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Complex reflection groups and K3 surfaces I

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In memory of Gianandrea

Abstract. We construct here many families of K3 surfaces that one can obtain as quotients of algebraic surfaces by some subgroups of the rank four complex reflection groups. We find in total 15 families with at worst ADE-singularities. In particular we classify all the K3 surfaces that can be obtained as quotients by the derived subgroup of the previous complex reflection groups. We prove our results by using the geometry of the weighted projective spaces where these surfaces are embedded and the theory of Springer and Lehrer-Springer on properties of complex reflection groups. This construction generalizes a previous construction by W. Barth and the second author.

Keywords. K3 surfaces; complex reflection groups

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[Français]

Groupes de réflexions complexes et surfaces K3 I

Résumé. Nous construisons ici de nombreuses familles de surfaces K3 que l'on peut obtenir comme quotients de surfaces algébriques par certains sous-groupes des groupes de réflexions complexes de rang 4. Nous trouvons au total 15 familles avec au plus des singularités ADE. En particulier, nous classifions toutes les surfaces K3 pouvant être obtenues comme quotients par le sous-groupe dérivé des groupes de réflexions complexes susmentionnés. Nous démontrons nos résultats en utilisant la géométrie des espaces projectifs à poids dans lesquels ces surfaces sont plongées et la théorie de Springer et Lehrer-Springer sur les propriétés des groupes de réflexions complexes. Cette construction généralise une construction précédente introduite par W. Barth et la seconde auteure.

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Contents

1. Introduction

In this paper we describe a relation between complex reflection groups and K3 surfaces. A relation already appeared recently in the paper [BS21] by the authors, where they use the reflection group denoted by G_{29} in the Shepard–Todd classification [ST54] to describe K3 surfaces with maximal finite automorphism groups containing the Mathieu group M_{20}. The motivation for our paper is an early paper of the second author [Sar01] and of W. Barth and the second author [BS03] where they study first one parameter families of surfaces of general type in the three dimensional complex projective space containing four surfaces with a high number of nodes (i.e. A_1–singularities). Then they study the quotients of these families by some groups related to the platonic solids: tetrahedron, octahedron and icosahedron and which they call bipolyhedral groups. These turn out to be subgroups of some complex reflection groups and they show that the quotients are K3 surfaces with ADE–singularities. In this paper we show that these examples are only a few examples of K3 surfaces that one can produce by using complex reflection groups. Moreover the theory of Springer and Lehrer–Springer and some technical lemmas allow a deep understanding of the reason why the quotients have trivial dualizing sheaf and admit only ADE–singularities. This allows then to conclude that the minimal resolution are K3 surfaces. We find in total 15 families of K3 surfaces.

More precisely we consider a complex reflection group W acting on a four dimensional complex vector space V. By Shephard–Todd/Chevalley/Serre Theorem [Bro10, Theorem 4.1] there exist 4 algebraically independent polynomials which are invariant under the action of W and which generate algebraically the ring of all W-invariant polynomials. We assume furthermore that W is generated by reflections of order 2 and in Table 1 we give the list of the degrees of the four invariant polynomials (observe that these degrees do not depend on the polynomials) and of the codegrees corresponding to the degrees of four invariant derivatives which generate the module of all W-invariant derivatives. The aim of the paper is to study the quotient of the projective zero set $\mathcal{Z}(f)$ of an homogeneous fundamental invariant f of W by some subgroup Γ of W: the derived subgroup W' and the group $W^{\mathrm{SL}} = \mathrm{Ker}(\det) \cap W$. A reason for this choice is that the simple structure of the invariant ring of W and the fact that W is generated by reflections of order 2 imply that $\mathcal{Z}(f)/\Gamma$ is a complete intersection in a weighted projective space. More precisely the quotient surface $\mathcal{Z}(f)/W^{\mathrm{SL}}$ is a double cover of a weighted projective plane whereas if W' is different from W^{SL} then

$\mathcal{Z}(f)/W'$ is a complete intersection in a four dimensional weighted projective space. We also explain how to obtain explicit equations for $\mathcal{Z}(f)/W'$ or $\mathcal{Z}(f)/W^{\mathrm{SL}}$.

It turns out that if the degree d of f is "well chosen" and $\mathcal{Z}(f)$ has only ADE-singularities, then $\mathcal{Z}(f)/\Gamma$ is a K3 surface with ADE-singularities ("well chosen" means that the sum of the degrees of the equations of the surface is equal to the sum of the weights of the ambient weighted projective space: the set of pairs (W,d) such that d is "well-chosen" is denoted by $\mathcal{K}_3$ and is described in §5.2). To show that we have ADE-singularities on the double covers one has to study carefully the singularities of the branching curve as well as the singularities that one gets from the singularities of the weighted projective plane. Our main Theorem 5.4 is the following:

Theorem 1.1. *Assume that* $(W,d) \in \mathcal{K}_3$. *Let* Γ *be the subgroup* W' *or* W^{SL} *of* W *and let* f *be a fundamental invariant of* W *of degree* d *whose projective zero set* $\mathcal{Z}(f)$ *has only* ADE*-singularities.*

Then $\mathcal{Z}(f)/\Gamma$ *is a K3 surface with* ADE*-singularities.*

In particular the theorem allows us to classify all the K3 surfaces that can be obtained as quotient by W' or W^{SL}. Theorem 5.4 is a qualitative result, that insures that one can build from invariants of some complex reflection groups of rank 4 several families of K3 surfaces with ADE-singularities. However, it does not say anything about the types of the singularities and important invariants of their minimal resolution (rank of the Picard number, description of the transcendental lattice).

Theorem 5.4 (and its proof) improves previous works by Barth and the second author [BS03] for two reasons:

- By looking at all complex reflection groups of rank 4, it enlarges considerably the class of examples of K3 surfaces that can be constructed as above. It shows moreover that the discovery of families of K3 surfaces in [BS03] is not just "an accident" but it is strictly related to polynomial invariants of complex reflection groups and their action on these.
- The main difficulty is to prove that $\mathcal{Z}(f)/\Gamma$ has only ADE singularities. Our proof involves some general facts about singularities (see Appendix B) and more complex reflection group theory (as the theory of Springer and Lehrer–Springer on eigenspaces of elements of complex reflection groups): as a consequence, our proof avoids as much as possible (but not completely) a case-by-case analysis, and so may also be viewed not only as a generalization but also as an enlightenment of results from [BS03].

An important feature of the K3 surfaces constructed in Theorem 5.4 is that most of them have big Picard number, and generally as big as possible compare to the number of moduli of the family they belong to. In particular, we can build in this way several (more than thirty) K3 surfaces with Picard number 20, often called *singular* K3 surfaces, whose moduli space is a 0-dimensional subspace of the 20-dimensional moduli space of K3 surfaces. This will be explained in the sequel to this paper [BS-II], [BS-III], where we aim to complete the qualitative results of this first part by quantitative results whenever W is assumed to be *primitive* (i.e. $W = G_{28}$, G_{29}, G_{30} or G_{31}). We will for instance compute the transcendental lattice of the singular K3 surfaces, and describe explicit elliptic fibrations in most cases. Note that the case where $(W,d) = (G_{28},6)$ or $(G_{30},12)$ and $\Gamma = W'$ was already treated by Barth and the second author [BS03]: these examples will be revisited thanks to our new techniques, and more geometrical informations will be given. Note also that, by taking Galois invariant models for complex reflection groups as in [MM10], it turns out that all our families of K3 surfaces are defined over $\mathbb{Q}$ and, in particular, all the singular K3 surfaces we obtain are defined over $\mathbb{Q}$: this fact is only checked case-by-case [BS-II], [BS-III] but would deserve a general explanation.

The paper is organized as follows: Section 2 contains basic facts on the action of groups of matrices on homogeneous polynomials and Section 3 recalls facts on reflection groups, in particular we find equations for the quotient surfaces and we recall basic facts from Springer and Lehrer-Springer theory, that we use in the next sections (and in the sequel to this paper [BS-II], [BS-III]) to describe the singularities that we have on the quotient surfaces. In Section 4 we give several useful facts to describe the singularities of the quotient surfaces in particular in the case that these are a double cover of a weighted projective plane. Note that Sections 2-4 are written in a greater generality (reflection groups acting on vector spaces of any finite dimension) as they might be of general interest. In Section 5 we describe how to obtain K3 surfaces. In Table 2 we recall the degrees of the equations and the weighted projective spaces where the (singular) surfaces are embedded. We give in this section the main part of the proof of our main Theorem 5.4. We finish the proof in Section 6 where we show that the quotient K3 surfaces have at worst ADE-singularities.

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2. Notation, preliminaries

If $d \geq 1$, we denote by $\boldsymbol{\mu}_d$ the group of d-th roots of unity in $\mathbb{C}^\times$ and we fix a primitive d-th root of unity ζ_d (we will use the standard notation $i = \zeta_4$). If $l_1, \dots, l_r$ are positive integers, then $\mathbb{P}(l_1, \dots, l_r)$ denotes the corresponding weighted projective space.

We fix an n-dimensional $\mathbb{C}$-vector space V and we denote by $\mathbb{P}(V)$ its associated projective space. If $v \in V \setminus \{0\}$, we denote by $[v] \in \mathbb{P}(V)$ the line it defines (i.e. $[v] = \mathbb{C}v$). The group $\mathbf{GL}_{\mathbb{C}}(V)$ acts on the algebra $\mathbb{C}[V]$ of polynomial functions on V as follows: if $g \in \mathbf{GL}_{\mathbb{C}}(V)$ and $f \in \mathbb{C}[V]$, we write

$$^{g}f(v) = f(g^{-1} \cdot v).$$

If $g \in \mathbf{GL}_{\mathbb{C}}(V)$ and $\zeta \in \mathbb{C}^\times$, we denote by $V(g,\zeta)$ the ζ-eigenspace of g. If $v \in V(g,\zeta)$ and $f \in \mathbb{C}[V]^g$, then

$$^{g}(\partial_v f) = \zeta \partial_v f. \tag{2.1}$$

If G is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we write PG for its image in $\mathbf{PGL}_{\mathbb{C}}(V)$. Recall that a subgroup G of $\mathbf{GL}_{\mathbb{C}}(V)$ is called ***primitive*** if there does not exist a decomposition $V = V_1 \oplus \cdots \oplus V_r$ with $V_i \neq 0$ and $r \geq 2$ such that G permutes the V_i's. If S is a subset of V, we denote by G_S (resp. $G(S)$) the setwise (resp. pointwise) stabilizer of S (so that $G(S)$ is a normal subgroup of G_S and $G_S/G(S)$ acts faithfully on S). Note that $G_S = G_{\mathbb{C}S}$ and $G(S) = G(\mathbb{C}S)$, where $\mathbb{C}S$ denotes the linear span of S. The derived subgroup of G will be denoted by G', and we set $G^{\mathrm{SL}} = G \cap \mathbf{SL}_{\mathbb{C}}(V)$. Note that $G' \subset G^{\mathrm{SL}}$ and that the inclusion might be strict. We state here a totally trivial result which will be used extensively and freely along this series of papers:

Lemma 2.2. *Let $g \in \mathbf{GL}_{\mathbb{C}}(V)$, let ζ be a root of unity of order d, let $v \in V$ be such that $g(v) = \zeta v$ and let $f \in \mathbb{C}[V]^g$ be homogeneous of degree e not divisible by d. Then $f(v) = 0$.*

Proof. As $f \in \mathbb{C}[V]^g$, we have $f(g(v)) = f(v)$. But $f(g(v)) = f(\zeta v) = \zeta^e f(v)$ because f is homogeneous of degree e. So the result follows from the fact that $\zeta^e \neq 1$. □

If X is a complex algebraic variety and $x \in X$, we denote by $\mathrm{T}_x(X)$ the tangent space of X at x. If $f \in \mathbb{C}[V]$ is homogeneous, we will denote by $\mathcal{Z}(f)$ the projective (possibly non-reduced) hypersurface in $\mathbb{P}(V) \simeq \mathbb{P}^{n-1}$ defined by f. Its singular locus will be denoted by $\mathcal{Z}_{\mathrm{sing}}(f)$.

Lemma 2.3. *Let G be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, let $f \in \mathbb{C}[V]^G$ be homogeneous and let $v \in V^G \setminus \{0\}$ be such that $f(v) = 0$. Then G acts trivially on $\mathrm{T}_{[v]}(\mathbb{P}(V))/\mathrm{T}_{[v]}(\mathcal{Z}(f))$.*

Proof. Since G is finite, there exists a G-stable subspace E of V such that $V = E \oplus \mathbb{C}v$. Let $\alpha \in V^*$ be such that $\alpha(v) = 1$ and $\alpha(E) = 0$. The affine chart U_α of $\mathbb{P}(V)$ defined by $\alpha \neq 0$ is identified with $v + E$ and, after translation, is identified with E: through this identification, $\mathcal{Z}(f) \cap U_\alpha$ is the affine hypersurface defined by the polynomial $F \in \mathbb{C}[E]$, where $F(e) = f(v+e)$. Since v is G-invariant, F is also G-invariant. Let us denote by F_i its i-th homogeneous component: it is G-invariant. Then $F_0 = 0$ because $f(v) = 0$. Moreover, $\mathrm{T}_{[v]}(\mathbb{P}(V)) = E$ and $\mathrm{T}_{[v]}(\mathcal{Z}(f)) = \mathrm{Ker}(F_1)$ (and these identifications are G-equivariant since $v \in V^G$).

But $\mathbb{C}F_1$ is the dual space to $E/\mathrm{Ker}(F_1)$: since G acts trivially on $\mathbb{C}F_1$, this shows that G acts trivially on $E/\mathrm{Ker}(F_1) = \mathrm{T}_{[v]}(\mathbb{P}(V))/\mathrm{T}_{[v]}(\mathcal{Z}(f))$. □

The next result is just an easy generalization of [Sar01, §6].

Corollary 2.4. *Let G be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ such that $\dim V^G = 1$ and let $f \in \mathbb{C}[V]^G$ be non-zero and homogeneous. We assume that f vanishes at V^G, viewed as a point of $\mathbb{P}(V)$. Then V^G is a singular point of $\mathcal{Z}(f)$.*

Proof. Let $v \in V^G \setminus \{0\}$. We keep the notation of the proof of the previous Lemma 2.3 (E, α, F, F_i). Since $V^G = \mathbb{C}v$, we have $E^G = 0$ and so, by semisimplicity, we have that $(E/\mathrm{Ker}(F_1))^G = 0$. But G acts trivially on $E/\mathrm{Ker}(F_1)$ by Lemma 2.3. Therefore, $E/\mathrm{Ker}(F_1) = 0$ so $\mathrm{T}_{[v]}(\mathcal{Z}(f)) = \mathrm{Ker}(F_1) = E$ and so $\mathcal{Z}(f)$ is singular at $[v] = V^G$. □

Remark 2.5. The previous lemma might be used to explain the construction of several singular curves and surfaces constructed by the two authors [Sar01], [Bon19]. Let us explain how to proceed.

Let G be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, and let $H_1, \dots, H_r$ be a set of representatives of conjugacy classes of maximal subgroups of G among the subgroups H satisfying $\dim(V^H) = 1$. Let $N_k = N_G(H_k)$, let $v_k \in V^{H_k} \setminus \{0\}$ and let Ω_k denote the G-orbit of $[v_k]$ in $\mathbb{P}(V)$. Then

$$|\Omega_k| = \frac{|G|}{|N_k|}. \tag{2.6}$$

For this, it is sufficient to prove that $N_k = G_{[v_k]}$. First, N_k stabilizes $V^{H_k} = [v_k]$, which proves that $N_k \subset G_{[v_k]}$. Conversely, $G_{[v_k]}$ normalizes G_{v_k}. But $H_k \subset G_{v_k}$ by construction and, by the maximality of H_k, this implies that $H_k = G_{v_k}$.

Now, we fix two linearly independent homogeneous polynomials f_1 and f_2 of the same degree such that $f_1(v_k) \neq 0$ for all k. We also set $\lambda_k = f_2(v_k)/f_1(v_k)$. Then it follows from Corollary 2.4 that

$$\mathcal{Z}(f_2 - \lambda_k f_1) \textit{ contains } \Omega_k \textit{ in its singular locus.} \tag{2.7}$$

It also shows that, if G is defined over a subfield K of $\mathbb{C}$, then the points of Ω_k (which are singular points of $\mathcal{Z}(f_2 - \lambda_k f_1)$) have coordinates in K.

Corollary 2.8. *Let G be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ such that $\dim V^G = 2$, let $f \in \mathbb{C}[V]^G$ be homogeneous and non-zero and let $v \in V^G \setminus \{0\}$ be such that $f(v) = 0$. Let L be the line $\mathbb{P}(V^G)$ and assume that $[v]$ is a smooth point of $\mathcal{Z}(f)$. Then the intersection of L with $\mathcal{Z}(f)$ is transverse at $[v]$.*

Proof. We keep again the notation of the proof of Lemma 2.3 (E, α, F, F_i). Since $\dim V^G = 2$, this forces $\dim E^G = 1$. Since $[v]$ is smooth, this means that $F_1 \neq 0$. It then follows from Lemma 2.3 that $E = E^G \oplus \mathrm{Ker}(F_1)$. But $E^G = \mathrm{T}_{[v]}(L)$ and $\mathrm{Ker}(F_1) = \mathrm{T}_{[v]}(\mathcal{Z}(f))$. This shows the result. □

Corollary 2.9. *Let G be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ such that $\dim V^G = 2$, and let $f \in \mathbb{C}[V]^G$ be homogeneous and non-zero. Let L be the line $\mathbb{P}(V^G)$ and assume that $L \subset \mathcal{Z}(f)$. Then $L \subset \mathcal{Z}_{\mathrm{sing}}(f)$.*

Proof. Let $v \in V^G \setminus \{0\}$ and assume that $[v]$ is a smooth point of $\mathcal{Z}(f)$. Then the intersection of L with $\mathcal{Z}(f)$ is not transverse at $[v]$ because $L \subset \mathcal{Z}(f)$: this contradicts Corollary 2.8. □

3. Reflection groups

We fix a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$ and we set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim(V^s) = n - 1\}.$$

Hypothesis. *We assume throughout this paper that*

$$W = \langle \mathrm{Ref}(W) \rangle.$$

In other words, W is a ***complex reflection group****. The number* $\mathrm{codim}(V^W)$ *is called the* ***rank*** *of W.*

Standard arguments allow to reduce most questions about reflection groups to questions about *irreducible* reflection groups. These last ones have been classified by Shephard-Todd and we refer to Shephard-Todd numbering [ST54] for such groups: there is an infinite family $G(de,e,r)$ with d, e, $r \geq 0$ (they are of rank r if $(d,e) \neq (1,1)$ and of rank $r-1$ otherwise) and 34 exceptional ones numbered from G_4 to G_{37} (they are exactly the primitive complex reflection groups). If W can be realized over the field of real numbers, then it is a Coxeter group and we will also use the notation $\mathrm{W}(X_i)$ where X_i is the type of some Coxeter graph. For instance, the group G_{30} in Shephard-Todd numbering is the Coxeter group $\mathrm{W}(H_4)$.

3.1. Invariants

By Shephard-Todd/Chevalley/Serre Theorem [Bro10, Theorem 4.1], there exist n algebraically independent homogeneous elements $f_1, f_2, \dots, f_n$ of $\mathbb{C}[V]^W$ such that

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, f_2, \dots, f_n].$$

Let $d_i = \deg(f_i)$. A family $(f_1, f_2, \dots, f_n)$ satisfying the above property is called a *family of fundamental invariants* of W. Observe that by a result of Marin-Michel [MM10], these polynomials can be defined over the rational numbers (for more details, see [BS-II]). A homogeneous element $f \in \mathbb{C}[V]^W$ is called a *fundamental invariant* if it belongs to a family of fundamental invariants. Whereas such a family is not uniquely defined even up to permutation, the list $(d_1, d_2, \dots, d_n)$ is well-defined up to permutation and is called the list of *degrees* of W: it will be denoted by $\mathrm{Deg}(W)$.

Notation. *From now on, and until the end of this paper, we fix a family* $\mathbf{f} = (f_1, f_2, \dots, f_n)$ *of fundamental invariants and we set* $d_i = \deg(f_i)$.

The following equalities are well-known [Bro10, Theorem 4.1]:

$$|W| = d_1 d_2 \cdots d_n \qquad \text{and} \qquad |\mathrm{Ref}(W)| = \sum_{i=1}^{n} (d_i - 1). \tag{3.1}$$

Also, as W acts irreducibly on V, its center $|\mathrm{Z}(W)|$ consists of homotheties, so it is cyclic. Moreover by [Bro10, Proposition 4.6],

$$|\mathrm{Z}(W)| = \mathrm{Gcd}(d_1, d_2, \dots, d_n). \tag{3.2}$$

The $\mathbb{C}[V]$-module $\mathrm{Der}(\mathbb{C}[V])$ of derivatives of the algebra $\mathbb{C}[V]$ is naturally graded in such a way that ∂_v has degree -1 for all $v \in V$. By Solomon Theorem [Bro10, Theorem 4.44 and §4.5.4], the graded $\mathbb{C}[V]^W$-module $\mathrm{Der}(\mathbb{C}[V])^W$ of invariant derivatives is free of rank n, hence it admits a homogeneous $\mathbb{C}[V]^W$-basis $(D_1, \dots, D_n)$ whose respective degrees are denoted by $d_1^* \dots, d_n^*$. Again, the family $(D_1, \dots, D_n)$ is not uniquely defined even up to permutation, but the list $(d_1^*, d_2^*, \dots, d_n^*)$ is well-defined up to permutation and is called the list of *codegrees* of W: it will be denoted by $\mathrm{Codeg}(W)$.

We conclude this subsection by a general easy result which follows immediately from the fact that $\mathbb{C}[V]^W$ is a graded polynomial algebra whose weights are given by $\mathrm{Deg}(W)$.

Proposition 3.3. *The map*

$$\begin{array}{rccc} \pi_{\mathbf{f}}: & \mathbb{P}(V) & \longrightarrow & \mathbb{P}(d_1,d_2,\dots,d_n) \\ & [v] & \longmapsto & [f_1(v):f_2(v):\cdots:f_n(v)] \end{array}$$

is well-defined and induces an isomorphism

$$\mathbb{P}(V)/W \xrightarrow{\sim} \mathbb{P}(d_1,d_2,\dots,d_n).$$

Moreover, $\pi_{\mathbf{f}}$ induces by restriction an isomorphism

$$\mathcal{Z}(f_1)/W \xrightarrow{\sim} \mathbb{P}(d_2,\dots,d_n).$$

3.2. Reflecting hyperplanes

If $s \in \operatorname{Ref}(W)$, then the hyperplane V^s is called the reflecting hyperplane of s (or a reflecting hyperplane of W). We denote by $\mathcal{A}$ the set of *reflecting hyperplanes* of W. If X is a subset of V, then, by Steinberg-Serre Theorem [Bro10, Theorem 4.7], $W(X)$ is generated by reflections and so is generated by the reflections whose reflecting hyperplane contains X: such a subgroup is called a *parabolic subgroup* of W.

If $H \in \mathcal{A}$, then the group $W(H)$ is cyclic (indeed, by semisimplicity, it acts faithfully on V/H which has dimension 1) and we denote its order by e_H. Note that $W_H \setminus \{1\}$ is the set of reflections of W whose reflecting hyperplane is H, so

$$|\operatorname{Ref}(W)| = \sum_{H \in \mathcal{A}} (e_H - 1). \tag{3.4}$$

We denote by α_H an element of V^* such that $H = \operatorname{Ker}(\alpha_H)$. In particular, if all the reflections have order 2, then $|\operatorname{Ref}(W)| = |\mathcal{A}|$. Finally, note the following equality [Bro10, Remark 4.48]

$$|\mathcal{A}| = \sum_{i=1}^{n} (d_i^* + 1). \tag{3.5}$$

If Ω is a W-orbit in $\mathcal{A}$, then we denote by e_Ω the common value of the e_H's for $H \in \Omega$. We then set

$$J_\Omega = \prod_{H \in \Omega} \alpha_H.$$

Then there exists a unique polynomial $P_{\mathbf{f},\Omega}$ in variables $x_1,\dots,x_n$ of respective weights $d_1,\dots,d_n$ such that

$$J_\Omega^{e_\Omega} = P_{\mathbf{f},\Omega}(f_1,\dots,f_n). \tag{3.6}$$

Note that $P_{\mathbf{f},\Omega}$ is homogeneous of degree $e_\Omega|\Omega|$. Then (see [Sta77, Theorem 2.3 and Corollary 4.3] or [LT09, Theorem 9.19 and Corollary 9.21])

$$\mathbb{C}[V]^{W'} = \mathbb{C}[f_1,\dots,f_n,(J_\Omega)_{\Omega \in \mathcal{A}/W}] \tag{3.7}$$

and a presentation of $\mathbb{C}[V]^{W'}$ is given by the relations (3.6). Consequently:

Proposition 3.8. *Let $\Omega_1,\dots,\Omega_r$ denote the W-orbits in $\mathcal{A}$. Then the map*

$$\begin{array}{rccc} \pi'_{\mathbf{f}}: & \mathbb{P}(V) & \longrightarrow & \mathbb{P}(d_1,d_2,\dots,d_n,|\Omega_1|,\dots,|\Omega_r|) \\ & [v] & \longmapsto & [f_1(v):f_2(v):\cdots:f_n(v):J_{\Omega_1}(v):\cdots:J_{\Omega_r}(v)] \end{array}$$

is well-defined and induces an isomorphism

$$\mathbb{P}(V)/W' \xrightarrow{\sim} \{[x_1:\cdots:x_n:j_1:\cdots:j_r] \in \mathbb{P}(d_1,\dots,d_n,|\Omega_1|,\dots,|\Omega_r|) \mid \forall\ 1 \le k \le r,\ j_k^{e_{\Omega_k}} = P_{\mathbf{f},\Omega_k}(x_1,\dots,x_n)\}.$$

Moreover, $\pi'_{\mathbf{f}}$ induces by restriction an isomorphism

$$\mathcal{Z}(f_1)/W' \xrightarrow{\sim} \{[x_2:\cdots:x_n:j_1:\cdots:j_r] \in \mathbb{P}(d_2,\dots,d_n,|\Omega_1|,\dots,|\Omega_r|) \mid$$

$$\forall\ 1 \le k \le r,\ j_k^{e_{\Omega_k}} = P_{\mathbf{f},\Omega_k}(0, x_2, \dots, x_n)\}.$$

Note that W/W^{SL} is cyclic but W^{SL} is not necessarily equal to the derived subgroup W' of W. We have $W' = W^{\mathrm{SL}}$ if and only if $|\mathcal{A}/W| = 1$. Now, let

$$J = \prod_{H \in \mathcal{A}} \alpha_H = \prod_{\Omega \in \mathcal{A}/W} J_\Omega \in \mathbb{C}[V].$$

It is well-defined up to a scalar and homogeneous of degree $|\mathcal{A}|$. It is the generator of the ideal of the reduced subscheme of the ramification locus of the morphism $V \to V/W$. Then by [Bro10, Remark 3.10 and Proposition 4.4]

$$(3.9) \qquad {}^wJ = \det(w)^{-1} J$$

for all $w \in W$. In particular $J \in \mathbb{C}[V]^{W^{\mathrm{SL}}}$ and $J^{|W/W^{\mathrm{SL}}|} \in \mathbb{C}[V]^W$. So there exists a unique polynomial $P_\mathbf{f} \in \mathbb{C}[X_1, \dots, X_n]$, which is homogeneous of degree $|W/W^{\mathrm{SL}}| \cdot |\mathcal{A}|$ if we assign to X_i the degree d_i, and such that

$$(3.10) \qquad J^{|W/W^{\mathrm{SL}}|} = P_\mathbf{f}(f_1, \dots, f_n).$$

Proposition 3.11. *Assume that the map $H \mapsto e_H$ is constant on $\mathcal{A}$ (and let e denote this constant value, which coincides with $|W/W^{\mathrm{SL}}|$). Then $\mathbb{C}[V]^{W^{\mathrm{SL}}} = \mathbb{C}[f_1, f_2, \dots, f_n, J]$ and a presentation is given by the single equation (3.10).*

So the map

$$\begin{array}{rccc} \pi_\mathbf{f}^{\mathrm{SL}} : & \mathbb{P}(V) & \longrightarrow & \mathbb{P}(d_1, d_2, \dots, d_n, |\mathcal{A}|) \\ & [v] & \longmapsto & [f_1(v) : f_2(v) : \cdots : f_n(v) : J(v)] \end{array}$$

is well-defined and induces an isomorphism

$$\mathbb{P}(V)/W^{\mathrm{SL}} \xrightarrow{\sim} \{[x_1 : \cdots : x_n : j] \in \mathbb{P}(d_1, \dots, d_n, |\mathcal{A}|) \mid j^e = P_\mathbf{f}(x_1, \dots, x_n)\}.$$

Moreover, $\pi_\mathbf{f}^{\mathrm{SL}}$ induces by restriction an isomorphism

$$\mathcal{Z}(f_1)/W^{\mathrm{SL}} \xrightarrow{\sim} \{[x_2 : \cdots : x_n : j] \in \mathbb{P}(d_2, \dots, d_n, |\mathcal{A}|) \mid j^e = P_\mathbf{f}(0, x_2, \dots, x_n)\}.$$

3.3. Eigenspaces, Springer theory

We now recall the basics of Springer and Lehrer-Springer theory: all the results stated in this subsection can be found in [Spr74], [LS99a], [LS99b]. Note that some of the proofs have been simplified in [LM03]. Let us fix now a natural number e. We set

$$d(e) = \{1 \le k \le n \mid e \text{ divides } d_k\},$$

$$d^*(e) = \{1 \le k \le n \mid e \text{ divides } d_k^*\},$$

$$\delta(e) = |d(e)| \qquad \text{and} \qquad \delta^*(e) = |d^*(e)|.$$

With this notation, we have

$$(3.12) \qquad \delta(e) = \max_{w \in W}\Big(\dim V(w, \zeta_e)\Big).$$

In particular, ζ_e is an eigenvalue of some element of W if and only if $\delta(e) \neq 0$ that is, if and only if e divides some degree of W. In this case, we fix an element w_e of W such that

$$\dim V(w_e, \zeta_e) = \delta(e).$$

We set for simplification $V(e) = V(w_e, \zeta_e)$ and $W(e) = W_{V(e)}/W(V(e))$: this subquotient of W acts faithfully on $V(e)$.

If $f \in \mathbb{C}[V]$, we denote by $f^{[e]}$ its restriction to $V(e)$. Note that if $i \not\in d(e)$, then $f_i^{[e]} = 0$ by Lemma 2.2. Let us recall here the results of Springer and Lehrer-Springer we will need:

Theorem 3.13 (Springer, Lehrer-Springer). *Assume that $\delta(e) \neq 0$. Then:*

(a) *If $w \in W$, then there exists $x \in W$ such that $x(V(w, \zeta_e)) \subset V(e)$.*

(b) *$W(e)$ acts (faithfully) on $V(e)$ as a group generated by reflections.*

(c) *The family $(f_k^{[e]})_{k \in d(e)}$ is a family of fundamental invaraints of $W(e)$. In particular, the list of degrees of $W(e)$ consists of the degrees of W which are divisible by e (i.e. $\operatorname{Deg}(W(e)) = (d_k)_{k \in d(e)}$).*

(d) *We have*

$$\bigcup_{w \in W} V(w, \zeta_e) = \bigcup_{x \in W} x(V(e)) = \{v \in V \mid \forall\ k \in \{1, 2, \ldots, n\} \setminus d(e),\ f_k(v) = 0\}.$$

(e) *$\delta^*(e) \geq \delta(e)$ with equality if and only if $W(V(w_e, \zeta_e)) = 1$.*

(f) *If $\delta^*(e) = \delta(e)$, then $W(e) = W_{V(e)} = C_W(w_e)$ and the family of eigenvalues (with multiplicity) of w_e is equal to $(\zeta_e^{1-d_k})_{1 \leq k \leq n}$. Moreover, if w is such that $\dim V(w, \zeta_e) = \delta(e)$, then w is conjugate to w_e.*

Remark 3.14. Let $k \in \{1, 2, \ldots, n\}$ be such that $\delta(d_k) = \delta^*(d_k) = 1$. Then $z_k = V(d_k)$ is a line in V, so we can view it as an element of $\mathbb{P}(V)$. By Theorem 3.13(f), the stabilizer W_{z_k} of z_k acts faithfully on $V(d_k)$, so it is cyclic and contains w_{d_k}. In fact,

$$W_{z_k} = \langle w_{d_k} \rangle.$$

For proving this, let $e = |W_{z_k}|$. We just need to verify that $e = d_k$. But d_k divides e and ζ_e is the eigenvalue of some elements of W. So e divides some d_j by the remark following (3.12). Therefore, d_k divides d_j and so $d_k = d_j$ because $\delta(d_k) = 1$. This proves that $e = d_k$, as desired.

Corollary 3.15. *Assume that $\delta(e) = \delta^*(e) \neq 0$ and let $k_0 \in \{1, 2, \ldots, n\}$ be such that d_{k_0} is divisible by e. Let $v \in V(e) \setminus \{0\}$ and let $z = [v]$.*

(a) *The family of eigenvalues of w_e for its action on the tangent space $\mathrm{T}_z(\mathbb{P}(V))$ is equal to $(\zeta_e^{-d_k})_{k \neq k_0}$.*

(b) *Let $f \in \mathbb{C}[V]^W$ be homogeneous of degree d and assume that $f(v) = 0$. Then:*

(b1) *If $d \not\equiv d_k \mod e$ for all $k \neq k_0$, then $\mathcal{Z}(f)$ is singular at z.*

(b2) *Assume that $\mathcal{Z}(f)$ is smooth at z and let $k_1 \neq k_0$ be such that $d \equiv d_{k_1} \mod e$ (the existence of k_1 is guaranted by* (b1)*). Then the family of eigenvalues of w_e for its action on the tangent space $\mathrm{T}_z(\mathcal{Z}(f))$ is equal to $(\zeta_e^{-d_k})_{k \neq k_0, k_1}$.*

Proof. By permuting if necessary the degrees, we may assume that $k_0 = 1$. Note that $\zeta_e^{1-d_1} = \zeta_e$. Choose a basis $(v_1, \ldots, v_n)$ of V such that $v = v_1$ and $w(v_k) = \zeta_e^{1-d_k} v_k$ for all $k \in \{1, 2, \ldots, n\}$ (see Theorem 3.13(e)).

(a) Identify $\mathbb{P}(V)$ with $\mathbb{P}^{n-1}(\mathbb{C})$ through the choice of this basis. Then the action of w_e is transported to

$$w_e \cdot [x_1 : x_2 : \cdots : x_n] = [\zeta_e x_1 : \zeta_e^{1-d_2} x_2 : \cdots : \zeta_e^{1-d_n} x_n] = [x_1 : \zeta_e^{-d_2} x_2 : \cdots : \zeta_e^{-d_n} x_n].$$

Since $z = [1 : 0 : \cdots : 0]$, this shows (a).

(b) Let us work in the affine chart “$x_1 = 1$”, identified with $\mathbf{A}^{n-1}(\mathbb{C})$ through the coordinates $(x_2, \ldots, x_n)$. The equation of the tangent space $\mathrm{T}_z(\mathcal{Z}(f))$ is given in this chart by

$$\sum_{k=2}^{n} (\partial_{v_k} f)(v) x_k = 0.$$

By (2.1),

$${}^w(\partial_{v_k} f)(v) = \zeta_e^{1-d_k} (\partial_{v_k} f)(v).$$

But

$${}^w(\partial_{v_k} f)(v) = (\partial_{v_k} f)(w^{-1}(v)) = (\partial_{v_k} f)(\zeta_e^{-1} v).$$

As $\partial_{v_k} f$ is homogeneous of degree $d-1$, this implies that

$$\zeta_e^{1-d_k}(\partial_{v_k} f)(v) = \zeta_e^{1-d}(\partial_{v_k} f)(v).$$

Therefore, if $d \not\equiv d_k \mod e$ for all $k \in \{2,\dots,n\}$, we get $(\partial_{v_k} f)(v) = 0$ for all $k \in \{2,\dots,n\}$, and so $\mathcal{Z}(f)$ is singular ar $[v]$. This shows (b1).

Now, if $\mathcal{Z}(f)$ is smooth at z, then there exists $k_1 \in \{2,\dots,n\}$ such that $\partial_{v_{k_1}} f(v) \neq 0$, and in particular $d \equiv d_{k_1} \mod e$. Then there exists $k \in \{2,\dots,n\}$ such that $\partial_{v_k} f(v) \neq 0$. This shows that the action of w_e on the one-dimensional space $\mathrm{T}_z(\mathbb{P}(V))/\mathrm{T}_{[v]}(\mathcal{Z}(f))$ is given by multiplication by $\zeta_e^{-d_{k_1}} = \zeta_e^{-d}$. The proof of (b2) is complete. □

4. Determining singularities

An important step for analyzing the properties of the K3 surfaces constructed in the next section is to determine the singularities of the variety $\mathcal{Z}(f)/\Gamma$ in the cases we are interested in (here, f is a fundamental invariant of W and Γ is a subgroup of W). We provide in this section two different tools that will be used in the sequel to this paper [BS-II], [BS-III], where particular examples will be studied.

4.1. Stabilizers

The singularity of $\mathcal{Z}(f)/\Gamma$ at the Γ-orbit of $z \in \mathcal{Z}(f)$ depends on the singularity of $\mathcal{Z}(f)$ at z and the action of Γ_z on this (eventually trivial) singularity. We investigate here some facts about the stabilizers W_z and their action on the tangent space $\mathrm{T}_z(\mathcal{Z}(f))$.

Let f denote a homogeneous invariant of W, let d denote its degree and let $v \in V \setminus \{0\}$ be such that $f(v) = 0$. We set $z = [v] \in \mathcal{Z}(f) \subset \mathbb{P}(V)$. We denote by $\theta_z : W_z \longrightarrow \mathbb{C}^\times$ the linear character defined by $w(v) = \theta_z(w)v$ for all $w \in W_z$. Then $W_v = \mathrm{Ker}(\theta_z)$ and we denote by $e_z = |\mathrm{Im}(\theta_z)|$. So there exists $w \in W_z$ such that $\theta_z(w) = \zeta_{e_z}$. In other words, $v \in V(w, \zeta_{e_z})$ and so, by Theorem 3.13(a), we may, and we will, assume that $v \in V(w_{e_z}, \zeta_{e_z}) = V(e_z)$. This shows that

$$W_z = W_v \langle w_{e_z} \rangle. \tag{4.1}$$

Recall from §3.2 that W_v is a parabolic subgroup of W and so is generated by reflections. Note the following useful facts:

(a) Let $m = |\mathrm{Z}(W)|$ (recall from (3.2) that $m = \gcd(\mathrm{Deg}(W))$). Since $\boldsymbol{\mu}_m = \mathrm{Z}(W) \subset W_z$, m divides e_z.

(b) If $\delta(e_z) = \delta^*(e_z)$, $f(v) = 0$ and $\mathcal{Z}(f)$ is smooth at z, then the eigenvalues of w_{e_z} on the tangent space $\mathrm{T}_z(\mathcal{Z}(f))$ are given by Corollary 3.15.

(c) Let P be a parabolic subgroup of W of rank $n-2$ and assume that $\mathcal{Z}(f)$ is *smooth*. Then $\dim V^P = 2$ and so $L = \mathbb{P}(V^P)$ is a line in $\mathbb{P}(V)$. Then L intersects $\mathcal{Z}(f)$ transversally by Corollary 2.8, so $|L \cap \mathcal{Z}(f)| = d$ because f has degree d. Moreover,

$$\textit{If } z \in L \cap \mathcal{Z}(f)\textit{, then } W_v = P. \tag{4.2}$$

Indeed, $P \subset W_v$ by construction and, if this inclusion is strict, this means that W_v has rank $n-1$ or n. But it cannot have rank n for otherwise $W_v = W$ and $v = 0$ (which is impossible). And it cannot have rank $n-1$ because Corollary 2.4 would imply that $\mathcal{Z}(f)$ is singular at z, contrarily to the hypothesis. This implies for instance that two smooth points in $L \cap \mathcal{Z}(f)$ are in the same Γ-orbit if and only if they are in the same $\mathrm{N}_\Gamma(P)$-orbit.

Moreover, in this case, we have a P-equivariant isomorphism

$$\mathrm{T}_{[v]}(\mathcal{Z}(f)) \simeq V/V^P \tag{4.3}$$

(see the proof of Corollary 2.8).

W	$\lvert W\rvert$	$\lvert W/\mathrm{Z}(W)\rvert$	$\lvert W'\rvert$	$\mathrm{Deg}(W)$ $\mathrm{Codeg}(W)$
$G(1,1,5) = \mathrm{W}(A_4) \simeq \mathfrak{S}_5$	120	120	60	$2,3,4,5$ $0,1,2,3$
$G(e,e,4), e \geq 2$	$24e^3$	$24e^3/\mathrm{gcd}(e,4)$	$12e^3$	$e,2e,3e,4$ $0,e,2e,3e-4$
$G(2e,e,4), e \geq 1$	$384e^3$	$192e^3/\mathrm{gcd}(e,4)$	$96e^3$	$2e,4e,6e,8$ $0,2e,4e,6e$
$G_{28} = \mathrm{W}(F_4)$	1 152	576	288	$2,6,8,12$ $0,4,6,10$
G_{29}	7 680	1 920	3 840	$4,8,12,20$ $0,8,12,16$
$G_{30} = \mathrm{W}(H_4)$	14 400	7 200	7 200	$2,12,20,30$ $0,10,18,28$
G_{31}	46 080	11 520	23 040	$8,12,20,24$ $0,12,16,28$

Table 1. Irreducible complex reflection groups of rank 4 generated by reflections of order 2

4.2. Singularities of double covers

If $n = 4$, $\Gamma = W^{\mathrm{SL}}$ and W is generated by reflections of order 2, then the surface $\mathcal{Z}(f)/\Gamma$ is the double cover of a weighted projective plane. Most of (but not all) the singularities of $\mathcal{Z}(f)/\Gamma$ may be then analyzed through the singularities of the branch locus of this cover.

So we fix a double cover $\pi : Y \to X$ between two irreducible algebraic surfaces and we assume that Y is normal and X is smooth. By the purity of the branch locus, the branch locus R of π is empty or pure of codimension 1 (i.e. pure of dimension 1). The next well-known fact (see for instance [BPVdV84, Part III, §7]) will help us in our explicit computations:

Proposition 4.4. *Let $y \in Y$ be such that $x = \pi(y)$ belongs to R. We assume that x is an ADE curve singularity of the branch locus R. Then y is an ADE surface singularity of the same type.*

5. Invariant K3 surfaces

Hypothesis. *In this section, and only in this section, we assume that $n = 4$ and that W is irreducible and generated by reflections of order 2.*

5.1. Classification

We provide in Table 1 the list of irreducible complex reflection groups W of rank 4 which are generated by reflections of order 2 together with the following informations: the order of W, the order of $W/\mathrm{Z}(W)$ (which is the group that acts faithfully on $\mathbb{P}(V)$), the order of W' and the lists of degrees and codegrees. We also recall their notation in Shephard-Todd classification [ST54] as well as their Coxeter name whenever they are real.

Recall that $G(2,1,4) = \mathrm{W}(B_4)$ and $G(2,2,4) = \mathrm{W}(D_4)$. Note that the hypothesis on the order of the reflections implies that

(5.2) $$|W^{\mathrm{SL}}| = \frac{|W|}{2}.$$

In particular, $W' \neq W^{\mathrm{SL}}$ if and only if $W = G_{28}$ or $W = G(2e,e,4)$ for some $e \geq 1$. Also, note the following diagram of non-trivial inclusions between those of the complex reflection groups which are contained in a primitive one (here, $H \overset{\triangleleft}{\hookrightarrow} G$ means that H is a normal subgroup of G).

(5.3)
$$\begin{tikzcd}
& G(2,1,4) \arrow[r, hook] \arrow[dr, hook] & G_{29} \arrow[dr, hook] & \\
G(2,2,4) \arrow[ur, hook, "\triangleleft"] \arrow[rr, hook, "\triangleleft"] \arrow[dr, hook] & & G_{28} \arrow[r, hook] & G_{31} \\
& G(4,4,4) \arrow[r, hook, "\triangleleft"] & G(4,2,4) \arrow[ur, hook] &
\end{tikzcd}$$

5.2. K3 surfaces

Equations of surfaces of the form $\mathcal{Z}(f)/W'$ or $\mathcal{Z}(f)/W^{\mathrm{SL}}$ (where f is a fundamental invariant of degree d) in a weighted projective space are provided by Propositions 3.8 and 3.11. Whenever some arithmetic conditions on d and the degrees of W are satisfied, it can then be proven thanks to results of Appendix A (and particularly Corollary A.3) that the canonical sheaf of $\mathcal{Z}(f)/W'$ or $\mathcal{Z}(f)/W^{\mathrm{SL}}$ is trivial (provided that $\mathcal{Z}(f)$ is normal, so that the quotient is also normal and the canonical sheaf is well-defined): it turns out that, in most cases, the quotient $\mathcal{Z}(f)/W'$ or $\mathcal{Z}(f)/W^{\mathrm{SL}}$ has only ADE singularities and positive Euler characteristic so that their minimal resolution are K3 surfaces. A particular feature of these examples is that their minimal resolution have always a big Picard number, as big as possible compare to the number of moduli of the family. Note that some of these examples were already studied by Barth and the second author [BS03]: we revisit these cases and simplify some arguments using more theory about complex reflection groups.

We denote by $\mathcal{K}_3$ the set of pairs (W,d) where W is an irreducible complex reflection group of rank 4 and d is a positive integer satisfying one of the following conditions:

- $W = G(1,1,5)$, $G(4,2,4)$ or G_{29}, and $d = 4$.
- $W = G(2e,2e,4)$ with e odd, and $d \in \{4e, 6e\}$.
- $W = G(4e,4e,4)$, and $d = 4e$.
- $W = G(2,1,4)$, and $d \in \{4,6\}$.
- $W = G_{28}$, and $d \in \{6,8\}$.
- $W = G_{30}$, and $d = 12$.
- $W = G_{31}$, and $d = 20$.

Theorem 5.4. *Assume that $(W,d) \in \mathcal{K}_3$. Let Γ be the subgroup W' or W^{SL} of W and let f be a fundamental invariant of W of degree d such that $\mathcal{Z}(f)$ has only ADE singularities.*

Then $\mathcal{Z}(f)/\Gamma$ is a K3 surface with ADE singularities.

The proof of Theorem 5.4 will be given in §5.5 and Section 6. As an immediate consequence, we get:

Corollary 5.5. *Under the hypotheses of Theorem 5.4, the minimal resolution of $\mathcal{Z}(f)/\Gamma$ is a smooth projective K3 surface.*

5.3. Numerical informations

Before proving this Theorem 5.4, let us make some remarks. By Propositions 3.8 and 3.11, the variety $\mathcal{Z}(f)/\Gamma$ is a weighted complete intersection (see [Dol82, §3.2] for the definition) in a weighted projective space (it is defined by one or two equations). If $\Gamma = W^{\mathrm{SL}}$, then $\mathcal{Z}(f)/\Gamma$ is a weighted hypersurface in a weighted projective space of dimension 3 (see Proposition 3.11). If $\Gamma = W'$, then $\mathcal{Z}(f)/\Gamma$ is a codimension 2 weighted complete intersection in a weighted projective space of dimension 4 (see Proposition 3.8). We give in Table 2 the list of the weights of the ambient projective space as well as the list of the degrees of the equations in all the different cases (we also give the description of $\mathcal{Z}(f)/W$ as a weighted projective space).

By looking at this Table 2, the reader might think that we have build infinitely many families of K3 surfaces, by letting the integer e vary in the fourth and fifth group considered. However, as it will be explained in §5.4 (see the isomorphisms (5.12), (5.13) and (5.15)), the general group with parameter e and the particular group for $e = 1$ give exactly the same families of surfaces.

Also, it turns out that $G(2,2,4)' = G(2,1,4)'$, and since invariants of $G(2,1,4) = \mathrm{W}(B_4)$ are invariants for $G(2,2,4) = \mathrm{W}(D_4)$, this shows that two of the families of K3 surfaces constructed with $G(2,1,4)$ are contained in families built from $G(2,2,4)$: note however that, for these particular examples, having the two points of view give different embeddings in weighted projective spaces.

As a consequence, we have build 15 families of K3 surfaces (note that the families corresponding to the groups $G(4,2,4)$ and G_{29} are 0-dimensional, as there is, up to scalar, a single quartic polynomial invariant by each of these groups). If we exclude the “easy” case of the quotient of a quartic by a finite subgroup of $\mathbf{SL}_4(\mathbb{C})$ (see §6.1), it remains 8 non-zero dimensional families of K3 surfaces whose construction is non-trivial.

Hypothesis and notation. *From now on, and until the end of this paper, we assume that $(W,d) \in \mathcal{K}_3$, that Γ is the subgroup W' or W^{SL} of W, and that f is a fundamental invariant of W of degree d such that $\mathcal{Z}(f)$ has only ADE singularities. We also fix a family $\mathbf{f} = (f, f_1, f_2, f_3)$ of fundamental invariants containing f and we set $d_i = \deg(f_i)$.*

Proof of the results given in Table 2. The proof follows from Table 1 and a case-by-case analysis. We will not give details for all cases, we will only treat two cases (the reader can easily check that all other cases can be treated similarly).

• Assume that $(W,d) = (G_{31}, 20)$ and that $\Gamma = W'$ $(= W^{\mathrm{SL}})$. Then $(d_1,d_2,d_3) = (8,12,24)$ by Table 1 and $|\mathcal{A}| = 60$ by (3.1) and (3.4). It then follows from Proposition 3.11 that

$$\mathcal{Z}(f)/\Gamma = \{[x_1 : x_2 : x_3 : j] \in \mathbb{P}(8,12,24,60) \mid j^2 = P_{\mathbf{f}}(0,x_1,x_2,x_3)\}.$$

But $\mathbb{P}(8,12,24,60) = \mathbb{P}(2,3,6,15) = \mathbb{P}(2,1,2,5)$, so that

$$\mathcal{Z}(f)/\Gamma = \{[y_1 : y_2 : y_3 : j] \in \mathbb{P}(2,1,2,5) \mid j^2 = Q(y_1,y_2,y_3)\}$$

for some polynomial $Q \in \mathbb{C}[Y_1,Y_2,Y_3]$. Hence $\mathcal{Z}(f)/\Gamma$ is defined by an equation of degree 10 in $\mathbb{P}(2,1,2,5)$, as expected.

Finally, by Proposition 3.11, $\mathcal{Z}(f)/W \simeq \mathbb{P}(8,12,24) = \mathbb{P}(2,3,6) \simeq \mathbb{P}(2,1,2) \simeq \mathbb{P}(1,1,1)$, as expected.

• Assume that $(W,d) = (G_{28}, 6)$ and $\Gamma = W'$. Then $(d_1,d_2,d_3) = (2,8,12)$ by Table 1. Note that $W' \neq W^{\mathrm{SL}}$ and that there are two W-orbits Ω_1 and Ω_2 of reflecting hyperplanes, which are both of cardinality 12. It then follows from Proposition 3.8, that

$$\begin{aligned}\mathcal{Z}(f)/\Gamma = \{[x_1 : x_2 : x_3 : j_1 : j_2] \in \mathbb{P}(2,8,12,12,12) \mid j_1^2 = P_{\mathbf{f},\Omega_1}(0,x_1,x_2,x_3)& \\ \text{and } j_2^2 = P_{\mathbf{f},\Omega_2}(0,x_1,x_2,x_3)\}&.\end{aligned}$$

But $\mathbb{P}(2,8,12,12,12) = \mathbb{P}(1,4,6,6,6) = \mathbb{P}(1,2,3,3,3)$, so that

$$\mathcal{Z}(f)/\Gamma = \{[y_1 : y_2 : y_3 : j_1 : j_2] \in \mathbb{P}(1,2,3,3,3) \mid j_1^2 = Q_1(y_1,y_2,y_3) \text{ and } j_2^2 = Q_2(y_1,y_3,y_4)\}$$

for some polynomials Q_1 and Q_2 in $\mathbb{C}[Y_1,Y_2,Y_3]$. So $\mathcal{Z}(f)/\Gamma$ is defined by two equations of degree 6 in $\mathbb{P}(1,2,3,3,3)$, as expected.

W	d	Γ	Ambient space	degree(s) of equation(s)	$\mathcal{Z}(f)/W$
$\mathrm{W}(A_4) \simeq \mathfrak{S}_5$	4	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(2,3,5,10)$	20	$\mathbb{P}(2,3,5)$
$G(2,1,4)$ $= \mathrm{W}(B_4)$	4	W'	$\mathbb{P}(1,3,4,6,2)$	12, 4	$\mathbb{P}(1,3,4)$
		W^{SL}	$\mathbb{P}(1,3,4,8)$	16	
	6	W'	$\mathbb{P}(1,1,2,3,1)$	6, 2	$\mathbb{P}(1,1,2)$
		W^{SL}	$\mathbb{P}(1,1,2,4)$	8	
$G(4,2,4)$	4	W'	$\mathbb{P}(2,3,2,1,6)$	2, 12	$\mathbb{P}(1,3,1)$
		W^{SL}	$\mathbb{P}(2,3,2,7)$	14	
$G(2e,2e,4)$ e odd	$4e$	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(1,3,2,6)$	12	$\mathbb{P}(1,3,2)$
	$6e$	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(1,1,1,3)$	6	$\mathbb{P}(1,1,1)$
$G(4e,4e,4)$	$4e$	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(2,3,1,6)$	12	$\mathbb{P}(2,3,1)$
G_{28} $= \mathrm{W}(F_4)$	6	W'	$\mathbb{P}(1,2,3,3,3)$	6, 6	$\mathbb{P}(1,2,3)$
		W^{SL}	$\mathbb{P}(1,2,3,6)$	12	
	8	W'	$\mathbb{P}(1,1,2,2,2)$	4, 4	$\mathbb{P}(1,1,2)$
		W^{SL}	$\mathbb{P}(1,1,2,4)$	8	
G_{29}	4	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(2,3,5,10)$	20	$\mathbb{P}(2,3,5)$
$G_{30} = \mathrm{W}(H_4)$	12	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(1,2,3,6)$	12	$\mathbb{P}(1,2,3)$
G_{31}	20	$W' = W^{\mathrm{SL}}$	$\mathbb{P}(2,1,2,5)$	10	$\mathbb{P}(1,1,1)$

Table 2. Weights of ambient projective spaces and degree(s) of equation(s) of $\mathcal{Z}(f)/\Gamma$

Finally, by Proposition 3.8, $\mathcal{Z}(f)/W \simeq \mathbb{P}(2,8,12) = \mathbb{P}(1,4,6) \simeq \mathbb{P}(1,2,3)$, as expected. □

Remark 5.7. The arithmetic of degrees and the classification of reflection groups imply that it does not seem possible to find a complex reflection W and a degree d of W such that W is not generated by reflections of order 2 and $\mathcal{Z}(f)/\Gamma$ has a trivial canonical sheaf, except whenever $d = 4$. But this is in some sense the less exciting case, as it is shown by the argument given in §6.1 below.

Also, note that if $e \notin \{1,2,4\}$, then $G(e,e,4)$ has a unique invariant of degree 4 that defines a quartic in $\mathbb{P}^3(\mathbb{C})$, but this invariant is equal to $xyzt$, and so $\mathcal{Z}(f)$ is not irreducible and does not fulfill the hypothesis of Theorem 5.4. That is why this case does not appear in the list $\mathcal{K}_3$.

Remark 5.8. If $(W,d) = (G_{28},6)$ or $(G_{30},12)$, and $\Gamma = W'$, then the above result was obtained by Barth and the second author [BS03]: the group Γ was denoted by G_d in their paper (this must not be confused with Shephard-Todd notation).

5.4. About the families attached to $G(2e,2e,4)$

Assume in this subsection, and only in this subsection, that $W = G(2e,2e,4)$ for some e. Recall that $G(2e,2e,4)$ is the group of monomial matrices in $\mathbf{GL}_4(\mathbb{C})$ with coefficients in $\boldsymbol{\mu}_{2e}$ and such that the product of the non-zero coefficients is equal to 1. Note that this implies that $W' = W^{\mathrm{SL}}$.

If $1 \le k \le 4$, we denote by σ_k the j-th elementary symmetric function in the variables x, y, z, t, and let

$$J_1 = (x-y)(x-z)(x-t)(y-z)(y-t)(z-t).$$

If $p \in \mathbb{C}[x,y,z,t]$ and $l \ge 1$ is an integer, we set $p[l] = p(x^l,y^l,z^l,t^l) \in \mathbb{C}[x,y,z,t]$. For instance, $\sigma_1[l] = \Sigma(x^l)$. Then $(\sigma_1, \sigma_1[2], \sigma_1[3], \sigma_4)$ is a family of fundamental invariants of $G(1,1,4) \simeq \mathfrak{S}_4$. So there exists a unique polynomial $P \in \mathbb{C}[x_1,x_2,x_3,x_4]$ such that

$$J_1^2 = P(\sigma_1, \sigma_1[2], \sigma_1[3], \sigma_4). \tag{5.9}$$

We do not need here the explicit form of P. If a, b, $c \in \mathbb{C}$, we set

$$F_{a,b,c} = a\sigma_1[4] + b\sigma_4 + c\sigma_1[2]^2 = a\Sigma(x^4) + bxyzt + c(\Sigma(x^2))^2,$$

$$G_{a,b,c} = \sigma_1[6] + a\sigma_1[4]\sigma_1[2] + b\sigma_1[2]^3 + c\sigma_1[2]\sigma_4.$$

With this notation, we may, and we will, choose as a family of fundamental invariants of W the family $(\sigma_1[2e], \sigma_1[4e], \sigma_1[6e], \sigma_4)$. Note that the element $J \in \mathbb{C}[x,y,z,t]^{W'}$ defined in §3.2 is equal to $J_1[2e]$ (up to a scalar). Applying the endomorphism $p \mapsto p[2e]$ of $\mathbb{C}[x,y,z,t]$ to the formula (5.9) and using Proposition 3.11 gives

$$\mathbb{P}(V)/W' \simeq \{[x_1:x_2:x_3:x_4:j] \in \mathbb{P}(2e,4e,6e,4,12e) | j^2 = P(x_1,x_2,x_3,x_4^{2e})\}, \tag{5.10}$$

because $\sigma_4[2e] = \sigma_4^{2e}$. Let us examine the situation according to the parity of e.

5.4.1. The case where e is odd Assume here that e is odd. Then

$$\mathbb{P}(2e,4e,6e,4,12e) = \mathbb{P}(e,2e,3e,2,6e) \simeq \mathbb{P}(1,2,3,2,6).$$

So it follows from (5.10) that

$$\mathbb{P}(V)/W' \simeq \{[x_1:x_2:x_3:x_4:j] \in \mathbb{P}(1,2,3,2,6) \mid j^2 = P(x_1,x_2,x_3,x_4^2)\}. \tag{5.11}$$

Now, a fundamental invariant of degree $6e$ of W if of the form $G_{a,b,c}[e]$. We deduce from (5.11) that

$$\mathcal{Z}(G_{a,b,c}[e])/W' \simeq \{[x_1:x_2:x_4:j] \in \mathbb{P}(1,2,2,6) \mid j^2 = P(x_1,x_2,-ax_1x_2 - bx_1^3 - cx_1x_4, x_4^2)\}$$

and in particular

$$\mathcal{Z}(G_{a,b,c}[e])/G(2e,2e,4)' = \mathcal{Z}(G_{a,b,c})/G(2,2,4)'. \tag{5.12}$$

Similarly, a fundamental invariant of degree $4e$ of W is of the form $F_{a,b,c}[e]$ with $(a,b) \ne 0$ and

$$\mathcal{Z}(F_{a,b,c}[e])/G(2e,2e,4)' = \mathcal{Z}(F_{a,b,c})/G(2,2,4)'. \tag{5.13}$$

This shows that the varieties $\mathcal{Z}(G_{a,b,c}[e])/G(2e,2e,4)'$ and $\mathcal{Z}(F_{a,b,c}[e])/G(2e,2e,4)'$ do not depend on e.

5.4.2. The case where e is even Assume here that $e = 2e'$ for some $e' \geq 1$. Then

$$\mathbb{P}(2e, 4e, 6e, 4, 12e) = \mathbb{P}(e', 2e', 3e', 1, 6e') = \mathbb{P}(1, 2, 3, 1, 6).$$

So it follows from (5.10) that

$$\mathbb{P}(V)/W' \simeq \{[x_1 : x_2 : x_3 : x_4 : j] \in \mathbb{P}(1,2,3,1,6) \mid j^2 = P(x_1, x_2, x_3, x_4^4)\}. \tag{5.14}$$

Now, a fundamental invariant of $G(4e', 4e', 4)$ of degree $4e'$ is of the form $F_{a,b,c}[e']$ and a similar argument as before shows that

$$\mathcal{Z}(F_{a,b,c}[e'])/G(4e', 4e', 4)' = \mathcal{Z}(F_{a,b,c})/G(4, 4, 4)'. \tag{5.15}$$

Again, the variety $\mathcal{Z}(F_{a,b,c}[e'])/G(4e', 4e', 4)'$ does not depend on e'.

5.4.3. Complements Note for future reference (see §6.3) the following fact:

Lemma 5.16. *If $\mathcal{Z}(F_{a,b,c})$ is irreducible, then it is smooth or has only A_1 singularities.*

Proof. Assume that $\mathcal{Z}(F_{a,b,c})$ is irreducible and singular. Let us first assume that $a = 0$. Then we may assume that $b = 1$ and the irreducibility of $\mathcal{Z}(F_{0,1,c})$ forces $c \neq 0$. An easy computation then shows that the only singular points of $\mathcal{Z}(F_{0,1,c})$ are the ones belonging to the $G(2, 2, 4)$-orbit of $p = [0 : 0 : i : 1]$. But the homogeneous component of degree 2 of $F_{0,1,c}(x, y, i + z, 1)$ is $ixy - 4z^2$, which is a non-degenerate quadratic form in x, y, z. So p is an A_1 singularity of $\mathcal{Z}(F)$, as expected.

Le us now assume that $a \neq 0$, and even that $a = 1$. Assume that we have found $(b_0, c_0) \in \mathbb{C}^2$ such that $\mathcal{Z}(F_{1,b_0,c_0})$ admits a singular point $q = [x_0 : y_0 : z_0 : t_0]$ which is not an A_1 singularity. Then, by permuting the coordinates if necessary, we may assume that $t_0 \neq 0$. So let us work in the affine chart $t \neq 0$ and set $F^\circ_{b,c} = F_{1,b,c}(x, y, z, 1)$. Let $H_{b,c}$ denote the Hessian matrix of $F^\circ_{b,c}$. Then $(x_0, y_0, z_0, b_0, c_0)$ belongs to the variety

$$\mathcal{X} = \{(x, y, z, b, c) \in \mathbb{A}^5(\mathbb{C}) \mid F^\circ_{b,c}(x, y, z) = \frac{\partial F^\circ_{b,c}}{\partial x}(x, y, z) = \frac{\partial F^\circ_{b,c}}{\partial y}(x, y, z) = \frac{\partial F^\circ_{b,c}}{\partial z}(x, y, z) = \det(H_{b,c}(x, y, z)) = 0\}.$$

Now let $\pi : \mathbb{A}^5(\mathbb{C}) \longrightarrow \mathbb{A}^2(\mathbb{C})$, $(x, y, z, \beta, \gamma) \mapsto (\beta, \gamma)$. A Magma [BCP97] computation:

```
> A5<x,y,z,b,c>:=AffineSpace(Rationals(),5);
> A2<b0,c0>:=AffineSpace(Rationals(),2);
> pi:=map<A5 -> A2 | [b,c]>;
> Fbco:=(x^4+y^4+z^4+1)+b*x*y*z+c*(x^2+y^2+z^2+1)^2;
> Hbc:=Matrix(CoordinateRing(A5),3,3,
>   [[Derivative(Derivative(Fbco,k),l) :
>     k in [1..3]] : l in [1..3]]);
> X:=Scheme(A5,[Fbco] cat [Derivative(Fbco,k) : k in [1,2,3]]
>   cat [Determinant(Hbc)]);
> MinimalBasis(ReducedSubscheme(pi(X)));
[
    c0 + 1/2,
    b0^2 - 16
]
```

shows that $\pi(\mathcal{X}) = \{(4, -1/2), (-4, -1/2)\}$. This implies that $(b_0, c_0) \in \{(4, -1/2), (-4, -1/2)\}$. But $F_{4,-1/2}$ and $F_{-4,-1/2}$ are not irreducible (they are divisible by $x - y - z + t$ and $x + y + z - t$ respectively): this contradicts the hypothesis. □

Remark 5.17. Observe that the previous family $\mathcal{Z}(F_{a,b,c})$ with $(a,b) = (1,0)$ was studied in [Sar04], where it is shown that the family contains exactly four singular surfaces with $4, 8, 12, 16$ A_1–singularities. In particular the surface with 16 A_1–singularities is a Kummer K3 surface.

5.5. Proof of Theorem 5.4

The rest of this paper is devoted to the proof of this Theorem 5.4. Note that it proceeds by a case-by-case analysis, but this case-by-case analysis is widely simplified by the general facts about complex reflection groups recalled in the previous sections.

Proof of Theorem 5.4. Assume that the hypotheses of Theorem 5.4 are satisfied. For proving that $\mathcal{Z}(f)/\Gamma$ is a K3 surface with ADE singularities, we need to show the following facts:

(S) The surface $\mathcal{Z}(f)/\Gamma$ has only ADE singularities.

(E) The Euler characteristic of $\mathcal{Z}(f)/\Gamma$ is positive.

(C) The canonical divisor of $\mathcal{Z}(f)/\Gamma$ is trivial.

Indeed, if $\tilde{X}$ denotes the minimal resolution of $\mathcal{Z}(f)/\Gamma$ and if (S), (E) and (C) are proved, then $\tilde{X}$ has a trivial canonical sheaf by (S) and (C), so by the classification of smooth algebraic surfaces $\tilde{X}$ is a K3 surface or an abelian surface. But, by (S), the Euler characteristic of $\tilde{X}$ is greater than or equal to the one of $\mathcal{Z}(f)/\Gamma$, so is also positive by (E). Since the Euler characteristic of an abelian surface is 0, we deduce that $\tilde{X}$ is a smooth K3 surface.

The technical step is to prove (S), namely that $\mathcal{Z}(f)/\Gamma$ has only ADE singularities. This will be postponed to the next Section 6. So assume here that (S) is proved.

Let us now prove the statement (E), namely that the Euler characteristic of $\mathcal{Z}(f)/\Gamma$ is positive. Since $\mathcal{Z}(f)$ has only isolated singularities by (S), it follows from [Dim92, Theorem 4.3] that $\mathrm{H}^1(\mathcal{Z}(f),\mathbb{C}) = 0$. Since $\mathcal{Z}(f)$ has only ADE singularities, it is rationally smooth [KL79, Definition A1]. As it is also projective, one can apply Poincaré duality and so $\mathrm{H}^3(\mathcal{Z}(f),\mathbb{C})$ is the dual of $\mathrm{H}^1(\mathcal{Z}(f),\mathbb{C})$, hence is equal to 0. So $\mathcal{Z}(f)$ has no odd cohomology and since $\mathrm{H}^j(\mathcal{Z}(f)/\Gamma,\mathbb{C}) = \mathrm{H}^j(\mathcal{Z}(f),\mathbb{C})^\Gamma$, this shows that $\mathcal{Z}(f)/\Gamma$ has no odd cohomology. So its Euler characteristic is positive.

Now it remains to prove (C), namely that the canonical sheaf of $X = \mathcal{Z}(f)/\Gamma$ is trivial. For this, we use Corollary A.3, so we need to prove that X satisfies the hypotheses (H1), (H2), (H3), (H4) and (H5) of Appendix A. Statements (H1), (H2) and (H4) are easily checked thanks to Table 2 while (H5) follows from (S). So it remains to prove (H3), namely that X is a well-formed weighted complete intersection. There are two cases:

- If $\Gamma = W^{\mathrm{SL}}$, then X is a weighted hypersurface of degree e in some $\mathbb{P}(l_0,l_1,l_2,l_3)$, and, according to [IF00, §6.10], X is well-formed if, for all $0 \le a < b \le 3$, $\gcd(l_a,l_b)$ divides e. This is easily checked with Table 2.
- If $\Gamma = W' \neq W^{\mathrm{SL}}$, then X is a weighted complete intersection defined by two equations of degree e_1 and e_2 in some $\mathbb{P}(l_0,l_1,l_2,l_3,l_4)$, and, according to [IF00, §6.11], X is well-formed if the following two properties are satisfied:
  - For all $0 \le a < b \le 4$, $\gcd(l_a,l_b)$ divides e_1 or e_2.
  - For all $0 \le a < b < c \le 4$, $\gcd(l_a,l_b,l_c)$ divides e_1 and e_2.

  Again, this is easily checked with Table 2.

The proof of Theorem 5.4 is complete, up to the proof of (S). □

6. Singularities of $\mathcal{Z}(f)/\Gamma$

The aim of this section is to complete the proof of Theorem 5.4, by proving that, under its hypotheses, the surface $\mathcal{Z}(f)/\Gamma$ has only ADE singularities. The first subsection is devoted to the (easy) case where $d = 4$ while the third subsection deals with the case where $(W, d) = (G(2e, 2e, 4), 4e)$ for some odd e by reduction to the $d = 4$ case. The other cases are treated in the second and fourth section.

6.1. The case $d = 4$

Assume in this subsection, and only in this subsection, that $d = 4$. This case is somewhat particular and requires its own treatment. It is also well known in the literature, but we recall the discussion for convenience of the reader. First, note that the hypothesis implies that $\mathcal{Z}(f)$ is already a K3 surface (with eventually ADE singularities) and we denote by ω a non-degenerate global holomorphic 2-form on the smooth locus (it is well-defined up to a scalar). By hypothesis, $\Gamma \subset \mathbf{SL}_4(\mathbb{C})$, so Γ preserves ω. So $\mathcal{Z}(f)/\Gamma$ inherits a non-degenerate global holomorphic 2-form ω_Γ on its smooth locus.

Now, let $p : X \to \mathcal{Z}(f)$ denote a minimal resolution of $\mathcal{Z}(f)$, and let ω_X denote the unique non-degenerate 2-form on X extending ω. Then X is a K3 surface which inherits an action of Γ which stabilizes ω: so this is a symplectic action and so X/Γ is a K3 surface with ADE singularities [Nik76, §5]. Let $q : Y \to X/\Gamma$ denote a minimal resolution. We have a commutative diagram

$$\begin{tikzcd} & X \arrow[r, "p"] \arrow[d] & \mathcal{Z}(f) \arrow[d] \\ Y \arrow[r, "q"] & X/\Gamma \arrow[r, "p_\Gamma"] & \mathcal{Z}(f)/\Gamma, \end{tikzcd}$$

where p_Γ is induced by p. This shows that $p_\Gamma \circ q : Y \to \mathcal{Z}(f)/\Gamma$ is a symplectic resolution, so $\mathcal{Z}(f)/\Gamma$ is a K3 surface with ADE singularities (and Y is its minimal resolution). This completes the proof of Theorem 5.4 whenever $d = 4$.

6.2. The case where $\mathcal{Z}(f)/W$ is smooth

Assume in this subsection, and only in this subsection, that $\mathcal{Z}(f)/W$ is smooth. By examining Table 2, this occurs only if $(W, d) = (G(2e, 2e, 4), 6e)$ or $(G_{31}, 20)$. In both cases, $\Gamma = W' = W^{\mathrm{SL}}$ is of index 2 in W, and so the result follows from Corollary B.7.

6.3. The case where $W = G(2e, 2e, 4)$

Assume in this subsection, and only in this subsection, that $W = G(2e, 2e, 4)$. The case where e is odd and $d = 6e$ is treated in the previous subsection 6.2. If e is odd and $d = 4e$, then it follows from the isomorphism (5.13) and Lemma 5.16 that we may assume that $e = 1$. Then $d = 4$ and this case is treated in §6.1. If $e = 2e'$ is even and $d = 4e'$, then it follows from the isomorphism (5.13) and Lemma 5.16 that we may assume that $e' = 1$. Then $d = 4$ and this case is treated in §6.1.

6.4. Remaining cases

According to the cases treated in §6.1, §6.2 and §6.3, we may now work under the following hypothesis:

Hypothesis. *From now on, and until the end of this section, we assume that $d \neq 4$ and $W \in \{G(2, 1, 4), G_{28}, G_{30}\}$.*

6.4.1. The case where $\Gamma = W^{\mathrm{SL}}$ Assume in this subsection, and only in this subsection, that $\Gamma = W^{\mathrm{SL}}$. In this case, Γ is a subgroup of index 2 of W. Recall from Propositions 3.3 and 3.11 that

$$\mathcal{Z}(f)/W \simeq \mathbb{P}(d_1, d_2, d_3) \tag{6.1}$$

and

$$\mathcal{Z}(f)/W^{\mathrm{SL}} \simeq \{[x_1 : x_2 : x_3 : j] \in \mathbb{P}(d_1, d_2, d_3, |\mathcal{A}|) \mid j^2 = P_{\mathbf{f}}(0, x_1, x_2, x_3)\}. \tag{6.2}$$

We denote by $\rho : \mathcal{Z}(f)/W^{\mathrm{SL}} \longrightarrow \mathcal{Z}(f)/W$ the canonical map, let $\mathcal{U}$ denote the smooth locus of $\mathbb{P}(d_1, d_2, d_3)$ and let $\mathcal{S}$ denote the set of singular points of $\mathbb{P}(d_1, d_2, d_3)$.

Now, $\mathcal{U} = \pi_{\mathbf{f}}^{-1}(\mathcal{U})/W$ is smooth so $\rho^{-1}(\mathcal{U}) = \pi_{\mathbf{f}}^{-1}(\mathcal{U})/\Gamma$ contains only ADE singularities by Corollary B.7 (because $\Gamma = W^{\mathrm{SL}}$ has index 2 in W). Hence, it remains to show that the points in $\rho^{-1}(\mathcal{S})$ are smooth or ADE singularities. Let $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and $p_3 = [0 : 0 : 1]$ in $\mathbb{P}(d_1, d_2, d_3)$. Then $\mathcal{S} \subset \{p_1, p_2, p_3\}$. The following fact is checked by a case-by-case analysis:

Lemma 6.3. *Assume that $d \neq 4$ and $W \in \{G(2, 1, 4), G_{28}, G_{30}\}$. If $p_k \in \mathbb{P}(d_1, d_2, d_3)$ is singular, then:*

(a) $\delta(d_k) = \delta^*(d_k) = 1$.

(b) $\det(w_{d_k}) = 1$.

(c) *p_k is an A_j singularity of $\mathbb{P}(d_1, d_2, d_3)$ for some $j \geq 1$.*

The proof will be given below. Let us first explain why this lemma might help to check that the points in $\rho^{-1}(\mathcal{S})$ are smooth or ADE singularities. So let $p_k \in \mathcal{S}$ and let $\Omega_k = \pi_{\mathbf{f}}^{-1}(p_k)$. Then

$$\Omega_k = \{p \in \mathbb{P}(V) \mid \forall\ 1 \leq j \neq k \leq 3,\ f_j(p) = f_k(p) = 0\}.$$

By Lemma 6.3, $\dim V(d_k) = 1$, so we might view $V(d_k) \in \mathbb{P}(V)$ as a point $z_k \in \mathcal{Z}(f)$. We denote by z_k^{SL} the image of z_k in $\mathcal{Z}(f)/\Gamma$. By Theorem 3.13(d), we have that Ω_k is the W-orbit of z_k. But the stabilizer of z_k in W is $\langle w_{d_k} \rangle$ by Remark 3.14, so it is contained in Γ by Lemma 6.3. So the map ρ is étale at z_k^{SL}, and so the singularity of $\mathcal{Z}(f)/\Gamma$ at z_k^{SL} is equivalent to the singularity of $\mathbb{P}(d_1, d_2, d_3)$ at p_k, hence is an A_j singularity by Lemma 6.3. This completes the proof of Theorem 5.4 whenever $\Gamma = W^{\mathrm{SL}}$ and $(W, d) \neq (G(2e, 2e, 4), 4e)$, provided that Lemma 6.3 is proved. This is done just below:

Proof of Lemma 6.3. Let us examine the different cases:

- *Type* $G(2, 1, 4)$. Assume here that $W = G(2, 1, 4)$. Then $d = 6$ and $(d_1, d_2, d_3) = (2, 4, 8)$. But $\mathbb{P}(2, 4, 8) \simeq \mathbb{P}(1, 2, 4) \simeq \mathbb{P}(1, 1, 2)$, so $\mathcal{S} = \{p_3\}$ and p_3 is an A_1 singularity. Moreover, $d_3 = 8$ and it follows from Table 1 that $\delta(8) = \delta^*(8) = 1$. Also, Theorem 3.13(f) implies that the eigenvalues of w_8 are $(\zeta_8^{-5}, \zeta_8^{-1}, \zeta_8^{-3}, \zeta_8)$, so $\det(w_8) = 1$.

- *Type* G_{28}. Assume here that $W = G_{28}$. If $d = 6$, then $(d_1, d_2, d_3) = (2, 8, 12)$: but in that case $\mathbb{P}(2, 8, 12) = \mathbb{P}(1, 4, 6) = \mathbb{P}(1, 2, 3)$ so $\mathcal{S} = \{p_2, p_3\}$ and so $d_k \in \{8, 12\}$ (and note that p_2 is an A_1 singularity, while p_3 is an A_2 singularity). If $d = 8$, then $(d_1, d_2, d_3) = (2, 6, 12)$ and we have $\mathbb{P}(2, 6, 12) = \mathbb{P}(1, 3, 6) = \mathbb{P}(1, 1, 2)$ so $\mathcal{S} = \{p_3\}$ and so $d_k = 12$ (and note that p_3 is an A_1 singularity). It follows fom Table 1 that $\delta(8) = \delta^*(8) = 1 = \delta(12) = \delta^*(12)$. Also, Theorem 3.13(f) implies that $\det(w_8) = \zeta_8^{4-d-d_1-d_2-d_3} = \zeta_8^{-24} = 1$ and $\det(w_{12}) = \zeta_{12}^{4-d-d_1-d_2-d_3} = \zeta_{12}^{-24} = 1$.

- *Type* G_{30}. Assume here that $W = G_{30}$. Then $d = 12$ and $(d_1, d_2, d_3) = (2, 20, 30)$. But we have then $\mathbb{P}(2, 20, 30) = \mathbb{P}(1, 10, 15) = \mathbb{P}(1, 2, 3)$ so $\mathcal{S} = \{p_2, p_3\}$, so $d_k \in \{20, 30\}$. Note also that p_2 is an A_1-singularity while p_3 is an A_2-singularity. It follows from Table 1 that $\delta(20) = \delta^*(20) = 1 = \delta(30) = \delta^*(30)$. Also, Theorem 3.13(f) implies that $\det(w_{20}) = \zeta_{20}^{4-d-d_1-d_2-d_3} = \zeta_{20}^{-60} = 1$ and $\det(w_{30}) = \zeta_{30}^{4-d-d_1-d_2-d_3} = \zeta_{30}^{-60} = 1$.

The proof of Lemma 6.3 is complete. □

6.4.2. The case where $\Gamma = W' \neq W^{\mathrm{SL}}$ This case can only occur if $W = G(2,1,4)$ or G_{28}.

- *Type* $G(2,1,4)$. Assume here that $W = G(2,1,4)$. Then $d = 6$ and f is also an invariant of degree 6 of $G(2,2,4)$. Since $W' = G(2,2,4)' = G(2,2,4)^{\mathrm{SL}}$, the result follows from the previous subsection.

- *Type* G_{28}. Then $d \in \{6,8\}$. Then $d_1 = 2$, $d_3 = 12$ and d_2 is the unique element of $\{6,8\} \setminus \{d\}$. Then
$$\mathcal{Z}(f)/W' = \{[x_1 : x_2 : x_3 : j_1 : j_2] \in \mathbb{P}(2, d_2, 12, 12, 12) \mid$$
$$j_1^2 = P_{\mathbf{f},\Omega_1}(0, x_1, x_2, x_3) \text{ and } j_2^2 = P_{\mathbf{f},\Omega_2}(0, x_1, x_2, x_3)\}.$$
The group W^{SL}/W' has order 2 (we denote by σ its non-trivial element) and it acts on $\mathcal{Z}(f)/W'$ as follows:
$$\sigma([x_1 : x_2 : x_3 : j_1 : j_2]) = [x_1 : x_2 : x_3 : -j_1 : -j_2].$$
So one can check that the ramification locus $\mathcal{R}$ of the morphism $\theta : \mathcal{Z}(f)/W' \longrightarrow \mathcal{Z}(f)/W^{\mathrm{SL}}$ is defined by $j_1 = j_2 = 0$ in both cases. We only need to prove that $\mathcal{R}$ is finite: indeed, if it is finite, then θ is unramified in codimension 1 and $\mathcal{Z}(f)/W^{\mathrm{SL}}$ has only ADE singularities as it was shown in §6.4.1, so $\mathcal{Z}(f)/W'$ has only ADE singularities by Lemma B.4.
 Now,
$$\pi_{\mathbf{f}}'^{-1}(\mathcal{R}) = \{p \in \mathbb{P}(V) \mid j_1(p) = j_2(p) = f(p) = 0\}.$$
We only need to prove that $\pi_{\mathbf{f}}'^{-1}(\mathcal{R})$ is finite. First, let
$$\mathcal{H} = \{p \in \mathbb{P}(V) \mid j_1(p) = j_2(p) = 0\}.$$
Then the irreducible components of $\mathcal{H}$ are lines of the form $\mathbb{P}(H_1 \cap H_2)$, where $H_1 \in \Omega_1$ and $H_2 \in \Omega_2$. This means that we only need to prove that such a line cannot be entirely contained in $\mathcal{Z}(f)$. So, let $H_1 \in \Omega_1$ and $H_2 \in \Omega_2$ and let s_k denote the reflection of W whose reflecting hyperplane is H_k. Let $G = \langle s_1, s_2 \rangle$. Then $V^G = H_1 \cap H_2$ so $\dim V^G = 2$. If $\mathbb{P}(V^G)$ is entirely contained in $\mathcal{Z}(f)$, it then follows from Corollary 2.9 that it is contained in $\mathcal{Z}_{\mathrm{sing}}(f)$: but this contradicts the fact that $\mathcal{Z}(f)$ has only ADE singularities.

The proof of Theorem 5.4 is complete. □

Appendix A. Surfaces in weighted projective spaces

Let $m \geq 3$ and let $l_0, l_1, \ldots, l_m$ be positive integers. We denote by $x_0, x_1, \ldots, x_m$ the coordinates in the weighted projective space $\mathbb{P}(l_0, l_1, \ldots, l_m)$ and we fix $m-2$ polynomials $F_1, \ldots, F_{m-2}$ in the variables x_0, $x_1, \ldots, x_m$ which are homogeneous of degree $e_1, \ldots, e_{m-2}$ (where x_k is given the degree l_k). We consider the variety
$$X = \{[x_0 : x_1 : \cdots : x_m] \in \mathbb{P}(l_0, l_1, \ldots, l_m) \mid \forall\ 1 \leq j \leq m-2,\ F_j(x_0, x_1, \ldots, x_m) = 0\}.$$
Let $\mathbb{P}_{\mathrm{sm}}$ (resp. $\mathbb{P}_{\mathrm{sing}}$) denote the smooth (resp. singular) locus of $\mathbb{P}(l_0, l_1, \ldots, l_m)$. We assume throughout this section that the following hold:

(H1) The weighted projective space $\mathbb{P}(l_0, l_1, \ldots, l_m)$ is well-formed, i.e.
$$\gcd(l_0, \ldots, l_{j-1}, l_{j+1}, \ldots, l_m) = 1$$
for all $j \in \{0, 1, \ldots, m\}$.

(H2) The variety X is a weighted complete intersection, i.e. $\dim(X) = 2$.

(H3) The variety X is *well-formed*, i.e. $\mathrm{codim}_X(X \cap \mathbb{P}_{\mathrm{sing}}) \geq 2$.

(H4) $l_0 + l_1 + \cdots + l_m = e_1 + \cdots + e_{m-2}$.

(H5) X has only ADE singularities.

Note that we do not assume that X is ***quasi-smooth*** (i.e. we do not assume that the affine cone of X in $\mathbb{C}^{m+1}$ is smooth outside the origin [Dol82, §3.1.5]). The following result is certainly well-known but, due to the lack of an appropriate reference (particularly in the non-quasi-smooth case), we provide here an explicit proof:

Lemma A.1. *Under the hypotheses* (H1), (H2), (H3), (H4) and (H5), *the smooth locus of the surface X has a non-degenerate 2-form.*

Proof. We set $\mathbb{P} = \mathbb{P}(l_0, l_1, \dots, l_m)$ for simplification. Let U denote the smooth locus of $X \cap \mathbb{P}_{\mathrm{sm}}$. By (H3), $X \cap \mathbb{P}_{\mathrm{sing}}$ has codimension ≥ 2 in X and so $X \setminus U$ has codimension ≥ 2 in X by (H5). Again by (H5), it is sufficient to prove that U admits a non-degenerate 2-form.

If $0 \leq a \leq m$ and $1 \leq j \leq m-2$, we denote $\mathbb{P}^{(a)}$ the open subset of $\mathbb{P}$ defined by $x_a \neq 0$: we identify it with $\mathbb{C}^m/\boldsymbol{\mu}_{l_a}$, where the coordinates in $\mathbb{C}^m$ are denoted by $(x_0, \dots, x_{a-1}, x_{a+1}, \dots, x_m)$ and $\boldsymbol{\mu}_{l_a}$ acts through $\zeta \cdot (x_0, \dots, x_{a-1}, x_{a+1}, \dots, x_m) = (\zeta^{l_0} x_0, \dots, \zeta^{l_{a-1}} x_{a-1}, \zeta^{l_{a+1}} x_{a+1}, \dots, \zeta^{l_m} x_m)$, and we set

$$F_j^{(a)}(x_0, \dots, x_{a-1}, x_{a+1}, \dots, x_m) = F_j(x_0, \dots, x_{a-1}, 1, x_{a+1}, \dots, x_m).$$

The smooth locus of $\mathbb{P}^{(a)}$ will be denoted by $\mathbb{P}^{(a)}_{\mathrm{sm}}$ and, since $\mathbb{P}$ is well-formed by (H1), the above action of $\boldsymbol{\mu}_{l_a}$ on $\mathbb{C}^m$ contains no reflection and so $\mathbb{P}^{(a)}_{\mathrm{sm}}$ is the unramified locus of the morphism $\mathbb{C}^m \to \mathbb{C}^m/\boldsymbol{\mu}_{l_a}$.

Let $J^{(a)} = \left(\partial F_j^{(a)}/\partial x_k\right)_{1 \leq j \leq m-2, 0 \leq k \neq a \leq m}$ denote the Jacobian matrix of the family $(F_1^{(a)}, \dots, F_{m-2}^{(a)})$. If b, c are two different elements of $\{0, 1, \dots, m\}$ which are different from a, we denote by $J_{b,c}^{(a)}$ the $(m-2) \times (m-2)$ minor of $J^{(a)}$ obtained by removing the two columns numbered by b and c. We set

$$\mathbb{P}^{(a)}_{\mathrm{sm},b,c} = \{p \in \mathbb{P}^{(a)}_{\mathrm{sm}} \mid J_{b,c}^{(a)}(p) \neq 0\}$$

and
$$U_{b,c}^{(a)} = X \cap \mathbb{P}^{(a)}_{\mathrm{sm},b,c}.$$

By the above description of $\mathbb{P}^{(a)}_{\mathrm{sm}}$ and (H2), we get

$$U = \bigcup_{\substack{0 \leq a,b,c \leq m \\ |\{a,b,c\}|=3}} U_{b,c}^{(a)}.$$

We now define a 2-form $\omega_{b,c}^{(a)}$ on $\mathbb{P}^{(a)}_{\mathrm{sm},b,c}$ by

$$\omega_{b,c}^{(a)} = l_a \frac{dx_b \wedge dx_c}{J_{b,c}^{(a)}}. \tag{A.2}$$

Let us first explain why this defines a 2-form on $\mathbb{P}^{(a)}_{\mathrm{sm},b,c}$. This amounts to show that $\omega_{b,c}^{(a)}$ is invariant under the action of $\boldsymbol{\mu}_{l_a}$ on the variables $(x_k)_{0 \leq k \neq a \leq m}$ given by $\xi \cdot x_k = \xi^{l_k} x_k$. But, if M is a monomial in $J_{b,c}^{(a)}$ of degree e in the variables $(x_k)_{0 \leq k \neq a \leq m}$, then

$$e \equiv e_1 + \dots + e_{m-2} - (l_0 + l_1 + \dots + l_m - l_a - l_b - l_c) \mod l_a \tag{\#}$$

because the variable x_a is specialized to 1. So $\xi \in \boldsymbol{\mu}_{l_a}$ acts on $J_{b,c}^{(a)}$ by multiplication by $\xi^{l_b + l_c}$ by (H4). So it acts trivially on $\omega_{b,c}^{(a)}$.

We now denote by $\omega_{U,b,c}^{(a)}$ the restriction of $\omega_{b,c}^{(a)}$ to $U_{b,c}^{(a)}$. Note that $U_{b,c}^{(a)} = U_{c,b}^{(a)}$ but that $\omega_{U,b,c}^{(a)} = -\omega_{U,c,b}^{(a)}$, so we have to make some choice. We denote by $\mathcal{E}$ the set of triples (a,b,c) of elements of $\{0, 1, \dots, m\}$ such that $a < b < c$ or $c < a < b$ or $b < c < a$. Then again

$$U = \bigcup_{(a,b,c) \in \mathcal{E}} U_{b,c}^{(a)}$$

and we want to show that the family of 2-forms $(\omega_{U,b,c}^{(a)})_{(a,b,c)\in\mathcal{E}}$ glue together to define a 2-form on U. The argument is standard and will be done in two steps.

First step: glueing inside an affine chart. We fix $a \in \{0,1,\dots,m\}$ and we set $U^{(a)} = U \cap \mathbb{P}^{(a)}$. Let b, c, b', $c' \in \{0,1,\dots,m\}$ be such that (a,b,c), $(a,b',c') \in \mathcal{E}$. We need to prove that

$$(\sharp) \qquad \omega_{U,b,c}^{(a)}\Big|_{U_{b,c}^{(a)}\cap U_{b',c'}^{(a)}} = \omega_{U,b',c'}^{(a)}\Big|_{U_{b,c}^{(a)}\cap U_{b',c'}^{(a)}}.$$

Proving $(\sharp)$ is a computation in $\mathbb{C}^m$ and amounts to prove that

$$(\sharp') \qquad J_{b',c'}^{(a)} dx_b \wedge dx_c = J_{b,c}^{(a)} dx_{b'} \wedge dx_{c'}$$

on the variety $\hat{X}^{(a)}$ defined by $F_1^{(a)} = \cdots = F_{m-2}^{(a)} = 0$ inside $\mathbb{C}^m$. By applying a power of the cyclic permutation $(0,1,\dots,m)$ to the coordinates, we may (and we will) assume that $a=0$ (so that $0<b<c$ and $0<b'<c'$). Since $F_j^{(0)}$ vanishes on $\hat{X}^{(0)}$, its differential vanishes also on X, which implies that

$$\forall 1 \le j \le m-2, \ \sum_{k=1}^{m} \frac{\partial F_j^{(0)}}{\partial x_k} dx_k = 0 \quad \text{on } \hat{X}^{(0)}.$$

Then $(\sharp')$ is an easy application of generalized Cramer's rule [GA95].

Second step: glueing affine charts. We denote by $\omega_U^{(a)}$ the glueing of the 2-forms $\omega_{U,b,c}^{(a)}$, where b, c are such that $(a,b,c)\in\mathcal{E}$. Let a, $a' \in \{0,1,\dots,m\}$. We need to prove that

$$(\flat) \qquad \omega^{(a)}|_{U^{(a)}\cap U^{(a')}} = \omega^{(a')}|_{U^{(a)}\cap U^{(a')}}.$$

For simplifying the notation, we will assume that $(a,a')=(0,1)$, the general case being treated similarly. We will denote by $(x_k)_{0\le k\ne a\le m}$ the coordinates on $\mathbb{P}^{(a)}$ and $(x'_k)_{0\le k\ne a'\le m}$ the coordinates on $\mathbb{P}^{(a')}$. Also for simplifying the notation, we will assume that $U_{1,2}^{(0)} \cap U_{(2,0)}^{(1)} \neq \varnothing$. So, for proving $(\flat)$, we only need to prove that

$$(\flat') \qquad l_0 J_{2,0}^{(1)} dx_1 \wedge dx_2 = l_1 J_{1,2}^{(0)} dx'_2 \wedge dx'_0 \qquad \text{on } \mathbb{P}^{(0)} \cap \mathbb{P}^{(1)}.$$

The variables $(x_k)_{0\le k\ne a\le m}$ and $(x'_k)_{0\le k\ne a'\le m}$ are related as follows:

$$\begin{cases} x'_0 = \dfrac{1}{x_1^{l_0/l_1}}, \\ \forall 2 \le k \le m, \ x'_k = \dfrac{x_k}{x_1^{l_k/l_1}}. \end{cases}$$

Therefore $dx'_0 = -(l_0/l_1) x_1^{-1-l_0/l_1} dx_1$ and so

$$(\flat'') \qquad l_1 dx'_2 \wedge dx'_0 = l_0 x_1^{-(l_0+l_1+l_2)/l_1} dx_1 \wedge dx_2.$$

Moreover, since F_j is homogeneous of degree e_j, we get

$$F_j^{(0)}(x_1,x_2,\dots,x_m) = x_1^{e_j/l_1} F_j^{(1)}(x'_0,x'_2,\dots,x'_m).$$

We deduce that

$$\frac{\partial F_j^{(0)}}{\partial x_k} = x_1^{(e_j-l_k)/l_1} \frac{\partial F_j^{(1)}}{\partial x'_k}$$

for all $k \ge 3$ and then

$$(\flat''') \qquad J_{1,2}^{(0)} = x_1^{((e_1+\cdots+e_{m-2})-(l_3+\cdots+l_m))/l_1} J_{2,0}^{(1)}.$$

So $(\flat')$ follows from $(\flat'')$ and $(\flat''')$ since $e_1+\cdots+e_{m-2} = l_0+l_1+\cdots+l_m$ by (H4). $\square$

Corollary A.3. *Under the hypotheses* (H1), (H2), (H3), (H4) and (H5), *the smooth locus of the surface X has a trivial canonical sheaf and its minimal resolution is a smooth K3 surface or an abelian variety.*

Appendix B. Around ADE singularities

The results of this appendix are certainly well-known. Here, we let $\mathbf{GL}_2(\mathbb{C})$ act on the ring of formal power series $\mathbb{C}[[t,u]]$ naturally by linear changes of the variables. We set $B = \operatorname{Spec} \mathbb{C}[[t,u]]$ and we denote by 0 its unique closed point. We set $B^{\#} = B \setminus \{0\}$: it is an open subscheme of B. Since B is normal, it follows from Hartog's Lemma that

$$\mathcal{O}_B(B^{\#}) = \mathbb{C}[[t,u]]. \tag{B.1}$$

Lemma B.2. *Let G be a finite subgroup of $\mathbf{GL}_2(\mathbb{C})$ containing no reflection and let σ be an automorphism of $\mathbb{C}[[t,u]]^G$. Then σ lifts to an automorphism of $\mathbb{C}[[t,u]]$.*

Remark B.3. If G is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, then G contains no reflection. This shows that the above lemma applies to ADE singularities $\mathbb{C}[[t,u]]^G$.

Proof. First, note that B has a trivial fundamental group by [Gro71, exposé I, théorème 6.1]. As it is regular of dimension 2, its open subset $B^{\#}$ has also a trivial fundamental group by [Gro71, exposé X, corollaire 3.3]. Therefore, the natural map

$$\pi : B^{\#} \longrightarrow B^{\#}/G$$

is a universal covering: indeed, the morphism $B \to B/G$ is ramified only at 0 because G does not contain any reflection. In particular, $\pi \circ \sigma$ is also a universal covering, which means that σ lifts to an automorphism of $B^{\#}$ since $B^{\#}/G$ is connected. Taking global sections and using (B.1) yields the result. ☐

Lemma B.4. *Let $\pi : Y \to X$ be a finite morphism of normal surfaces which is unramified in codimension 1. We assume moreover that X has only ADE singularities. Then Y has only ADE singularities.*

Proof. Let $y \in Y$ and let $x = \pi(y)$. Then there exists a finite subgroup G of $\mathbf{SL}_2(\mathbb{C})$ such that the completion of the local ring $\mathcal{O}_{X,x}$ of X at x is given by $\hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[t,u]]^G$. Therefore, the morphism of schemes

$$\pi_y : (\operatorname{Spec} \hat{\mathcal{O}}_{Y,y}) \setminus \{y\} \longrightarrow B^{\#}/G$$

induced by π is unramified by hypothesis, so there exists a morphism of schemes

$$B^{\#} \longrightarrow (\operatorname{Spec} \hat{\mathcal{O}}_{Y,y}) \setminus \{y\}$$

whose composition with π_y is a universal covering of $B^{\#}/G$ (see the proof of Lemma B.2).

Consequently, there exists a subgroup H of G such that

$$(\operatorname{Spec} \hat{\mathcal{O}}_{Y,y}) \setminus \{y\} = B^{\#}/H.$$

Taking global sections and applying Hartog's Lemma together with (B.1) yields that $\hat{\mathcal{O}}_{Y,y} = \mathbb{C}[[t,u]]^H$. ☐

Recall that, if $G \subset \mathbf{GL}_2(\mathbb{C})$, then the only point of $\mathbb{C}^2/G$ that might be singular is the G-orbit of 0 (denoted by $\bar{0}$). The next result is certainly well-known:

Lemma B.5. *Let G be a finite subgroup of $\mathbf{GL}_2(\mathbb{C})$ which is generated by $\operatorname{Ref}(G)$ and let Γ be a subgroup of G of index 1 or 2. Then $\bar{0} \in V/\Gamma$ is smooth or an ADE singularity.*

Proof. We argue by induction on the order of G, the case where $|G| = 1$ being trivial. Also, if $\Gamma = G$, then V/Γ is smooth so we may assume that $\Gamma \neq G$. As Γ is of index 2, it is normal and we denote by $\tau : G \to \boldsymbol{\mu}_2$ the unique morphism such that $\Gamma = \operatorname{Ker}(\tau)$. Let Γ_r be the subgroup of Γ generated by reflections belonging to Γ. It is a normal subgroup of G and

$$V/G = (V/\Gamma_r)/(G/\Gamma_r) \qquad \text{and} \qquad V/\Gamma = (V/\Gamma_r)/(\Gamma/\Gamma_r).$$

Two cases may occur:

- If $\Gamma_r \neq 1$, then V/Γ_r is isomorphic to a vector space on which G/Γ_r acts linearly as a reflection group since V/G is smooth (see also [BBR02, proposition 3.5]). So the result follows from the induction hypothesis.
- If $\Gamma_r = 1$, then $\tau(s) \neq 1$ for all $s \in \mathrm{Ref}(G)$. This implies that all reflections of G have order 2. Indeed, if $s \in \mathrm{Ref}(G)$, then $\tau(s^2) = 1$ so s^2 cannot be a reflection, hence is equal to 1. This shows in particular that $\tau(s) = \det(s)$ for all $s \in \mathrm{Ref}(G)$ and so $\tau(w) = \det(w)$ for all $w \in W$. In particular, $\Gamma \subset \mathbf{SL}_2(\mathbb{C})$ and the result follows.

The proof of the lemma is complete. □

Remark B.6. We explain here why the general result stated in Lemma B.5 is in some sense optimal. Let $d \geq 3$. Then there exists a reflection group G in $\mathbf{GL}_2(\mathbb{C})$ admitting a normal subgroup Γ such that W/Γ is cyclic of order d and $\mathbb{C}^2/\Gamma$ admits a non-simple singularity. Take for instance $G = \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d$ embedded through diagonal matrices, and $\Gamma \simeq \boldsymbol{\mu}_d$ embedded through scalar multiplication. Then $\bar{0}$ is not an ADE singularity of $\mathbb{C}^2/\Gamma$.

In the same spirit, there exists a reflection group G in $\mathbf{GL}_2(\mathbb{C})$ admitting a normal subgroup Γ such that $G/\Gamma \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ and such that V/Γ admits a non-simple singularity. Take for instance $G = G(4, 2, 2)$ and $\Gamma = Z(W)$. Then G/Γ is indeed isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ and Γ is isomorphic to $\boldsymbol{\mu}_4$ acting through scalar multiplication. So $\bar{0}$ is not an ADE singularity of $\mathbb{C}^2/\Gamma$.

Corollary B.7. *Let X be a surface with only ADE singularities and let G be a group acting on X such that X/G is smooth. Let Γ be a subgroup of G of index 2. Then X/Γ has only ADE singularities.*

Proof. Let $x \in X$. It is sufficient to show that X/Γ_x has an ADE singularity at the image of x. Note that we know that X/G_x is smooth at the image of x. Also, Γ_x has index 1 or 2 in G_x. This shows that we may, and we will, assume that $G = G_x$ (and so $\Gamma = \Gamma_x$).

By hypothesis, there exists a subgroup H of $\mathbf{SL}_2(\mathbb{C})$ such that the complete local ring $\hat{\mathcal{O}}_{X,x}$ is isomorphic to $\mathbb{C}[[t,u]]^H$. Let us identify $\hat{\mathcal{O}}_{X,x}$ with $\mathbb{C}[[t,u]]^H$, so that the group G acts on $\mathbb{C}[[t,u]]^H$. By Lemma B.2, the action of an element $g \in G$ on $\mathbb{C}[[t,u]]^H$ lifts to an automorphism $\tilde{g}$ of $\mathbb{C}[[t,u]]$: we fix such a $\tilde{g}$ for all $g \in G$. Note that $\{\tilde{g}h \mid h \in H\}$ is the set of all lifts of g to $\mathbb{C}[[t,u]]$. Let

$$\tilde{G} = \{\tilde{g}h \mid g \in G \text{ and } h \in H\}.$$

Then $\tilde{G}$ is a group (as $\tilde{g}h\tilde{g}'h'$ is a lift of gg' so belongs to $\tilde{G}$) and we have an exact sequence

$$1 \longrightarrow H \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Let $\tilde{\Gamma}$ denote the inverse image of Γ in $\tilde{G}$. Note that $(\mathbb{C}[[t,u]]^H)^G = \mathbb{C}[[t,u]]^{\tilde{G}}$ and $(\mathbb{C}[[t,u]]^H)^\Gamma = \mathbb{C}[[t,u]]^{\tilde{\Gamma}}$.

Now, by hypothesis, $(\mathbb{C}[[t,u]]^H)^G$ is regular. This shows that $\tilde{G}$ acts as a reflection group on the tangent space of $\operatorname{Spec} \mathbb{C}[[t,u]]$ at its unique closed point, and so the result follows from Lemma B.5 because $\tilde{\Gamma}$ has index 1 or 2 in $\tilde{G}$. □

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