# Białynicki-Birula schemes in Hilbert schemes of points and monic functors 

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#### Abstract

The Bialynicki-Birula strata on the Hilbert scheme $H^{n}\left(\mathbb{A}^{d}\right)$ are smooth in dimension $d=2$. We prove that there is a schematic structure in higher dimensions, the Bialynicki-Birula scheme, which is natural in the sense that it represents a functor. Let $\rho_{i}: H^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)$ be the Hilbert-Chow morphism of the $i^{\text {th }}$ coordinate. We prove that a Bialynicki-Birula scheme associated with an action of a torus $T$ is schematically included in the fiber $\rho_{i}^{-1}(0)$ if the $i^{\text {th }}$ weight of $T$ is non-positive. We prove that the monic functors parametrizing families of ideals with a prescribed initial ideal are representable.


Keywords. Hilbert scheme of points; standard sets; Bialynicki-Birula decomposition; stratification; torus action; representable functor; Gröbner basis

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## [Français]

## Schémas de Bialynicki-Birula dans les schémas de Hilbert de points et foncteurs moniques

Résumé. Les strates de Bialynicki-Birula sur le schéma de Hilbert $H^{n}\left(\mathbb{A}^{d}\right)$ sont lisses en dimension $d=2$. Nous démontrons qu'il existe une structure schématique en dimension supérieure, le schéma de Bialynicki-Birula, qui est naturelle au sens où elle représente un foncteur. Considérons $\rho_{i}: H^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)$ le morphisme de Hilbert-Chow de la $i^{\text {ème }}$ coordonnée. Nous prouvons qu'un schéma de Bialynicki-Birula associé à l'action d'un tore $T$ est schématiquement inclus dans la fibre $\rho_{i}^{-1}(0)$ si le $i^{\text {ème }}$ poids de $T$ est négatif ou nul. Nous prouvons que les foncteurs moniques qui paramètrent les familles d'ideaux ayant un idéal initial donné sont représentables.

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## 1. Introduction

Let $H^{n}\left(\mathbb{A}^{d}\right)$ be the Hilbert scheme parametrizing zero-dimensional subschemes of length $n$ in the affine $d$-space $\mathbb{A}^{d}$ over a field $k$. This scheme is mostly called the Hilbert scheme of points, sometimes also the punctual Hilbert scheme. There is a natural action of the $d$-dimensional split torus on $\mathbb{A}^{d}$, which induces a natural action on $H^{n}\left(\mathbb{A}^{d}\right)$. If $T$ is a one-dimensional split subtorus of the $d$-dimensional torus, then $T$ defines the Bialynicki-Birula strata $H^{B B(T, \Delta)}$ parametrizing the subschemes of $\mathbb{A}^{d}$ converging to some fixed point $Z^{\Delta}$ under the action of $T$, where $Z^{\Delta}$ is a monomial subscheme with staircase $\Delta$. When $T$ is general, any $T$-fixed point is monomial and is a $Z^{\Delta}$ for some staircase $\Delta$. The case where $T$ is general is thus of particular interest, however we will consider $H^{B B(T, \Delta)}$ for any $T$.

These stratifications are preeminent in most studies of the punctual Hilbert scheme in dimension two. For instance, they appear in the computation of the Betti numbers (see [ES87], [ES88]), in the determination of the irreducible components of (multi)graded Hilbert schemes (see [Eva04], [MS10]), or in the study of the ring of symmetric functions via symmetric products of embedded curves (see [Gro96], [Nak99]).

The Białynicki-Birula strata in $H^{n}\left(\mathbb{A}^{2}\right)$ are affine spaces. In contrast, not much is known on these strata for higher dimensional $\mathbb{A}^{d}$, and the difficulty to control and describe the Bialynicki-Birula strata is probably one of the reasons why the Hilbert scheme of points is still mysterious in higher dimensions.

In dimension three, the Bialynicki-Birula strata are not irreducible (Proposition 7.1). In higher dimensions, they are not reduced either [Jel20]. It is therefore necessary to define them with their natural scheme structure as representing a functor. Apart from the necessity to define them schematically, it is desirable to have functorial descriptions of Hilbert schemes at hand, as these descriptions are known to be both powerful and easy to handle.

In the present paper, we introduce the Bialynicki-Birula functor parametrizing families of subschemes $Z$ such that $\lim _{t \rightarrow 0, t \in T} t \cdot Z=Z^{\Delta}$ for some fixed monomial subscheme $Z^{\Delta}$. We will prove (Theorem 5.4):

Theorem. The Biatynicki-Birula functor is representable by a locally closed subscheme $H^{B B(T, \Delta)}\left(\mathbb{A}^{d}\right)$ of the Hilbert scheme $H^{n}\left(\mathbb{A}^{d}\right)$.

The theorem is constructive: if a flat family is given, the Bialynicki-Birula strata are computable by the algorithms encapsulated in the proofs.

During the proof, a linchpin construction is to consider ideals with a prescribed initial ideal for a total order. The total order considered is any order refining the partial order on the monomials induced by their weight for the torus action. We prove the representability of the corresponding functor (Theorem 4.6).

Theorem. The monic functor $\mathcal{H}^{\text {mon(<, }()}$ parametrizing ideals with initial ideal $I^{\Delta}=I\left(Z^{\Delta}\right)$ is representable.
There are results in the same circle of ideas when Gröbner basis theory is workable [Led11, LR16]. In the context of Theorem 4.6, Gröbner basis theory does not apply because of possible negative weights. As far as we know, monic functors have never been considered in this context. We develop an original approach for the proof as the ideas from [Led11, LR16] are not easily adjustable.

Our sign convention for the weight $\xi=\left(\xi_{1}, \ldots, \xi_{d}\right)$ is that the action of the one-dimensional subtorus $T$ on $\mathbb{A}^{d}$ is given by $t \cdot\left(a_{1}, \ldots, a_{d}\right):=\left(t^{\xi_{1}} a_{1}, \ldots, t^{\xi_{d}} a_{d}\right)$. If $\xi_{i} \leq 0$, then the closed points of $H^{B B(T, \Delta)}\left(\mathbb{A}^{d}\right)$ correspond to subschemes $Z$ whose support is in the hyperplane $x_{i}=0$. This follows from the naive observation that if $t \cdot Z$ tends to $Z^{\Delta}$, then the support of $t . Z$ tends to the support of $Z^{\Delta}$. A much more subtle question is to ask whether this remains true at the schematic level, when we consider the Bialynicki-Birula scheme with its possibly non-reduced structure. The answer is positive. Recall that there is a Hilbert-Chow morphism $\rho_{i}: H^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right) \simeq \mathbb{A}^{n}$ which sends a point $p$ parametrizing a subscheme $Z$ to (the coefficients of) the characteristic polynomial of the multiplication by the $i^{\text {th }}$ coordinate $x_{i}$ in $\mathcal{O}(Z)$. We will prove (see Theorem 6.1):

Theorem. Let $\rho_{i}$ be the Hilbert-Chow morphism associated with the $i^{\text {th }}$ coordinate. If $\xi_{i} \leq 0$, then $H^{B B(T, \Delta)}\left(\mathbb{A}^{d}\right)$ is schematically included in the fiber $\rho_{i}^{-1}(0)$.

For simplicity, we have considered a field $k$ in this introduction. But throughout the paper, we shall work over a ring $k$ of arbitrary characteristic.

After the first version of this article appeared, several authors dealt with the Bialynicki-Birula decomposition for singular schemes, algebraic spaces, stacks... with methods, scope, computability specific to each approach and important applications [Dri15, AHR20, Ric19, Jel19, Jel20, Kam19].

Let Z be an algebraic $k$-space of finite type equipped with a $\mathbb{G}_{m}$-action. Drinfeld [Dril5] defines the attractor $Z^{+}$, which informally is the functor whose points are the points $z \in Z(k)$ with an existing limit $\lim _{t \rightarrow 0} t \cdot z$. He proves that $Z^{+}$is representable (Theorem 1.4.2, ibid.). Using the results by Drinfeld, it is possible to recover some results of the present paper (see Section 7).

Alper-Hall-Rydh [AHR20] work in the context of algebraic stacks. They prove (Theorem 5.27, ibid.) the existence of Białynicki-Birula decomposition for Deligne-Mumford stack of finite type over a field $k$, equipped with a $\mathbb{G}_{m}$-action. They recover some results by Drinfeld ( $c f$. Remark 5.28, ibid.).

Richarz extends some representability results by Drinfeld to algebraic spaces with an étale locally linearizable $\mathbb{G}_{m}$-action [Ric19, Theorem A]

An important application of the Białynicki-Birula decomposition is due to Jelisiejew. In [Jel19, Proposition 3.1], he realizes the Białynicki-Birula decomposition in the multigraded Hilbert scheme of HaimanSturmfels [HS02]. Then, in [Jel20], the decomposition is used to prove that the Hilbert scheme of points on a higher dimensional affine space is non-reduced and has components lying entirely in characteristic $p$ for all primes $p$, and that Vakil's Murphy's Law (every singularity type of finite type appears) holds up to retraction for this scheme.

Proofs. Let us say a word about the proofs. The action of $T$ on $\mathbb{A}^{d}=\operatorname{Spec} k[\mathbf{x}]=\operatorname{Spec} k\left[x_{1}, \ldots, x_{d}\right]$ with weight $\xi$ induces a partial order $<_{\xi}$ on the monomials of $k[\mathbf{x}]$ : the monomials are ordered according to their weight for the $T$-action. To control the Bialynicki-Birula strata $H^{B B(T, \Delta)}$ we prove that, roughly speaking, a subscheme $Z$ is in $H^{B B(T, \Delta)}$ if and only if its initial ideal $\mathrm{in}_{<_{\xi}}(I(Z))$ equals the monomial ideal $I\left(Z^{\Delta}\right)$ (Proposition 2.7). Therefore, a natural strategy could be to introduce a monic functor $\mathcal{H}^{\text {mon( }(<\xi, \Delta)}$
parametrizing ideals with a prescribed initial ideal and to show that this functor is representable and isomorphic to the Białynicki-Birula functor.

However, a technical barrier is that the initial ideal in our context is a poor analog of the same notion used in the context of Gröbner basis, for instance in [Eis99, Chapter 15] for two reasons: the order on the monomials is a only a partial order, and when some weights of the action are negative, the division algorithm may not terminate. This makes the above strategy inefficient.

The modified strategy is the following. We consider total orders rather than partial orders when we introduce the monic functors mainly because this condition ensures functoriality (Remark 3.1). However, because of the possible negative weights, we still don't have in general a monomial order in the sense of [Eis99]. We prove that the monic functor $\mathcal{H}^{\operatorname{mon}(<, \Delta)}$ parametrizing ideals with initial ideal $I^{\Delta}$ is representable (Theorem 4.6).

In this modified strategy, we realize the Bialynicki-Birula functor $\mathcal{H}^{B B(T, \Delta)}$ as the intersection of two wellchosen monic functors $\mathcal{H}^{\text {mon }\left(<_{-}, \Delta\right)} \cap \mathcal{H}^{\text {mon }\left(<_{+}, \Delta\right)}$ where $<_{-}$and $<_{+}$are total orders refining $<_{\xi}$ (Proposition 5.3). Realizing the Bialynicki-Birula functor as an intersection is the functorial counterpart to the following remark : having a prescribed initial ideal for the partial order $<_{\xi}$ is equivalent to having the same prescribed initial ideal for both $<_{-}$and $<_{+}$. The representability of the Bialynicki-Birula functors then follows from the representability of the monic functors.

When dealing with representability of functors, constructions for individual subschemes often require uniformity lemmas when one passes to families. The paragon of this situation is the Castelnuovo-Mumford regularity, in the construction of the Hilbert scheme. In contrast, punctual subschemes localized at a fixed point $p$ of the projective space are not representable by a closed subscheme of the Hilbert scheme because the families lack a uniformity property : The smallest infinitesimal neighborhood of $p$ containing a family of such subschemes may be of arbitrary large order.

When the weights of the action are non-positive, the families that we consider are supported on the origin. It may be surprising that we get the representability in this context. The reason is that Bialynicki-Birula families are included in an infinitesimal neighborhood of uniform order. This uniformity is settled in Lemma 2.11 which says that the families of the Białynicki-Birula functor are included in $x_{i}^{n}$ if $x_{i}$ is a coordinate with a non-positive weight, where $n$ is the length of the parameterized punctual subschemes. In some sense, these families are uniformly bounded. As to the monic functors, some boundedness condition is included in their definition, i.e. the families are included in $x_{i}^{r}$ for some $r$, and one proves that $r$ may be chosen uniform in Proposition 4.5 to get the representability.

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## 2. Bialynicki-Birula functors and $\Delta$-monic families

In this section, we introduce the Bialynicki-Birula functors whose geometric points are subschemes having a prescribed limit under the action of a torus $T$. These subschemes are characterized by their initial ideal for some order. Accordingly, we reformulate Białynicki-Birula families in terms of $\Delta$-monic ideals (Proposition 2.7). Finally, we prove an important uniformity lemma for Bialynicki-Birula families (Corollary 2.10).

In this paper, we consider schemes over a commutative ring $k$ of arbitrary characteristic. We denote by $k[\mathbf{x}]$ the polynomial ring $k\left[x_{1}, \ldots, x_{d}\right]$. Similarly, for $e=\left(e_{1}, \ldots, e_{d}\right) \in \mathbb{N}^{d}$, we use the multi-index notation $\mathbf{x}^{e}:=x_{1}^{e_{1}} \cdots x_{d}^{e_{d}}$.

A standard set, or staircase, is a subset $\Delta \subset \mathbb{N}^{d}$ whose complement $C:=\mathbb{N}^{d} \backslash \Delta$ satisfies $C+\mathbb{N}^{d}=C$. We call the minimal elements of $C$ the outer corners of $\Delta$. All standard sets under consideration will be of finite cardinality $n$. The ideal generated by the monomials $\mathbf{x}^{e}, e \in C$ is denoted by $I^{\Delta}$. The notation $I^{\Delta}$ makes sense in $k[\mathbf{x}]$, but more generally in any ring containing the monomials $\mathbf{x}^{e}$, such as the ring $B[\mathbf{x}]$ introduced below. We shall freely identify the monomials $\mathbf{x}^{e}$ with their exponent $e$. In particular, the notion of a staircase of monomials makes sense.

If $B$ is a $k$-algebra, then the tensor product $B \otimes_{k} k[\mathbf{x}]$ is just $B[\mathbf{x}]$, the ring of polynomials with coefficients in $B$. Similarly, we write $B\left[t, t^{-1}, \mathbf{x}\right]:=B[\mathbf{x}] \otimes_{k} k\left[t, t^{-1}\right]$. Let $\xi \in \mathbb{R}^{d}$, we denote by $f_{\xi} \in\left(\mathbb{R}^{d}\right)^{*}$ the linear form defined by $f_{\xi}\left(\alpha_{1}, \ldots, \alpha_{d}\right)=\sum \alpha_{i} \xi_{i}$. If $\xi \in \mathbb{Z}^{d}$, and $I \subset B[\mathbf{x}], I_{\xi}\left[t, t^{-1}\right] \subset B\left[t, t^{-1}, \mathbf{x}\right]$ denotes the ideal generated by the elements $t \cdot g:=\sum t^{-f_{\xi}(e)} c_{e} \mathbf{x}^{e}$ where $g=\sum c_{e} \mathbf{x}^{e} \in I$. We denote by $I_{\xi}[t] \subset B[t, \mathbf{x}]$ the ideal $I_{\xi}\left[t, t^{-1}, \mathbf{x}\right] \cap B[t, \mathbf{x}]$. When $a \in k$, we denote by $I_{\xi}[a] \subset B[\mathbf{x}]$ the ideal $\phi_{a}\left(I_{\xi}[t]\right)$ where $\phi_{a}: B[t, \mathbf{x}] \rightarrow B[\mathbf{x}]$ is the evaluation morphism sending $t$ to $a$. In particular $I=I_{\xi}[1]$.

Definition 2.1. We denote by $\mathcal{H}^{B B(\Delta, \xi)}(B)$, or more simply by $\mathcal{H}^{B B(\Delta)}(B)$ when $\xi$ is obvious, the set of ideals $I \subset B[\mathbf{x}]$ such that $\lim _{t \rightarrow 0} t \cdot I=I^{\Delta}$, which means:

- $B[t, \mathbf{x}] / I_{\xi}[t]$ is a locally free $B[t]$-module of rank $n=\# \Delta$.
- $I_{\xi}[0]=I^{\Delta} \subset B[\mathbf{x}]$
$\mathcal{H}^{B B(\Delta)}$ is a covariant functor from the category of $k$-algebras to the category of sets. We call it the Bialynicki-Birula functor.

Remark that the sign convention for the action is consistent with the choices made in the introduction. The action of the torus was $t \cdot\left(a_{1}, \ldots, a_{d}\right):=\left(t^{\xi_{1}} a_{1}, \ldots, t^{\xi_{d}} a_{d}\right)$ on $\mathbb{A}^{d}$. This corresponds to the torus action on the polynomial ring $k[\mathbf{x}]$ which is trivial on scalars and is given by $t \cdot \mathbf{x}^{e}:=t^{-f_{\xi}(e)} \mathbf{x}^{e}$ on monomials.

The first bulleted item of the definition says that $\operatorname{Spec} B[t, \mathbf{x}] / I_{\xi}[t] \rightarrow \operatorname{Spec} B[t]$ is a finite flat family. The second bulleted item says that its fiber over $t=0$ is the monomial subscheme of $\operatorname{Spec} B[t, \mathbf{x}] /\langle t\rangle=\operatorname{Spec} B[\mathbf{x}]$ defined by $I^{\Delta}$. The limit $\lim _{t \rightarrow 0} t \cdot I$ is therefore a well-defined flat limit.

Definition 2.2. Let $\xi \in \mathbb{R}^{d}$. We define the partial order $<_{\xi}$ on monomials in $k[\mathbf{x}]$ by setting $\mathbf{x}^{e}<_{\xi} \mathbf{x}^{g}$ if $f_{\xi}(e)<f_{\xi}(g)$, and letting $\mathbf{x}^{e}$ and $\mathbf{x}^{g}$ be incomparable if $f_{\xi}(e)=f_{\xi}(g)$. Since we identify monomials and exponents, we adopt the convention $f_{\xi}\left(\mathbf{x}^{e}\right):=f_{\xi}(e)$.
A variable $x_{i}$ is called positive (resp. negative, non-positive, non negative) for $<_{\xi}$ if $\xi_{i}>0$ (resp $\xi_{i}<0$, $\xi_{i} \leq 0, \xi_{i} \geq 0$ ).

If the weights $\xi_{i}$ are linearly independent over $\mathbb{Q}$, then $<_{\xi}$ is a total order on monomials. Otherwise, the order is only partial. The most interesting case for us is when $\xi \in \mathbb{Z}^{d}$ is the weight vector of the action.

## Definition 2.3.

- Let $\xi \in \mathbb{R}^{d},<\xi$ be the associated order, and $f=\sum a_{e} \mathbf{x}^{e} \in B[\mathbf{x}]$. Let $\mathbf{x}^{e_{1}}, \ldots, \mathbf{x}^{e_{l}}$ be the maximal monomials appearing in $f$ (with non-vanishing coefficients $a_{e_{i}}, i=1 \ldots l$ ). The initial form of $f$ for $<_{\xi}$ is $\operatorname{in}(f):=\sum a_{e_{i}} \mathbf{x}^{e_{i}}$. This is a unique term when $<_{\xi}$ is a total order, but may be a sum of terms otherwise. We denote by in $(I)$ the ideal generated by the elements $\operatorname{in}(f), f \in I$.
- Let $I \subset B[\mathbf{x}]$ and $m=\mathbf{x}^{e}$ be a monomial. We denote by $\mathrm{in}_{m}(I) \subset B$ the ideal generated by the elements $b \in B$ such that $\operatorname{in}(f)=b m$ for some $f \in I$.
- Let $\Delta$ be a standard set of cardinality $n$. The ideal $I$ is called $\Delta$-monic if $\operatorname{in}_{m}(I)=\langle 1\rangle$ for all $m \notin \Delta$ and $\mathrm{in}_{m}(I)=0$ for all $m \in \Delta$.

The following proposition connects Białynicki-Birula families and the order $<_{\xi}$ : an element $g$ is in the limit ideal $I[0]=\lim _{t \rightarrow 0} t \cdot I$ iff its homogeneous components $g_{i}$ are initial parts of elements of $I$.

Proposition 2.4. Let $I \subset B[\mathbf{x}]$. Let $g=\sum_{i} g_{i} \in B[\mathbf{x}]$, with $g_{i}=\sum_{g_{\xi}(e)=i} c_{e} \mathbf{x}^{e}$. Then the following conditions are equivalent:
(1) $g \in I_{\xi}[0]$
(2) $\forall i, \exists l_{i} \in I, \operatorname{in}\left(l_{i}\right)=g_{i}$ for the order $<_{\xi}$.

Proof. To prove $2 \Rightarrow 1$, let $s_{i}:=t^{i}\left(t \cdot l_{i}\right)$. We have $s_{i} \in I_{\xi}[t]$ and $s_{i}(0)=\operatorname{in}\left(l_{i}\right)=g_{i}$. Thus $g_{i} \in I_{\xi}[0]$ and $g=\sum g_{i} \in I_{\xi}[0]$ too.
Conversely, if $g \in I_{\xi}[0]$, then $g=h(0)$ with $h \in I_{\xi}[t]$. We decompose $h$ as $h=\sum_{j} s_{j}$ with $s_{j}=P_{j}\left(t \cdot l_{j}\right)$ for some $l_{j} \in I$ and $P_{j} \in B\left[t, t^{-1}, \mathbf{x}\right]$. By linearity, we may suppose that $P_{j}=c_{j} \mathbf{x}^{a_{j}} t^{b_{j}}$. Since $P_{j}\left(t \cdot l_{j}\right)=t^{b_{j}+f_{\xi}\left(a_{j}\right)}\left(t \cdot\left(c_{j} \mathbf{x}^{a_{j}} l_{j}\right)\right)$ we may suppose that $a_{j}=0$ and $c_{j}=1$. Grouping terms by linearity, we may suppose that the exponents $b_{j}$ are pairwise distinct. If we put the weights on the variables with the formulas $\operatorname{deg}\left(\mathbf{x}^{e}\right)=f_{\xi}(e)$ and $\operatorname{deg}(t)=1$, then $s_{j}=t^{b_{j}}\left(t \cdot l_{j}\right)$ is the only term with degree $b_{j}$ in the sum $h=\sum_{j} s_{j}$. Since $h \in B[t, \mathbf{x}]$ it follows that $s_{j} \in B[t, \mathbf{x}]$ and $b_{j} \geq f_{\xi}(e)$ for every term $c_{e} \mathbf{x}^{e}$ in $l_{j}$. If $b_{j}>f_{\xi}(e)$ for every term, then $s_{j}(0)=0$ and we may ignore $s_{j}$ in the definition of $g=\sum s_{j}(0)$. If the equality occurs for some term, then $s_{j}(0)=\operatorname{in}\left(l_{j}\right)$. This shows that the homogeneous components of $g$ are initial terms.

Lemma 2.5. Let $A$ be a commutative ring, $M$ be an $A$-module, $n \geq 1$ be an integer and $f: A^{n} \oplus M \rightarrow A^{n}$ be an injective morphism of $A$-modules. Then $M=0$.

Proof. First, we suppose that $A$ is Noetherian. Let $M^{i}$ be the direct sum $M \oplus \cdots \oplus M$ with $i$ copies of $M$. Let $f_{i}: A^{n} \oplus M^{i+1} \rightarrow A^{n} \oplus M^{i}$ be the morphism defined by $f_{i}\left(a, m_{0}, m_{1}, \ldots, m_{i}\right)=\left(f\left(a, m_{0}\right), m_{1}, \ldots, m_{i}\right)$. In particular $f_{0}=f$. Let $c_{i}=f_{0} \circ f_{1} \circ \cdots \circ f_{i}: A^{n} \oplus M^{i+1} \rightarrow A^{n}$. Since $f$ is injective, $f_{i}$ and $c_{i}$ are injective for all $i \geq 0$. Let $M_{i+1}^{i} \subset M^{i+1}$ be the set of elements $\left(m_{0}, \ldots, m_{i}\right)$ with $m_{0}=0$. If $M$ is not zero, the inclusion $M_{i+1}^{i} \subset M^{i+1}$ is strict . The inclusions $0 \oplus M^{i}=f_{i}\left(0 \oplus M_{i+1}^{i}\right) \subset f_{i}\left(0 \oplus M^{i+1}\right)$, and $c_{i-1}\left(0 \oplus M^{i}\right) \subset c_{i-1} \circ f_{i}\left(0 \oplus M^{i+1}\right)=c_{i}\left(0 \oplus M^{i+1}\right)$ are strict too by injectivity. Thus $M=0$, otherwise we would have a non stationary increasing sequence $c_{i-1}\left(0 \oplus M^{i}\right)$ of submodules in the Noetherian module $A^{n}$.

To prove the general case, let $N$ be the matrix of $g: A^{n} \rightarrow A^{n}$, where $g$ is the restriction of $f$ to $A^{n} \oplus 0$, and let $m$ be any element of $M$. Let $B$ be the $\mathbb{Z}$-subalgebra of $A$ generated by the entries of $N$ and the entries of the vector $f(0, m)$. Let $h: B^{n} \oplus B m \rightarrow B^{n}$ be the morphism of $B$-modules obtained by restriction of $f$. Since $B$ is Noetherian and $h$ is injective, we get $m=0$.

Lemma 2.6. Let $I \subset B[\mathbf{x}]$ be an ideal with $\operatorname{in}(I)=I^{\Delta}$ for the order $<_{\xi}$ and $B[\mathbf{x}] / I$ free of rank $\# \Delta$ as a B-module. Then the monomials $\mathbf{x}^{e}, e \in \Delta$ form a basis of $B[\mathbf{x}] / I$.

Proof. The monomials $\mathbf{x}^{e}, e \in \Delta$ are independent $\bmod I$ since $\operatorname{in}(I)=I^{\Delta}$. We argue by contradiction and we suppose that they generate a strict $B$-submodule $C \subset B[\mathbf{x}] / I$. Then there exists $f \in B[\mathbf{x}]$ whose class $\dot{f} \in B[\mathbf{x}] / I$ satisfies $\dot{f} \notin C$. Let $B\left[\mathbf{x}^{\Delta}\right] \subset B[\mathbf{x}]$ be the $B$-submodule generated by the monomials $\mathbf{x}^{e}, e \in \Delta$. Replacing $f$ by $f-h, h \in B\left[\mathbf{x}^{\Delta}\right]$, one may suppose that $f$ has no terms in $\Delta$. In particular, there exists $g \in I$, with in $(g)=\operatorname{in}(f)$. Replacing $f$ by $f-g$ lowers the initial form of $f$. Repeating the process of killing the terms of $f$ in $\Delta$ and of lowering the initial form $\operatorname{in}(f)$, one may suppose that $\operatorname{in}(f)$ is smaller than any monomial in the finite set $\Delta$. It follows that any term $\lambda_{e} \mathbf{x}^{e}$ in $f=\sum \lambda_{e} \mathbf{x}^{e}$ is smaller than any monomial in $\Delta$ too, hence not in $\Delta$. In particular $f \in I^{\Delta}$ and there is a direct sum

$$
B f \oplus B\left[\mathbf{x}^{\Delta}\right] \subset B[\mathbf{x}]=I^{\Delta} \oplus B\left[\mathbf{x}^{\Delta}\right]
$$

Since in $(I)=I^{\Delta}$, the intersection of $I$ with $B f \oplus B\left[\mathbf{x}^{\Delta}\right]$ contains only elements of the form $b f \oplus 0$. Quotienting the displayed inclusion by $I$ yields an injection $B f /(I \cap B f) \oplus B\left[\mathbf{x}^{\Delta}\right] \rightarrow B[\mathbf{x}] / I$. This contradicts Lemma 2.5 since $\operatorname{rank}(B[\mathbf{x}] / I)=\operatorname{rank}\left(B\left[\mathbf{x}^{\Delta}\right]\right)=\# \Delta$.

Proposition 2.7. Let $\xi \in \mathbb{Z}^{d}$ and $<_{\xi}$ the associated partial order. Then $I \in \mathcal{H}^{B B(\Delta, \xi)}(B)$ if, and only if, the following conditions are satisfied:
(1) $\operatorname{in}(I)=I^{\Delta}$
(2) $B[\mathbf{x}] / I$ is a locally free $B$-module of rank \# $\Delta$.

Proof. We only prove that the conditions imply that $I \in \mathcal{H}^{B B(\Delta, \xi)}$, the converse being easy using Proposition 2.4.

Let us temporarily assume that $B[t, \mathbf{x}] / I_{\xi}[t]$ is a locally free $B[t]$-module. Then the flat limit $\lim _{t \rightarrow 0} t \cdot I$ exists and Proposition 2.4 implies that this limit is $I^{\Delta}$. For proving that $I \in \mathcal{H}^{B B(\Delta, \xi)}(B)$, it therefore remains to show that $P[t]:=B[t, \mathbf{x}] / I_{\xi}[t]$ is a locally free $B[t]$-module.

Upon localizing $B$ (that is to say, upon replacing $\operatorname{Spec} B$ with an affine open subset), we may assume that $B[\mathbf{x}] / I$ is $B$-free of rank \# $\Delta$.

The monomials $\mathbf{x}^{e}, e \in \Delta$ are a basis of $B[\mathbf{x}] / I$ by Lemma 2.6. They are therefore also a basis of the $B\left[t, t^{-1}\right]$-module $B[\mathbf{x}] / I \otimes_{k} k\left[t, t^{-1}\right] \simeq B\left[\mathbf{x}, t, t^{-1}\right] / I_{\xi}\left[t, t^{-1}\right]$ the later isomorphism being given by the torus action $t \cdot \mathbf{x}^{e}=t^{-\xi \cdot e} \mathbf{x}^{e}$ on the polynomial ring.

The monomials $\mathbf{x}^{e}$, for $e \in \Delta$, remain linearly independent in the $B[t]$-module $P[t]$. It remains to prove that $P[t]$ is a finite $B[t]$-module generated by the elements $\mathbf{x}^{e}, e \in \Delta$. It suffices to exhibit for every monomial $m \in B[\mathbf{x}], m \notin \Delta$ a polynomial $P=m+\sum_{e \in \Delta} c_{e} \mathbf{x}^{e}$ with $P \in I_{\xi}[t]$. Since the elements $\mathbf{x}^{e}, e \in \Delta$ form a basis of $B[\mathbf{x}] / I$, the decomposition of $m$ yields an expression $m=\sum c_{e} \mathbf{x}^{e} \bmod I$. By condition 1$), c_{e}=0$ if $x^{e} \geq_{\xi} m$. Thus we can take $P=t_{\xi}^{f_{\xi}(m)}\left(t \cdot\left(m-\sum c_{e} \mathbf{x}^{e}\right)\right)$.

Remark 2.8. According to the intuition from Gröbner bases, one could think that the second condition is a consequence of the first one. This is not the case, as is shown by the example $d=1, I=\left\langle x^{2}+x^{3}\right\rangle, \xi=-1$, $\Delta=\{1, x\}$.

Definition 2.9. Let $\xi \in \mathbb{R}^{d}$ and $<\xi$ the corresponding order. An ideal $I \subset B[\mathbf{x}]$ with $B[\mathbf{x}] / I$ finite is called bounded for $<_{\xi}$ if for every non-positive variable $x_{i}$, there exists an integer $r_{i} \geq 0$ with $x_{i}^{r_{i}} \in I$.
Lemma 2.10. Let $I \in \mathcal{H}^{B B(\xi, \Delta)}(B)$ be an ideal. Then I is bounded for $<\xi$.
Proof. Up to reordering the components of $\xi$, one can assume that $\xi_{i} \geq \xi_{i+1}$. We denote by $k, l$ the integers such that $\xi_{i}>0 \Leftrightarrow i<k$ and $\xi_{i}<0 \Leftrightarrow i \geq l$.

The monomials with exponents in $\Delta$ form a basis of $B[\mathbf{x}] / I$ by Proposition 2.6. In particular, every monomial $\mathbf{x}^{e}$ leads to an element $\mathbf{x}^{e}+\sum_{m \in \Delta} c_{m} \mathbf{x}^{m}$ in $I$. For $i \geq l, r_{i} \gg 0$ and $\mathbf{x}^{e}=x_{i}^{r_{i}}$, we have $c_{m}=0$ since $\operatorname{in}(I)=I^{\Delta}$. Thus $h_{i}:=x_{i}^{r_{i}} \in I$.

Let $x_{i}$ be a non-positive variable of weight 0 . To prove that $x_{i}^{r} \in I$ for large $r$, we shall prove that for any monomial $m \in B\left[x_{l}, \ldots, x_{d}\right], x_{i}^{r} m \in I$ for large $r$, and subsequently apply that to the monomial $m=1$. If the exponent $\left(e_{l}, \ldots, e_{d}\right)$ of $m=x_{l}^{e_{l}} \cdots x_{d}^{e_{d}}$ is large (in concrete terms, if $e_{i} \geq r_{i}$ for some $i$ ), then $m \in I$ by the above. We are therefore left with a finite collection of $m$ for which the claim has to be checked. We proceed by induction over $m$. Let $m$ be a minimal element of the finite collection for which the claim has not been proved yet. By our hypothesis on $I$, for large $r$, the monomial $x_{i}^{r} m$ lies in the limit ideal $I_{\xi}[0]$, thus $I$ contains an element $f=x_{i}^{r} m+R$, where all terms in $R$ are strictly smaller with respect to $<_{\xi}$ than $x_{i}^{r} m$. We write the monomials appearing in $R$ as $m_{1} m_{2} m_{3}$, with $m_{1} \in k\left[x_{1} \ldots, x_{k-1}\right], m_{2} \in k\left[x_{k} \ldots, x_{l-1}\right]$, $m_{3} \in k\left[x_{l}, \ldots, x_{d}\right]$. Such a product satisfies $m_{1} m_{2} m_{3}<_{\xi} x_{i}^{r} m$ only if $m_{3}<_{\xi} m$. By induction we know that when multiplying by an adequate power $x_{i}^{p}$, we get $x_{i}^{p} m_{3} \in I$, hence $x_{i}^{r+p} m=x_{i}^{p} f-x_{i}^{p} R \in I$, as required.
Proposition 2.11. Let $I \in \mathcal{H}^{B B(\xi, \Delta)}(B)$ be an ideal. Then for every non-positive variable $x_{i}$, we have $x_{i}^{n} \in I$, where $n:=\# \Delta$ is the cardinality of $\Delta$.

Proof. After applying a suitable permutation, we may assume that $x_{1}, \ldots, x_{l}$ (resp. $x_{l+1}, \ldots, x_{d}$ ) are the non-positive (resp. positive) variables, and we shall prove that $x_{1}^{n} \in I$. For every monomial $m \in k\left[x_{2}, \ldots, x_{d}\right]$,
there exists a unique $h(m) \in \mathbb{N}$, call it the height, such that $m^{-}:=x_{1}^{h(m)-1} m \in \Delta$ and $m^{+}:=x_{1}^{h(m)} m \notin \Delta$. In particular, $h(m)=0$ iff $m \notin \Delta$, and all heights of all $\mathbf{x}^{e} \in k\left[x_{2}, \ldots, x_{d}\right]$ sum up to $n$, the cardinality of $\Delta$.

Let $M \subset k\left[x_{2}, \ldots, x_{l}\right]$ be the set of monomials with exponents in $\Delta^{\prime}:=\mathbb{N}^{l-1} \cap \Delta$. The set $M$ is finite, and we number its elements such that $m_{1}^{+} \leq_{\xi} m_{2}^{+} \leq_{\xi} \cdots \leq_{\xi} m_{\# M}^{+}$. For any monomial $m \in k\left[x_{2}, \ldots, x_{d}\right]$, we define $H(m):=\sum_{m_{i}^{+} \leq \xi m^{+}} h\left(m_{i}\right)$. Note that $H(m) \leq \sum_{\mathbf{x}^{e} \in k\left[x_{2}, \ldots, x_{d}\right]} h\left(\mathbf{x}^{e}\right)=n$. In particular, if we prove

$$
\begin{equation*}
\forall m=\mathbf{x}^{e} \in k\left[x_{2}, \ldots, x_{l}\right], x_{1}^{H(m)} m \in I \tag{1}
\end{equation*}
$$

then for the particular choice $m:=1$, (1) implies $x_{1}^{n} \in I$, which concludes the proof.
Since $I$ is bounded by Lemma 2.10, for each $i$ with $2 \leq i \leq l$, there exists some $r_{i}$ such that $x_{i}^{r_{i}} \in I$. Thus (1) is true if $x_{i}^{r_{i}}$ divides $m$. It follows that the set $C:=\left\{m \in k\left[x_{2}, \ldots, x_{l}\right] \mid m\right.$ is a monomial, $\left.x_{1}^{H(m)} m \notin I\right\}$ is finite. If (1) is not true, then $C \neq \emptyset$ and $C$ contains an element $m_{0}$ which is minimal in the sense that $m_{0}^{+} \leq_{\xi} m^{+}$for every $m \in C$, $m \neq m_{0}$.

The decomposition of $m_{0}^{+}$on the basis $\mathbf{x}^{e}, e \in \Delta$ of $B[\mathbf{x}] / I$ yields an expression $f=m_{0}^{+}+\sum_{e \in \Delta} c_{e} \mathbf{x}^{e}$ with $f \in I$. By Proposition 2.7, $c_{e}=0$ if $x^{e} \geq_{\xi} m_{0}^{+}$. Thus

$$
\begin{equation*}
f=m_{0}^{+}+\sum_{\mathbf{x}^{e}<m_{0}^{+}} c_{e} \mathbf{x}^{e} \tag{2}
\end{equation*}
$$

We have

$$
\mathbf{x}^{e}=x_{1}^{e_{1}} \cdots x_{d}^{e_{d}} \in \Delta \Rightarrow x_{1}^{e_{1}} \cdots x_{l}^{e_{l}} \in \Delta \Rightarrow e_{1}<h\left(x_{2}^{e_{2}} \cdots x_{l}^{e_{l}}\right)
$$

Since $x_{1}$ is a non-positive variable, and $x_{l+1}, \ldots, x_{d}$ are positive, we get:

$$
\left(x_{2}^{e_{2}} \cdots x_{l}^{e_{l}}\right)^{+}=x_{1}^{h\left(x_{2}^{e_{2}} \cdots x_{l}^{e_{l}}\right)}\left(x_{2}^{e_{2}} \cdots x_{l}^{e_{l}}\right) \leq_{\xi} x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{l}^{e_{l}} x_{l+1}^{e_{l+1}} \cdots x_{d}^{e_{d}}=\mathbf{x}^{e}
$$

Here is the upshot of the above: if $C \neq \emptyset, m_{0} \in C$ is its minimum, and a monomial $\mathbf{x}^{e}=x_{1}^{e_{1}} \cdots x_{d}^{e_{d}}$ appears in a term of $f$, then $m_{e}:=x_{2}^{e_{2}} \cdots x_{l}^{e_{l}}$ satisfies $m_{e}^{+} \leq_{\xi} \mathbf{x}^{e}<_{\xi} m_{0}^{+}$. Thus $H\left(m_{e}\right) \leq H\left(m_{0}\right)-h\left(m_{0}\right)$, and by minimality of $m_{0}, m_{e} x_{1}^{H\left(m_{e}\right)} \in I$. It follows that the multiple $\mathbf{x}^{e} x_{1}^{H\left(m_{0}\right)-h\left(m_{0}\right)}$ lies in $I$. The product of the expression (2) with $x_{1}^{H\left(m_{0}\right)-h\left(m_{0}\right)}$ yields

$$
m_{0} x_{1}^{H\left(m_{0}\right)}=m_{0}^{+} x_{1}^{H\left(m_{0}\right)-h\left(m_{0}\right)}=f x_{1}^{H\left(m_{0}\right)-h\left(m_{0}\right)}-\sum c_{e} \mathbf{x}^{e} x_{1}^{H\left(m_{0}\right)-h\left(m_{0}\right)} \in I,
$$

a contradiction. It follows that $C=\emptyset$, and (1) is true.
Using the order $<_{\xi}$, it not in general possible to make a division like in Gröbner basis theory because the algorithm may not terminate, due to the possible negative signs in the coordinates of $\xi$. However, for Białynicki-Birula families, a substitute of a division is possible. This proposition will not be used in the sequel, but we include it for itself.

Proposition 2.12. Let $I \in \mathcal{H}^{B B(\xi, \Delta)}(B)$. Let $o_{1}, \ldots, o_{u}$ be the outside corners of $\Delta$, and $f_{1}, \ldots, f_{u} \in I$ be elements with $\operatorname{in}_{<_{\xi}}\left(f_{i}\right)=o_{i}$. Then for all $f \in B[\mathbf{x}]$, there exists a division

$$
f=\sum \lambda_{i} f_{i}+R_{\Delta}+R^{\prime}
$$

such that

- each term $\tau=b \mathbf{x}^{e}$ of $R_{\Delta}$ satisfies $e \in \Delta$,
- for every term $c \mathbf{x}^{s}$ of $R^{\prime}$ and for every $m \in \Delta, \mathbf{x}^{s}<_{\xi} \mathbf{x}^{m}$.

For every such division,

- $R_{\Delta}$ is independent of the choice of the division,
- $f \in I$ if, and only if, $R_{\Delta}=0$,
- the map $f \mapsto R_{\Delta}$ is a homomorphism of $B$-modules $B[\mathbf{x}] \rightarrow B[\mathbf{x}] / I$, where we identify the latter module with $B[\Delta]:=\oplus_{e \in \Delta} B \mathbf{x}^{e}$ using Lemma 2.6.

Proof. To construct the expected expression $f=\sum \lambda_{i} f_{i}+R_{\Delta}+R^{\prime}$, we proceed in several steps. At each step $j$, we have an expression $f=\sum \lambda_{i j} f_{i}+T_{j}$. For $j=0$, we take $\lambda_{i 0}:=0, T_{0}:=f$. We decompose $T_{j}=R_{\Delta, j}+R_{j}^{\prime}$, with $R_{\Delta, j}:=\sum_{m \in \Delta} c_{m j} m$ and $R_{j}^{\prime}:=\sum_{m \notin \Delta} c_{m j} m$. The initial part in $n_{<_{\xi}}\left(R_{j}^{\prime}\right)$ contains a term $\mu_{j} \mathrm{in}_{<_{\xi}}\left(f_{i_{j}}\right)$ for some $f_{i_{j}}$ by hypothesis. We set $\lambda_{i_{j}(j+1)}:=\lambda_{i_{j} j}+\mu_{j}$ and $\lambda_{i^{\prime}(j+1)}:=\lambda_{i^{\prime} j}$ for $i^{\prime} \neq i_{j}$. Then $T_{j+1}:=f-\sum \lambda_{i(j+1)} f_{i}=T_{j}-\mu_{j} f_{i_{j}}$ decomposes, analogously as above, into $T_{j+1}=R_{\Delta, j+1}+R_{j+1}^{\prime}$.

If, for some $j$, it happens that $R_{j}^{\prime}=0$, then we define $R_{\Delta}:=R_{\Delta, j}$ and $R^{\prime}:=0$, and have constructed the expected expression. Otherwise, after a finite number of steps, the terms in $R_{j}^{\prime}$ are arbitrarily small and the condition of the second bullet is satisfied with $R^{\prime}=R_{j}^{\prime}$ and $R_{\Delta}=R_{\Delta, j}$.

If $f \in I$, then in the expression $f=\sum \lambda_{i} f_{i}+R_{\Delta}+R^{\prime}$, we have $R_{\Delta}=0$ since in ${\iota_{\varepsilon}}_{( }\left(f-\sum \lambda_{i} f_{i}\right)$ cannot lie in $\Delta$ by hypothesis. In particular, if $f=\sum \mu_{i} f_{i}+S_{\Delta}+S^{\prime}$ is another division, we take their difference and obtain a division of $0 \in I$, which implies $R_{\Delta}=S_{\Delta}$.

It is obvious that $f \mapsto R_{\Delta}$ is a homomorphism of $B$-modules as it is possible to add divisions, or to multiply them with a scalar $\lambda \in B$.

Let us now consider the $B$-submodule $B[\Delta]$ of $B[\mathbf{x}]$. The identity on $B[\Delta]$ factors as $B[\Delta] \rightarrow B[\mathbf{x}] \rightarrow B[\Delta]$ where the first arrow is the inclusion and the second is the morphism $R_{\Delta}$. The above implies that this factorization induces a factorization $B[\Delta] \rightarrow B[\mathbf{x}] / I \rightarrow B[\Delta]$. This composition is surjective between locally free modules of the same rank, so it is an isomorphism. In particular, we obtain that $R_{\Delta}=0$ implies $f \in I$.

## 3. Definition of monic functors

The goal of this section is to define monic functors, which parameterize ideals with a prescribed initial ideal. In general, the initial ideal does not commute with arbitrary base change, but only with flat base change (see [BGS93]). However, we prove that $\Delta$-monic families are stable by arbitrary (non flat) base change (Proposition 3.3) and functoriality follows.

Remark 3.1. When working with monic functors, we will consider total orders rather than the partial orders used in the previous section. The reason is that we control the base changes and the functoriality for the graded parts $\operatorname{in}_{m}(I)$. With a total order, $\operatorname{in}(I)=\bigoplus_{m \text { monomial }} \mathrm{in}_{m}(I) m$, thus we control the base change of the initial ideal through its graded pieces. The following proposition shows that this control may fail for a partial order.

Proposition 3.2. Let $I \subset B[\mathbf{x}]$ be a $\Delta$-monic ideal for an order $<$ refining the partial order $<\xi$, $i e$. in $\lim _{<, m}(I)=B$ if $m \notin \Delta$ and $\mathrm{in}_{<, m}(I)=0$ if $m \in \Delta$. If $<$ is a total order, then $\operatorname{in}(I)=I^{\Delta}$. This may not be true if $<$ is not a total order.

Proof. If the order is total, then the initial ideal is generated by terms thus it is graded by the degrees of the corresponding monomials. As for the counterexample, we take $I \subset k[x, y]$ generated by $\left\langle x+y, x^{2}, x y, y^{2}\right\rangle$, $\Delta=\{1, x, y\}, \xi=(1,1)$ and $<$ identical to $<_{\xi}$ (trivial refinement). Then the sum of the graded parts $\oplus_{m}$ monomial $\mathrm{in}_{m}(I) m=I^{\Delta}$ but in $(x+y)=x+y \in \operatorname{in}(I)$ is not in $I^{\Delta}$.

Proposition 3.3. Let $<$ be a total order refining $<\xi$. Let $I \subset B[\mathbf{x}]$ be a bounded ideal. Let $m$ be a monomial. If for all $m^{\prime} \geq m$, either $\mathrm{in}_{m^{\prime}}(I)=0$ or $\mathrm{in}_{m^{\prime}}(I)=\langle 1\rangle$ holds, then for every base change $B \rightarrow A$, the equality $\mathrm{in}_{m}(I A[\mathbf{x}])=\mathrm{in}_{m}(I) A$ holds.
 that $\mathrm{in}_{m}(I A[\mathbf{x}]) \neq 0$. Choose $f \in I A[\mathbf{x}]$ with $\operatorname{in}(f)=a m, a \neq 0$. Let $x_{j}$ be a non-positive variable. Then
$x_{j}^{d_{j}} \in I$ for $d_{j}$ large. In particular, replacing $f$ with $f-\lambda x_{j}^{e}$, we may assume that $f$ has no term divisible by $x_{j}^{d_{j}}$. Choose an expression

$$
\begin{equation*}
f=\sum a_{i} f_{i} \tag{3}
\end{equation*}
$$

with $a_{i} \in A$ and $f_{i} \in I \subset B[\mathbf{x}]$. As above, we may assume that none of the $f_{i}$ contains a term divisible by $x_{j}^{d_{j}}$. Let $m^{\prime}$ be the maximal monomial appearing in the $f_{i}$. Note that $m^{\prime} \geq m$ and, more precisely, $m^{\prime}>m$ since $\mathrm{in}_{m}(I)=0$. Among all possible expressions (3), choose one with minimal $m^{\prime}$. (Although $<$ is not necessarily a monomial order, a minimal $m^{\prime}$ exists since we have bounded the exponent of the non-positive variables.) Let $J:=\left\{i: \operatorname{in}\left(f_{i}\right)=\lambda_{i} m^{\prime}\right\}$ be the set of indices of $f_{i}$ with initial monomial $m^{\prime}$. The coefficient of $m^{\prime}$ vanishes on the right hand side of (3), thus $\sum_{i \in J} a_{i} \operatorname{in}\left(f_{i}\right)=\sum_{i \in J}\left(a_{i} \lambda_{i}\right) m^{\prime}=0$. Thus $\sum_{i \in J} a_{i} \lambda_{i}=0$. Let $g \in I$ with $\operatorname{in}(g)=m^{\prime}$ and such that $g$ has no term divisible by $x_{j}^{d_{j}}$. Let $f_{i}^{\prime}:=f_{i}$ if $i \notin J$ and $f_{i}^{\prime}:=f_{i}-\lambda_{i} g$ if $i \in J$. Then

$$
f=\sum a_{i} f_{i}^{\prime}
$$

and this expression contradicts the maximality of $m^{\prime}$.
Let $X=\operatorname{Spec} A$ be the affine scheme corresponding to the ring $A$. The ideal $\mathrm{in}_{m}(I) \subset A$ defines a closed subscheme of $X$. If $X$ is a scheme which is not affine, we wish to glue the local constructions we have been working with so far. Since open immersions are flat, the following proposition implies that gluing is possible and that for any sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X}[x]$, there is a well-defined sheaf of ideals $\mathrm{in}_{m}(\mathcal{I}) \subset \mathcal{O}_{X}$ on a possibly non-affine scheme $X$. This allows us to speak of bounded and monic, resp., ideal sheaves rather than ideals.

Proposition 3.4. Let $<$ be a total order refining $<\xi$. Let $I \subset B[\mathbf{x}]$ be a bounded ideal. Let $B \rightarrow A$ be a ring homomorphism which makes A a flat B-module. Then $\mathrm{in}_{m}(I A[\mathbf{x}])=\mathrm{in}_{m}(I) A$.

Proof. Theorem 3.6 of [BGS93] proves the statement in the case where < is a monomial order. The same proof also goes through in our context, provided that we take care of the high powers of the non-positive variables as we did in the proof of Proposition 3.3.

Definition 3.5. Let $<$ be a total order refining $<_{\xi}$ and $\Delta$ be a standard set. Let $B$ be a $k$-algebra and $\mathcal{H}^{\text {mon }(<, \Delta)}(B)$ be the set of ideals $I \subset B[\mathbf{x}]$ such that

- $I$ is bounded and $\Delta$-monic
- $B[\mathbf{x}] / I$ is $B$ locally free of rank \# $\Delta$

This defines a covariant functor $\mathcal{H}^{\text {mon }(<, \Delta)}$, which we call a monic functor, from the category of $k$-algebras to the category of sets.

Remark 3.6. The above definition makes sense. Indeed, it follows from the base change Proposition 3.3 that being a $\Delta$-monic ideal is stable by base change. The boundedness property and the local freeness are also stable by base change.

## 4. Monic functors are representable

The goal of this section is to prove that the monic functors are representable (Theorem 4.6).
Proposition 4.1. Let $<$ be a total order refining $<\xi$. Let $I \subset B[\mathbf{x}]$ be a bounded ideal with $B[\mathbf{x}] / I$ a locally free $B$-module of rank $n$, and let $\mathcal{I} \subset \mathcal{O}_{X}[\mathbf{x}]$ be the ideal sheaf on $X:=\operatorname{Spec} B$ defined by $I$. Let $\Delta$ be a standard set of cardinality $n$. There exists a locally closed subscheme $Z \subset X$ such that

- the restriction of $\mathcal{I}$ to $Z$ is a bounded $\Delta$-monic family,
- any morphism $f: \operatorname{Spec} A \rightarrow \operatorname{Spec} B$ such that $I A[\mathbf{x}]$ is a bounded $\Delta$-monic family factors through $Z$.

Proposition 4.1 is a particular case of Proposition 4.3, which we shall prove by induction.
Lemma 4.2. Let $I \subset B[\mathbf{x}]$ be a bounded ideal for the order $<_{\xi}$ such that $B[\mathbf{x}] / I$ is a finite $B$-module. Let $C:=\left\{x^{e} \mid e_{i}<r_{i}\right\}$ be a "hypercuboid of monomials" with edges of lengths $r_{i}$. Then it is possible to choose the integers $r_{1}, \ldots, r_{d}$ such that:

- the monomials in $C$ generate $B[\mathbf{x}] / I$ and $B[\mathbf{x}, t] / I_{\xi}[t]$
- if $m$ is a monomial with $m \notin C$, there exists $f \in I$ with $f=m+\sum_{e \in C} c_{e} x^{e}$, and $\mathrm{in}_{<_{\xi}}(f)=m$.

Proof. Since $I$ is bounded, if $x_{i}$ is non-positive, we can choose $r_{i}$ such that $x_{i}^{r_{i}} \in I$. Let $S$ be a set of monomials generators of $B[\mathbf{x}] / I$. We may suppose that for every $m$ in $S$ and every non positive variable $x_{i}$, the exponent of $x_{i}$ in $m$ is smaller than $r_{i}$. For $x_{i}$ positive, we can choose $r_{i}$ large, such that any monomial $m$ multiple of $x_{i}^{r_{i}}$ whose exponent in any non-positive variable $x_{j}$ is less than $r_{j}$ satisfies $m>_{\xi} s$ for any $s \in S$. Moreover, we may choose $r_{i}$ larger than the exponent of $x_{i}$ in any monomial of $S$. The monomials in $C$ generate $B[\mathbf{x}] / I$ as $C \supset S$.

Let $m \notin C$ be a monomial. Then $m$ is a multiple of some $x_{i}^{r_{i}}$. If one can take $x_{i}$ non-positive, $m \in I$ and we take $f=m$. If not, $x_{i}$ is positive and the decomposition in $B[\mathbf{x}] / I$ of $m$ on $S$ yields an expression $f=m-\sum_{s \in S} c_{s} \mathbf{x}^{s}$ in $I$ with $\mathrm{in}_{<_{\xi}}(f)=m$.

Then $I_{\xi}[t]$ contains the elements $t^{f_{\xi}(m)}(t \cdot f)=m-\sum_{s \in S} t_{\xi}(m)-f_{\xi}(s){ }_{c} \mathbf{c}_{s}^{s}$. It follows that the quotient $B[t, \mathbf{x}] / I_{\xi}[t]$ is a finite $B[t]$-module generated by the monomials in $C$.

Proposition 4.3. Let $<$ be a total order refining $<\xi$. Let $I \subset B[\mathbf{x}]$ be a bounded ideal such that $B[\mathbf{x}] / I$ is a locally free B-module of rank $n$, and let $\mathcal{I} \subset \mathcal{O}_{X}[\mathbf{x}]$ be the ideal sheaf on $X:=\operatorname{Spec} B$ defined by $I$. Let $C:=\left\{x^{e} \mid e_{i}<r_{i}\right\}$ be a "hypercuboid of monomials" of edge lengths $r_{i}$ and therefore, of cardinality $s:=\prod_{i=1}^{d} r_{i}$, satisfying the conditions of Lemma 4.2. We number the monomials $m_{i} \in C$ such that $m_{1}>m_{2}>\cdots>m_{s}$. Let $r \leq s$, and fix a map $\mu:\{1, \ldots, r\} \rightarrow\{0,1\}$. Then there exists a locally closed subscheme $Z_{r} \subset X$ (possibly empty) such that

- The sheaf of ideals $\mathcal{I}_{r}$, which we define as the restriction of $\mathcal{I}$ to $Z_{r}$, is a bounded monic family with $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m}=\langle 1\rangle$ for $m \notin C$ and $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{i}}=\langle\mu(i)\rangle$ for $1 \leq i \leq r$.
- Let $f: \operatorname{Spec} A \rightarrow \operatorname{Spec} B$ be a morphism and $K:=I A[\mathbf{x}]$. Then $\operatorname{in}(K)_{m}=\langle 1\rangle$ for $m \notin C$ and we have $\operatorname{in}(K)_{m_{i}}=\langle\mu(i)\rangle$ for $1 \leq i \leq r$ if, and only if, $f$ factors through $Z_{r}$.

Proof. We start with the first item, which we prove by induction on $r \geq 0$.
When $r=0$, we may take $Z_{0}=X$ since the condition in $\left(\mathcal{I}_{r}\right)_{m}=\langle 1\rangle$ for $m \notin C$ is true by construction of $C$. We may assume that $Z_{r-1}$ is adequately defined. Let $F_{r} \subset Z_{r-1}$ the closed subscheme defined by the sheaf of ideals $\mathcal{I}\left(F_{r}\right):=\operatorname{in}\left(\mathcal{I}_{r-1}\right)_{m_{r}}$. Let $O_{r}:=Z_{r-1} \backslash F_{r}$. We define

$$
Z_{r}:= \begin{cases}F_{r} & \text { if } \mu(r)=0 \\ O_{r} & \text { otherwise. }\end{cases}
$$

By Proposition 3.3 and the induction hypothesis, we have in $\left(\mathcal{I}_{r}\right)_{m}=\langle 1\rangle$ for $m \notin C$ and $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{i}}=\langle\mu(i)\rangle$ for $1 \leq i \leq r-1$. We have to prove that $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{r}}=\langle\mu(r)\rangle$. This is a local problem, so we may assume that $Z_{r-1} \subset X$ is a closed subscheme defined by an ideal $J_{r-1} \subset B$. If $\mu(r)=1$, the base change $Z_{r} \hookrightarrow Z_{r-1}$ is open, thus flat. Proposition 3.4 therefore shows that $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{r}}=\langle 1\rangle$ on a neighborhood of any $p \in Z_{r}$.

Assume now that $\mu(r)=0$. We claim that $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{r}}=0$. The problem is local, so we may assume that both $Z_{r}$ and $Z_{r-1}$ are affine. Accordingly, we replace the sheaves $\mathcal{I}_{r}$ and $\mathcal{I}_{r-1}$ by their respective ideals of global sections, which we denote by $I_{r}$ and $I_{r-1}$, respectively. We will argue by contradiction, supposing that there exists some $f \in I_{r}$ with $\operatorname{in}(f)=c m_{r}, c \neq 0$. Take some $g \in I_{r-1}$ that restricts to $f$ over $Z_{r}$. Using Lemma 4.2, we may assume that both $f$ and $g$ are linear combinations of monomials in $C, f=\sum_{m_{i} \in C} a_{i} m_{i}$ and $g=\sum_{m_{i} \in C} b_{i} m_{i}$, resp. Among all possible $g$, choose one which minimizes in $(g)$. Then in $(g)=d m$ with
$m \geq m_{r}$. Suppose that $m>m_{r}$. Since in $\left(I_{r-1}\right)_{m}=\langle 0\rangle$ or $\langle 1\rangle$ by induction hypothesis and since $d \neq 0$, we obtain that in $\left(I_{r-1}\right)_{m}=\langle 1\rangle$. Choose $h \in I_{r-1}$ with in $(h)=m$. Then $g^{\prime}:=g-d h$ contradicts the minimality of $g$. Thus $m=m_{r}$, and $d \in\left(I_{r-1}\right)_{m_{r}}=I\left(F_{r}\right)$ vanishes on $F_{r}$. Since $d$ restricts to $c$ on $F_{r}$, it follows that $c=0$. This is a contradiction, which finishes the proof of $\operatorname{in}\left(\mathcal{I}_{r}\right)_{m_{r}}=0$.

We now come to the second point. If $f$ factors through $Z_{r}$, then in $(K)_{m}=\langle 1\rangle$ for $m \notin C$ and in $(K)_{m_{i}}=$ $\langle\mu(i)\rangle$ for $1 \leq i \leq r$, as these properties are inherited from $Z_{r}$ by Proposition 3.3.

If, on the other hand, $f$ does not factor through $Z_{r}$, then we want to prove that $\operatorname{in}(K)_{m_{i}} \neq\langle\mu(i)\rangle$ for some $i, 1 \leq i \leq r$. We may assume that $f$ factors through $Z_{r-1}$, since in the complementary case, we are done by induction.

We first consider the case $\mu(r)=0$. By the factorization property of $f$ through $Z_{r-1}$, we may assume that $B$ is the coordinate ring of the scheme $Z_{r-1}$. We denote by $f^{\#}: B \rightarrow A$ the morphism associated with $f$. Since $f$ does not factor through $Z_{r}$, there exists some $g \in I_{r-1}$ with $\operatorname{in}(g)=c m_{r}, f^{\#}(c) \neq 0$. The pullback of $g$ to $A[\mathbf{x}]$ shows that $\operatorname{in}(I A[\mathbf{x}])_{m_{r}} \neq\langle 0\rangle$, and we are done.

Now consider the case $\mu(r)=1$. Since $Z_{r} \subset Z_{r-1}$ is open in this case, the factorization property of $f$ implies the existence of a point $p \in \operatorname{Spec} A$ with $f(p) \in Z_{r-1}$ and $f(p) \notin Z_{r}=Z_{r-1} \backslash F_{r}$. In particular, $\operatorname{in}(I \cdot k(p)[\mathbf{x}])_{m_{r}}=0$ by Proposition 3.3. Thus in $(K)_{m_{r}} \neq\langle 1\rangle$, as expected, since otherwise we would have in $(I \cdot k(p)[\mathbf{x}])_{m_{r}}=\langle 1\rangle$ by Proposition 3.3.

Remark 4.4. It is natural to try to formulate the last proposition in terms of commutative algebra, without sheafs of ideals. However, the subscheme $Z_{r}$ is in general not an affine scheme. For instance, suppose that $k$ is an algebraically closed field, let $B=k[a, b], I=I(a, b)=\left\langle x_{1}^{3}, x_{2}+a x_{1}+b x_{1}^{2}\right\rangle$ and choose the order such that $x_{1} \gg x_{2}$. Then for every closed point $\left(a_{0}, b_{0}\right) \neq(0,0)$, the ideal $I\left(a_{0}, b_{0}\right)$ contains an element with initial term $x_{1}^{2}$. Since $I(0,0)=\left\langle x_{2}, x_{1}^{3}\right\rangle$, it follows that the locus where $I$ admits $x_{1}^{2}$ as an initial term is the complement of the origin in the plane, which is not affine.
Proposition 4.5. Let $<$ be a total order refining $<_{\xi}$, Let $I \subset B[\mathbf{x}]$ be an ideal in $\mathcal{H}^{\operatorname{mon}(<, \Delta)}(B)$, then for every non-positive variable $x_{i}$, we have $x_{i}^{n} \in I($ with $n=\# \Delta)$.
Proof. By definition, $I$ is bounded and $\operatorname{in}_{<}(I)=I^{\Delta}$. Then we can run exactly the same proofs as in Lemma 2.6 and Proposition 2.11. In these statements, the order is partial, but we can a fortiori consider the proof for a total order.

Theorem 4.6. Let $<$ be a total order refining $<_{\xi}$. Then the monic functor $\mathcal{H}^{\operatorname{mon}(<, \Delta)}$ is representable by a locally closed scheme $H^{\operatorname{mon}(<, \Delta)}$ of $H^{n}\left(\mathbb{A}^{d}\right)$, where $n=\# \Delta$.
Proof. Let $I \subset B[\mathbf{x}]$ be an ideal defining a flat family $\operatorname{Spec} B[\mathbf{x}] / I \rightarrow \operatorname{Spec} B$ of relative length $n$ with $I$ bounded and $\Delta$-monic. Let $L_{i} \subset H^{n}\left(\mathbb{A}^{d}\right)$ be the closed subscheme of $H^{n}\left(\mathbb{A}^{d}\right)$ parametrizing the subschemes $Z$ included in the subscheme $\left\{x_{i}^{n}=0\right\} \subset \mathbb{A}^{d}$. Let $L:=\cap_{x_{i} \text { negative }} L_{i}$. Since the universal ideal of the Hilbert scheme (see [Ledll] for the construction and properties of that universal ideal) is bounded over $L$, there is a locally closed subscheme $L_{\Delta} \subset L$ parametrizing $\Delta$-monic ideals (Proposition 4.1). By the universal property of the Hilbert scheme, the ideal $I$ corresponds to a unique morphism $\phi: \operatorname{Spec} B \rightarrow H^{n}\left(\mathbb{A}^{d}\right)$. Proposition 4.5 implies that the morphism $\phi$ factors through $L$. Proposition 4.1 (the universal property of $L_{\Delta}$ ) implies that $\phi$ even factors through $L_{\Delta}$. Conversely, any morphism Spec $B \rightarrow L_{\Delta}$ yields a $\Delta$-monic bounded ideal by pullback of the universal ideal over the Hilbert scheme. Upon defining $H^{\text {mon }(<, \Delta)}:=L_{\Delta}$, we thus get the required result.

## 5. Bialynicki-Birula functors are representable

The goal of this section is to prove that Białynicki-Birula functors are representable.
So far, we have identified Białynicki-Birula families with families with an appropriate initial ideal for a partial order. On the other hand, we have proved that functors of families with a prescribed initial ideal for
a total order are representable. To conclude, we will prove that having an initial ideal for a partial order is equivalent to having an initial ideal for two ad hoc total orders. The representability of the Bialynicki-Birula functors will follow.

The total orders that we need are introduced in the following definition. We call them signed orders. They refine the partial orders $<_{\xi}$ used in Section 2. Like the orders $<_{\xi}$, they are not monomial orders in the sense of [Eis99, Chapter 15] if some components of $\xi$ are negative.

Definition 5.1. Suppose given a map $\epsilon:\{1, \ldots, d\} \rightarrow\{-1,1\}$. The partial order $<_{\xi}$ on the monomials can be refined to a total order $<$ as follows: For all monomials $\mathbf{x}^{e}$ and $\mathbf{x}^{f}$ with $f_{\xi}(e)=f_{\xi}(f)$, we have $\mathbf{x}^{e}<\mathbf{x}^{f}$ if, and only if, $\left(\epsilon(1) e_{1}, \ldots, \epsilon(d) e_{d}\right)<\left(\epsilon(1) f_{1}, \ldots, \epsilon(d) f_{d}\right)$ in the lexicographic order.

We call such an order $<$ a signed order refining $<\xi$.
Remark 5.2. Special cases of signed orders are implicit in [ES87, ES88, Gro96, Nak99], for studying the Hilbert scheme of points in the two-dimensional case.

Proposition 5.3. Let $\xi \in \mathbb{Z}^{d}$. Let $<_{+}$be a signed order which refines $<_{\xi}$ and $\epsilon$ the function that defines $<_{+}$. Let $<_{-}$be the "opposite" signed order, i.e. the signed order defined by the opposite function $-\epsilon$. Then $\mathcal{H}^{B B(\Delta, \xi)}=\mathcal{H}^{\operatorname{mon}\left(<_{+}, \Delta\right)} \cap \mathcal{H}^{\operatorname{mon}\left(<_{-, \Delta)}\right)}$.

Proof. It is clear that $\mathcal{H}^{B B(\Delta, \xi)} \subset \mathcal{H}^{\text {mon }\left(<_{+}, \Delta\right)} \cap \mathcal{H}^{\text {mon }\left(<_{-}, \Delta\right)}$ since by Proposition 2.7, $\mathcal{H}^{B B(\Delta, \xi)} \subset \mathcal{H}^{\text {mon }(<, \Delta)}$ for any refinement $<$ of $<_{\xi}$. Conversely, take $I \in \mathcal{H}^{\operatorname{mon}\left(<_{+}, \Delta\right)}(B) \cap \mathcal{H}^{\operatorname{mon}\left(<_{-}, \Delta\right)}(B)$. For proving that $I \in \mathcal{H}^{B B(\Delta, \xi)}(B)$, we first prove that $\mathrm{in}_{<\xi, m}(I)=\langle 1\rangle$ for $m \notin \Delta$. This will imply by Proposition 2.4 that a monomial $m \notin \Delta$ satisfies $m \in I_{\xi}[0]$.

We argue by contradiction. Suppose that the set $C:=\left\{m \notin \Delta \mid \mathrm{in}_{<_{\varepsilon}, m}(I) \neq\langle 1\rangle\right\}$ is non empty. Since $I$ is bounded, $C$ is finite and the quotient $B[t, \mathbf{x}] / I_{\xi}[t]$ is a finite $B[t]$-module, by Lemma 4.2. Let $m^{\prime}$ be the smallest element of $C$ for the order $<_{+}$. Since $m^{\prime} \notin \Delta$, there is an $f \in I$ with $\mathrm{in}_{<_{+}}(f)=m^{\prime}$. We write $f=m^{\prime}+r+s+t$, with

$$
r=\sum_{f_{\xi}(e)<f_{\xi}\left(m^{\prime}\right)} c_{e} \mathbf{x}^{e}, \quad s=\sum_{\substack{e \in \Delta \\ f_{\xi}(e)=f_{\xi}\left(m^{\prime}\right) \\ e<_{+} m^{\prime}}} c_{e} \mathbf{x}^{e}, \quad \text { and } t=\sum_{\substack{e \notin \Delta \\ f_{\xi}(e)=f_{\xi}\left(m^{\prime}\right) \\ e<_{+} m^{\prime}}} c_{e} \mathbf{x}^{e} .
$$

By minimality of $m^{\prime}$, for any term $c_{e} \mathbf{x}^{e}$ in $t$, there exists some $g_{e} \in I$ of shape $g_{e}=\mathbf{x}^{e}+\sum_{f_{\xi}(f)<f_{\xi}(e)} d_{f} \mathbf{x}^{f}$.
Let

$$
h:=f-\sum_{c_{e} \mathbf{x}^{e} \text { a term of } t} c_{e} g_{e} .
$$

Then $h$ reads $h=m^{\prime}+r+s+u$ with

$$
u=\sum_{\substack{e \notin \Delta \\ f_{\xi}(e)=f_{\xi}\left(m^{\prime}\right) \\ e<_{+} m^{\prime}}} c_{e}\left(\mathbf{x}^{e}-g_{e}\right) .
$$

The only terms $c_{e} \mathbf{x}^{e}$ in $h$ with $f_{\xi}(e)=f_{\xi}\left(m^{\prime}\right)$ are the terms in $s$. Note that the orders $<_{+}$and $<_{-}$have been chosen such that any pair of monomials $m, m^{\prime}$ satisfies the following:

- if $f_{\xi}(m) \neq f_{\xi}\left(m^{\prime}\right)$, then $m<_{+} m^{\prime} \Leftrightarrow m<_{-} m^{\prime}$
- if $f_{\xi}(m)=f_{\xi}\left(m^{\prime}\right)$, then $m<_{+} m^{\prime} \Leftrightarrow m^{\prime}<_{-} m$.

Thus, if $s \neq 0$, then $\operatorname{in}_{<_{-}}(h)$ is a term $c_{e} \mathbf{x}^{e}$ with $e \in \Delta$, contradicting the assumption that $I \in \mathcal{H}^{\text {mon }\left(<_{-}, \Delta\right)}(B)$. We thus obtain that $s=0$ and $\mathrm{in}_{<_{\varepsilon}}(h)=m^{\prime}$, contradicting the definition of $C$.

Summing things up, Spec $B[t, \mathbf{x}] / I_{\xi}[t]$ is a finite family over Spec $B[t]$. By construction, this is a flat family of relative length $n$ over the open set $t \neq 0$. The fiber $B[\mathbf{x}] / I_{\xi}[0]$ over $t=0$ is a quotient of $B[\mathbf{x}] / I^{\Delta}$ which is a flat family of relative length $n$ over Spec $B$. It follows by semi-continuity that $B[t, \mathbf{x}] / I_{\xi}[t]$ is a locally
free $B[t]$-module of rank $n$ with $I_{\xi}[0]=I^{\Delta}$. Indeed, let Fitt ${ }_{i} \subset B[t]$ be the $i^{\text {th }}$ Fitting ideal of $B[t, \mathbf{x}] / I_{\xi}[t]$. The local freeness of rank $n$ of $B[t, \mathbf{x}] / I_{\xi}[t]$ is equivalent to the equalities Fitt $_{n}=B[t]$ and Fitt $_{n-1}=0$. The equality $\operatorname{Fitt}_{n}=B[t]$ is true since the fiber of the family has length at most $n$ over each closed point. Since the Fitting ideals commute with the base change $B[t] \rightarrow B\left[t, t^{-1}\right]$, we obtain Fitt ${ }_{n-1} B\left[t, t^{-1}\right]=0$, hence Fitt $_{n-1}=0$ and $B[t, x] / I_{\xi}[t]$ has rank $n$. Since a surjective morphism between free modules of the same rank is an isomorphism, the equality $I_{\xi}[0]=I^{\Delta}$ follows.

Theorem 5.4. The Bialynicki-Birula functor $\mathcal{H}^{B B(\Delta, \xi)}$ is representable.
Proof. By the above, the Bialynicki-Birula functor is an intersection of two functors $\mathcal{H}^{\text {mon }\left(<_{+}, \Delta\right)}$ and $\mathcal{H}^{\operatorname{mon}\left(<_{-}, \Delta\right)}$, both representable by locally closed subschemes $H^{\operatorname{mon}\left(<_{+}, \Delta\right)}$ and $H^{\operatorname{mon}\left(<_{-}, \Delta\right)}$, respectively, of the Hilbert scheme. The Białynicki-Birula functor is therefore representable by the schematic intersection $H^{\text {mon }\left(<_{+}, \Delta\right)} \cap H^{\text {mon }\left(<_{-}, \Delta\right)}$.

### 5.1. Signed orders are limit orders

The results of this section are not directly used in the proof of our main results. However, they give some intuition on the role of the signed orders which may appear unnatural at first glance: a signed order $<$ is obtained from $<_{\xi}$ with an infinitesimal deformation of $\xi$, and the monic functors are stable under infinitesimal deformations. These facts explain our strategy to replace $<\xi$ with a signed order $<$ when studying the representability of monic functors.

## Definition 5.5.

- A sequence of partial orders $<_{j}$ converges to the total order $<$ if for every pair of monomials $a, b$, we have $a<b$ if, and only if, $a<_{j} b$ for $j$ large enough.
- Let < be a signed order defined by the function $\epsilon$. A sequence compatible with $\epsilon$ is a sequence $\xi^{j}$ in $\mathbb{R}^{d}$ converging to 0 such that the sign of $\xi_{k}^{j}$ is $\epsilon(k)$ and such that the quotient $\xi_{l}^{j} / \xi_{k}^{j}$ tends to 0 if, and only if, $k<l$.

The connection between signed orders and convergence is given by the next proposition (the proof of which, being easy, is left to the reader).

Proposition 5.6. Let $<$ be a refinement of the order $<_{\xi}$. Then the following conditions are equivalent:

- The order < is the signed order defined by the function $\epsilon$
- For every sequence $\xi^{j}$ compatible with $\epsilon$, the sequence of orders $<\xi_{+} \xi^{j}$ converges to $<$.

The following proposition tells that monic functors are stable under small deformations of the order as long as we consider families parameterized by noetherian rings.

Proposition 5.7. Let $<_{,}<_{j}$ be total orders refining $<_{\xi}$. We suppose that the sequence of orders $<_{j}$ converges to $<$. Then for $j$ large, the functors $\mathcal{H}_{\text {noeth }}^{\operatorname{mon}\left(<_{j}, \Delta\right)}$ and $\mathcal{H}_{\text {noeth }}^{\operatorname{mon}(<, \Delta)}$ are isomorphic.

Proof. This is Proposition 22 in the first arxiv version of this paper [EL12], where noetherianity is always assumed. The proof is thus omitted.

## 6. Białynicki-Birula schemes and Hilbert-Chow morphisms

The goal of this section is to prove the following theorem:
Theorem 6.1. If $\xi_{i} \leq 0$, then $H^{B B(\xi, \Delta)}$ is schematically included in the fiber over the origin $\rho_{i}^{-1}(0)$, where $\rho_{i}: H^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)$ is the Hilbert-Chow morphism associated with the $i^{\text {th }}$ coordinate.

To be precise, let $\rho: H^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{d}\right)$ be the usual Hilbert-Chow morphism. The projection $p_{i}: \mathbb{A}^{d} \rightarrow \mathbb{A}^{1}$ to the $i^{\text {th }}$-coordinate induces a morphism $p_{i}^{n}: \operatorname{Sym}^{n}\left(\mathbb{A}^{d}\right) \rightarrow \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)$. Recall that $\operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right) \simeq \mathbb{A}^{n}$ via elementary symmetric functions. We denote by $\rho_{i}:=p_{i}^{n} \circ \rho$ the Hilbert-Chow morphism associated with the $i^{\text {th }}$ coordinate. We denote by $0 \in \operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)$ the point corresponding to $n$ copies of the origin of $\mathbb{A}^{1}$.

Lemma 6.2. Let $I \in \mathcal{H}^{\mathrm{BB}(\xi, \Delta)}(B)$. Suppose that $x_{i}$ is a non-positive variable. Let $m_{i}$ be the multiplication by $x_{i}$ in $B[\mathbf{x}] / I$. Then there exists a basis of $B[\mathbf{x}] / I$ such that the matrix of $m_{i}$ is strictly lower triangular.

Proof. We consider any signed order $<$ refining $<_{\xi}$ and defined by $(\epsilon, o)$ with $o=$ Identity and $\epsilon(i)=-1$. The monomials $b_{i}$ with exponents in $\Delta$ are a basis of $B[\mathbf{x}] / I$. We number them such that $b_{1}>b_{2}>\cdots>b_{n}$. Then, if $x_{i} b_{j} \in \Delta$, we get $x_{i} b_{j}=b_{l}, l>j$. If $x_{i} b_{j} \notin \Delta$. The decomposition of $x_{i} b_{j} \in B[\mathbf{x}] / I$ yields $x_{i} b_{j}=\sum_{b_{k} \in \Delta} c_{k} b_{k}$ $\bmod I$. Since $\operatorname{in}_{<_{\xi}}(I)=I^{\Delta}, c_{k} \neq 0$ implies $b_{k}<_{\xi} x_{i} b_{j} \leq_{\xi} b_{j}$ and $k>j$.

Proof of Theorem 6.1. We recall the observation by Bertin [Ber10] that the Hilbert-Chow morphism is given by the linearized determinant of Iversen. Let $I \in \mathcal{H}^{\operatorname{mon}(<, \Delta)}(B)$ and $b_{1}, \ldots, b_{n}$ a basis of $B[\mathbf{x}] / I$. If $P \in k\left[x_{i}\right]$, we denote by $C_{P}^{j}$ the $j^{\text {th }}$ column of the matrix (with respect to our fixed basis) of multiplication by $P$. If $P_{1} \otimes \cdots \otimes P_{n}$ is a pure tensor in $k\left[x_{i}\right]^{\otimes n}$, we put $\operatorname{ld}\left(P_{1} \otimes \cdots \otimes P_{n}\right):=\operatorname{det}\left(C_{P_{1}}^{1}, \ldots, C_{P_{n}}^{n}\right)$. The symmetric group $S_{n}$ acts on $k\left[x_{i}\right]^{\otimes n}$; we denote by $k\left[x_{i}\right]^{(n)} \subset k\left[x_{i}\right]^{\otimes n}$ the invariant part. Iversen [Ive70] proved that ld : $k\left[x_{i}\right]^{(n)} \rightarrow B$ is a $k$-algebra homomorphism. As was remarked by Bertin, this homomorphism corresponds to the Hilbert-Chow morphism $\rho_{i}$. The ideal of the origin is generated by the elementary symmetric polynomials, which have degree at least one. For proving the theorem, it therefore suffices to show that $\operatorname{det}\left(C_{P_{1}}^{1}, \ldots, C_{P_{n}}^{n}\right)=0$ if $x_{i}$ divides some $P_{j}$. According to the lemma above, that determinant is the determinant of a lower triangular matrix, and that triangular matrix has a zero term on the diagonal if $x_{i}$ divides some $P_{j}$.

## 7. Relations to literature

After the first version of this paper was posted on the arXiv, Drinfeld has proved in [Dril5] results that allow to retrieve several results of this paper, with a different approach.

Drinfeld settles his constructions in the category of algebraic spaces of finite type over a field. Using Drinfelds's language ( $\S 0.2$, ibid.), let $H^{+}$be the attractor of the Hilbert scheme of points $H=H^{n}\left(\mathbb{A}^{d}\right)$, then $H^{+}$is represented by a finite type scheme, and the limit map $q^{+}: H^{+} \rightarrow H^{T}$ is affine (Theorem 1.4.2, ibid.). Theorem 1.4.2 deals with algebraic spaces, but since $H^{T}$ is a scheme and $q^{+}$is affine, it follows that $H^{+}$ is a scheme as well. The functor $H^{B B(\Delta, \xi)}$ is canonically identified with the fiber $\left(q^{+}\right)^{-1}\left(\left[Z^{\Delta}\right]\right)$, hence is representable by an affine scheme.

The Hilbert-Chow morphism is considered in Theorem 6.1 under the condition $\xi_{i} \leq 0$. In the special case $\xi_{i}<0$, Drinfeld's results can be applied : $\left(\operatorname{Sym}^{n}\left(\mathbb{A}^{1}\right)\right)^{+}=\{0\}$ by [Dri15, §1.3.4]; it follows by functoriality of $(-)^{+}$that the whole $H^{+}$is schematically included in the fiber over $\{0\}$.

To get the locally closed embedding, note that $H$ is quasi-projective and fix a $T$-equivariant locally closed embedding $H \subset \mathbb{P}^{N}$. Then $(H)^{+} \subset\left(\mathbb{P}^{N}\right)^{+}$is also a locally closed embedding by Lemmas 1.4.7 and 1.4.9 from [Dri15]. Moreover, by the classical Białynicki-Birula decomposition, each component of $\left(\mathbb{P}^{N}\right)^{+}$is locally closed in $\mathbb{P}^{N}$. Putting everything together, a component of $H^{+}$is a locally closed subscheme of $\mathbb{P}^{N}$ that factors through the locally closed $H$. It follows that $H^{B B(\Delta, \xi)} \rightarrow H$ is a locally closed embedding.

We conclude by proving that some Bialynicki-Birula strata are reducible for the Hilbert scheme of $\mathbb{A}^{3}$. It is a consequence of the reducibility of the Hilbert scheme proved by Iarrobino in [Iarr72].

Proposition 7.1. Let $n$ be such that the Hilbert scheme $H^{n}\left(\mathbb{A}^{3}\right)$ is reducible. Let $n_{1} \gg n_{2} \gg n_{3} \gg 0$ and let $T$ be the one-dimensional torus acting on $\mathbb{A}^{3}$ via $t \cdot(x, y, z)=\left(t^{n_{1}} x, t^{n_{2}} y, t^{n_{3}} z\right)$. Then there exists $\Delta$ such that the Bialynicki-Birula stratum $H^{B B(T, \Delta)} \subset H^{n}\left(\mathbb{A}^{3}\right)$ is reducible.

Proof. Let $C_{1} \subset H^{n}\left(\mathbb{A}^{3}\right)$ be the component containing the unions of $n$ distinct points and $C_{2} \subset H^{n}\left(\mathbb{A}^{3}\right)$ be any other component. By the choice of the weights, the fixed points for the torus action are the monomial schemes $Z^{\Delta}$ with ideal $I^{\Delta}$. Thus, if $Z \in C_{2} \backslash C_{1}, \lim _{t \rightarrow 0} t \cdot Z=Z^{\Delta}$ for some $\Delta$. In particular, $\operatorname{dim}\left(C_{2} \cap H^{B B(T, \Delta)}\right) \geq 1$.

We have $\operatorname{dim}\left(C_{1} \cap H^{B B(T, \Delta)}\right) \geq 1$ too. Indeed, let $Z$ be the disjoint union of points with integer coordinates $(i, j, k)$ with $(i, j, k) \in \Delta$. If $x^{a} y^{b} z^{c} \in I^{\Delta}$, any point in the support of $Z$ has coordinates $0 \leq x<a$ or $0 \leq y<b$ or $0 \leq z<c$, thus $f=x(x-1) \ldots(x-(a-1)) y(y-1) \ldots(y-(b-1)) z(z-1) \ldots(z-(c-1))$ is in $I(Z)$. We have $\lim _{t \rightarrow 0} t^{n_{1} a+n_{2} b+n_{3} c}(t \cdot f)=x^{a} y^{b} z^{c}$, thus $\lim _{t \rightarrow 0} t \cdot Z \subset Z^{\Delta}$. The equality $\lim _{t \rightarrow 0} t \cdot Z=Z^{\Delta}$ follows since the two schemes have the same length.

Thus $H^{B B(T, \Delta)}$ contains at least two components, one in $C_{1}$ and one in $C_{2}$, meeting in $Z^{\Delta}$.

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