

Rationally connected rational double covers of primitive Fano varieties

Aleksandr V. Pukhlikov

Abstract. We show that for a Zariski general hypersurface V of degree $M + 1$ in \mathbb{P}^{M+1} for $M \geq 5$ there are no Galois rational covers $X \dashrightarrow V$ of degree $d \geq 2$ with an abelian Galois group, where X is a rationally connected variety. In particular, there are no rational maps $X \dashrightarrow V$ of degree 2 with X rationally connected. This fact is true for many other families of primitive Fano varieties as well and motivates a conjecture on absolute rigidity of primitive Fano varieties.

Keywords. Fano variety; rational map; free family of rational curves; canonical singularity; branch divisor

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[Français]

Revêtements rationnels doubles rationnellement connexes des variétés de Fano primitives

Résumé. Nous montrons qu'une hypersurface V de degré $M + 1$ de \mathbb{P}^{M+1} avec $M \geq 5$ n'admet aucun revêtement rationnel galoisien $X \dashrightarrow V$ de degré $d \geq 2$ et de groupe de Galois abélien avec X rationnellement connexe. En particulier, V n'admet aucune application rationnelle $X \dashrightarrow V$ de degré 2 avec X rationnellement connexe. Ceci est également vérifié par beaucoup d'autres familles de variétés de Fano primitives et motive une conjecture concernant la rigidité absolue des variétés de Fano primitives.

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1. Statement of the main results

One of the most challenging problems in the modern birational geometry is the *unirationality problem*: for a given rationally connected projective variety V , is there a rational dominant map $\mathbb{P}^M \dashrightarrow V$? While in the past 50 years a huge progress has been made in solving the rationality problem, the unirationality still remains an unknown territory. There are lots of explicit examples and constructions of unirational varieties but not a single result about non-unirationality (in the rationally connected category). During the past century (starting with Fano himself and maybe even earlier) the non-unirationality was conjectured for various classes of algebraic varieties, but the questions remained unanswered.

The aim of the present paper is to prove a theorem that implies, in particular, that there are no rational maps of degree 2

$$\mathbb{P}^M \dashrightarrow V$$

for a Zariski general hypersurface $V \subset \mathbb{P}^{M+1}$ of degree $M+1$ (which is a Fano variety of index 1, in particular, a rationally connected variety). In fact, there are no rational maps of degree 2

$$X \dashrightarrow V$$

with X rationally connected for such hypersurfaces V . We now state the main theorem.

Let V be a projective factorial variety with at most terminal singularities, such that $\text{Pic } V = \mathbb{Z}K_V$ and the anticanonical class $(-K_V)$ is ample (that is, a primitive Fano variety). We work over the ground field \mathbb{C} of complex numbers.

Definition 1.1 (*cf.* [Puk05]). The Fano variety V , described above, is *divisorially canonical* if for every effective divisor $D \sim -nK_V$, $n \geq 1$, the pair $(V, \frac{1}{n}D)$ is canonical; that is to say, for every exceptional prime divisor E over V the inequality

$$\text{ord}_E D \leq n \cdot a(E),$$

where $a(E)$ is the discrepancy of E with respect to V , holds.

Apart from the divisorial canonicity, we will need the following technical conditions.

- (★1) For every anticanonical divisor $R \in |-K_V|$, every prime number $p \geq 2$ and any, possibly reducible, closed subset $Y \subset V$ of codimension ≥ 2 there is a non-singular curve $N \subset V$ such that

$$p \nmid (N \cdot K_V),$$

$N \cap Y = \emptyset$ and N meets R transversally at non-singular points.

- (★2) For every, possibly reducible, closed subset $Y \subset V$ of codimension ≥ 2 there is a non-singular rational curve $N \subset V$ such that $N \cap Y = \emptyset$.

Definition 1.2. We say that a rational dominant map $X \dashrightarrow Z$ of varieties of the same dimension is a *rational Galois cover*, if the corresponding field extension $\mathbb{C}(Z) \subset \mathbb{C}(X)$ is a Galois extension. If the corresponding Galois group is cyclic, we say that this rational map is a *rational cyclic cover*.

The main result of the present paper is the following claim.

Theorem 1.3. *Assume that the Fano variety V , introduced above, is divisorially canonical and satisfies the conditions (★1) and (★2). Then there are no rational Galois covers $X \xrightarrow{d:1} V$ with an abelian Galois group of order $d \geq 2$, where X is a rationally connected variety.*

Since the divisorial canonicity implies birational superrigidity, as an immediate consequence of Theorem 1.3, we get the following claim.

Corollary 1.4. *In the assumptions of Theorem 1.3, if $X \xrightarrow{d:1} V$ is a rational Galois cover with an abelian Galois group, then $d = 1$ and the MMP for X has the unique outcome V .*

Since every rational map of degree 2 is a Galois rational cover with the cyclic group C_2 as the Galois group, we obtain the following claim.

Corollary 1.5. *In the assumptions of Theorem 1.3, there are no rational maps $X \xrightarrow{2:1} V$ of degree 2 with X a rationally connected variety.*

Although a particular case of Theorem 1.3, the last corollary is especially important as it covers *all* rational maps of degree 2 and therefore motivates the following conjecture.

Conjecture 1.6 (on absolute rigidity). *If V is a divisorially canonical Fano variety, then every rational dominant map $X \dashrightarrow V$, where X is a rationally connected variety of dimension $\dim V$, is a birational map.*

2. Divisorially canonical varieties

Given that the main assumption for the variety V in Theorem 1.3 is divisorial canonicity, the natural question to ask now is how typical this property is in the class of Fano varieties? Let the symbol \mathbb{P} stand for the complex projective space \mathbb{P}^{M+1} , where $M \geq 5$. Set $\mathcal{F} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(M+1)))$ to be the space of hypersurfaces of degree $M+1$ in \mathbb{P} . If a hypersurface $V \in \mathcal{F}$ is factorial, then $\text{Pic } V = \mathbb{Z}H$, where H is the class of a hyperplane section. In [Puk05] it was shown that a Zariski general *non-singular* hypersurface V is divisorially canonical. Since it is not hard to check that the properties (★1) and (★2) are satisfied for a non-singular hypersurface V (this is done below in Section 3), we obtain the following claim.

Corollary 2.1. *For a Zariski general hypersurface $V \subset \mathbb{P}$ of degree $M+1$, where $M \geq 5$, there are no non-trivial rational Galois covers $X \xrightarrow{d:1} V$ with an abelian Galois group of order $d \geq 2$, where X is a rationally connected variety; in particular, there are no rational maps $X \dashrightarrow V$ of degree 2 with X rationally connected*

Note that in [Puk15] it was shown that for $M \geq 9$ there exists a Zariski open subset $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$, such that every hypersurface $V \in \mathcal{F}_{\text{reg}}$ has at most quadratic singularities of rank ≥ 8 , so is a factorial variety with

terminal singularities, and satisfies the property of divisorial canonicity. Moreover, for the complement $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$ the inequality

$$\text{codim}((\mathcal{F} \setminus \mathcal{F}_{\text{reg}}) \subset \mathcal{F}) \geq \frac{(M-6)(M-5)}{2} - 5$$

holds. The property $(\star 1)$ is easy to show for hypersurfaces $V \in \mathcal{F}_{\text{reg}}$, see Section 3.

Apart from Fano hypersurfaces of index 1, the divisorial canonicity was also shown for Zariski general varieties in the following families:

- double spaces of index 1 and dimension ≥ 3 , see [Puk05],
- a majority of the families of Fano complete intersections of index 1 in the projective space [Puk06, EP16, Puk18],
- finite, not necessarily cyclic, covers of index 1 of the projective spaces [Puk19].

This list is probably not complete: computing or estimating the (log) canonical thresholds has become a popular topic, see [CPW14, CS08, LZ19, Zhu18] and other works in this direction.

3. Proof of Theorem 1.3

Let us assume the converse and fix a non-trivial rational Galois cover $\sigma: X \dashrightarrow V$ with an abelian Galois group, where V is a divisorially canonical variety, satisfying $(\star 1)$ and $(\star 2)$, and X is rationally connected. Considering the field extension $\mathbb{C}(V) \subset \mathbb{C}(X)$, we can find an intermediate field which is a normal extension of $\mathbb{C}(V)$ with a cyclic group of a prime order $p \geq 2$ as its Galois group. Since the image of a rationally connected variety is rationally connected, we may assume that the Galois group of the original extension $\mathbb{C}(V) \subset \mathbb{C}(X)$ is a cyclic group of a prime order $p \geq 2$. Further, we may assume that $\sigma: X \rightarrow V$ is a morphism and X is a non-singular projective variety, $\dim X = \dim V$.

We say that a family \mathcal{L} of irreducible projective curves on a quasi-projective variety is *free*, if they sweep out a dense subset of that variety and for every subvariety Y of codimension ≥ 2 the subset

$$\{L \in \mathcal{L} \mid L \cap Y \neq \emptyset\}$$

is a proper closed subfamily of the family \mathcal{L} (that is to say, a curve $L \in \mathcal{L}$ of general position does not intersect Y). Let us fix a free family \mathcal{C}_X of non-singular rational curves on X .

This free family of rational curves is a crucial object in our proof. For the existence and basic properties of free families of rational curves, our reference is Kollár's book, [Kol96, Sections II.3 and IV.3]. The facts used below are well known and standard; here we explain briefly the easiest way to understand the basic geometry of such families. By Theorem 3.9 in [Kol96, Section IV.3] we have a family \mathcal{C}_X of non-singular rational curves C_X on X such that the vector bundle $T_X|_{C_X}$ is ample, that is, it is of the form

$$\oplus_{\mathbb{P}^1}(\alpha_i)$$

with all $\alpha_i \geq 1$. (This implies, in particular, that $(K_X \cdot C_X) < 0$, although that inequality holds under much weaker assumptions for the family of curves.) Therefore, the infinitesimal deformations of the curve C_X given by the sections of the vector bundle $T_X|_{C_X}$ are unobstructed, see Points 3.3 – 3.5.4 in [Kol96, Section II.3], and if for a particular subvariety $Y \subset X$ of codimension ≥ 2 and a particular curve $C_X \in \mathcal{C}_X$ the intersection $C_X \cap Y \neq \emptyset$, then a general deformation of C_X in the family \mathcal{C}_X does not meet Y : we can deform the curve away from Y , see Proposition 3.7 in [Kol96, Section II.3].

For the same reason, for every prime divisor $\Delta \subset X$, such that $\sigma_*: T_p X \rightarrow T_{\sigma(p)} V$ is not an isomorphism for a point of general position $p \in \Delta$ (this is true, in particular, if $\text{codim}(\sigma(\Delta) \subset V) \geq 2$), a general curve $C_X \in \mathcal{C}_X$ meets Δ transversally at points of general position. (There are finitely many such divisors, so the assumption that a general C_X meets every Δ at points of general position is justified by the family \mathcal{C}_X being

free; the transversality follows from the ampleness of the restriction of the tangent bundle T_X onto C_X , see the references above.)

The same deformation arguments (see the proof of Proposition 3.7 in [Kol96, Section II.3] and Theorems 1.7 and 1.8 in [Kol96, Section II.1] give us that for a general curve $C_X \in \mathcal{C}_X$ the morphism

$$\sigma|_{C_X}: C_X \rightarrow \sigma(C_X)$$

is birational (again, this comes from the ampleness of T_X restricted onto a general C_X , see Proposition 3.5 and Corollary 3.5.3 in [Kol96, Section II.3]: when we deform the curve C_X , any two distinct points $p \neq q$ on this curve vary independently, so that if for a particular curve C_X the morphism $\sigma|_{C_X}$ is not birational, choosing two points $p \neq q$ such that $\sigma(p) = \sigma(q)$, we can see that a general deformation of this curve satisfies the required property.)

Since in the subsequent arguments we need only the general curve $C_X \in \mathcal{C}_X$, we will remove from \mathcal{C}_X proper closed subsets when we need it, without special comments and keeping the same notation \mathcal{C}_X . Let $\mathcal{C}_V = \sigma_*\mathcal{C}_X$ be the image of that family on V . The family \mathcal{C}_V is, generally speaking, not free: if the σ -image of a prime divisor $\Delta \subset X$ is of codimension ≥ 2 , then the general curve $C_V \in \mathcal{C}_V$ meets $\sigma(\Delta)$.

Proposition 3.1. *There is a birational morphism $\varphi: V^+ \rightarrow V$, where V^+ is a non-singular projective variety, such that the strict transform \mathcal{C}_V^+ of the family \mathcal{C}_V on V^+ is a free family of curves.*

Proof. It is given in Section 4. (Here of course the strict transform of a family of irreducible curves is the family of irreducible curves, a general curve in which is the strict transform of a general curve in the original family. The parameterizing space of the new family is, generally speaking, a Zariski open subset of the parameterizing space of the original family.) \square

Proposition 3.2. *There is a non-singular quasi-projective variety U_X , a birational map $\varphi_X: U_X \dashrightarrow X$ and a Zariski open subset $U \subset V^+$, such that:*

(i) *the rational map*

$$\sigma_* = \varphi^{-1} \circ \sigma \circ \varphi_X: U_X \dashrightarrow V^+$$

extends to a morphism $\sigma_U: U_X \rightarrow V^+$, the image of which is U ,

(ii) *the inequality*

$$\text{codim}((V^+ \setminus U) \subset V^+) \geq 2$$

holds,

(iii) *the map $\sigma_U: U_X \rightarrow U$ is a cyclic cover of order p , branched over a non-singular hypersurface $W \subset U$.*

The proof is given in Section 5.

To make the statement of Proposition 3.2 more visual, we arrange the maps into the following commutative diagram:

$$\begin{array}{ccccc} U_X & & \xrightarrow{\varphi_X} & & X \\ \sigma_U \downarrow & & & & \downarrow \sigma \\ U & \subset & V^+ & \xrightarrow{\varphi} & V. \end{array}$$

Assuming Propositions 3.1 and 3.2, let us complete the proof of Theorem 1.3. By the inequality (ii) of Proposition 3.2 a general curve $C \in \mathcal{C}_V^+$ does not meet the closed set $V^+ \setminus U$ and for that reason is contained entirely in U . Therefore, for some open subfamily $\mathcal{C} \subset \mathcal{C}_V^+$ all curves $C \in \mathcal{C}$ are entirely contained in U , so that \mathcal{C} is a family of irreducible projective rational curves sweeping out U . By the construction of the family \mathcal{C} for a general curve $C \in \mathcal{C}$ its preimage

$$\sigma_U^{-1}(C) = C_1 \cup C_2 \cup \dots \cup C_p$$

is a union of p distinct rational curves on the quasi-projective variety U_X . Indeed, since for a general curve $C_X \in \mathcal{C}_X$ the morphism $\sigma|_{C_X}$ is birational, and by construction of the family \mathcal{C} , we may assume that C_1 belongs to a family of irreducible rational curves sweeping out U_X and the morphism $C_1 \rightarrow C$ is birational. Then the other curves C_i , $i \neq 1$, are the images of C_1 under the action of elements of the cyclic Galois group; in particular, they also belong to families of irreducible rational curves sweeping out U_X , so that

$$(C_i \cdot K_U) < 0,$$

$i = 1, \dots, p$, where K_U is the canonical class of the variety U_X . Write

$$K_V^+ = -\varphi^*H + \sum_{i \in I} a_i E_i$$

for the canonical class of the variety V^+ , where $E_i \subset V^+$ are all the prime φ -exceptional divisors and $a_i > 0$ are their discrepancies with respect to V and $H = -K_V$ is the ample anticanonical generator of $\text{Pic } V$. (The notation K_V^+ looks better than the standard symbol K_{V^+} .) Denoting the restrictions of the divisorial classes onto U by the same symbols and omitting the symbol φ^* , we get

$$(3.1) \quad K_U = \sigma_U^* \left(K_V^+ + \left(1 - \frac{1}{p}\right) W \right),$$

where $W \subset U$ is a non-singular hypersurface, over which the cyclic cover σ_U is branched. Collecting separately the components of the hypersurface W , which are divisorial on V and φ -exceptional, write

$$W = W_{\text{div}} + W_{\text{exc}},$$

where $W_{\text{div}} = nH - \sum_{i \in I} b_i E_i$ with $b_i \in \mathbb{Z}_+$ and $W_{\text{exc}} = \sum_{i \in I} c_i E_i$ with $c_i \in \{0, 1\}$. Obviously, the inequalities $(C \cdot K_V^+) < 0$, $(C \cdot W_{\text{div}}) \geq 0$ and $(C \cdot W_{\text{exc}}) \geq 0$ hold. At the same time,

$$(3.2) \quad (C_i \cdot K_U) = (C \cdot K_V^+) + \left(1 - \frac{1}{p}\right) (C \cdot W) < 0.$$

Assume first that $n \geq 2$. Adding to the left hand side of (3.2) the non-positive expression $[(1 - \frac{1}{p})n - 1](C \cdot K_V^+)$, we obtain the inequality

$$(3.3) \quad n(C \cdot K_V^+) + (C \cdot W) = \left(C \cdot \sum_{i \in I} (na_i - b_i + c_i) E_i \right) < 0,$$

so that for some $i \in I$ we have

$$b_i > na_i + c_i \geq n \cdot a_i$$

and the pair $(V, \frac{1}{n}\varphi_* W_{\text{div}})$ with $\varphi_* W_{\text{div}} \sim nH$ is not canonical, which contradicts the divisorial canonicity of the variety V . Therefore, $n = 1$ or 0 .

The case $n = 1$ is impossible: in that case $\varphi_* W_{\text{div}} = \varphi(W_{\text{div}}) \subset V$ is an anticanonical divisor on the variety V and we apply the assumption $(\star 1)$ for the prime divisor $R = \varphi(W_{\text{div}})$: we take a non-singular curve N (of arbitrary genus) such that $p \nmid (N \cdot K_V)$, which does not meet the set

$$(3.4) \quad \varphi(V^+ \setminus U) \cup \varphi \left(\bigcup_{i \in I} E_i \right)$$

and meets the divisor R transversally at points of general position. Its strict transform N^+ is contained entirely in U , and moreover,

$$\sigma_U^{-1}(N^+) \rightarrow N^+$$

is a cyclic cover of a non-singular curve, branched over a set of $(N \cdot R)$ points, the number of which is not divisible by p , which is impossible.

The case $n = 0$ is also impossible. Here we apply the assumption $(\star 2)$, taking on the variety V a non-singular rational curve N , which does not meet the set (3.4). Then the strict transform $N^+ \subset V^+$ is contained entirely in U , so that

$$\sigma_U^{-1}(N^+) \rightarrow N^+$$

is a non-ramified p -cyclic cover of a non-singular rational curve, which is impossible. \square

Proof of Corollary 2.1. Given that Zariski general hypersurfaces $V \in \mathcal{F}$ were shown in [Puk05] to be divisorially canonical, it remains to check that $(\star 1)$ and $(\star 2)$ hold for a general hypersurface V . In fact, they are satisfied for any non-singular hypersurface $V \subset \mathbb{P}$ of degree $(M + 1)$: $(\star 2)$ is true because V is a non-singular Fano variety, and for $(\star 1)$ we have the following simple argument. If $p \nmid \deg V$, then a section of V by a general 2-plane in \mathbb{P} does the job. Let us assume that $p \mid \deg V$.

A general line $L \subset V$ meets the hyperplane section R transversally at one point. However, it might happen that $L \cap Y \neq \emptyset$. In any case, since the lines on V sweep out a divisor, we may assume that $L \not\subset Y$, so that $L \cap Y$ is a finite set of points.

Now take a general plane $P \subset \mathbb{P}$ containing L , and let $N \subset P$ be the residual curve of the intersection $P \cap V = L + N$. Making sure that P is not contained in the hyperplanes in \mathbb{P} that are tangent to V at the points in $L \cap Y$ and taking into account that the image of Y with respect to the projection from the line L is a proper closed subset of \mathbb{P}^{M-1} , we may assume that $N \cap Y = \emptyset$. As

$$\deg N = \deg V - 1$$

and $p \mid \deg V$, the curve N is what we need. \square

Remark 3.3. The property $(\star 1)$ is easy to show for singular hypersurfaces $V \in \mathcal{F}_{\text{reg}}$, considered in [Puk15]. Recall [Puk15, Section 3, Subsection 3.2] that the conditions defining the open set \mathcal{F}_{reg} , include the following bound for the singularities of V : the hypersurface may have at most quadratic singularities of rank ≥ 8 , so that, in particular,

$$\text{codim}(\text{Sing } V \subset V) \geq 7.$$

Therefore, a general 2-plane in \mathbb{P} does not meet the closed set $\text{Sing } V$. Furthermore, the regularity conditions at every singular point $o \in \text{Sing } V$ ensure that there are finitely many lines through the point o on V . Therefore, a general line $L \subset V$ does not meet the singular locus of V and the proof of $(\star 1)$, given above for a non-singular hypersurface V , works for any $V \in \mathcal{F}_{\text{reg}}$ word for word.

4. Resolution of a family of curves

In order to prove Proposition 3.1, let us construct a sequence of birational morphisms

$$(4.1) \quad V_0 \xleftarrow{\varphi_1} V_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_{M-1}} V_{M-1}$$

of non-singular projective varieties, such that $V_0 = V$, $\mathcal{C}_0 = \mathcal{C}_V$ and for every closed subset $Y \subset V_i$ of dimension $\leq i - 1$ a general curve $C_i \in \mathcal{C}_i$ of the strict transform of the family \mathcal{C}_V on V_i does not meet the subset Y .

For $i = 0$ the last claim holds in a trivial way. We will explain in detail the first two steps of this construction: the morphisms φ_1 and φ_2 . The general step $\varphi_j: V_j \rightarrow V_{j-1}$ is very similar to φ_2 and will be easy to understand when φ_2 is clear.

Step 1. Let us consider the family $\mathcal{C}_0 = \mathcal{C}_V$ of curves, sweeping out the variety V_0 . It is clear that the set of points $\Xi_0 \subset V_0$, which are contained in all curves of the family \mathcal{C}_0 , is finite. We blow up this finite set of points:

$$\varphi_{1,1}: V_{0,1} \rightarrow V_{0,0} = V_0,$$

and look at the strict transform $\mathcal{C}_{0,1}$ of the family \mathcal{C}_0 on $V_{0,1}$. If there are no points that lie on all curves in $\mathcal{C}_{0,1}$, we stop. Otherwise, there is a finite set $\Xi_{0,1} \subset V_{0,1}$ of such points (obviously, $\Xi_{0,1}$ is contained in the exceptional divisor of $\varphi_{1,1}$) and we blow it up. Repeating, if necessary, we get a finite sequence of blow ups of finite sets of points,

$$V_0 = V_{0,0} \xleftarrow{\varphi_{1,1}} V_{0,1} \xleftarrow{\varphi_{1,2}} \cdots \xleftarrow{\varphi_{1,e(1)}} V_{0,e(1)},$$

such that for every point $p \in V_{0,e(1)}$ the curves $C_{0,e(1)} \in \mathcal{C}_{0,e(1)}$ (the last symbol means the strict transform of the family \mathcal{C}_0 on $V_{0,e(1)}$), containing the point p , form a proper closed subset of the family $\mathcal{C}_{0,e(1)}$. (That our procedure can not be infinite and must terminate, follows from considering some formal parameterizations of all branches of a general curve $C_0 \in \mathcal{C}_0$ at the points of the set Ξ_0 .) Set $V_1 = V_{0,e(1)}$ and

$$\varphi_1 = \varphi_{1,1} \circ \cdots \circ \varphi_{1,e(1)}: V_1 \rightarrow V_0.$$

Step 2. For the next step of our construction, consider the set of curves $\Xi_1 \subset V_1$, which intersect all curves of the family \mathcal{C}_1 . This is a finite set of curves. Indeed, embedding the quasi-projective variety, parameterizing the curves of the family \mathcal{C}_1 , in some projective space, and intersecting with the appropriate number of general hyperplanes, we obtain a sufficiently mobile family of curves in that quasi-projective variety, which gives us a family of surfaces \mathcal{S}_1 on V_1 , such every surface $S_1 \in \mathcal{S}_1$ contains all curves of the set Ξ_1 . Now it is clear that the set Ξ_1 is finite.

If D_1 is a general divisor of a very ample system on V_1 , then, intersecting the surfaces $S_1 \in \mathcal{S}_1$ with D_1 , we obtain a family of curves on D_1 , each of which contains the points of the set $\Xi_1 \cap D_1$. Now, arguing as at the previous step, for the family of curves $(S_1 \cap D_1, S_1 \in \mathcal{S}_1)$, we see that there is a finite sequence of birational morphisms

$$V_1 = V_{1,0} \xleftarrow{\varphi_{2,1}} V_{1,1} \xleftarrow{\varphi_{2,2}} \cdots \xleftarrow{\varphi_{2,e(2)}} V_{1,e(2)},$$

where $\varphi_{2,1}$ is a composition of two birational maps:

- a desingularization of the (reducible) 1-dimensional subset Ξ_1 , that is, a composition of finitely many blow ups of points, transforming Ξ_1 into a disjoint union of non-singular curves, and
- the blow up of the non-singular strict transform of Ξ_1 ,

and, similarly, $\varphi_{2,i}$ first resolves the singularities of the 1-dimensional set $\Xi_{1,i-1} \subset V_{1,i-1}$, which is the union of all irreducible curves on $V_{1,i-1}$, intersecting all curves of the family $\mathcal{C}_{1,i-1}$ (the strict transform of the family \mathcal{C}_1 on $V_{1,i-1}$), and then blows up the non-singular strict transform of the reducible curve $\Xi_{1,i-1}$. Note that by construction the image of every curve in $\Xi_{1,i-1}$ on V_1 is one of the curves of the set Ξ_1 (otherwise, all curves of the family \mathcal{C}_1 would have passed through some point, which is not true). Our procedure terminates for the same reason as at Step 1 (looking at the family of curves $(S_1 \cap D_1, S_1 \in \mathcal{S}_1)$ that was used above). Finally, on $V_{1,e(2)}$ there are no curves meeting all curves of the family $\mathcal{C}_{1,e(2)}$. We set $V_2 = V_{1,e(2)}$, $\mathcal{C}_2 = \mathcal{C}_{1,e(2)}$ and

$$\varphi_2 = \varphi_{2,1} \circ \cdots \circ \varphi_{2,e(2)}: V_2 \rightarrow V_1.$$

Carrying on in the same spirit, we construct the whole sequence (4.1). Let us consider briefly the general step of our construction.

Step j . By the previous arguments, if an irreducible subvariety intersects all curves of the family \mathcal{C}_{j-1} on V_{j-1} , it has dimension $\geq j-1$. If there are no irreducible subvarieties of dimension $j-1$, intersecting all curves in \mathcal{C}_{j-1} , then there is nothing to do: we set $V_j = V_{j-1}$ and φ_j is the identity morphism. Otherwise, let $\Xi_{j-1} \subset V_{j-1}$ be the union of all irreducible subvarieties of dimension $j-1$ meeting all curves in \mathcal{C}_{j-1} . Embedding the quasi-projective variety, parameterizing the curves of the family \mathcal{C}_{j-1} in a projective space and intersecting it with the appropriate number of general hyperplanes, we obtain a sufficiently mobile family of subvarieties of dimension $j-1$ in that quasi-projective variety, which gives us a family \mathcal{S}_{j-1} of

irreducible subvarieties of dimension j on V_{j-1} , such that every subvariety $S_{j-1} \in \mathcal{S}_{j-1}$ contains Ξ_{j-1} ; in particular, Ξ_{j-1} is a Zariski closed subset, and hence it contains finitely many irreducible components. If

$$D_1, \dots, D_{j-1}$$

is a general tuple of divisors in a very ample linear system on V_{j-1} , then, intersecting the subvarieties $S_{j-1} \in \mathcal{S}_{j-1}$ with $D_1 \cap \dots \cap D_{j-1}$, we obtain a family of curves on $D_1 \cap \dots \cap D_{j-1}$, each of which contains the points of the set $\Xi_{j-1} \cap D_1 \cap \dots \cap D_{j-1}$. Now, arguing as at Step 2, we see that there is a finite sequence of birational morphisms

$$V_{j-1} = V_{j-1,0} \xleftarrow{\varphi_{j,1}} V_{j-1,1} \xleftarrow{\varphi_{j,2}} \dots \xleftarrow{\varphi_{j,e(j)}} V_{j-1,e(j)},$$

where $\varphi_{j,1}$ is a composition of two birational maps:

- a desingularization of the (reducible) $(j-1)$ -dimensional closed subset Ξ_{j-1} , that is, a composition of finitely many blow ups of subvarieties of dimension $\leq j-2$, transforming Ξ_{j-1} into a disjoint union of non-singular subvarieties of dimension $j-1$, and
- the blow up of the non-singular strict transform of Ξ_{j-1} .

Similarly, $\varphi_{j,i}$ first resolves the singularities of the closed set $\Xi_{j-1,i-1} \subset V_{j-1,i-1}$, which is the union of all irreducible subvarieties on $V_{j-1,i-1}$, intersecting all curves of the family $\mathcal{C}_{j-1,i-1}$ (the strict transform of the family \mathcal{C}_{j-1} on $V_{j-1,i-1}$), and then blows up the non-singular strict transform of the closed set $\Xi_{j-1,i-1}$. Again, as at Step 2, the image of every component of the closed set $\Xi_{j-1,i-1}$ on V_{j-1} is one of the components of the set Ξ_{j-1} — otherwise, every curve in \mathcal{C}_{j-1} would have met some irreducible subvariety of dimension $\leq j-2$, which is not true by the previous steps of our construction. Our procedure terminates for the same reason as at Steps 1 and 2 (using the family of curves

$$(S_{j-1} \cap D_1 \cap \dots \cap D_{j-1}, S_{j-1} \in \mathcal{S}_{j-1})$$

on $D_1 \cap \dots \cap D_{j-1}$). Finally, on $V_{j-1,e(j)}$ there are no subvarieties of dimension $\leq j-1$, meeting all curves of the family $\mathcal{C}_{j-1,e(j)}$. We set $V_j = V_{j-1,e(j)}$, $\mathcal{C}_j = \mathcal{C}_{j-1,e(j)}$ and

$$\varphi_j = \varphi_{j,1} \circ \dots \circ \varphi_{j,e(j)}: V_j \rightarrow V_{j-1}.$$

Having constructed the sequence (4.1), set $V^+ = V_{M-1}$ and

$$\varphi = \varphi_0 \circ \varphi_1 \circ \dots \circ \varphi_{M-1}: V^+ \rightarrow V.$$

Obviously, V^+ has the property, described in the statement of the proposition. \square

5. Construction of the cyclic cover

Let us show Proposition 3.2. The field extension $\mathbb{C}(V) \subset \mathbb{C}(X)$ is generated by some element $\xi \in \mathbb{C}(X)$, satisfying the equation

$$\xi^p - q = 0$$

for some rational function $q \in \mathbb{C}(V) = \mathbb{C}(V^+)$. For some effective divisor R on V^+ and sections $a_0, a_1 \in \mathcal{O}_{V^+}(R)$ we can write $q = a_0/a_1$. Consider the hypersurface

$$\{G = 0\} \subset V^+ \times \mathbb{P}^1_{(x_0:x_1)},$$

where

$$G = a_1 x_1^p - a_0 x_0^p \in H^0(V^+ \times \mathbb{P}^1, \text{pr}_V^* \mathcal{O}_{V^+}(R) \otimes \text{pr}_{\mathbb{P}}^* \mathcal{O}_{\mathbb{P}^1}(p)),$$

the symbols pr_V and $\text{pr}_{\mathbb{P}}$ mean the projections of $V^+ \times \mathbb{P}^1$ onto the first and second factor, respectively. This hypersurface is, generally speaking, reducible, but has a unique irreducible component X_0 , such that $\text{pr}_V(X_0) = V^+$.

If for some prime divisor $T \subset V^+$ the hypersurface $\{G = 0\}$ contains $\text{pr}_V^{-1}(T)$, then $a_0, a_1|_T \equiv 0$, so that, replacing R by $R - mT$, where

$$m = \min\{\text{ord}_T(a_0), \text{ord}_T(a_1)\},$$

and $a_0, a_1 \in \mathcal{O}_{V^+}(R)$ by

$$\frac{a_0}{s_T^m}, \frac{a_1}{s_T^m} \in \mathcal{O}_{V^+}(R - mT),$$

where $s_T \in \mathcal{O}_{V^+}(T)$ is the section, corresponding to the divisor T , we remove the component $\text{pr}_V^{-1}(T)$. Therefore, we may assume from the beginning that the sections $a_0, a_1 \in \mathcal{O}_{V^+}(R)$ do not vanish simultaneously on any prime divisor $T \subset V^+$ and $\{G = 0\} = X_0 \subset V^+ \times \mathbb{P}^1$ is an irreducible hypersurface, by construction birational to the original variety X .

Now let us consider the singularities of the variety X_0 . Assume that for some prime divisor $T \subset V^+$ there is a subvariety $T_X \subset \text{Sing } X_0$, such that

$$\text{pr}_V(T_X) = T.$$

Let \mathcal{T} be the set of all prime divisors on V^+ with that property. By what was said above, we may assume that say, $a_1|_T \not\equiv 0$, which implies that

$$\text{ord}_T a_0 \geq 2.$$

Removing the set of common zeros of the sections $a_0, a_1 \in \mathcal{O}_{V^+}(R)$, the pairwise intersections of the divisors $T \in \mathcal{T}$ (if $\#\mathcal{T} \geq 2$) and the sets of singular points $\text{Sing } T$ for all $T \in \mathcal{T}$, we obtain a Zariski open set $U \subset V^+$, such that

- $\text{codim}((V^+ \setminus U) \subset V^+) \geq 2$,
- the morphism $X_0 \cap \text{pr}_V^{-1}(U) \rightarrow U$ is a finite morphism of degree p and
- for every divisor $T \in \mathcal{T}$ the quasi-projective varieties $T \cap U$ and $T_X \cap \text{pr}_V^{-1}(U)$ are non-singular, and moreover, the projection pr_V gives an isomorphism of these varieties, which we for simplicity of notations write down again as T and T_X .

Now distinct varieties in \mathcal{T} are disjoint.

Furthermore, we may assume that for every $T \in \mathcal{T}$ the section $a_i, i = 0, 1$, which does not vanish identically on T , is everywhere non-zero on $T \cap U$, so that if say $a_1|_T \not\equiv 0$, then the hypersurface $X_0 \cap \text{pr}_V^{-1}(U)$ over a neighborhood of the divisor $T \cap U$ is contained in the open subset

$$\{x_0 \neq 0\} = U \times \mathbb{A}_z^1,$$

with $z = x_1/x_0$ and given by the equation

$$a_1 z^p - a_0 = 0,$$

so that the corresponding subvariety $T_X \subset \text{Sing } X_0$ over $T \cap U$ is given in $T \times \mathbb{A}_z^1$ by the equation $z = 0$.

Thus we have constructed a locally trivial \mathbb{P}^1 -bundle \mathcal{X}_1 over a non-singular quasi-projective variety U (of course, $\mathcal{X}_1 = U \times \mathbb{P}^1$) and a hypersurface $X_1 = X_0 \cap \text{pr}_V^{-1}(U) \subset \mathcal{X}_1$ which is a cyclic cover of the prime order p over U . The projection $\mathcal{X}_1 \rightarrow U$ will be denoted by the symbol π_1 . The cyclic cover $X_1 \rightarrow U$ has all properties required in Proposition 3.2 except for X_1 being non-singular. The rest of our proof of the proposition is removing the singularities of X_1 . This has to be done carefully, preserving the cyclic cover.

We desingularize the covering variety in two steps. The first step is constructing a sequence of locally trivial \mathbb{P}^1 -bundles over U :

$$\mathcal{X}_1 \xleftarrow{\beta_1} \mathcal{X}_2 \xleftarrow{\beta_2} \dots \xleftarrow{\beta_{k-1}} \mathcal{X}_k,$$

where β_i is an elementary birational transformation over U ,

$$\begin{array}{ccc} \mathcal{X}_{i+1} & \xrightarrow{\beta_i} & \mathcal{X}_i \\ \pi_{i+1} \downarrow & & \downarrow \pi_i \\ U & = & U, \end{array}$$

defined in the following way. Assume that the strict transform $X_i \subset \mathcal{X}_i$ of the hypersurface X_1 is singular along a subvariety $\pi_i^{-1}(T) \cap X_i = T_i$ for some $T \in \mathcal{T}$. With respect to a certain trivialization of the \mathbb{P}^1 -bundle \mathcal{X}_i/U over an open subset, intersecting the divisor T , the hypersurface X_i is given by the equation

$$a_{i,1}x_1^p - a_{i,0}x_0^p = 0,$$

where, say, $a_{i,1}|_T \neq 0$ and $\text{ord}_T a_{i,0} \geq 2$. Assume, furthermore, that in fact the inequality

$$\text{ord}_T a_{i,0} \geq p$$

holds. Then the birational transformation $\beta_i: \mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$ is the composition of the blow up of the subvariety T_i (by construction, this non-singular subvariety is a section of the \mathbb{P}^1 -bundle $\pi_i^{-1}(T) \rightarrow T$) and subsequent contraction of the strict transform of the hypersurface $\pi_i^{-1}(T)$. Elementary computations with local parameters, using the explicit presentation described above, show that locally in a neighborhood of the generic point of the divisor T the hypersurface X_{i+1} is given by the equation

$$a_{i+1,1}x_1^p - a_{i+1,0}x_0^p = 0,$$

where $a_{i+1,1}|_T \neq 0$ and $\text{ord}_T a_{i+1,0} = \text{ord}_T a_{i,0} - p$. Therefore, after finitely many elementary transformations of the ambient \mathbb{P}^1 -bundle we obtain a locally trivial \mathbb{P}^1 -bundle

$$\pi_k: \mathcal{X}_k \rightarrow U,$$

such that if the strict transform $X_k \subset \mathcal{X}_k$ of the hypersurface X_1 is singular along a subvariety $\pi_k^{-1}(T) \cap X_k = \widetilde{T}$ for a $T \in \mathcal{T}$, then over the general point of T the hypersurface $X_k \subset \mathcal{X}_k$ is given by the equation

$$a_{T,1}x_1^p - a_{T,0}x_0^p = 0,$$

where, say, $a_{T,1}$ does not vanish on T , and

$$\text{ord}_T a_{T,0} = l_T \in \{2, \dots, p-1\}.$$

This means that if $p = 2$, then the proof of Proposition 3.2 is completed: the variety X_k is a non-singular cyclic cover of U .

Assume now that $p \geq 3$. Then along the subvariety $\pi_k^{-1}(T) \cap X_k$ the variety X_k has a cuspidal singularity of the type

$$t^p - s^{l_T} = 0.$$

Taking the normalization of the variety X_k or applying the obvious sequence of blow ups, we obtain the required variety U_X that covers U cyclically, and complete the proof of Proposition 3.2 for a cyclic cover of degree $p \geq 3$. This completes the proof of Theorem 1.3. \square

6. Generalizations

First of all, given that the divisorial canonicity has been shown in [Puk06, EP16, Puk18] for many families of Fano complete intersections of index 1 in the projective space, we obtain the following generalization of Corollary 2.1.

Corollary 6.1. *For a divisorially canonical non-singular Fano complete intersection $V \subset \mathbb{P}^{M+k}$ of codimension k and index 1, there are no non-trivial rational Galois covers $X \xrightarrow{d:1} V$ with an abelian Galois group of order $d \geq 2$, where X is a rationally connected variety; in particular, there are no rational maps $X \dashrightarrow V$ of degree 2 with X rationally connected.*

Proof. We need only to check that the properties $(\star 1)$ and $(\star 2)$ are satisfied for a non-singular complete intersection V . The property $(\star 2)$ is automatic and in order to show $(\star 1)$, we argue as in the proof of

Corollary 2.1: if $p \nmid \deg V$, then a section of V by a general linear $(k+1)$ -subspace in \mathbb{P}^{M+k} works as N , and if $p \mid \deg V$, we obtain N as the residual curve,

$$P \cap V = L + N,$$

where $L \subset V$ is a general line and P a general linear $(k+1)$ -subspace in \mathbb{P}^{M+k} , containing L . \square

Remark 6.2. It seems that the claim of Theorem 1.3 is true for rationally connected Galois rational covers with an arbitrary Galois group, not necessarily an abelian one. We may assume that X is a non-singular projective variety and the Galois group of the extension $\mathbb{C}(V) \subset \mathbb{C}(X)$ is a finite subgroup of the group $\text{Aut } X$. Now the ramification divisor of the finite cover $U_X \rightarrow U$ is invariant with respect to the action of the Galois group and for that reason is pulled back from U , so that the proof of Theorem 1.3 must work in the non-abelian case, too.

Conjecture 6.3. *The claim of Theorem 1.3 holds without the assumption that the Galois group is abelian.*

Remark 6.4. The fact that there are no rationally connected rational double covers for a hypersurface $V \subset \mathbb{P}$ of degree $M+1$, where $M \geq 5$, was obtained from the property that every pair $(V, \frac{1}{n}D)$, where $D \sim nH$, is canonical. However, for certain special hypersurfaces V this is not true: for instance, if $V \cap T_o V$ is a cone with the vertex at the point o . On the other hand, it seems probable that the claim of Theorem 1.3 is true for any smooth hypersurface of index 1 for $M \geq 5$ — and, possibly, for four-dimensional quintics and three-dimensional quartics, for which the technique of [Puk05] does not prove the property of divisorial canonicity. Note also that it is hard to see any obstructions for the Conjecture 1.6 to be true at least for any smooth hypersurface of index 1 and dimension ≥ 4 . At the same time, as Segre's example shows, for certain smooth three-dimensional quartics Conjecture 1.6 does not hold because they are unirational, see [IM71].

7. Historical remarks and acknowledgements

As far as I know, the first time when a conjecture on non-unirationality of certain rationally connected three-folds (in that case, conic bundles) was explicitly stated, was in [Fan31] by Gino Fano himself. In [Kol98] Kollár suggested an approach to proving the non-unirationality of primitive Fano hypersurfaces: if the variety parameterizing rational curves of some arbitrary fixed degree passing through a general point $o \in V$ contains no rational curves, then there are no rational surfaces on V containing this point, which of course implies non-unirationality. The suggested approach motivated a far reaching investigation of the space of rational curves on hypersurfaces in [BS08, BK13]. For the various explicit results on unirationality (solving the unirationality problem affirmatively) see, for instance, [HMP98, CM98, Con01, CMM09, CM06-08]. A topic that can be linked to the conjecture on absolute rigidity is endomorphisms, especially rational endomorphisms of rationally connected varieties; there are quite a few papers on that subject, see, for instance, [AKP08, HN11, ABR11, Zha12, Zha14].

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