 \otimes_k k'$ for a finite extension k' of k. Thus the action of Φ on G induces a well defined automorphism Φ of $\mathcal{S}_{\bar{\rho}}$. By the construction of the Zariski topology on $\mathcal{S}_{\bar{\rho}}$ via Proposition 2.1 the automorphism Φ is a homeomorphism.

We define the *arithmetic points* of $\mathcal{S}_{\bar{\rho}}$ as the fixed points of powers of Φ

$$\mathcal{A}_{\bar{\rho}} := \bigcup_{n>0} \mathcal{S}_{\bar{\rho}}^{\Phi^n}.$$

Remark 3.1. The arithmetic points in $\mathcal{S}_{\bar{\rho}}$ correspond to those continuous semi-simple representations $\rho\colon G \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ which can be extended to a continuous representation

$$\pi_1^{\text{ét}}(X_0 \otimes_k k', x) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$$

for some finite extension $k' \subset \bar{k}$ of k, see [Del80, § 1.1.14].

Conjecture 3.2 (Weak Conjecture). *The space $\mathcal{S}_{\bar{\rho}}$ is the Zariski closure of its arithmetic points $\mathcal{A}_{\bar{\rho}}$.*

Conjecture 3.3 (Strong Conjecture). *A Zariski closed subset $Z \subset \mathcal{S}_{\bar{\rho}}$ with $\Phi^n(Z) = Z$ for some integer $n > 0$ is the Zariski closure of its arithmetic points $Z \cap \mathcal{A}_{\bar{\rho}}$.*

Note that the formulation of the conjectures depends only on X and not on the choice of X_0 or the base point x.

Remark 3.4. If $r = 1$ and X is projective or X is a torus, then the strong conjecture is true by virtue of [EK21, Theorem 1.7 and Lemma 3.1].

Remark 3.5. If we endow $\mathcal{S}_{\bar\rho}$ with the ℓ-adic topology as in Remark 2.2, then the subset of arithmetic points $\mathcal{A}_{\bar\rho}$ is discrete and closed, see [Lit21, Theorem 1.1.3].

Using the Lefschetz theorem on fundamental groups and the Belyi principle we reduce in Section 6 the Strong Conjecture to the case where X is a curve.

Proposition 3.6. *For varieties over the fixed field $\bar k$ we have the implications:*

(1) *If for fixed r the Strong Conjecture holds for $\dim(X) = 1$, then it holds for any X and the given degree r.*
(2) *If the Strong conjecture holds for all $r > 0$ for tame representations $\bar\rho$ on the variety $X = \mathbb{P}^1_{\bar k} \setminus \{0, 1, \infty\}$, then it holds in general.*

The main results of our note are the following.

Theorem 3.7 (Theorem A). *Assume that $\ell > 2$. If $\bar\rho$ is absolutely irreducible and X is a curve, then the weak conjecture holds.*

Theorem 3.8 (Theorem B). *If $X = \mathbb{P}^1_{\bar k} \setminus \{0, 1, \infty\}$, $r = 2$ and $\bar\rho$ is tame, then the strong conjecture holds. The arithmetic local systems are then precisely those with quasi-unipotent monodromies at infinity.*

The only reason why we assume $\ell > 2$ in Theorem 3.7 is that de Jong's conjecture [deJ01, Conjecture 2.3] is known only under this assumption at the moment, see [Gai07, § 1.4]. In fact our proof of Theorem 3.8 yields a geometric proof of de Jong's conjecture on $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ in rank 2 for any ℓ, without any use of the Langlands program, see Section 9.2.

4. The deformation space of pseudorepresentations

In this section we recall some properties of the deformation space of pseudorepresentations $\mathrm{PD}_{\bar\rho}$ following [Che14]. The reason why we work with pseudorepresentations is that they naturally give rise to a parametrization of the semi-simple representations $\mathcal{S}_{\bar\rho}$ defined in Section 2. As in Section 2, G is a profinite group satisfying Mazur's ℓ-finiteness property.

Let $\mathcal{C}$ be the category of complete local $\mathcal{O}$-algebras $(A, \mathfrak{m}_A)$ such that $\mathcal{O} \to A/\mathfrak{m}_A$ identifies the residue fields of A and $\mathcal{O}$. Following [Che14, Section 3] we define the *functor of pseudodeformations* of $\bar\rho$

$$\mathrm{PD}_{\bar\rho} : \mathcal{C} \to \mathrm{Sets}$$

which assigns to A the set of continuous r-dimensional A-valued determinants $D : A[G] \to A$ such that $D \otimes_A \mathbb{F} : \mathbb{F}[G] \to \mathbb{F}$ is the $\mathbb{F}$-valued determinant induced by $\bar\rho$.

Recall that a determinant is given by a compatible collection of maps $D_B : B[G] \to B$, where B runs through all commutative A-algebras, see [Che14, § 1.2]. Every continuous representation $\rho : G \to \mathrm{GL}_r(A)$ gives rise to a continuous determinant $\mathrm{Det}(\rho) : A[G] \to A$. We define the *coefficients of the characteristic polynomial* of D as the maps $\Lambda_i : G \to A$ determined by the formula

$$D_{A[t]}(t - [g]) = \sum_{i=0}^{r} (-1)^i \Lambda_i(g) t^{d-i},$$

see [Che14, § 1.3]. Recall that from the Λ_i we can reconstruct the whole determinant D by means of Amitsur's formula [Che14, § 1.3].

With a slight abuse of notation we define $\mathrm{PD}_{\bar\rho}(\bar{\mathbb{Z}}_\ell)$ to be the set of r-dimensional $\bar{\mathbb{Z}}_\ell$-valued determinants $D : \bar{\mathbb{Z}}_\ell[G] \to \bar{\mathbb{Z}}_\ell$ with $D \otimes_{\bar{\mathbb{Z}}_\ell} \bar{\mathbb{F}}_\ell = \mathrm{Det}(\bar\rho) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$, which are induced by base change from $\mathcal{O}'$ to $\bar{\mathbb{Z}}_\ell$ by $\mathcal{O}'$-valued continuous determinants $D^{\mathcal{O}'} : \mathcal{O}'[G] \to \mathcal{O}'$, where $\mathcal{O}' \subset \bar{\mathbb{Z}}_\ell$ runs over all finite extensions of $\mathcal{O}$.

Proposition 4.1.

(1) *The functor $\mathrm{PD}_{\bar\rho}$ is representable by $R^P_{\bar\rho} \in \mathcal{C}$ with universal determinant $D^{R^P_{\bar\rho}} : R^P_{\bar\rho}[G] \to R^P_{\bar\rho}$.*

(2) *The complete local ring $R_{\bar{\rho}}^P$ is Noetherian and topologically generated as an $\mathcal{O}$-algebra by the finitely many elements $\Lambda_j(g_i)$ where $1 \leq j \leq r$ and $g_1, \ldots, g_m \in G$ is a suitable family.*

(3) *If $\bar{\rho}$ is absolutely irreducible, $R_{\bar{\rho}}^P$ coincides with Mazur's universal deformation ring and $D^{R_{\bar{\rho}}^P}$ is the determinant of the universal deformation.*

We refer to [Che14, Proposition 7.59] for part (1), to [Che14, Remark 7.61] for part (2), to [Che14, Example 7.60] for part (3). Recall that Mazur's deformation functor $\mathcal{C} \to$ Sets assigns to $A \in \mathcal{C}$ the set of isomorphism classes of continuous representations $\rho \colon G \to \mathrm{GL}_r(A)$ such that $\rho \otimes_A \mathbb{F}$ is isomorphic to $\bar{\rho}$, see for example [Til96, Section 3]. Note that for any $\bar{\rho}$ we have by definition

$$R_{\bar{\rho}}^P = R_{\bar{\rho}^{\mathrm{ss}}}^P$$

where $^{\mathrm{ss}}$ indicates the semi-simplification. We define the *universal deformation space of pseudorepresentations* of $\bar{\rho}$ by

$$\mathrm{PD}_{\bar{\rho}} = \mathrm{Spf} R_{\bar{\rho}}^P.$$

Remark 4.2. The construction of the universal deformation ring $R_{\bar{\rho}}^P$ is compatible with any finite base change of local rings $\mathcal{O} \subset \mathcal{O}'$, *i.e.* the universal deformation space of pseudorepresentations over $\mathcal{O}'$ with residue field $\mathbb{F}' \subset \bar{\mathbb{F}}$ a finite extension of $\mathbb{F}$ and residual condition $D \otimes_{\mathcal{O}'} \mathbb{F}' = \mathrm{Det}(\bar{\rho}) \otimes_{\mathbb{F}} \mathbb{F}'$ is given by $R_{\bar{\rho}}^P \otimes_{\mathcal{O}} \mathcal{O}'$, see [Che14, Proposition 7.59]. In particular the canonical map

$$\mathrm{Hom}_{\mathcal{O}}(R_{\bar{\rho}}^P, \bar{\mathbb{Z}}_\ell) \xrightarrow{\sim} \mathrm{PD}_{\bar{\rho}}(\bar{\mathbb{Z}}_\ell)$$

is bijective.

Proposition 4.3. *Sending a continuous representation $\rho \colon G \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ to its determinant induces a bijection*

$$\mathrm{Det} \colon \mathcal{S}_{\bar{\rho}} \xrightarrow{\sim} \mathrm{PD}_{\bar{\rho}}(\bar{\mathbb{Z}}_\ell).$$

Proof. By [Che14, Theorem A] there is a bijection between the isomorphism classes of not necessarily continuous semi-simple representations $\rho \colon G \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ and the not necessarily continuous determinants $D \colon \bar{\mathbb{Q}}_\ell[G] \to \bar{\mathbb{Q}}_\ell$. We combine this with the simple fact from representation theory that a semi-simple representation $\rho \colon G \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$ is continuous precisely when its character $\mathrm{tr} \circ \rho$ has image in a finite extension of $\mathrm{Frac}(\mathcal{O})$ inside $\bar{\mathbb{Q}}_\ell$ and is continuous. $\square$

Combining Remark 4.2 and Proposition 4.3 we obtain a canonical identification

$$(1) \qquad \mathcal{S}_{\bar{\rho}} \xrightarrow{\sim} \mathrm{Spm}(R_{\bar{\rho}}^P \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell).$$

We shall see in the next section that (1) induces the same Zariski topology on $\mathcal{S}_{\bar{\rho}}$ as the one defined in Section 2.

5. Characteristic polynomials

In this section G is a profinite group satisfying Mazur's ℓ-finiteness property. Recall that for a family $\underline{p} = (p_1, \ldots, p_m)$ of m monic polynomials p_i of degree r over the finite field $\mathbb{F}$, we introduced the complete local deformation ring $R_{\underline{p}}$ in Section 2. When p_i is the characteristic polynomial of the matrix $\bar{\rho}(g_i)$ for a representation $\bar{\rho} \colon G \to \mathrm{GL}_r(\mathbb{F})$ and for $i = 1, \ldots, m$, we also write $R_{\bar{\rho}(\underline{g})}$ for $R_{\underline{p}}$. Let

$$\mathrm{D}_{\bar{\rho}(\underline{g})} = \mathrm{Spf} R_{\bar{\rho}(\underline{g})}$$

be the corresponding formal scheme over $\mathcal{O}$. We obtain a canonical morphism of formal schemes

$$\mathrm{PD}_{\bar{\rho}} \xrightarrow{\mathrm{char}_{\underline{g}}} \mathrm{D}_{\bar{\rho}(\underline{g})}$$

which sends a pseudorepresentation to the family of associated characteristic polynomials of $g_1, \ldots, g_m$. In view of the identification (1) this induces the map $\mathrm{char}_{\underline{g}} \colon \mathcal{S}_{\bar{\rho}} \to \mathrm{Spm}(R_{\bar{\rho}(\underline{g})} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell)$ from Section 2.

Lemma 5.1. *There is a finite family $\underline{\tilde{g}} \in G^{\tilde{m}}$ such that for any finite family $\underline{g} \in G^m$ containing $\underline{\tilde{g}}$, the morphism*

$$\mathrm{char}_{\underline{g}} \colon \mathrm{PD}_{\bar{\rho}} \to \mathrm{D}_{\bar{\rho}(\underline{g})}$$

is a closed immersion.

Proof. This is an immediate consequence of Proposition 4.1(2). □

Proof of Proposition 2.1. Take $\underline{\tilde{g}}$ as in Lemma 5.1 and use the identification (1). □

Assume given roots $\underline{\mu} = (\mu_i^{(j)})_{\substack{1 \le j \le m \\ 1 \le i \le r}} \in \mathbb{A}_{\mathcal{O}}^{rm}(\mathbb{F})$ of the m monic polynomials $\underline{p}$ of degree r, we let $\mathrm{D}_{\underline{\mu}}$ be the formal completion of the affine space $\mathbb{A}_{\mathcal{O}}^{rm}$ at $\underline{\mu}$. Sending a point

$$(\lambda_i^{(j)})_{i,j} \in \mathrm{D}_{\underline{\mu}}(A) \ \ \text{to} \ \ \left((t - \lambda_1^{(1)}) \cdots (t - \lambda_r^{(1)}), \ldots \right) \in \mathrm{D}_{\underline{p}}(A)$$

induces a finite "symmetrization" morphism of $\mathcal{O}$-formal schemes

$$\mathrm{poly} \colon \mathrm{D}_{\underline{\mu}} \to \mathrm{D}_{\underline{p}}.$$

For an integer n we write $\underline{\mu}^n$ for $((\mu_i^{(j)})^n)_{i,j}$. Sending

$$\underline{\lambda} \in \mathrm{D}_{\underline{\mu}}(A) \ \ \text{to} \ \ \underline{\lambda}^n \in \mathrm{D}_{\underline{\mu}^n}(A)$$

induces a finite morphism of $\mathcal{O}$-formal schemes

$$[n] \colon \mathrm{D}_{\underline{\mu}} \to \mathrm{D}_{\underline{\mu}^n}.$$

There exists a unique lower horizontal morphism $[n]$ of $\mathcal{O}$-formal schemes making the square

$$
\begin{array}{ccc}
\mathrm{D}_{\underline{\mu}} & \xrightarrow{\ [n]\ } & \mathrm{D}_{\underline{\mu}^n} \\
{\scriptstyle \mathrm{poly}}\downarrow & & \downarrow{\scriptstyle \mathrm{poly}} \\
\mathrm{D}_{\underline{p}} & \xrightarrow[\ [n]\ ]{} & \mathrm{D}_{[n]\underline{p}}
\end{array}
$$

commutative, where $[n]\underline{p}$ has the obvious meaning, that is its jth component is defined to be

$$(t - (\mu_1^{(j)})^n) \cdots (t - (\mu_r^{(j)})^n).$$

We now study the compatibility of the universal deformation space of pseudodeformations with restriction. Let $U \subset G$ be an open subgroup, $\bar{\rho}|_U \colon U \to \mathrm{GL}_r(\mathbb{F})$ be the restriction of ρ to U. It induces a morphism

$$\mathrm{rest} \colon \mathrm{PD}_{\bar{\rho}} \to \mathrm{PD}_{\bar{\rho}|_U}$$

by sending $D \colon A[G] \to A$ to its restriction $D|_U \colon A[U] \to A$.

Lemma 5.2. *The morphism* rest *is finite.*

Proof. Fix a family $\underline{g}$ in G as in Lemma 5.1. Choose an integer $n > 0$ such that $g_i^n \in U$ for all the g_i of the family and denote by $\underline{g}^n$ the family $(g_1^n, \ldots, g_m^n)$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{PD}_{\bar{\rho}} & \xrightarrow{\ \mathrm{char}_{\underline{g}}\ } & \mathrm{D}_{\bar{\rho}(\underline{g})} \\
{\scriptstyle \mathrm{rest}}\downarrow & & \downarrow{\scriptstyle [n]} \\
\mathrm{PD}_{\bar{\rho}|_U} & \xrightarrow[\ \mathrm{char}_{\underline{g}^n}\ ]{} & \mathrm{D}_{\bar{\rho}(\underline{g}^n)}
\end{array}
$$

in which the upper horizontal arrow $\mathrm{char}_{\underline{g}}$ is a closed immersion and the right vertical arrow $[n]$ is finite. This implies that the left vertical arrow rest is finite as well. □

It is likely that the notion of induction for pseudorepresentations with respect to an open subgroup $U \subset G$ can be defined and induces a finite morphism of universal deformation spaces of pseudorepresentations. Unfortunately, this is not documented in the literature.

We now describe a weak form of induction which is sufficient for our purpose. For simplicity assume that $U \subset G$ is a normal subgroup of index n. Let $\bar{\rho}_U \colon U \to \mathrm{GL}_r(\mathbb{F})$ be a continuous representation and set $\bar{\rho} = \mathrm{Ind}_U^G \bar{\rho}_U$. Let $\bar{\rho}|_U \colon U \to \mathrm{GL}_{nr}(\mathbb{F})$ be the restriction of $\bar{\rho}$ to U. We have

$$
\bar{\rho}|_U = \bigoplus_{i=1}^n (\bar{\rho}_U)^{s_i}
$$

where $\Sigma := \{s_1, \ldots, s_n\} \subset G$ is a set of representatives of G/U, and

$$
(\bar{\rho}_U)^{s_i}(u) = \bar{\rho}_U(s_i u s_i^{-1}) \text{ for } u \in U.
$$

Similarly, sending a pseudorepresentation $D \colon A[U] \to A$ to the pseudorepresentation $\oplus_{i=1}^n D^{s_i} \colon A[U] \to A$ where $D^{s_i}(u) = D(s_i u s_i^{-1})$ defines the horizontal morphism ξ in the triangle

(2)
$$
\begin{array}{ccc}
\mathrm{PD}_{\bar{\rho}_U} & \xrightarrow{\ \xi\ } & \mathrm{PD}_{\bar{\rho}|_U} \\
{\scriptstyle ?}\Big\downarrow & \nearrow{\scriptstyle \mathrm{rest}} & \\
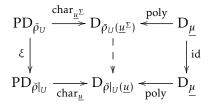\mathrm{PD}_{\bar{\rho}} & &
\end{array}
$$

We expect that (2) can be extended to a commutative diagram by the indicated dashed induction arrow. As ξ is finite by Lemma 5.3, the dashed arrow, if it exists, is automatically finite.

Lemma 5.3. *The morphism ξ in diagram* (2) *is finite.*

Proof. Let $\underline{u} = (u_1, \ldots, u_m)$, $u_i \in U$ be a family as in Lemma 5.1 for $\bar{\rho}_U$. We consider the family

$$
\underline{u}^\Sigma = \left(s_1 u_1 s_1^{-1}, \ldots, s_1 u_m s_1^{-1}, s_2 u_1 s_2^{-1}, \ldots, s_n u_m s_n^{-1} \right)
$$

and the associated roots $\underline{\mu}$ of the characteristic polynomials of the matrices $\bar{\rho}(\underline{u}^\Sigma)$. We have the diagram

$$
\begin{array}{ccccc}
\mathrm{PD}_{\bar{\rho}_U} & \xrightarrow{\ \mathrm{char}_{\underline{u}^\Sigma}\ } & \mathrm{D}_{\bar{\rho}_U(\underline{u}^\Sigma)} & \xleftarrow{\ \mathrm{poly}\ } & \mathrm{D}_{\underline{\mu}} \\
{\scriptstyle \xi}\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{PD}_{\bar{\rho}|_U} & \xrightarrow[\ \mathrm{char}_{\underline{u}}\ ]{} & \mathrm{D}_{\bar{\rho}|_U(\underline{u})} & \xleftarrow[\ \mathrm{poly}\ ]{} & \mathrm{D}_{\underline{\mu}}
\end{array}
$$

where we now define the dashed arrow as the quotient of the identity map id as follows. We label the λ coordinates of $D_{\underline{\mu}}$

$$\left(\lambda_i^{(j)}(a) \mid a = 1,\ldots,n, \ j = 1,\ldots,m, \ i = 1,\ldots,r \right).$$

The upper poly map is defined by the elementary symmetric functions on r letters $\left(\mathrm{sym}\left(\lambda_i^{(j)}(a) \right)_{1 \le i \le r} \right)_{a,j}$. In other words, it is the quotient by the product $\prod_{a,j} \Sigma_r$ where Σ_r is the symmetric group in r letters. The lower poly map is defined by $\left(\mathrm{sym}\left(\lambda_i^{(j)}(a) \right)_{1 \le a \le n, 1 \le i \le r} \right)_j$ where sym are the elementary symmetric functions on rn letters. In other words, it is the quotient by the product $\prod_j \Sigma_{rn}$. The embedding $\prod_a \Sigma_r \subset \Sigma_{rn}$ induces the embedding $\prod_{a,j} \Sigma_r \subset \prod_j \Sigma_{rn}$, and thus defines the requested dashed arrow

$$D_{\underline{\mu}} / \prod_{a,j} \Sigma_r \dashrightarrow D_{\underline{\mu}} / \prod_j \Sigma_{rn}$$

which is finite. This finishes the proof. $\qquad\square$

Remark 5.4. When evaluated on $\bar{\mathbb{Z}}_\ell$ the diagram (2) becomes commutative if we define the dashed arrow on $\bar{\mathbb{Z}}_\ell$-points as the induction

$$\mathrm{PD}_{\bar{\rho}_U}(\bar{\mathbb{Z}}_\ell) \cong \mathcal{S}_{\bar{\rho}_U} \xrightarrow{\mathrm{Ind}_U^G} \mathcal{S}_{\bar{\rho}} \cong \mathrm{PD}_{\bar{\rho}}(\bar{\mathbb{Z}}_\ell)$$

on representations. Here the isomorphisms are coming from (1) and induction is understood up to semi-simplification.

We now study the case of two-dimensional pseudorepresentations. A two-dimensional determinant $D \colon A[G] \to A$ has characteristic polynomial

$$g \mapsto t^2 - \tau(g)t + \delta(g) \in A[t],$$

here for simplicity of notation we write $\tau(g)$ for $\Lambda_1(g)$ and $\delta(g)$ for $\Lambda_2(g)$. For elements $g_0, g_1 \in G$ we have (see [Che14, Lemma 7.7])

$$\tau(g_0 g_1) = \tau(g_1 g_0),$$
$$\delta(g_0 g_1) = \delta(g_0)\delta(g_1),$$
$$\tau(g_0 g_1) = \tau(g_0)\tau(g_1) - \delta(g_0)\tau(g_0^{-1} g_1).$$

From these formulae we deduce that if F_2 is the free group on 2 elements g_0, g_1, for any element h in F_2 there exists

$$F \in \mathbb{Z}[X_0, X_1, X_2, Y_1, Y_2, Y_1^{-1}, Y_2^{-1}]$$

with $\tau(h) = F(\tau(g_0), \tau(g_1), \tau(g_0 g_1), \delta(g_0), \delta(g_1))$. This proves:

Lemma 5.5. *If G is topologically generated by g_0 and g_1 then with*

$$\underline{g} = (g_0, g_1, g_0 g_1)$$

the morphism $\mathrm{char}_{\underline{g}} \colon \mathrm{PD}_{\bar{\rho}} \to D_{\bar{\rho}(\underline{g})}$ *is a closed immersion.*

6. Compatibility with restriction and induction

In this section we prove some reductions and compatibilities which enable us to prove Proposition 3.6. So as there G is a profinite group satisfying Mazur's ℓ-finiteness property. As we are interested in the density of the fixed points of powers of an automorphism on a topological space, we formulate the simple Lemma 6.1 in this context. For a topological space S and a homeomorphism $\Phi \colon S \to S$, we define

$$S^{\Phi^\infty} = \bigcup_{n>0} S^{\Phi^n}$$

and study the following density property.

> $(\mathbf{D})_{S,\Phi}$: For any closed subset $Z \subset S$ with $\Phi^n(Z) = Z$ for some integer $n > 0$ the intersection $Z \cap S^{\Phi^\infty}$ is dense in Z.

If Φ is clear from the context we omit it in our notation.

Let $\psi^\sharp \colon R_2 \to R_1$ be a homomorphism of Noetherian Jacobson rings. Let $\Phi_1 \colon R_1 \xrightarrow{\sim} R_1$ and $\Phi_2 \colon R_2 \xrightarrow{\sim} R_2$ be compatible ring automorphisms. Endow $S_i = \mathrm{Spm}\, R_i$ with the Zariski topology ($i = 1, 2$). Let $\psi \colon S_1 \to S_2$ be the induced morphism.

Lemma 6.1.

(1) *If ψ is surjective then*

$$(\mathbf{D})_{S_1,\Phi_1} \Rightarrow (\mathbf{D})_{S_2,\Phi_2}.$$

(2) *If the ring homomorphism $\psi^\sharp \colon R_2 \to R_1$ is finite then*

$$(\mathbf{D})_{S_2,\Phi_2} \Rightarrow (\mathbf{D})_{S_1,\Phi_1}.$$

Proof. Part (1) is obvious. To show part (2) consider $Z \subset S_1$ closed with $\Phi_1^n(Z) = Z$ for some $n > 0$. Then, replacing n by m for some $m > 0$ the latter is true for each irreducible component of Z, so in order to show that $Z \cap S_1^{\Phi_1^\infty}$ is dense in Z we can assume without loss of generality that Z is irreducible.

We assume that the closure Z' of $Z \cap S_1^{\Phi_1^\infty}$ is not equal to Z and we are going to deduce a contradiction. Incomparability, see [Bou98, Section V.2.1, Corollary 1], tells us that we get a proper inclusion $\psi(Z') \subsetneq \psi(Z)$ of closed subsets of S_2. As the fibres of ψ are finite we have $\psi^{-1}(S_2^{\Phi_2^\infty}) = S_1^{\Phi_1^\infty}$, so $S_2^{\Phi_2^\infty} \cap \psi(Z) \subset \psi(Z')$. But then $(\mathbf{D})_{S_2,\Phi_2}$ applied to the closed subset $\psi(Z)$ says that $\psi(Z') = \psi(Z)$, which is a contradiction. $\qquad\square$

Let $U \subset G$ be an open subgroup and let $\bar\rho \colon G \to \mathrm{GL}_r(\mathbb{F})$ be a continuous representation. Let $\Phi \colon G \to G$ be an automorphism with $\Phi(U) = U$ and with $\Phi(\bar\rho) \simeq \bar\rho$. We can then deduce compatibility of our density property with restriction and induction.

Proposition 6.2.

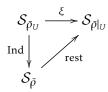(1) *We have the implication $(\mathbf{D})_{\mathcal{S}_{\bar\rho|_U}} \Rightarrow (\mathbf{D})_{\mathcal{S}_{\bar\rho}}$.*

(2) *If $U \subset G$ is normal and $\bar\rho = \mathrm{Ind}_U^G \bar\rho_U$ with $\Phi(\bar\rho_U) \simeq \bar\rho_U$ we have the implication $(\mathbf{D})_{\mathcal{S}_{\bar\rho}} \Rightarrow (\mathbf{D})_{\mathcal{S}_{\bar\rho_U}}$.*

Proof. For part (1) we observe that the restriction map $\mathcal{S}_{\bar\rho} \to \mathcal{S}_{\bar\rho|_U}$ is induced via the identification (1) by the finite homomorphism of Noetherian Jacobson rings

$$R_{\bar\rho|_U}^P \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell \to R_{\bar\rho}^P \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell.$$

The Noetherian Jacobson property of these rings follows from the general fact that for a Noetherian complete local $\mathcal{O}$-algebra R with finite residue field, the ring $R \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is Noetherian and Jacobson, see [GL96, Propositions A.2.2.2 and A.2.2.3.(ii)]. The finiteness of the homomorphism is Lemma 5.2. Then part (1) follows from Lemma 6.1(2) with $S_1 = \mathcal{S}_{\bar\rho}$, $S_2 = \mathcal{S}_{\bar\rho|_U}$ and $\psi = \mathrm{rest}$.

We prove part (2). In view of Remark 5.4 taking $\bar{\mathbb{Z}}_\ell$-points in the diagram (2) we get the commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}_{\bar{\rho}_U} & \xrightarrow{\xi} & \mathcal{S}_{\bar{\rho}|_U} \\
\text{Ind} \downarrow & \nearrow \text{rest} & \\
\mathcal{S}_{\bar{\rho}} & &
\end{array}
$$

By Lemma 5.2 the image $S = \mathrm{rest}(\mathcal{S}_{\bar{\rho}}) \subset \mathcal{S}_{\bar{\rho}|_U}$ is closed, so it is naturally a maximal spectrum. By Lemma 6.1(1) our assumption $(\mathbf{D})_{\mathcal{S}_{\bar{\rho}}}$ implies that $(\mathbf{D})_S$ holds. As by Lemma 5.3 the map $\xi \colon \mathcal{S}_{\bar{\rho}_U} \to S$ is induced by a finite ring homomorphism on maximal spectra, we can apply Lemma 6.1(2) and deduce that $(\mathbf{D})_{\mathcal{S}_{\bar{\rho}_U}}$ holds. $\qquad\square$

Let X be a smooth connected variety over an algebraically closed field $\bar{k}$. Fix a geometric point $x \in X(\bar{k})$. We apply the results from the previous sections to the case $G = \pi_1^{\text{ét}}(X, x)$. As in Section 3, we consider a continuous representation $\bar{\rho} \colon G \to \mathrm{GL}_r(\mathbb{F})$.

For a morphism of smooth connected varieties $\iota \colon Y \to X$ and a geometric point $y \in Y(\bar{k})$ mapping to x, we let $\iota^*\bar{\rho} \colon \pi_1^{\text{ét}}(Y, y) \to \mathrm{GL}_r(\mathbb{F})$ be the composition of $\iota_* \colon \pi_1^{\text{ét}}(Y, y) \to \pi_1^{\text{ét}}(X, x)$ with $\bar{\rho}$.

Proposition 6.3. *There is a smooth connected one-dimensional C and a locally closed immersion $\iota \colon C \to X$ such that the induced morphism $\iota^* \colon \mathrm{PD}_{\bar{\rho}} \to \mathrm{PD}_{\iota^*\bar{\rho}}$ is a closed immersion.*

Proof. Let $\mathfrak{m} \subset R_{\bar{\rho}}^P$ be the maximal ideal. By [Che14, Lemma 7.52] there exists an open normal subgroup $U \subset G$ such that the composed determinant

$$
R_{\bar{\rho}}^P[G] \xrightarrow{D^{R_{\bar{\rho}}^P}} R_{\bar{\rho}}^P \to R_{\bar{\rho}}^P/\mathfrak{m}^2
$$

factors through a determinant $R_{\bar{\rho}}^P/\mathfrak{m}^2[G/U] \to R_{\bar{\rho}}^P/\mathfrak{m}^2$.

It is sufficient to choose ι such that

$$
\iota^* \colon R_{\iota^*\bar{\rho}}^P \to R_{\bar{\rho}}^P/\mathfrak{m}^2
$$

is surjective. Recall from Proposition 4.1(2) that the $\mathcal{O}$-algebra $R_{\bar{\rho}}^P/\mathfrak{m}^2$ is generated by the coefficients of the characteristic polynomials $\Lambda_j(\bar{g})$ with $1 \le j \le r$ and $\bar{g} \in G/U$. So any closed immersion $\iota \colon Y \to X$ such that the composition

$$
\pi_1^{\text{ét}}(Y, y) \xrightarrow{\iota_*} G \to G/U
$$

is surjective will suffice.

To find such a ι we can assume without loss of generality that $X \hookrightarrow \mathbb{A}_{\bar{k}}^N$ is affine and use Bertini's theorem [Jou83, Theorem 6.3] applied to the étale covering X' of X corresponding to $U \subset G$ and the unramified map $X' \to \mathbb{A}_{\bar{k}}^N$. In fact Bertini tells us that a generic affine line $L \subset \mathbb{A}_{\bar{k}}^N$ has the property that $X' \times_{\mathbb{A}_{\bar{k}}^N} L$ is smooth connected and one-dimensional, so one can take $C = X \times_{\mathbb{A}_{\bar{k}}^N} L$. $\qquad\square$

Remark 6.4. The above argument shows in addition that if $X^\circ \xrightarrow{\iota} X$ is an open embedding, then the induced morphism $\iota^* \colon \mathrm{PD}_{\bar{\rho}} \to \mathrm{PD}_{\iota^*\bar{\rho}}$ is a closed immersion. Indeed $\pi_1^{\text{ét}}(X^\circ, y) \xrightarrow{\iota_*} G$ is surjective, so a fortiori

$$
\pi_1^{\text{ét}}(X^\circ, y) \xrightarrow{\iota_*} G \to G/U
$$

and one argues as above.

Proof of Proposition 3.6. For part (1), we choose ι as in Proposition 6.3. Then via the identification (1) one sees that $\iota^* \colon \mathcal{S}_{\bar{\rho}} \hookrightarrow \mathcal{S}_{\iota^*\bar{\rho}}$ is a closed embedding of topological spaces. One can descend ι to a morphism $\iota_0 \colon Y_0 \to X_0$ of varieties over a finite field k. Then ι^* is Φ-equivariant. So $(\mathbf{D})_{\mathcal{S}_{\iota^*\bar{\rho}}} \Rightarrow (\mathbf{D})_{\mathcal{S}_{\bar{\rho}}}$.

We prove part (2). By part (1) we may assume that X_0 is a smooth geometrically connected curve. Let $\pi\colon Y_0 \to X_0$ be a finite étale cover trivializing $\bar{\rho}$. Let $\bar{1}$ be the trivial rank r representation on Y with value in $\mathrm{GL}_r(\mathbb{F})$. By Proposition 6.2(1), we may assume that $\bar{\rho} = \bar{1}$ on the one-dimensional X_0.

Let $X_0 \hookrightarrow \bar{X}_0$ be the normal compactification. By [Sai97, Theorem 5.6] if $p \geq 3$ and [SY20, Theorem 1.2] if $p = 2$, there is a tame finite Belyi map $\pi\colon \bar{X}_0 \to \mathbb{P}^1$ with ramification in $\{0, 1, \infty\}$. Let $\Sigma \subset \mathbb{P}^1$ be the union of $\pi(\bar{X}_0 \setminus X_0)$ with $\{0, 1, \infty\}$. Let z be the coordinate on $\mathbb{P}^1$ with value 0 at 0, 1 at 1 and ∞ at ∞. Let us denote by a_i the z coordinate of the other closed points of Σ. Then a_i lies in the units of a finite field extension of $\mathbb{F}_q$, thus there is an integer $n > 0$ prime to p such that the morphism $z^n\colon \mathbb{P}^1 \to \mathbb{P}^1$, which is defined over $\mathbb{F}_q$, sends Σ to $\{0, 1, \infty\}$. It follows that the finite morphism $\tau = z^n \circ \pi\colon \bar{X}_0 \to \mathbb{P}^1$ has the property that $\bar{X}_0 \setminus \tau^{-1}(\{0, 1, \infty\}) \subset X_0$. By Remark 6.4 we can replace X_0 by $\bar{X}_0 \setminus \tau^{-1}(\{0, 1, \infty\})$. Moreover, using Proposition 6.2(1) again, we can replace X_0 by the Galois hull of τ. We finally apply Proposition 6.2(2) in order to reduce to the case of the curve $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and to the representation $\bar{\rho} = \mathrm{Ind}_{\pi_1^{\text{ét}}(X)}^{\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})} \bar{1}$, which is tame as τ is tame. This finishes the proof. $\qquad\square$

7. Proof of Theorem B

The aim of this section is to prove Theorem B. Let $\mathcal{X}$ be the scheme $\mathbb{P}_W^1 \setminus \{0, 1, \infty\}$ over the ring of Witt vectors $W = W(k)$. Set $K = \mathrm{Frac}(W)$ and fix an algebraic closure $\bar{K}$ of K together with an embedding $\bar{K} \hookrightarrow \mathbb{C}$ and an isomorphism of the residue field of $\bar{K}$ with $\bar{k}$. We also fix a lift $x^\diamond \in \mathcal{X}(\bar{K})$ of our base point $x \in X_0(\bar{k})$.

Fix an orientation for $\mathbb{C}$ and let $\gamma_0, \gamma_1, \gamma_\infty \in \pi_1^{\text{top}}(\mathcal{X}(\mathbb{C}), x^\diamond)$ be suitable "simple" loops around 0, 1 and ∞ such that

$$\gamma_0 \cdot \gamma_1 \cdot \gamma_\infty = 1.$$

Then $\pi_1^{\text{top}}(\mathcal{X}(\mathbb{C}), x^\diamond)$ is a free group with generators γ_0, γ_1.

The étale fundamental group $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}, x^\diamond)$ is the pro-finite completion of the topological fundamental group $\pi_1^{\text{top}}(\mathcal{X}(\mathbb{C}), x^\diamond)$ and there is a canonical outer action of $H = \mathrm{Gal}(\bar{K}/K)$ on the former. Let $\chi\colon H \to \widehat{\mathbb{Z}}^\times$ be the cyclotomic character.

Claim 7.1. For $h \in \mathrm{Gal}(\bar{K}/K)$ and $a \in \{0, 1, \infty\}$ the element $h(\gamma_a)$ (which is well-defined up to conjugation) is conjugate to $\gamma_a^{\chi(h)}$ in $\pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}, x^\diamond)$.

Proof. Up to inner automorphisms one can replace the base point $x^\diamond$ in the étale fundamental group by the base point $y_a\colon \mathrm{Spec}(\bar{K}_a) \to \mathcal{X}_K$, where $\bar{K}_a$ is an algebraic closure of the fraction field K_a of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1, a}^h$. As explained in [Del73, § 1.1.10] there is a corresponding generalized topological base point of $\mathcal{X}(\mathbb{C})$, which we simply write as $\tilde{D}_a^*$. Here D_a^* is a small punctured disk around the point $a \in \mathbb{P}_{\mathbb{C}}^1(\mathbb{C})$ and $\tilde{D}_a^*$ is its universal covering. Then by *loc. cit.* we have a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Z} & == & \pi_1^{\text{top}}(D_a^*, \tilde{D}_a^*) & \longrightarrow & \pi_1^{\text{top}}(\mathcal{X}(\mathbb{C}), \tilde{D}_a^*) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{\mathbb{Z}}(1) & == & \mathrm{Gal}(\bar{K}_a/K_a) & \longrightarrow & \pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}, y_a)
\end{array}
$$

where the vertical maps are pro-finite completions. $\qquad\square$

By [Gro71, Exposé XIII, § 2.10 and 4.7] the specialization homomorphism

$$\mathrm{sp}\colon \pi_1^{\text{ét}}(\mathcal{X}_{\bar{K}}, x^\diamond) \to \pi_1^{\text{t}}(X, x)$$

is surjective and compatible with the action of the Frobenius lift Φ. Here the codomain is the tame fundamental group which we also write G^{t} in the following.

We set $g_a = \mathrm{sp}(\gamma_a)$ for $a \in \{0, 1, \infty\}$, so we have

$$g_0 \cdot g_1 \cdot g_\infty = 1 \text{ in } G^{\mathrm{t}}.$$

In addition, G^{t} is topologically generated by g_0, g_1. By Lemma 5.5 the map

$$(3) \qquad\qquad \mathrm{char}_{\underline{g}} \colon \mathrm{PD}_{\bar\rho} \to \mathrm{D}_{\bar\rho(\underline{g})}$$

is a closed immersion, where $\underline{g} = (g_0, g_1, g_\infty^{-1})$ and where $\mathrm{PD}_{\bar\rho}$ classifies pseudorepresentations of the group G^{t}. From Claim 7.1 we conclude that

$$\Phi(g_a) \text{ is conjugate to } g_a^q,$$

for $a = 0, 1, \infty$. In particular, as $\bar\rho$ is fixed by Φ, one has

$$\mathrm{char}(\bar\rho(\underline{g}^q)) = \mathrm{char}(\bar\rho(\underline{g})).$$

We assume without loss of generality, by replacing k by a finite extension and thus Φ by a power, that the family of roots $\underline{\mu}$ of the polynomials $\mathrm{char}(\bar\rho(\underline{g}))$ satisfy $\underline{\mu}^q = \underline{\mu}$, thus the isomorphism $\mathrm{D}_{\underline{\mu}} \xrightarrow{[q]} \mathrm{D}_{\underline{\mu}}$ is well defined. This implies that with the notation as in Section 5 we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{PD}_{\bar\rho} & \xrightarrow{\ \Phi\ } & \mathrm{PD}_{\bar\rho} \\
{\scriptstyle \mathrm{char}_{\underline{g}}} \downarrow & & \downarrow {\scriptstyle \mathrm{char}_{\underline{g}}} \\
\mathrm{D}_{\bar\rho(\underline{g})} & \xrightarrow{\ [q]\ } & \mathrm{D}_{\bar\rho(\underline{g})} \\
{\scriptstyle \mathrm{poly}} \uparrow & & \uparrow {\scriptstyle \mathrm{poly}} \\
\mathrm{D}_{\underline{\mu}} & \xrightarrow{\ [q]\ } & \mathrm{D}_{\underline{\mu}}
\end{array}
$$

which is Φ-equivariant.

As $\mathrm{char}_{\underline{g}}$ is a closed immersion we have the implication

$$(\mathbf{D})_{\mathrm{D}_{\bar\rho(\underline{g})}(\bar{\mathbb{Z}}_\ell)} \Rightarrow (\mathbf{D})_{\mathcal{S}_{\bar\rho}},$$

see Section 6.

As the morphism poly is surjective on $\bar{\mathbb{Z}}_\ell$-points, we have by Lemma 6.1(1) the implication

$$(\mathbf{D})_{\mathrm{D}_{\underline{\mu}}(\bar{\mathbb{Z}}_\ell)} \Rightarrow (\mathbf{D})_{\mathrm{D}_{\bar\rho(\underline{g})}(\bar{\mathbb{Z}}_\ell)}.$$

Property $(\mathbf{D})_{\mathrm{D}_{\underline{\mu}}(\bar{\mathbb{Z}}_\ell)}$ is a consequence of [EK21, Theorem 1.7] by noting that translating $\mathrm{D}_{\underline{\mu}}$ by the Teichmüller lift of $\underline{\mu}$ we can assume that $\underline{\mu} = (1, \ldots, 1)$, so that $\mathrm{D}_{\underline{\mu}}(\bar{\mathbb{Z}}_\ell)$ consists of the $\bar{\mathbb{Q}}_\ell$-points of the multiplicative formal Lie group associated to $\pi = \mathbb{Z}_\ell^6$. Indeed, we apply *loc. cit.* with $\sigma = [q]$. This implies that the closed subset Z in Property $(\mathbf{D})_{\mathrm{D}_{\underline{\mu}}(\bar{\mathbb{Z}}_\ell)}$ is a finite union of torsion translates of formal subtori.

Finally, the points of $\mathrm{D}_{\underline{\mu}}$ invariant under $[q]^n$ for integers $n > 0$ are precisely the roots of unity. It follows that those points located on Z are dense on Z by [EK21, Lemma 3.1]. In particular a point of $\mathcal{S}_{\bar\rho}$ is arithmetic if and only if its local monodromies at $0, 1, \infty$ are quasi-unipotent. This finishes the proof of Theorem B. $\qquad\square$

Remark 7.2. What makes our argument work is the particular property due to Riemann that rank two local systems on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ are rigid, see [Kat96, p. 1].

8. Proof of Theorem A

The aim of this section is to prove Theorem A. So $\bar{\rho}$ is supposed to be absolutely irreducible. The arguments rely on [deJ01], and are partly similar to [BK06] and [BHKT19, Section 5].

As recalled in Proposition 4.1(3) $R_{\bar{\rho}}^{P}$ is then Mazur's universal deformation ring, which parametrizes isomorphism classes of continuous representations $\rho\colon G \to \mathrm{GL}_r(A)$ for $A \in \mathcal{C}$ such that $\rho \otimes_A \mathbb{F}$ is isomorphic to $\bar{\rho}$. In this case we simply write $R_{\bar{\rho}}$ for $R_{\bar{\rho}}^{P}$ and $\mathrm{D}_{\bar{\rho}}$ for $\mathrm{PD}_{\bar{\rho}}$.

Lemma 8.1. *The $\mathcal{O}$-algebra $R_{\bar{\rho}}$ is formally smooth, i.e. there is a non-canonical $\mathcal{O}$-isomorphism*

$$R_{\bar{\rho}} \cong \mathcal{O}[[t_1,\ldots,t_b]].$$

Proof. Let $\mathrm{D}_{\det\bar{\rho}} = \mathrm{Spf}\, R_{\det\bar{\rho}}$ be the universal deformation space of the degree one representation

$$\det\bar{\rho}\colon G \to \mathbb{F}^{\times}.$$

Let $0 \to I \to B \to A \to 0$ be an extension in $\mathcal{C}$ such that $I \cdot \mathfrak{m}_B = 0$ (so the B-module structure on I factors through $\mathbb{F}$). By [Maz89, § 1.6], there exists a canonical commutative obstruction diagram with exact rows

(4)
$$
\begin{array}{ccccc}
\mathrm{D}_{\bar{\rho}}(B) & \longrightarrow & \mathrm{D}_{\bar{\rho}}(A) & \overset{O}{\longrightarrow} & H^2(X,\mathcal{E}nd(\mathcal{F})) \\
\downarrow{\scriptstyle\det} & & \downarrow{\scriptstyle\det} & & \downarrow{\scriptstyle\mathrm{tr}} \\
\mathrm{D}_{\det\bar{\rho}}(B) & \longrightarrow & \mathrm{D}_{\det\bar{\rho}}(A) & \overset{O}{\longrightarrow} & H^2(X,\mathbb{F})
\end{array}
$$

Here $\mathcal{F}$ is the lisse étale sheaf on X corresponding to $\bar{\rho}$ and we use that the canonical map

$$H^2(G,\mathrm{Ad}_{\bar{\rho}}) \hookrightarrow H^2(X,\mathcal{E}nd(\mathcal{F}))$$

induced by the Hochschild-Serre spectral sequence of the universal covering $\tilde{X} \to X$ is injective. The latter injectivity is due to the fact that $\mathcal{F}$ is trivialized on $\tilde{X}$ and that its first cohomology on $\tilde{X}$ vanishes.

First case: X is affine.
By [Art73, Corollary 3.5] we have $H^2(X,\mathcal{E}nd(\mathcal{F})) = 0$, so the first exact row in (4) tells us that $\mathrm{D}_{\bar{\rho}}$ is formally smooth.

Second case: X is projective.
This case follows from the following two claims and a chase in the diagram (4). □

Claim 8.2. $\mathrm{D}_{\det\bar{\rho}}$ is formally smooth over $\mathcal{O}$.

Claim 8.3. The map tr in (4) is an isomorphism.

Proof of Claim 8.2. We know that if $\det\bar{\rho} = \bar{1}$ is trivial, then $\mathrm{D}_{\bar{1}} = \mathcal{O}[[G^{\mathrm{ab},\ell}]]$, where $G^{\mathrm{ab},\ell}$ is the abelian, ℓ-adic étale fundamental group of X. In general, $\mathrm{D}_{\det\bar{\rho}}$ is isomorphic to $\mathrm{D}_{\bar{1}}$ by translating with the Teichmüller lift of $\det\bar{\rho}$, see [Maz89, § 1.4]. Thus $\mathrm{D}_{\det\bar{\rho}}$ is formally smooth, since $G^{\mathrm{ab},\ell}$ is torsion free. □

Proof of Claim 8.3. As the trace map of étale sheaves

$$\mathcal{E}nd(\mathcal{F}) \to \mathbb{F}$$

is surjective and X has dimension one, the map tr in (4) is surjective as well. So it suffices to show that both $\mathbb{F}$-vector spaces have dimension one. For $H^2(X,\mathbb{F})$ this is immediate from Poincaré duality. In order to apply Poincaré duality to $H^2(X,\mathcal{E}nd(\mathcal{F}))$ we recall that the trace pairing induces an isomorphism $\mathcal{E}nd(\mathcal{F})^{\vee} \cong \mathcal{E}nd(\mathcal{F})$. So we obtain from duality an isomorphism $H^2(X,\mathcal{E}nd(\mathcal{F})) \cong \mathrm{End}_G(\bar{\rho})^{\vee} = \mathbb{F}$. The equality comes from the absolute irreducibility of $\bar{\rho}$ and Schur's lemma. □

For an integer $n > 0$ we consider the quotient ring $(R_{\bar{\rho}})_{\Phi^n} = R_{\bar{\rho}}/I_n$, where I_n is the ideal generated by $\Phi^n(\alpha) - \alpha$ for all $\alpha \in R_{\bar{\rho}}$. Based on the presentation of Lemma 8.1, we see that I_n is generated by the b elements

(5) $$\Phi^n(t_1) - t_1, \ldots, \Phi^n(t_b) - t_b.$$

By definition

$$\mathcal{S}_{\bar{\rho}} \supset \mathcal{A}_{\bar{\rho}} = \bigcup_{n>0} \mathrm{Spm}((R_{\bar{\rho}})_{\Phi^n} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell).$$

We use the following two propositions.

Proposition 8.4. *The ring $(R_{\bar{\rho}})_{\Phi^n}$ is finite, flat and a complete intersection over $\mathcal{O}$ for any $n > 0$.*

Proposition 8.5. *The generic fibre $(R_{\bar{\rho}})_{\Phi^n} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is reduced for any $n > 0$.*

Proposition 8.5 is the same as Proposition 5.12 in [BHKT19].

Sketch of proof of Proposition 8.4. As in [deJ01, § 3.14], we have to show that the images of the elements (5) form a regular sequence in $R_{\bar{\rho}} \otimes_{\mathcal{O}} \mathbb{F}$. The latter is equivalent to $(R_{\bar{\rho}})_{\Phi} \otimes_{\mathcal{O}} \mathbb{F}$ being zero-dimensional. This is deduced by verbatim the same argument as *loc. cit.* in view of the fact that de Jong's conjecture is known for $\ell > 2$ by [Gai07]. □

Proof of Proposition 8.5. Consider a continuous representation $\rho\colon G \to \mathrm{GL}_r(\mathcal{O}')$ corresponding to a homomorphism $(R_{\bar{\rho}})_{\Phi^n} \to \mathcal{O}'$, where $\mathcal{O}'$ is a discrete valuation ring which is a finite extension of $\mathcal{O}$. Then up to replacing k by a finite extension, ρ can be extended to a continuous representation $\rho_0\colon \pi_1^{\text{ét}}(X_0, x) \to \mathrm{GL}_r(\mathcal{O}')$, see Remark 3.1. As $\bar{\rho}$ is absolutely irreducible, $\rho_0 \otimes_{\mathcal{O}'} \bar{\mathbb{Q}}_\ell$ is irreducible. After a suitable twist we can assume without loss of generality that $\det(\rho_0)$ is finite, see [Del80, Proposition 1.3.4]. Then by the Langlands correspondence [Laf02, Theorem VII.6] the lisse sheaf $\mathcal{F}_0$ corresponding to ρ_0 is pure of weight zero. The tangent space to ρ in $(R_{\bar{\rho}})_{\Phi} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is given by

$$H^1\left(G, \mathrm{Ad}_\rho[\tfrac{1}{\ell}]\right)^{\Phi} = H^1\left(X, \mathcal{E}nd(\mathcal{F})[\tfrac{1}{\ell}]\right)^{\Phi} = 0,$$

where the last equality follows from the fact that $H^1(X, \mathcal{E}nd(\mathcal{F}))$ has weight one as $\mathcal{E}nd(\mathcal{F}_0)$ has weight zero. This finishes the proof. □

Proof of Theorem A. We have to show that an element $\alpha \in R_{\bar{\rho}} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ which vanishes on the points $\mathcal{A}_{\bar{\rho}}$ is zero. After replacing $\mathcal{O}$ by a finite extension we may assume without loss of generality that $\alpha \in R_{\bar{\rho}}$. The vanishing condition means that α is contained in all the maximal ideals corresponding to the points of $\mathcal{A}_{\bar{\rho}}$, *i.e.* that the image of α in the ring $(R_{\bar{\rho}})_{\Phi^n} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is contained in its nilpotent radical for all $n > 0$. As $(R_{\bar{\rho}})_{\Phi^n} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ is reduced by Proposition 8.5, this means that the image of α vanishes in $(R_{\bar{\rho}})_{\Phi^n} \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_\ell$ for all $n > 0$. By the flatness in Proposition 8.4 it actually vanishes in $(R_{\bar{\rho}})_{\Phi^n}$ for any $n > 0$. By Claim 8.6 this implies that $\alpha = 0$. □

Claim 8.6. The canonical map $R_{\bar{\rho}} \to \lim_n (R_{\bar{\rho}})_{\Phi^n}$ is injective.

Proof. Let $\mathfrak{m}$ be the maximal ideal of $R_{\bar{\rho}}$. For any integer $m > 0$, there is an integer $n > 0$ such that Φ^n acts trivially on $R_{\bar{\rho}}/\mathfrak{m}^m$ as the latter ring is finite, so $I_n \subset \mathfrak{m}^m$. Thus $\cap_{n>0} I_n \subset \cap_{m>0} \mathfrak{m}^m = \{0\}$. □

9. Some Applications

In this section we make two remarks concerning applications.

9.1. The Hard Lefschetz theorem in positive characteristic.

This application of our Strong Conjecture is motivated by [Dri01]. Let $f : X \to Y$ be a projective morphism of separated schemes of finite type over an algebraically closed field $\bar{k}$. Let $\eta \in H^2(X, \mathbb{Q}_\ell)$ be the Chern class of a relative ample line bundle. Here we omit Tate twists for simplicity of notation.

One conjectures (see [EK21, Remark 1.4]) that if $\mathcal{F} \in D_c^b(X, \bar{\mathbb{Q}}_\ell)$ is a semi-simple perverse sheaf, then the Hard Lefschetz property holds, *i.e.* the cup-product

$$(6) \qquad \qquad \cup \eta^i : {}^p H^{-i} f_* \mathcal{F} \to {}^p H^i f_* \mathcal{F}$$

is an isomorphism for all $i \geq 0$. It is known that this holds if

(i) $\mathcal{F}$ is of geometric origin in the sense of [BBD82, § 6.2.4–6.2.5]
(ii) $\bar{k}$ is the algebraic closure of a finite field k and f, η and $\mathcal{F}$ descend to schemes X_0, Y_0 over the field k.

For part (ii) one combines [BBD82, Theorem 6.2.10] and the Langlands correspondence of Drinfeld–Lafforgue [Laf02, Theorem VII.6]. By [EK21, Theorem 1.1] the Hard Lefschetz property is also known if $\mathcal{F}$ is a rank 1 $\bar{\mathbb{Q}}_\ell$-local system $\mathcal{L}$, and more generally if it is a twist of such an $\mathcal{L}$ by a sheaf as in (i) (see Theorem 5.4 in *loc. cit.*).

Proposition 9.1. *If the irreducible constituents of $\mathcal{F}$ have generic rank at most r and the Strong Conjecture 3.3 holds for any representation of degree $\leq r$ then the map* (6) *is an isomorphism.*

Sketch of proof. Similar to [BBD82, Lemma 6.1.9] one uses a spreading argument in order to reduce to the case in which $\bar{k}$ is the algebraic closure of a finite field k_0 and f and η are defined over k_0. Then $\mathcal{F}$ corresponds to an irreducible representation $\rho_{\mathcal{F}} : \pi_1^{\text{ét}}(U) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$, where $U \subset X$ is a smooth locally closed geometrically irreducible subvariety (over which $\mathcal{F}$ is a shifted smooth sheaf).

Let $\bar{\rho} : \pi_1^{\text{ét}}(U) \to \mathrm{GL}_r(\mathbb{F})$ be the semi-simple reduction of $\rho_{\mathcal{F}}$. In fact each representation $\rho \in \mathcal{S}_{\bar{\rho}}$ gives rise via the intermediate extension of the associated smooth sheaf to a $\bar{\mathbb{Q}}_\ell$-perverse sheaf $\mathcal{F}_\rho$ on X.

Similarly to [EK21, Corollary 4.3] one shows

Claim 9.2. The subset $Z^\circ \subset \mathcal{S}_{\bar{\rho}}$ of those ρ for which the Hard Lefschetz property for the perverse sheaf $\mathcal{F}_\rho$ fails to hold, is constructible.

In order to give a complete proof of the claim one would need a theory of perverse étale adic sheaves over fields more general than $\bar{\mathbb{Q}}_\ell$. Unfortunately, such a theory does not exist in the literature at the moment, but should be rather formal in terms of the pro-étale topology.

As Z° is also stabilized by the Frobenius Φ, we can apply the Strong Conjecture to the Zariski closure Z of Z°. This implies that Z° contains an arithmetic point, which contradicts (ii) above. $\qquad \square$

Remark 9.3. Using Proposition 3.6, it would be enough in Proposition 9.1 to prove the Strong Conjecture in rank $\leq r$ on all curves or in any rank on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ for a tame $\bar{\rho}$.

9.2. Our proof of Theorem B on $X_0 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ for $\bar{\rho}$ tame and $r = 2$ implies de Jong's conjecture [deJ01, Conjecture 2.3] in this case

Let $\rho_0 : \pi_1(X_0, x) \to \mathrm{GL}_2(\mathbb{F}[[t]])$ be an arithmetic representation and ρ be its restriction to G. By [deJ01, Proposition 2.4 and Lemma 2.10] we may assume that $\rho \otimes_{\mathbb{F}[[t]]} \mathbb{F}((t))$ is absolutely irreducible. By the proof of Theorem B in Section 7, the Φ-invariant point $\mathrm{Det}(\rho) \in \mathrm{PD}_{\bar{\rho}}(\mathbb{F}[[t]])$ has a $[q]$-invariant image in $\mathrm{D}_{\bar{\rho}(\underline{g})}(\mathbb{F}[[t]])$. Thus it lies in $\mathrm{D}_{\bar{\rho}(\underline{g})}(\mathbb{F}') \subset \mathrm{D}_{\bar{\rho}(\underline{g})}(\mathbb{F}'[[t]])$ for a finite extension $\mathbb{F}' \supset \mathbb{F}$. Thus $\rho \otimes_{\mathbb{F}[[t]]} \overline{\mathbb{F}((t))}$ comes from a continuous representation $G \to \mathrm{GL}_2(\mathbb{F}'')$ for a finite extension $\mathbb{F}'' \supset \mathbb{F}'$ ([Bas80, Proposition 2.2]) and thus has finite monodromy. This finishes the proof. $\qquad \square$

We observe that our proof avoids the use of the geometric Langlands correspondence, which is used in [deJ01, Theorem 1.2] to establish the degree two case of de Jong's conjecture.

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