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The conjectures of Artin–Tate and Birch–Swinnerton-Dyer

Stephen Lichtenbaum, Niranjan Ramachandran, and Takashi Suzuki

Abstract. We provide two proofs that the conjecture of Artin–Tate for a fibered surface is equivalent to the conjecture of Birch–Swinnerton-Dyer for the Jacobian of the generic fibre. As a byproduct, we obtain a new proof of a theorem of Geisser relating the orders of the Brauer group and the Tate–Shafarevich group.

Keywords. Birch–Swinnerton-Dyer conjecture; finite fields; zeta functions; Tate conjecture

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Stephen Lichtenbaum
Department of Mathematics, Brown University, Providence, RI 02912
e-mail: stephen_lichtenbaum@brown.edu
Niranjan Ramachandran
Department of Mathematics, University of Maryland, College Park, MD 20742 USA.
e-mail: atma@math.umd.edu
Takashi Suzuki
Department of Mathematics, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
e-mail: tsuzuki@gug.math.chuo-u.ac.jp

Contents

1. Introduction and statement of results

Let $k = \mathbb{F}_q$ be a finite field of characteristic p and let S be a smooth projective (geometrically connected) curve over $T = \operatorname{Spec} k$ and let $F = k(S) = \mathbb{F}_q(S)$ be the function field of S. Let X be a smooth proper surface over T with a flat proper morphism $\pi : X \to S$ with smooth geometrically connected generic fiber X_0 over Spec F. The Jacobian J of X_0 is an Abelian variety over F.

Our first main result is a proof of the following statement conjectured by Artin and Tate [Tat66, Conjecture (d)]:

Theorem 1.1. *The Artin–Tate conjecture for X is equivalent to the Birch–Swinnerton-Dyer conjecture for J.*

Recall that these conjectures concern two (conjecturally finite) groups: the Tate–Shafarevich group $Ш(J/F)$ of J and the Brauer group $\operatorname{Br}(X)$ of X. A result of Artin–Grothendieck [Gor79, Theorem 2.3] [Gro68, §4] is that $Ш(J/F)$ is finite if and only if $\operatorname{Br}(X)$ is finite.

Our second main result is a new proof of a beautiful result (2.18) of Geisser [Gei20, Theorem 1.1] that relates the conjectural finite orders of $Ш(J/F)$ and $\operatorname{Br}(X)$; special cases of (2.18) are due to Milne–Gonzales-Aviles [Mil81, GA03].

We actually provide two proofs of Theorem 1.1; while our first proof uses Geisser's result (2.18), the second (and very short) proof in §4, completely due to the third-named author, does not.

1.1. History

Artin and Tate regarded Theorem 1.1 as easier to prove as opposed to the other conjectures in [Tat66]. They proved Theorem 1.1 when π is smooth and has a section ([Tat66, p.427]) using the equality

$$[Ш(J/F)] = [\operatorname{Br}(X)] \tag{1.1}$$

between the orders of the groups $Ш(J/F)$ and $\operatorname{Br}(X)$ which follows from Artin's theorem [Tat66, Theorem 3.1], [Gor79, Theorem 2.3]: if π is generically smooth with connected fibers and admits a section, then $Ш(J/F) \cong \operatorname{Br}(X)$. Gordon [Gor79, Theorem 6.1] used (1.1) to prove Theorem 1.1 when[1] π is cohomologically flat with a section (see [Gor79, Theorem 2.3]). Building on Gordon [Gor79], Liu–Lorenzini–Raynaud [LLR04] proved several new cases of Theorem 1.1 by eliminating the condition of cohomological flatness of π; their proof [LLR04, Theorem 4.3] proceeds by proving that Theorem 1.1 is equivalent to a precise relation generalizing (1.1) between $[\operatorname{Br}(X)]$ and $[Ш(J/F)]$ which in their case had been proved by Milne and Gonzales-Aviles [Mil81, GA03].

[1] There is another proof (up to p-torsion) in this case due to Z. Yun [Yun15].

As Liu–Lorenzini–Raynaud (and Milne) point out [LLR05, Theorem 2], Theorem 1.1 follows by combining [Tat66, Gro68, Mil75, KT03]:

$$AT(X) \overset{\text{Artin–Tate–Milne}}{\Longleftrightarrow} \mathrm{Br}(X) \text{ finite} \overset{\text{Artin–Grothendieck}}{\Longleftrightarrow} Ш(J/F) \text{ finite} \overset{\text{Kato–Trihan}}{\Longleftrightarrow} BSD(J).$$

In 2018, Geisser pointed out that a slight correction is necessary in the relation [LLR04, Theorem 4.3] between $[\mathrm{Br}(X)]$ and $[Ш(J/F)]$; Liu–Lorenzini–Raynaud [LLR18, Corrected Theorem 4.3] showed that Theorem 1.1 holds if and only if this slightly corrected version holds. This precise relation (Theorem 2.11) was then proved by Geisser [Gei20, Theorem 1.1] without using Theorem 1.1. Thus, combining [LLR18, Corrected Theorem 4.3] and [Gei20, Theorem 1.1] gives the second known proof of Theorem 1.1. But this proof relies heavily on the work of Gordon[2] [Gor79] as can be seen from [LLR18, §3, (3.9)].

1.2. Our approach

Our first proof depends on [Gor79] only for the elementary result (2.9). As in [Gor79, LLR04, LLR18], this proof also follows the strategy in [Tat66, §4]. We use the localization sequence to record a short proof[3] of the Tate–Shioda relation (Corollary 2.2). In turn, this gives a quick calculation (2.17) of the height pairing $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))$ on the Néron–Severi group of X. The same calculation in [Gor79, LLR18] requires a detailed analysis of various subgroups of $\mathrm{NS}(X)$. A beautiful introduction to these results is [Ulm14]; see [Lic83, Lic05, GS20] for Weil-étale analogues.

The second proof (§4) of Theorem 1.1 uses only (2.5) and the Weil-étale formulations of the two conjectures. In this proof, we do not compare each term of the two special value formulas and entirely work in derived categories.

Notations

Throughout, $k = \mathbb{F}_q$ is a finite field of characteristic p and $T = \mathrm{Spec}\ k$; if $\bar{k}$ is an algebraic closure of k, let $\bar{T} = \mathrm{Spec}\ \bar{k}$. The function field of S is $F = k(S)$. Let X be a smooth proper surface over T with a flat proper morphism $\pi : X \to S$ with smooth geometrically connected generic fiber X_0 over Spec F. The Jacobian J of X_0 is an Abelian variety over F.

1.3. The Artin–Tate conjecture

Let $k = \mathbb{F}_q$ and $F = k(S)$. For any scheme V of finite type over T, the zeta function $\zeta(V, s)$ is defined as

$$\zeta(V,s) = \prod_{v\in V} \frac{1}{(1-q_v^{-s})};$$

the product is over all closed points v of V and q_v is the size of the finite residue field $k(v)$ of v. If V is smooth proper (geometrically connected) of dimension d, then the zeta function $\zeta(V,s)$ factorizes as

$$\zeta(V,s) = \frac{P_1(V,q^{-s})\cdots P_{2d-1}(V,q^{-s})}{P_0(V,q^{-s})\cdots P_{2d}(V,q^{-s})}, \quad P_0 = (1-q^{-s}), \quad P_{2d} = (1-q^{d-s}),$$

where $P_i(V,t) \in \mathbb{Z}[t]$ is the characteristic polynomial of Frobenius acting on the ℓ-adic étale cohomology $H^i(V \times_T \bar{T}, \mathbb{Q}_\ell)$ for any prime ℓ not dividing q; by Grothendieck and Deligne, $P_j(V,t)$ is independent of ℓ. One has the factorization [Tat66, (4.1)] (the second equality uses Poincaré duality)

$$\zeta(X,s) = \frac{P_1(X,q^{-s})\cdot P_3(X,q^{-s})}{(1-q^{-s})\cdot P_2(X,q^{-s})\cdot(1-q^{2-s})} = \frac{P_1(X,q^{-s})\cdot P_1(X,q^{1-s})}{(1-q^{-s})\cdot P_2(X,q^{-s})\cdot(1-q^{2-s})}. \tag{1.2}$$

[2] Known to have several inaccuracies; see [LLR18, §3.3].

[3] This is similar to the ideas of Hindry–Pacheco and Kahn in [Kah09, §§3.2-3.3].

Let $\rho(X)$ be the rank of the finitely generated Néron–Severi group $\mathrm{NS}(X)$. The intersection $D\cdot E$ of divisors D and E provides a symmetric non-degenerate bilinear pairing on $\mathrm{NS}(X)$; the height pairing $\langle D,E\rangle_{\mathrm{ar}}$ [LLR18, Remark 3.11] on $\mathrm{NS}(X)$ is related to the intersection pairing as follows:

$$\mathrm{NS}(X)\times\mathrm{NS}(X)\to\mathbb{Q}(\log q),\qquad D,E\mapsto\langle D,E\rangle_{\mathrm{ar}}=(D\cdot E)\log q.$$

Let A be the reduced identity component $\mathrm{Pic}^{\mathrm{red},0}_{X/k}$ of the Picard scheme $\mathrm{Pic}_{X/k}$ of X. Let

$$\alpha(X)=\chi(X,\mathcal{O}_X)-1+\dim(A). \tag{1.3}$$

We write $[G]$ for the order of a finite group G.

Conjecture 1.2 (Artin–Tate [Tat66, Conjecture (C)]). *The Brauer group* $\mathrm{Br}(X)$ *is finite,* $\mathrm{ord}_{s=1}P_2(X,q^{-s})=\rho(X)$*, and the special value*

$$P_2^*(X,q^{-1}):=\lim_{s\to 1}\frac{P_2(X,q^{-s})}{(s-1)^{\rho(X)}}$$

of $P_2(X,t)$ *at* $t=1/q$ *(this corresponds to* $s=1$*) satisfies*

$$P_2^*(X,q^{-1})=[\mathrm{Br}(X)]\cdot\Delta_{\mathrm{ar}}(\mathrm{NS}(X))\cdot q^{-\alpha(X)}. \tag{1.4}$$

Here $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))$ *is the discriminant (see §1.4) of the height pairing on* $\mathrm{NS}(X)$.

Remark. The discriminant $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))$ of the height pairing on $\mathrm{NS}(X)$ is related to the discriminant $\Delta(\mathrm{NS}(X))$ of the intersection pairing as follows: $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))=\Delta(\mathrm{NS}(X))\cdot(\log q)^{\rho(X)}$.

1.4. Discriminants

For more details on the basic notions recalled next, see [Yun15, §2.8] and [Blo87]. Let N be a finitely generated Abelian group N and let $\psi:N\times N\to K$ be a symmetric bilinear form with values in any field K of characteristic zero. If $\psi:N/\mathrm{tor}\times N/\mathrm{tor}\to K$ is non-degenerate, the discriminant $\Delta(N)$ is defined as the determinant of the matrix $\psi(b_i,b_j)$ divided by $(N:N')^2$ where N' is the subgroup of finite index generated by a maximal linearly independent subset $\{b_i\}$ of N. Note that $\Delta(N)$ is independent of the choice of the subset $\{b_i\}$ and the subgroup N' and incorporates the order of the torsion subgroup of N. For us, $K=\mathbb{Q}$ or $\mathbb{Q}(\log q)$.

Given a short exact sequence $0\to N'\to N\to N''\to 0$ which splits over $\mathbb{Q}$ as an orthogonal direct sum $N_{\mathbb{Q}}\cong N'_{\mathbb{Q}}\oplus N''_{\mathbb{Q}}$ with respect to a definite pairing ψ on N, one has the following standard relation

$$\Delta(N)=\Delta(N')\cdot\Delta(N''). \tag{1.5}$$

Given a map $f:C\to C'$ of Abelian groups with finite kernel and cokernel, the invariant $z(f)=\frac{[\mathrm{Ker}(f)]}{[\mathrm{Coker}(f)]}$ [Tat66] extends to the derived category $\mathcal{D}$ of complexes in Abelian groups with bounded and finite homology: given any such complex $C_\bullet$, the invariant

$$z(C_\bullet)=\prod_i[H_i(C_\bullet)]^{(-1)^i}$$

is an Euler characteristic; for any triangle $K\to L\to M\to K[1]$ in $\mathcal{D}$, the following relation holds

$$z(K)\cdot z(M)=z(L). \tag{1.6}$$

One recovers $z(f)$ viewing $f:C\to C'$ as a complex in degrees zero and one. For any pairing $\psi:N\times N\to\mathbb{Z}$, the induced map $N\to R\mathrm{Hom}(N,\mathbb{Z})$ recovers $\Delta(N)$ above:

$$\Delta(N)=z(N\to R\mathrm{Hom}(N,\mathbb{Z}))^{-1}.$$

□

1.5. The Birch–Swinnerton-Dyer conjecture

For more details on the basic notions recalled next, see [GS20]. Let J be the Jacobian of X_0. Recall that the complete L-function [Ser70, Mil72], [GS20, §4] of J is defined as a product of local factors

$$L(J,s) = \prod_{v \in S} \frac{1}{L_v(J, q_v^{-s})}. \tag{1.7}$$

For any closed point v of S, the local factor $L_v(J,t)$ is the characteristic polynomial of Frobenius on

$$H^1_{\text{ét}}(J \times F_v^{\text{sep}}, \mathbb{Q}_\ell)^{I_v}, \tag{1.8}$$

where F_v is the complete local field corresponding to v and I_v is the inertia group at v. By [GS20, Proposition 4.1], $L_v(J,t)$ has coefficients in $\mathbb{Z}$ and is independent of ℓ, for any prime ℓ distinct from the characteristic of k. Let $\text{Ш}(J/F)$ be the Tate–Shafarevich group of J over Spec F and let r be the rank of the finitely generated group $J(F)$. Let $\Delta_{\text{NT}}(J(F))$ be the discriminant of the Néron–Tate pairing [Tat66, p. 419], [KT03, §1.5] on $J(F)$:

$$J(F) \times J(F) \to \mathbb{Q}(\log\ q), \quad (\gamma, \kappa) \mapsto \langle \gamma, \kappa \rangle_{\text{NT}}. \tag{1.9}$$

Let $\mathcal{J} \to S$ be the Néron model of J; for any closed point $v \in S$, define $c_v = [\Phi_v(k_v)]$ where Φ_v is the group of connected components of $\mathcal{J}_v$ and put $c(J) = \prod_{v\in S} c_v$; this is a finite product as $c_v = 1$ for all but finitely many v. Let Lie $\mathcal{J}$ be the locally free sheaf on S defined by the Lie algebra of $\mathcal{J}$. Recall the[4]

Conjecture 1.3 (Birch–Swinnerton-Dyer). *The group* $\text{Ш}(J/F)$ *is finite,* $\text{ord}_{s=1} L(J,s) = r$*, and the special value*

$$L^*(J,1) := \lim_{s\to 1} \frac{L(J,s)}{(s-1)^r}$$

satisfies

$$L^*(J,1) = [\text{Ш}(J/F)] \cdot \Delta_{\text{NT}}(J(F)) \cdot c(J) \cdot q^{\chi(S, \text{Lie}\,\mathcal{J})}. \tag{1.10}$$

The proof of Theorem 1.1, *i.e.* the equivalence of Conjectures 1.2 and 1.3, naturally divides into four parts:

- Br(X) is finite if and only if $\text{Ш}(J/F)$ is finite. This is known [Gro68, (4.41), Corollaire (4.4)].
- Comparison of $\chi(S, \text{Lie}\,\mathcal{J})$ and $\alpha(X)$ given in (2.5). This is known [LLR04, p. 483]. For the convenience of the reader, we recall it in §2.2.
- (Proposition 2.4) $\text{ord}_{s=1} P_2(X, q^{-s}) = \rho(X)$ if and only if $\text{ord}_{s=1} L(J,s) = r$.
- (§3) $P_2^*(X,1)$ satisfies (1.4) if and only if $L^*(J,1)$ satisfies (1.10).

The first two parts are not difficult and we provide elementary proofs of the last two parts.

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2. Preparations

2.1. Elementary identities and known results

The Néron–Severi group NS(X) is the group of k-points of the group scheme $\text{NS}_{X/k} = \pi_0(\text{Pic}_{X/k})$ of connected components of the Picard scheme $\text{Pic}_{X/k}$ of X. Let $A = \text{Pic}^{\text{red},0}_{X/k}$. The Leray spectral sequence for

[4] By [GS20, Corollary 4.5], this is equivalent to the formulation in [Tat66].

the morphism $X \to \operatorname{Spec} k$ and the étale sheaf $\mathbb{G}_m$ provides the first exact sequences [BLR90, Proposition 4, p. 204] below:

$$0 \longrightarrow \operatorname{Pic}(k) \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}_{X/k}(k) \longrightarrow \operatorname{Br}(k) \quad \text{and} \quad 0 \longrightarrow \operatorname{Pic}^0_{X/k} \longrightarrow \operatorname{Pic}_{X/k} \longrightarrow \pi_0(\operatorname{Pic}_{X/k}) \longrightarrow 0.$$

Since $\operatorname{Br}(k) = 0$, $H^1_{\text{ét}}(\operatorname{Spec} k, \operatorname{Pic}^0_{X/k}) = H^1_{\text{ét}}(\operatorname{Spec} k, \operatorname{Pic}^{\text{red},0}_{X/k})$ and $H^1_{\text{ét}}(\operatorname{Spec} k, A) = 0$ (Lang's theorem [Tat66, p. 209]), this provides

$$\operatorname{Pic}_{X/k}(k) = \operatorname{Pic}(X) \quad \text{and} \quad \operatorname{NS}(X) = \operatorname{NS}_{X/k}(k) = \frac{\operatorname{Pic}(X)}{A(k)}. \tag{2.1}$$

Let P be the identity component of the Picard scheme $\operatorname{Pic}_{S/k}$ of S. Let B be the cokernel of the natural injective map $\pi^* : P \to A$. So one has short exact sequences (using Lang's theorem [Tat66, p. 209] for the last sequence)

$$A = \operatorname{Pic}^{\text{red},0}_{X/k}, \quad P = \operatorname{Pic}^0_{S/k}, \quad 0 \longrightarrow P \longrightarrow A \longrightarrow B \longrightarrow 0, \quad \text{and} \quad 0 \longrightarrow P(k) \longrightarrow A(k) \longrightarrow B(k) \longrightarrow 0. \tag{2.2}$$

It is known that [Tat66, p. 428]

$$P_1(S, q^{-s}) = P_1(P, q^{-s}), \quad P_1(X, q^{-s}) = P_1(A, q^{-s}), \quad \text{and} \quad P_1(A, q^{-s}) = P_1(P, q^{-s}) \cdot P_1(B, q^{-s}). \tag{2.3}$$

For any Abelian variety G of dimension d over $k = \mathbb{F}_q$, it is well known that [Tat66, p. 429, top line] (or [Gor79, 6.1.3])

$$P_1(G, 1) = [G(k)] \quad \text{and} \quad P_1(G, q^{-1}) = [G(k)]q^{-d}. \tag{2.4}$$

2.2. Comparison of $\chi(S, \operatorname{Lie} \mathcal{J})$ and $\alpha(X)$

It is known [LLR04, p. 483] that

$$\chi(S, \operatorname{Lie} \mathcal{J}) - \dim(B) = -\alpha(X). \tag{2.5}$$

We include their proof here for the convenience of the reader. A special case of this is due to Gordon [Gor79, Proposition 6.5]. The Leray spectral sequence for π and $\mathcal{O}_X$ provides $H^0(S, \mathcal{O}_S) \cong H^0(X, \mathcal{O}_X)$,

$$0 \to H^1(S, \mathcal{O}_S) \to H^1(X, \mathcal{O}_X) \to H^0(S, R^1\pi_*\mathcal{O}_X) \to 0, \quad H^2(X, \mathcal{O}_X) \cong H^1(S, R^1\pi_*\mathcal{O}_X).$$

This proves $\chi(X, \mathcal{O}_X) = \chi(S, \mathcal{O}_S) - \chi(S, R^1\pi_*\mathcal{O}_X)$. Recall that $\mathcal{J}$ is the Néron model of the Jacobian J of X_0. As the kernel and cokernel of the natural map[5] $\phi : R^1\pi_*\mathcal{O}_X \to \operatorname{Lie} \mathcal{J}$ are torsion sheaves on S of the same length [LLR04, Theorem 4.2], we have [LLR04, p. 483]

$$\chi(S, R^1\pi_*\mathcal{O}_X) = \chi(S, \operatorname{Lie} \mathcal{J}). \tag{2.6}$$

Thus,

$$\begin{aligned}
\alpha(X) &\overset{(1.3)}{=} \chi(X, \mathcal{O}_X) - 1 + \dim(A) = \chi(S, \mathcal{O}_S) - \chi(S, R^1\pi_*\mathcal{O}_X) - 1 + \dim(A) \\
&= 1 - \dim(P) - \chi(S, \operatorname{Lie} \mathcal{J}) - 1 + \dim(A) = -\chi(S, \operatorname{Lie} \mathcal{J}) + \dim(A) - \dim(P) \\
&\overset{(2.2)}{=} -\chi(S, \operatorname{Lie} \mathcal{J}) + \dim(B).
\end{aligned}$$

2.3. The Tate–Shioda relation about the Néron–Severi group

The structure of $\operatorname{NS}(X)$ depends on the singular fibers of the morphism $\pi : X \to S$.

[5] The map ϕ is obtained by the composition of the maps $R^1\pi_*\mathcal{O}_X \to \operatorname{Lie} P$ [LLR04, Proposition 1.3 (b)] and $\operatorname{Lie} P \to \operatorname{Lie} Q$ [LLR04, Theorem 3.1] with $Q \xrightarrow{\sim} \mathcal{J}$ [LLR04, Facts 3.7 (a)]; it uses the fact that X is regular, $\pi : X \to S$ is proper flat, and $\pi_*\mathcal{O}_X = \mathcal{O}_S$.

2.3.1. Singular fibers.— Let $Z = \{v \in S \mid \pi^{-1}(v) = X_v \text{ is not smooth}\}$. For any $v \in S$, let G_v be the set of irreducible components Γ_i of X_v, let m_v be the cardinality of G_v, and $m := \sum_{v\in Z}(m_v - 1)$; for any $i \in G_v$, let r_i be the number of irreducible components of $\Gamma_i \times \overline{k(v)}$. Let R_v be the quotient

$$R_v = \frac{\mathbb{Z}^{G_v}}{\mathbb{Z}} \tag{2.7}$$

of the free Abelian group generated by the irreducible components of X_v by the subgroup generated by the cycle associated with $X_v = \pi^{-1}(v)$. If $v \notin Z$, then R_v is trivial.

Let $U = S - Z$; the map $X_U = \pi^{-1}(U) \to U$ is smooth. For any finite $Z' \subset S$ with $Z \subset Z'$, we consider $U' = S - Z'$ and $X_{U'} = X - \pi^{-1}(U')$. The following proposition provides a description of $\mathrm{NS}(X) \overset{(2.1)}{\cong} \mathrm{Pic}(X)/A(k)$.

Proposition 2.1.

(i) The natural maps $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ *and* $\pi^* : \mathrm{Pic}(U') \to \mathrm{Pic}(X_{U'})$ *are injective.*

(ii) There is an exact sequence

$$0 \longrightarrow \bigoplus_{v\in Z} R_v \longrightarrow \frac{\mathrm{Pic}(X)}{\pi^*\mathrm{Pic}(S)} \longrightarrow \mathrm{Pic}(X_0) \longrightarrow 0. \tag{2.8}$$

Proof. (i) From the Leray spectral sequence for $\pi : X \to S$ and the étale sheaf $\mathbb{G}_m$ on X, we get the exact sequence

$$0 \longrightarrow H^1_{et}(S, \pi_*\mathbb{G}_m) \longrightarrow H^1_{et}(X, \mathbb{G}_m) \longrightarrow H^0(S, R^1\pi_*\mathbb{G}_m) \longrightarrow \mathrm{Br}(S).$$

Now X_0 being geometrically connected and smooth over F implies [Mil81, Remark 1.7a] that $\pi_*\mathbb{G}_m$ is the sheaf $\mathbb{G}_m$ on S. This provides the injectivity of the first map. The same argument with U' in place of S provides the injectivity of the second.

(ii) The class group $\mathrm{Cl}(Y)$ and the Picard group $\mathrm{Pic}(Y)$ are isomorphic for regular schemes Y such as S and X. The localization sequences for $X_{U'} \subset X$ and $U' \subset S$ can be combined as

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Gamma(S,\mathbb{G}_m) & \longrightarrow & \Gamma(U',\mathbb{G}_m) & \longrightarrow & \bigoplus_{v\in Z'} \mathbb{Z} & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}(U') & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(X,\mathbb{G}_m) & \longrightarrow & \Gamma(X_{U'},\mathbb{G}_m) & \longrightarrow & \bigoplus_{v\in Z'} \mathbb{Z}^{G_v} & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X_{U'}) & \longrightarrow & 0.
\end{array}$$

Here $\Gamma(X,\mathbb{G}_m) = H^0_{et}(X,\mathbb{G}_m) = H^0_{Zar}(X,\mathbb{G}_m)$. The induced exact sequence on the cokernels of the vertical maps is

$$0 \longrightarrow \bigoplus_{v\in Z'} R_v \longrightarrow \frac{\mathrm{Pic}(X)}{\pi^*\mathrm{Pic}(S)} \longrightarrow \frac{\mathrm{Pic}(X_{U'})}{\pi^*\mathrm{Pic}(U')} \longrightarrow 0.$$

In particular, we get this sequence for Z and U. By assumption, X_v is geometrically irreducible for any $v \notin Z$; so $R_v = 0$ for any $v \notin Z$. So this means that, for any $U' = S - Z'$ contained in U, the induced maps

$$\frac{\mathrm{Pic}(X_U)}{\pi^*\mathrm{Pic}(U)} \longrightarrow \frac{\mathrm{Pic}(X_{U'})}{\pi^*\mathrm{Pic}(U')}$$

are isomorphisms. Taking the limit over Z' gives us the exact sequence in the proposition. □

Corollary 2.2.

(i) The Tate-Shioda relation [Tat66, (4.5)] $\rho(X) = 2 + r + m$ *holds.*

(ii) One has an exact sequence

$$0 \longrightarrow B(k) \longrightarrow \frac{\mathrm{Pic}(X)}{\pi^*\mathrm{Pic}(S)} \longrightarrow \frac{\mathrm{NS}(X)}{\pi^*\mathrm{NS}(S)} \longrightarrow 0.$$

Proof. (i) Since r is the rank of $J(F)$, the rank of $\mathrm{Pic}(X_0)$ is $r+1$. Since $\mathrm{Pic}(S)$ has rank one, $A(k)$ is finite and $m = \sum_{v\in Z}(m_v - 1)$, this follows from (2.1) and (2.8).
(ii) This follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P(k) & \xrightarrow{\pi^*} & A(k) & \longrightarrow & B(k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{\pi^*} & \mathrm{Pic}(X) & \longrightarrow & \frac{\mathrm{Pic}(X)}{\pi^*\mathrm{Pic}(S)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{NS}(S) & \xrightarrow{\pi^*} & \mathrm{NS}(X) & \longrightarrow & \frac{\mathrm{NS}(X)}{\pi^*\mathrm{NS}(S)} & \longrightarrow & 0.
\end{array}$$

□

2.4. Relating the order of vanishing at $s=1$ of $P_2(X, q^{-s})$ and $L(J,s)$

By[6] [Gor79, Proposition 3.3], one has

$$\zeta(X_v, s) = \frac{P_1(X_v, q_v^{-s})}{(1-q_v^{-s})\cdot P_2(X_v, q_v^{-s})}, \quad \text{and} \quad P_2(X_v, q_v^{-s}) = \left\{ \begin{array}{ll} (1-q_v^{1-s}), & \text{for } v \notin Z \\ \prod_{i\in G_v}(1-(q_v)^{r_i(1-s)}), & \text{for } v \in Z \end{array} \right\}, \tag{2.9}$$

see §2.3.1 for notation. Using

$$Q_2(s) = \prod_{v\in Z} \frac{P_2(X_v, q^{-s})}{(1-q_v^{1-s})}, \quad \zeta(S,s) = \frac{P_1(S, q^{-s})}{(1-q^{-s})\cdot(1-q^{1-s})}, \quad \text{and} \quad Q_1(s) = \prod_{v\in S} P_1(X_v, q_v^{-s}),$$

we can rewrite

$$\zeta(X,s) = \prod_{v\in S}\zeta(X_v,s) = \frac{1}{Q_2(s)}\cdot\prod_{v\in S}\frac{P_1(X_v,q_v^{-s})}{(1-q_v^{-s})\cdot(1-q_v^{1-s})} = \frac{\zeta(S,s)\cdot\zeta(S,s-1)\cdot Q_1(s)}{Q_2(s)}.$$

The precise relation between $P_2(X, q^{-s})$ and $L(J,s)$ is given by (2.11).

Proposition 2.3. *One has* $\mathrm{ord}_{s=1}\, Q_2(s) = m$ *and*

$$Q_2^*(1) = \lim_{s\to 1}\frac{Q_2(s)}{(s-1)^m} = \prod_{v\in Z}\left((\log q_v)^{(m_v-1)}\cdot\prod_{i\in G_v} r_i\right), \tag{2.10}$$

$$\frac{P_2(X, q^{-s})}{(1-q^{1-s})^2} = P_1(B, q^{-s})\cdot P_1(B, q^{1-s})\cdot L(J,s)\cdot Q_2(s). \tag{2.11}$$

Proof. Observe that (2.10) is elementary: for any positive integer r, one has

$$\lim_{s\to 1}\frac{(1-q_v^{r(1-s)})}{(s-1)} = \lim_{s\to 1}\frac{(1-q_v^{r(1-s)})}{(1-q_v^{1-s})}\cdot\frac{(1-q_v^{1-s})}{(s-1)} = \lim_{s\to 1}(1+q_v^{1-s}+\cdots+q_v^{(r-1)(1-s)})\cdot\log q_v = r\cdot\log q_v.$$

For each $v\in Z$, this shows that

$$\lim_{s\to 1}\frac{P_2(X_v, q^{-s})}{(s-1)^{m_v}} = (\log q_v)^{m_v}\cdot\prod_{i\in G_v} r_i.$$

Therefore, we obtain that

$$\lim_{s\to 1}\frac{Q_2(s)}{(s-1)^m} = \prod_{v\in Z}\lim_{s\to 1}\frac{\frac{P_2(X_v,q^{-s})}{(1-q_v^{1-s})}}{(s-1)^{m_v-1}} = \prod_{v\in Z}\lim_{s\to 1}\frac{\frac{P_2(X_v,q^{-s})}{(s-1)^{m_v}}}{\frac{(1-q_v^{1-s})}{s-1}} = \prod_{v\in Z}\left(\frac{(\log q_v)^{m_v}\cdot\prod_{i\in G_v} r_i\cdot}{\log q_v}\right).$$

[6] This proposition, first stated on Page 176 of [Gor79], has a typo in the formula for P_2 which is corrected in its restatement on Page 193. We only need the part about P_2 (and this is elementary).

We now prove (2.11). Simplifying the identity

$$\frac{P_1(X,q^{-s})\ \cdot\ P_1(X,q^{1-s})}{(1-q^{-s})\cdot P_2(X,q^{-s})\cdot(1-q^{2-s})} = \zeta(X,s) = \frac{P_1(S,q^{-s})}{(1-q^{-s})\cdot(1-q^{1-s})}\cdot\frac{P_1(S,q^{1-s})}{(1-q^{1-s})\cdot(1-q^{2-s})}\cdot\frac{Q_1(s)}{Q_2(s)}$$

from (1.2) using (2.3), one obtains

$$\frac{P_1(B,q^{-s})\cdot P_1(B,q^{1-s})}{P_2(X,q^{-s})} = \frac{1}{(1-q^{1-s})}\cdot\frac{1}{(1-q^{1-s})}\cdot\frac{Q_1(s)}{Q_2(s)}.$$

On reordering, this becomes

$$\frac{P_2(X,q^{-s})}{(1-q^{1-s})^2} = \frac{P_1(B,q^{-s})\cdot P_1(B,q^{1-s})\cdot Q_2(s)}{Q_1(s)}.$$

Let $T_\ell J$ be the ℓ-adic Tate module of the Jacobian J of X. For any $v\in S$, the Kummer sequence on X and J provides a $\mathrm{Gal}(F_v^{\mathrm{sep}}/F_v)$-equivariant isomorphism

$$H^1_{\text{ét}}(X\times_S F_v^{\mathrm{sep}},\mathbb{Z}_\ell(1)) \xrightarrow{\ \sim\ } T_\ell J \xleftarrow{\ \sim\ } H^1_{\text{ét}}(J\times_F F_v^{\mathrm{sep}},\mathbb{Z}_\ell(1)),$$

as J is a self-dual Abelian variety: this provides the isomorphisms

$$H^1_{\text{ét}}(J\times_F F_v^{\mathrm{sep}},\mathbb{Q}_\ell)\cong H^1_{\text{ét}}(X\times_S F_v^{\mathrm{sep}},\mathbb{Q}_\ell),\quad H^1_{\text{ét}}(J\times_F F_v^{\mathrm{sep}},\mathbb{Q}_\ell)^{I_v}\cong H^1_{\text{ét}}(X\times_S F_v^{\mathrm{sep}},\mathbb{Q}_\ell)^{I_v}.$$

From [Del80, Théorème 3.6.1, pp.213–214] (the arithmetic case is in [Blo87, Lemma 1.2]), we obtain an isomorphism

$$H^1_{\text{ét}}(X_v\times_{k(v)}\overline{k(v)},\mathbb{Q}_\ell) \xrightarrow{\ \sim\ } H^1_{\text{ét}}(X\times_S F_v^{\mathrm{sep}},\mathbb{Q}_\ell)^{I_v}.$$

The definition of $L_v(J,t)$ in (1.8) now implies that $P_1(X_v,q_v^{-s}) = L_v(J,q_v^{-s})$ and hence $Q_1(s)\cdot L(J,s)=1$. □

Proposition 2.4.

(i) $\operatorname{ord}_{s=1} P_2(X,q^{-s}) = \rho(X)$ *if and only if* $\operatorname{ord}_{s=1} L(J,s) = r$.

(ii) One has

$$P_2^*(X,\frac{1}{q}) = P_1(B,q^{-1})\cdot P_1(B,1)\cdot L^*(J,1)\cdot Q_2^*(1)\cdot(\log\ q)^2 \overset{(2.4)}{=} \frac{[B(k)]^2}{q^{\dim(B)}}\cdot L^*(J,1)\cdot Q_2^*(1)\cdot(\log\ q)^2. \tag{2.12}$$

Proof. As $P_1(B,q^{-s})\cdot P_1(B,q^{1-s})$ does not vanish at $s=1$ by (2.4), it follows from (2.11) that

$$\operatorname{ord}_{s=1} P_2(X,q^{-s}) - 2 = \operatorname{ord}_{s=1} L(J,s) + \operatorname{ord}_{s=1} Q_2(s).$$

Corollary 2.2 says $\rho(X) = r+m+2$; (i) follows as $\operatorname{ord}_{s=1} Q_2(s) = m$.

For (ii), use (2.4) and (2.11). □

2.5. Pairings on NS(X)

Our next task is to compute $\Delta(\mathrm{NS}(X))$.

Definition 2.5.

(i) Let $\mathrm{Pic}^0(X_0)$ be the kernel of the degree map $\deg : \mathrm{Pic}(X_0)\to\mathbb{Z}$; the order δ of its cokernel is, by definition, the index of X_0 over F.

(ii) Let α be the order of the cokernel of the natural map $\mathrm{Pic}^0(X_0)\hookrightarrow J(F)$.

(iii) Let H (horizontal divisor on X) be the Zariski closure in X of a divisor d on X_0, rational over F, of degree δ.

(iv) The (vertical) divisor V on X is $\pi^{-1}(s)$ for a divisor s of degree one on S. Such a divisor s exists as k is a finite field and so the index of the curve S over k is one. Writing $s=\sum a_i v_i$ as a sum of closed points v_i on S gives $V=\sum a_i\pi^{-1}(v_i)$. Note that V generates $\pi^*\mathrm{NS}(S)\subset\mathrm{NS}(X)$.

Remark. The definitions show that the intersections of the divisor classes H and V in $\mathrm{NS}(X)$ are given by

$$H \cdot V = \delta = V \cdot H \quad \text{and} \quad V \cdot V = 0. \tag{2.13}$$

Also, since $\pi : X \to S$ is a flat map between smooth schemes, the map $\pi^* : CH(S) \to CH(X)$ on Chow groups is compatible with intersection of cycles. Since $V = \pi^*(s)$ and the intersection $s \cdot s = 0$ in $CH(S)$, one has $V \cdot V = 0$.

Let $\mathrm{NS}(X)_0 = (\pi^* \mathrm{NS}(S))^{\perp}$; as V generates $\pi^* \mathrm{NS}(S)$, we see that $\mathrm{NS}(X)_0$ is the subgroup of divisor classes Y such that $Y \cdot X_v = 0$ for any fiber $\pi^{-1}(v) = X_v$ of π; let $\mathrm{Pic}(X)_0$ be the inverse image of $\mathrm{NS}(X)_0$ under the projection $\mathrm{Pic}(X) \to \mathrm{NS}(X) \cong \frac{\mathrm{Pic}(X)}{A(k)}$.

Lemma 2.6. *$\mathrm{NS}(X)_0$ is the subgroup of $\mathrm{NS}(X)$ generated by divisor classes whose restriction to X_0 is trivial.*

Proof. We need to show that $\mathrm{NS}(X)_0$ is equal to $K := \mathrm{Ker}(\mathrm{NS}(X) \to \mathrm{NS}(X_0))$. If D is a vertical divisor ($\pi(D) \subset S$ is finite), then D is clearly in K; by [Liu02, §9.1, Proposition 1.21], D is in $\mathrm{NS}(X)_0$.

If D has no vertical components, then $D \cdot V = \deg(D_0)$. To see this, clearly we may assume D is reduced and irreducible (integral) and so flat over S. So $\mathcal{O}_D$ is locally free over $\mathcal{O}_S$ of constant degree n since S is connected. But then $\deg(D_0)$ is equal to n as is the integer $D \cdot V$. □

Lemma 2.7. *Let us denote*

$$R = \bigoplus_{v \in Z} R_v \quad \textit{and} \quad E = B(k) \cap R \subset \frac{\mathrm{Pic}(X)_0}{\pi^* \mathrm{Pic}(S)}.$$

One has the exact sequences

$$0 \longrightarrow R \longrightarrow \frac{\mathrm{Pic}(X)_0}{\pi^* \mathrm{Pic}(S)} \longrightarrow \mathrm{Pic}^0(X_0) \longrightarrow 0, \quad \textit{and}$$

$$0 \longrightarrow \frac{R}{E} \longrightarrow \frac{\mathrm{NS}(X)_0}{\pi^* \mathrm{NS}(S)} \longrightarrow \frac{\mathrm{Pic}^0(X_0)}{B(k)/E} \longrightarrow 0. \tag{2.14}$$

Proof. Lemma 2.6 shows that $R \subset \frac{\mathrm{Pic}(X)_0}{\pi^* \mathrm{Pic}(S)}$. As $A(k)$ is the kernel of the map $\mathrm{Pic}(X) \to \mathrm{NS}(X)$, it follows that $A(k) \subset \mathrm{Pic}(X)_0$. Thus, $B(k)$ is a subgroup of $\frac{\mathrm{Pic}(X)_0}{\pi^* \mathrm{Pic}(S)}$.

The first exact sequence follows from Lemma 2.6; the second one follows from Corollary 2.2 (ii). □

Lemma 2.8. *One has the equality*

$$\Delta_{\mathrm{ar}}\left(\frac{\mathrm{NS}(X)_0}{\pi^* \mathrm{NS}(S)}\right) = [B(k)]^2 \cdot \alpha^2 \cdot \Delta_{\mathrm{NT}}(J(F)) \cdot \prod_{v \in Z} \Delta_{\mathrm{ar}}(R_v).$$

Proof. The exact sequence (2.14) splits orthogonally over $\mathbb{Q}$: for any divisor γ representing an element of $\mathrm{Pic}(X_0)$, consider its Zariski closure $\bar{\gamma}$ in X. Since the intersection pairing on R_v is negative-definite [Liu02, §9.1, Theorem 1.23], the linear map $R_v \to \mathbb{Z}$ defined by $\beta \mapsto \beta \cdot \bar{\gamma}$ is represented by a unique element

$$\psi_v(\gamma) \in R_v \otimes \mathbb{Q} \subset \frac{\mathrm{NS}(X)_0}{\pi^* \mathrm{NS}(S)} \otimes \mathbb{Q}.$$

Thus, the element

$$\tilde{\gamma} := \bar{\gamma} - \sum_{v \in Z} \psi_v(\gamma)$$

is *good* in the sense of [Gor79, §5, p. 185]: by construction, the divisor $\tilde{\gamma}$ on X intersects every irreducible component of every fiber of π with multiplicity zero. Fix $\gamma, \kappa \in \mathrm{Pic}^0(X_0)$: viewing them as elements of $J(F)$, one computes their Neron–Tate pairing (1.9); also, one can compute the height pairing of $\tilde{\gamma}$ and $\tilde{\kappa}$ in $\mathrm{NS}(X)$. These two are related by the identity [Tat66, p. 429] [LLR18, Remark 3.11]

$$\langle \gamma, \kappa \rangle_{\mathrm{NT}} = -\langle \tilde{\gamma}, \tilde{\kappa} \rangle_{\mathrm{ar}} = -(\tilde{\gamma} \cdot \tilde{\kappa}) \cdot \log q.$$

This says that

$$\Delta_{\mathrm{ar}}\left(\mathrm{Pic}^0(X_0)\right)=\Delta_{\mathrm{NT}}\left(\mathrm{Pic}^0(X_0)\right). \tag{2.15}$$

The map

$$\mathrm{Pic}^0(X_0)\otimes\mathbb{Q}\to\frac{\mathrm{NS}(X)_0}{\pi^*\mathrm{NS}(S)}\otimes\mathbb{Q},\qquad \gamma\mapsto\tilde\gamma$$

provides an orthogonal splitting of (2.14) (over $\mathbb{Q}$). So

$$\begin{aligned}\Delta_{\mathrm{ar}}\left(\frac{\mathrm{NS}(X)_0}{\pi^*\mathrm{NS}(S)}\right)&\overset{(1.5)}{=}\Delta_{\mathrm{ar}}\left(\frac{\mathrm{Pic}^0(X_0)}{B(k)/E}\right)\cdot\Delta_{\mathrm{ar}}\left(\frac{R}{E}\right)=\frac{[B(k)]^2}{e^2}\cdot\Delta_{\mathrm{ar}}\left(\mathrm{Pic}^0(X_0)\right)\cdot e^2\Delta_{\mathrm{ar}}(R)\\&\overset{(2.15)}{=}[B(k)]^2\cdot\Delta_{\mathrm{NT}}\left(\mathrm{Pic}^0(X_0)\right)\cdot\Delta_{\mathrm{ar}}(R)\end{aligned}$$

where $e=[E]$ as the size of E. As

$$\Delta_{\mathrm{NT}}(\mathrm{Pic}^0(X_0))=\alpha^2\cdot\Delta_{\mathrm{NT}}(J(F))\quad\text{and}\Delta_{\mathrm{ar}}(R)=\prod_{v\in Z}\Delta_{\mathrm{ar}}(R_v), \tag{2.16}$$

this proves the lemma. □

With Lemma 2.8 at hand we are almost ready to compute $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))$. As the intersection pairing on $\mathrm{NS}(X)$ is not definite (Hodge index theorem), we cannot apply (1.5). Instead, we use a variant of a lemma of Z. Yun [Yun15].

2.5.1. A lemma of Yun.— Given a non-degenerate symmetric bilinear pairing $\Lambda\times\Lambda\to\mathbb{Z}$ on a finitely generated Abelian group Λ, an isotropic subgroup Γ, a subgroup Γ' containing Γ and with finite index in $\Gamma^\perp$, let $\Lambda_0=\frac{\Gamma'}{\Gamma}$. We recall from §1.4 that $\Delta(\Lambda)=z(D)^{-1}$ where $D:=\Lambda\to R\mathrm{Hom}(\Lambda,\mathbb{Z})$ and $\Delta(\Lambda_0)=z(D_0)^{-1}$ where $D_0:=\Lambda_0\to R\mathrm{Hom}(\Lambda_0,\mathbb{Z})$. Let Δ be the discriminant of the induced non-degenerate pairing $\Gamma\times\frac{\Lambda}{\Gamma'}\to\mathbb{Z}$:

$$\Delta=\frac{1}{z(C)}=\frac{1}{z(C')},\quad C:=\Gamma\to R\mathrm{Hom}\left(\frac{\Lambda}{\Gamma'},\mathbb{Z}\right),\quad\text{and}\quad C':=\frac{\Lambda}{\Gamma'}\to R\mathrm{Hom}(\Gamma,\mathbb{Z}).$$

Lemma 2.9 (*cf.* [Yun15, Lemma 2.12]). *One has* $\Delta(\Lambda)=\Delta(\Lambda_0)\cdot\Delta^2$.

Proof. Applying (1.6) to the maps of triangles

$$\begin{array}{ccccccc}\Gamma & \longrightarrow & \Lambda & \longrightarrow & \frac{\Lambda}{\Gamma} & \longrightarrow & \Gamma[1]\\ \downarrow & & \downarrow & & \downarrow & & \downarrow\\ R\mathrm{Hom}\left(\frac{\Lambda}{\Gamma'},\mathbb{Z}\right) & \longrightarrow & R\mathrm{Hom}(\Lambda,\mathbb{Z}) & \longrightarrow & R\mathrm{Hom}(\Gamma',\mathbb{Z}) & \longrightarrow & R\mathrm{Hom}\left(\frac{\Lambda}{\Gamma'},\mathbb{Z}\right)[1]\end{array}$$

and

$$\begin{array}{ccccccc}\frac{\Gamma'}{\Gamma} & \longrightarrow & \frac{\Lambda}{\Gamma} & \longrightarrow & \frac{\Lambda}{\Gamma'} & \longrightarrow & \frac{\Gamma'}{\Gamma}[1]\\ \downarrow & & \downarrow & & \downarrow & & \downarrow\\ R\mathrm{Hom}\left(\frac{\Gamma'}{\Gamma},\mathbb{Z}\right) & \longrightarrow & R\mathrm{Hom}(\Gamma',\mathbb{Z}) & \longrightarrow & R\mathrm{Hom}(\Gamma,\mathbb{Z}) & \longrightarrow & R\mathrm{Hom}\left(\frac{\Gamma'}{\Gamma},\mathbb{Z}\right)[1]\end{array}$$

shows that $z(D)\cdot z(C)^{-1}=z(D_0)\cdot z(C')$. □

We can finally compute $\Delta_{\mathrm{ar}}(\mathrm{NS}(X))$.

Proposition 2.10. *The following relations hold*

$$\Delta_{\mathrm{ar}}(\mathrm{NS}(X))=\delta^2\cdot\Delta_{\mathrm{ar}}\left(\frac{\mathrm{NS}(X)_0}{\pi^*\mathrm{NS}(S)}\right)\cdot(\log q)^2\quad\textit{and}\quad\Delta(\mathrm{NS}(X))=\delta^2\cdot\Delta\left(\frac{\mathrm{NS}(X)_0}{\pi^*\mathrm{NS}(S)}\right).$$

Proof. Let $\mathbb{Z} \cong \Gamma = \pi^* \mathrm{NS}(S) \subset \mathrm{NS}(X) = \Lambda$ with $\Gamma' = \mathrm{NS}(X)_0$ and $\Lambda_0 = \frac{\mathrm{NS}(X)_0}{\pi^*\mathrm{NS}(S)}$. Lemma 2.6 implies that

$$\frac{\Lambda}{\Gamma'} = \frac{\mathrm{NS}(X)}{\mathrm{NS}(X)_0} \cong \mathbb{Z} \quad \text{and} \quad C = \Gamma \to \mathrm{Hom}\left(\frac{\mathrm{NS}(X)}{\mathrm{NS}(X)_0}, \mathbb{Z}\right),$$

with C as in Lemma 2.9. Now (2.13) shows that $\pi^*\mathrm{NS}(S)$ is isotropic and $\Delta = \delta$. The result follows from Lemma 2.9. □

Combining the previous proposition with Lemma 2.8 provides the identity

$$\Delta_{\mathrm{ar}}(\mathrm{NS}(X)) = \delta^2 \cdot [B(k)]^2 \cdot \alpha^2 \cdot \Delta_{\mathrm{NT}}(J(F)) \cdot \prod_{v\in Z} \Delta_{\mathrm{ar}}(R_v) \cdot (\log q)^2. \tag{2.17}$$

For $v \in S$, we put δ_v and δ'_v for the (local) index and period of $X \times F_v$ over the local field F_v.

Theorem 2.11. [Gei20, Theorem 1.1] *Assume that* $\mathrm{Br}(X)$ *is finite. The following equality holds:*

$$[\mathrm{Br}(X)]\alpha^2\delta^2 = [Ш(J/F)] \prod_{v\in S} \delta'_v \delta_v. \tag{2.18}$$

Remark 2.12. Note that for $v \in U$, one has $\delta_v = 1 = \delta'_v$ [LLR18, p. 603], [FS21, (74)] (for $\delta_v = 1$), [Gro68, Proposition (4.1) (a)] (δ'_v divides δ_v); the basic reason is that if $v \in U$, then X_v has a rational divisor of degree one as $k(v)$ is finite; this divisor lifts to a rational divisor of degree one on $X \times F_v$ by smoothness of X_v. Also, $c_v = 1$ [BLR90, Theorem 1, §9.5 p. 264]. So $c(J) := \prod_{v\in S} c_v$ satisfies

$$c(J) = \prod_{v\in Z} c_v. \tag{2.19}$$

Lemma 2.13. *One has*

$$c(J) \cdot Q_2^*(1) = \prod_{v\in Z} \delta_v \cdot \delta'_v \cdot \Delta_{\mathrm{ar}}(R_v). \tag{2.20}$$

Proof. By a result of Flach and Siebel [FS21, Lemma 17] (using Raynaud's theorem [Gor79, Theorem 5.2] in [BL99]), one has

$$\Delta_{\mathrm{ar}}(R_v) = \frac{c_v}{\delta_v \cdot \delta'_v} \cdot (\log q_v)^{m_v - 1} \cdot \prod_{i\in G_v} r_i.$$

So we find that

$$\begin{aligned}\prod_{v\in Z} \delta_v \cdot \delta'_v \cdot \Delta_{\mathrm{ar}}(R_v) &= \prod_{v\in Z}\left(c_v \cdot (\log q_v)^{m_v-1} \cdot \prod_{i\in G_v} r_i\right) = \prod_{v\in Z} c_v \cdot \prod_{v\in Z}\left((\log q_v)^{m_v-1} \cdot \prod_{i\in G_v} r_i\right) \\ &\overset{(2.19)}{=} c(J) \cdot \prod_{v\in Z}\left((\log q_v)^{m_v-1} \cdot \prod_{i\in G_v} r_i\right) \overset{(2.10)}{=} c(J) \cdot Q_2^*(1).\end{aligned}$$

□

3. First proof of Theorem 1.1

Proof of Theorem 1.1. By (2.17) and (2.20), we have

$$\Delta_{\mathrm{ar}}(\mathrm{NS}(X)) = \frac{\alpha^2\delta^2}{\prod_{v\in Z}\delta_v \cdot \delta'_v} \cdot \Delta_{\mathrm{NT}}(J(F)) \cdot c(J) \cdot [B(k)]^2 \cdot Q_2^*(1) \cdot (\log q)^2.$$

From Theorem 2.11, we have

$$[\mathrm{Br}(X)] \cdot \Delta_{\mathrm{ar}}(\mathrm{NS}(X)) = [Ш(J/F)] \cdot \Delta_{\mathrm{NT}}(J(F)) \cdot c(J) \cdot [B(k)]^2 \cdot Q_2^*(1) \cdot (\log q)^2.$$

Further with (2.5), we obtain

$$[\mathrm{Br}(X)]\cdot\Delta_{\mathrm{ar}}(\mathrm{NS}(X))\cdot q^{-\alpha(X)} = [Ш(J/F)]\cdot\Delta_{\mathrm{NT}}(J(F))\cdot c(J)\cdot q^{\chi(S,\mathrm{Lie}\,\mathcal{J})}\cdot[B(k)]^2\cdot Q_2^*(1)\cdot q^{-\dim(B)}\cdot(\log q)^2.$$

On the other hand, recall (2.12)

$$P_2^*(X,\frac{1}{q}) = L^*(J,1)\cdot[B(k)]^2\cdot Q_2^*(1)\cdot q^{-\dim(B)}\cdot(\log q)^2.$$

The ratio of the previous two equalities gives

$$\frac{P_2^*(X,\frac{1}{q})}{[\mathrm{Br}(X)]\cdot\Delta_{\mathrm{ar}}(\mathrm{NS}(X))\cdot q^{-\alpha(X)}} = \frac{L^*(J,1)}{[Ш(J/F)]\cdot\Delta_{\mathrm{NT}}(J(F))\cdot c(J)\cdot q^{\chi(S,\mathrm{Lie}\,\mathcal{J})}}.$$

This equality implies Theorem 1.1. □

4. Second proof of Theorem 1.1

We will give another more direct proof of Theorem 1.1 using Weil-étale cohomology. We refer the reader to [Lic05, Gei04, GS20] for basics about Weil-étale cohomology over finite fields. Throughout this section, we assume that $\mathrm{Br}(X)$ (and hence $Ш(J/F)$) is finite.

4.1. Setup

Let $C\in D^b(T_{\text{ét}})$ be an object of the bounded derived category of sheaves of Abelian groups on the small étale site $T_{\text{ét}}$. Let $D\in D^b(\mathrm{FDVect}_k)$ be an object of the bounded derived category of finite-dimensional vector spaces over k. Assume that the Weil-étale cohomology $H_W^*(T,C)$ is finitely generated and the cohomology sheaf $H^*(C\otimes^L\mathbb{Z}/l\mathbb{Z})$ is finite in all degrees for all prime numbers $l\nmid q$. Let $e\colon H_W^i(T,C)\to H_W^{i+1}(T,C)$ be the map defined by cup product with the arithmetic Frobenius $\in H_W^1(T,\mathbb{Z})$. It defines a complex

$$\cdots\xrightarrow{e} H_W^i(T,C)\xrightarrow{e} H_W^{i+1}(T,C)\xrightarrow{e}\cdots$$

with finite cohomology. Set $C_{\mathbb{Q}_l} = R\varprojlim_n(C\otimes^L\mathbb{Z}/l^n\mathbb{Z})\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$, whose cohomologies are finite-dimensional vector spaces over $\mathbb{Q}_l$ (by the finiteness of $H^*(C\otimes^L\mathbb{Z}/l\mathbb{Z})$) equipped with an action of the geometric Frobenius φ of k. Define

$$\begin{aligned}
Z(C,t) &= \prod_i \det(1-\varphi t\,|\,H^i(C_{\mathbb{Q}_l}))^{(-1)^{i+1}},\\
\rho(C) &= \sum_j(-1)^{j+1}\cdot j\cdot\mathrm{rank}\,H_W^j(T,C),\\
\chi_W(C) &= \chi(H_W^*(T,C),e),\quad\text{and}\\
\chi(D) &= \sum_j(-1)^j\dim H^j(D).
\end{aligned}$$

Assume that $Z(C,t)\in\mathbb{Q}(t)$ and is independent of l. Define $Q(C,D)\in\mathbb{Q}_{>0}^\times\times(1-t)^{\mathbb{Z}}$ to be the leading term of the $(1-t)$-adic expansion of the function

$$\pm\frac{Z(C,t)(1-t)^{\rho(C)}}{\chi_W(C)q^{\chi(D)}}$$

(the sign is the one that makes the coefficient positive). It is the defect of a zeta value formula of the form

$$\lim_{t\to1}Z(C,t)(1-t)^{\rho(C)} = \pm\chi_W(C)q^{\chi(D)}.$$

We mention $Q(C,D)$ only when $H^*_W(T,C)$ is finitely generated, $H^*(C\otimes^L \mathbb{Z}/l\mathbb{Z})$ is finite and $Z(C,t)\in\mathbb{Q}(t)$ is independent of l. These conditions are satisfied for the cases of interest below. We have

$$Q(C[1],D[1]) = Q(C,D)^{-1}.$$

If (C,D), (C',D') and (C'',D'') are pairs as above, and $C\to C'\to C''\to C[1]$ and $D\to D'\to D''\to D[1]$ are distinguished triangles, then $Q(C',D') = Q(C,D)Q(C'',D'')$.

4.2. Special cases

We give two special cases of the above constructions. First, let $\pi_X\colon X_{\text{ét}}\to T_{\text{ét}}$ be the structure morphism. Let $P_2^{\diamond}(X,1)(1-t)^{\rho(X)'}$ be the leading term of the $(1-t)$-adic expansion of $P_2(X,t/q)$.

Proposition 4.1. *Let* $(C,D) = (R\pi_{X,*}\mathbb{G}_m[-1], R\Gamma(X,\mathcal{O}_X))$. *Then* $H^*(C\otimes^{\mathrm{L}}\mathbb{Z}/l\mathbb{Z})$ *is finite,* $H^*_W(T,C)$ *is finitely generated,* $Z(C,q^{-s}) = \zeta(X,s+1)$ *and*

$$Q(C,D)^{-1} = \frac{P_2^{\diamond}(X,1)\cdot(1-t)^{\rho(X)'-\rho(X)}}{[\mathrm{Br}(X)]\cdot\Delta(\mathrm{NS}(X))\cdot q^{-\alpha(X)}}.$$

In particular, the statement $Q(C,D)=1$ *is equivalent to Conjecture 1.2.*

Proof. We have $H^*_W(T,C)\cong H^*_W(X,\mathbb{G}_m[-1])\cong H^*_W(X,\mathbb{Z}(1))$. The finiteness assumption on $\mathrm{Br}(X)$ implies the Tate conjecture for divisors on X and hence the finite generation of $H^*_W(X,\mathbb{Z}(1))$ by [Gei04, Theorems 8.4 and 9.3]. The object $C\otimes^{\mathrm{L}}\mathbb{Z}/l\mathbb{Z}\cong R\pi_{X,*}\mathbb{Z}/l\mathbb{Z}(1)\in D^b(T_{\text{ét}})$ is constructible and hence its cohomologies are finite. We have $H^i(C_{\mathbb{Q}_l})\cong R^i\pi_{X,*}\mathbb{Q}_l(1)$, which is the vector space $H^i_{\text{ét}}(X\times_k \bar{k},\mathbb{Q}_l(1))$ equipped with the natural Frobenius action. It follows that $Z(C,q^{-s}) = \zeta(X,s+1)$.

We calculate $Q(C,D)^{-1}$. By (1.2), (2.3) and (2.4), the leading term of the $(1-t)$-adic expansion of $Z(C,t)$ is

$$-\frac{[A(k)]^2}{P_2^{\diamond}(X,1)\cdot(q-1)^2\cdot q^{\dim A-1}\cdot(1-t)^{\rho(X)'}}. \tag{4.1}$$

By [Gei04, Theorems 7.5 and 9.1], we have

$$\chi_W(C) = \prod_i [H^i_W(X,\mathbb{Z}(1))_{\mathrm{tor}}]^{(-1)^i}\cdot R^{-1},$$

where R is the determinant of the pairing

$$H^2_W(X,\mathbb{Z}(1))\times H^2_W(X,\mathbb{Z}(1)) \xrightarrow{\cup} H^4_W(X,\mathbb{Z}(2)) \longrightarrow H^4_{\text{ét}}(X\times_k\bar{k},\mathbb{Z}(2))\cong \mathrm{CH}^2(X\times_k\bar{k}) \xrightarrow{\deg} \mathbb{Z}.$$

We have $H^n_W(X,\mathbb{Z}(1)) = 0$ for $n>5$ by [Gei04, Theorem 7.3] and for $n<1$ obviously. Also

$$H^1_W(X,\mathbb{Z}(1))\cong k^\times, \quad H^2_W(X,\mathbb{Z}(1))\cong \mathrm{Pic}(X), \quad \text{and} \quad H^3_W(X,\mathbb{Z}(1))_{\mathrm{tor}}\cong \mathrm{Br}(X)$$

by [Gei04, Proposition 7.4 (c) and (d)]. By [Gei18, Remark 3.3], the group $H^i_W(X,\mathbb{Z}(1))_{\mathrm{tor}}$ is Pontryagin dual to $H^{6-i}_W(X,\mathbb{Z}(1))_{\mathrm{tor}}$ for any i. The above pairing defining R can be identified with the intersection pairing $\mathrm{Pic}(X)\times\mathrm{Pic}(X)\to\mathbb{Z}$. Thus, with (2.1), we have

$$\chi_W(C) = \frac{[A(k)]^2}{[\mathrm{Br}(X)]\cdot\Delta(\mathrm{NS}(X))\cdot(q-1)^2}. \tag{4.2}$$

Since the rank of $H^i_W(X,\mathbb{Z}(1))$ is $\rho(X)$ for $i=2,3$ and zero otherwise by [Gei04, Proposition 7.4 (c) and (d)], we have

$$\rho(C) = \rho(X). \tag{4.3}$$

Combining (1.3), (4.1), (4.2) and (4.3), we get the desired formula for $Q(C,D)^{-1}$. □

Next, let $\pi_S\colon S_{\text{ét}} \to T_{\text{ét}}$ be the structure morphism. Let $L^{\diamond}(J,1)(1-q^{-s})^{r'}$ be the leading term of the $(1-q^{-s})$-adic expansion of $L(J,s+1)$. Let $\Delta(J(F))$ be the discriminant of the pairing $(\gamma,\kappa) \mapsto \langle \gamma,\kappa\rangle_{\mathrm{NT}}/\log q$ on $J(F)$.

Proposition 4.2. *Let* $(C,D) = (R\pi_{S,*}\mathcal{J}[-1], R\Gamma(S, \operatorname{Lie}\mathcal{J}))$. *Then* $H^*(C \otimes^{\mathrm{L}} \mathbb{Z}/l\mathbb{Z})$ *is finite,* $H_W^*(T,C)$ *is finitely generated,* $Z(C,q^{-s}) = L(J,s+1)$ *and*

$$Q(C,D) = \frac{L^{\diamond}(J,1)\cdot(1-t)^{r'-r}}{[\text{Ш}(J/F)]\cdot\Delta(J(F))\cdot c(J)\cdot q^{\chi(S,\operatorname{Lie}\mathcal{J})}}.$$

In particular, the statement $Q(C,D)=1$ *is equivalent to Conjecture 1.3.*

Proof. We have $H_W^*(T,C) \cong H_W^{*-1}(S,\mathcal{J})$. The finiteness assumption of $\text{Ш}(J/F)$ implies the finite generation of $H_W^*(S,\mathcal{J})$ by [GS20, Proposition 6.4]. We have $C \otimes^{\mathrm{L}} \mathbb{Z}/l\mathbb{Z} \cong R\pi_{S,*}(\mathcal{J}\otimes^{L}\mathbb{Z}/l\mathbb{Z})[-1]$. By the paragraph before the proof of [GS20, Proposition 9.2] and the first displayed equation in the proof of [GS20, Proposition 9.2], we know that $\mathcal{J} \otimes^{\mathrm{L}} \mathbb{Z}/l\mathbb{Z} \in D^b(S_{\text{ét}})$ is constructible. Hence $H^*(C \otimes^{\mathrm{L}} \mathbb{Z}/l\mathbb{Z})$ is finite. We also have $H^i(C_{\mathbb{Q}_l}) \cong R^i\pi_{S,*}V_l(\mathcal{J})$ (where V_l denotes the l-adic Tate modules tensored with $\mathbb{Q}_l$), which is the vector space $H^i_{\text{ét}}(S\times_k \bar{k}, V_l(\mathcal{J}))$ equipped with the natural Frobenius action. Hence we have $Z(C,q^{-s}) = L(J,s+1)$ by [Sch82, Satz 1]. We have

$$\chi_W(C) = [\text{Ш}(J/F)]\cdot\Delta(J(F))\cdot c(J)$$

by [GS20, Proposition 8.3]. By [GS20, Proposition 7.1], the rank of $H_W^i(S,\mathcal{J})$ is r for $i=0,1$ and zero otherwise. Hence $\rho(C) = -r$. The formula for $Q(C,D)$ follows. □

4.3. Comparison

Now Theorem 1.1 follows from the following

Proposition 4.3. *One has*

$$Q(R\pi_{X,*}\mathbb{G}_m[-1], R\Gamma(X,\mathcal{O}_X))^{-1} = Q(R\pi_{S,*}\mathcal{J}[-1], R\Gamma(S,\operatorname{Lie}\mathcal{J})).$$

Proof. We have $R^i\pi_*\mathbb{G}_m = 0$ over $S_{\text{ét}}$ for all $i\geq 2$ by [Gro68, Corollaire (3.2)]. Hence we have a distinguished triangle

$$R\pi_{S,*}\mathbb{G}_m \longrightarrow R\pi_{X,*}\mathbb{G}_m \longrightarrow R\pi_{S,*}\operatorname{Pic}_{X/S}[-1] \longrightarrow R\pi_{S,*}\mathbb{G}_m[1]$$

in $D(T_{\text{ét}})$.[7] Similarly, we have a distinguished triangle

$$R\Gamma(S,\mathcal{O}_S) \longrightarrow R\Gamma(X,\mathcal{O}_X) \longrightarrow R\Gamma(S,R^1\pi_*\mathcal{O}_X)[-1] \longrightarrow R\Gamma(S,\mathcal{O}_S)[1].$$

We have $Q(R\pi_{S,*}\mathbb{G}_m[-1], R\Gamma(S,\mathcal{O}_S)) = 1$ by the class number formula ([Gei04, Theorems 9.1 and 9.3], or [Lic05, Theorems 5.4 and 7.4] and the functional equation). Therefore

$$Q(R\pi_{X,*}\mathbb{G}_m[-1], R\Gamma(X,\mathcal{O}_X))^{-1} = Q(R\pi_{S,*}\operatorname{Pic}_{X/S}[-1], R\Gamma(S,R^1\pi_*\mathcal{O}_X)). \tag{4.4}$$

For a closed point $v\in S$, let $\iota_v\colon \operatorname{Spec} k(v) \hookrightarrow S$ be the inclusion. For any $i\in G_v$, let $k(v)_i$ be the algebraic closure of $k(v)$ in the function field of Γ_i. Let $\iota_{v,i}\colon \operatorname{Spec} k(v)_i \to S$ be the natural morphism. Set

$$E = \bigoplus_{v\in Z} \frac{\bigoplus_{i\in G_v}\iota_{v,i,*}\mathbb{Z}}{\iota_{v,*}\mathbb{Z}}.$$

Let $j\colon \operatorname{Spec} F \hookrightarrow S$ be the inclusion. Then we have a natural exact sequence

$$0 \longrightarrow E \longrightarrow \operatorname{Pic}_{X/S} \longrightarrow j_*\operatorname{Pic}_{X_0/F} \longrightarrow 0$$

[7] Here $\operatorname{Pic}_{X/S} = R^1\pi_*\mathbb{G}_m$ is only an étale sheaf. The fppf sheaf denoted by the same symbol is not an algebraic space in general.

over $S_{\text{ét}}$ by [Gro68, Equations (4.10 bis) and (4.21)] (where the assumption [Gro68, Equation (4.13)] is satisfied since $k(v)$ is finite and hence perfect for all closed $v \in S$). Therefore we have a distinguished triangle

$$R\pi_{S,*}E \longrightarrow R\pi_{S,*}\operatorname{Pic}_{X/S} \longrightarrow R\pi_{S,*}j_*\operatorname{Pic}_{X_0/F} \longrightarrow R\pi_{S,*}E[1].$$

Since E is skyscraper, we have $Q(R\pi_{S,*}E,0) = 1$ by [GS21, Theorem 3.1] (Step 3 of the proof is sufficient). Therefore

$$Q(R\pi_{S,*}\operatorname{Pic}_{X/S}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S,*}j_*\operatorname{Pic}_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)). \tag{4.5}$$

Applying j_* to the exact sequence

$$0 \longrightarrow J \longrightarrow \operatorname{Pic}_{X_0/F} \longrightarrow \mathbb{Z} \longrightarrow 0$$

over $\operatorname{Spec} F_{\text{ét}}$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow j_*\operatorname{Pic}_{X_0/F} \longrightarrow \mathbb{Z}$$

over $S_{\text{ét}}$. Let I be the image of the last morphism, so that we have an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow j_*\operatorname{Pic}_{X_0/F} \longrightarrow I \longrightarrow 0.$$

Then we have distinguished triangles

$$R\pi_{S,*}\mathcal{J} \longrightarrow R\pi_{S,*}j_*\operatorname{Pic}_{X_0/F} \longrightarrow R\pi_{S,*}I \longrightarrow R\pi_{S,*}\mathcal{J}[1], \quad \text{and}$$
$$R\pi_{S,*}I \longrightarrow R\pi_{S,*}\mathbb{Z} \longrightarrow R\pi_{S,*}(\mathbb{Z}/I) \longrightarrow R\pi_{S,*}I[1].$$

We have $Q(R\pi_{S,*}\mathbb{Z},0) = 1$ again by the class number formula ([Gei04, Theorems 9.1 and 9.2] or [Lic05, Theorem 7.4]). Since $\mathbb{Z}/I$ is skyscraper with finite stalks, we have $Q(R\pi_{S,*}(\mathbb{Z}/I),0) = 1$ by [GS21, Theorem 3.1] (Step 2 of the proof is sufficient). Therefore

$$Q(R\pi_{S,*}j_*\operatorname{Pic}_{X_0/F}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S,*}\mathcal{J}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)). \tag{4.6}$$

The complexes $R\Gamma(S, R^1\pi_*\mathcal{O}_X)$ and $R\Gamma(S, \operatorname{Lie}\mathcal{J})$ have the same Euler characteristic by (2.15). Hence

$$Q(R\pi_{S,*}\mathcal{J}[-1], R\Gamma(S, R^1\pi_*\mathcal{O}_X)) = Q(R\pi_{S,*}\mathcal{J}[-1], R\Gamma(S, \operatorname{Lie}\mathcal{J})). \tag{4.7}$$

Combining (4.4)—(4.7), we get the desired equality. □

4.4. A new proof of Geisser's formula

The above proposition, combined with the results of the previous sections, also gives a new proof of Theorem 2.11 as follows.

Proof of Theorem 2.11. By Proposition 4.3, we have

$$\frac{P_2^{\diamond}(X,1)}{[\operatorname{Br}(X)] \cdot \Delta(\operatorname{NS}(X)) \cdot q^{-\alpha(X)}} = \frac{L^{\diamond}(J,1)}{[\text{Ш}(J/F)] \cdot \Delta(J(F)) \cdot c(J) \cdot q^{\chi(S,\operatorname{Lie}\mathcal{J})}}.$$

By (2.12), we have

$$P_2^{\diamond}(X,1) = L^{\diamond}(J,1) \cdot q^{-\dim B} \cdot [B(k)]^2 \cdot Q_2^{\diamond}(1),$$

where $Q_2^{\diamond}(1)$ is the leading coefficient of the $(1-q^{-s})$-adic expansion of $Q_2(s+1)$. By (2.17) and (2.20), we have

$$\Delta(\operatorname{NS}(X)) = \frac{\alpha^2\delta^2}{\prod_{v\in Z}\delta'_v\delta_v} \cdot \Delta(J(F)) \cdot c(J) \cdot [B(k)]^2 \cdot Q_2^{\diamond}(1).$$

By (2.5), we have

$$q^{-\alpha(X)} = q^{\chi(S,\operatorname{Lie}\mathcal{J})} \cdot q^{-\dim B}.$$

Taking a suitable alternating product of these four equalities, we obtain (2.18). □

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