EPIGA

A characterization of finite étale morphisms in tensor triangular geometry

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Abstract. We provide a characterization of finite étale morphisms in tensor triangular geometry. They are precisely those functors which have a conservative right adjoint, satisfy Grothendieck–Neeman duality, and for which the relative dualizing object is trivial (via a canonically-defined map).

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1. Introduction

The purpose of this note is to give a characterization of "finite étale morphisms" in tensor triangular geometry. We follow the notation, terminology, and perspective of [BDS16]. In particular, we will work in the context of rigidly-compactly generated tensor-triangulated categories [BDS16, Definition 2.7]. The kind of characterization we have in mind is analogous to the following well-known characterization of smashing localizations:

Theorem 1.1. *Smashing localizations of a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ are precisely those geometric functors $f^*\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tensor-triangulated categories whose right adjoint f_* is fully faithful.*

We will recall a proof in Remark 3.10 below. Smashing localizations include, for example, restriction to a quasi-compact open subset of the Balmer spectrum. More generally, the tensor-triangular analogue of an étale morphism is extension-of-scalars with respect to a commutative separable algebra (with smashing localizations being the special case of idempotent algebras). Finite étale morphisms are, by definition, extension-of-scalars with respect to a *compact* commutative separable algebra (see Definition 4.2). Most smashing localizations are not *finite* étale morphisms, just as most open immersions are not proper. We will prove:

Theorem 1.2. *Finite étale extensions of a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ are precisely those geometric functors $f^*\colon \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tensor-triangulated categories which satisfy the following three properties:*

(a) f^* satisfies Grothendieck–Neeman duality;
(b) the right adjoint f_* is conservative;
(c) the canonical map $\mathbb{1}_\mathcal{S} \to \omega_f$ is an isomorphism.

The terminology and notation will be explained in Section 4. We just remark that under hypothesis (a), the algebra $f_*(\mathbb{1}_\mathcal{S})$ is rigid (a.k.a. dualizable) and hence has an associated trace map. This corresponds by adjunction to a canonical map $\mathbb{1}_\mathcal{S} \to \omega_f$ from the unit to the relative dualizing object, which hypothesis (c) asserts is an isomorphism.

The keys to the theorem are the robust monadicity theorems which hold for triangulated categories and a deeper understanding of strongly separable algebras. Indeed, we begin the paper in Section 2 with a treatment of strongly separable algebras in arbitrary symmetric monoidal categories which may be of independent interest. We prove, in particular, that a rigid commutative algebra is separable if and only if

it is strongly separable if and only if its canonically-defined trace form is nondegenerate (Corollary 2.38). We then turn in Section 3 to tensor-triangulated categories and the role separable algebras play in that setting. A key tool is a strengthened version of separable monadicity (Proposition 3.8). We define finite étale morphisms and prove the main theorem (Theorem 4.8) in Section 4. We also show that if the target category is locally monogenic, then the conservativity condition (b) is implied by the other two conditions (Corollary 4.20). In Section 5, we illustrate the theorem by giving some examples and non-examples of finite étale morphisms in equivariant homotopy theory, algebraic geometry, and derived algebra.

Acknowledgments

2. Strongly separable algebras

We begin with a discussion of separable algebras in an arbitrary symmetric monoidal category. Although separable algebras are well-understood at this level of generality, we would like to clarify the notion of strongly separable algebra. Our main goal is to show that the equivalent characterizations of classical strongly separable algebras over fields established by [Agu00] have suitable generalizations to arbitrary symmetric monoidal categories. The main punch line is that a rigid commutative algebra is separable if and only if it is strongly separable if and only if its trace form is nondegenerate (see Corollary 2.38). Moreover, this is the case if and only if it has the (necessarily unique) structure of a special symmetric Frobenius algebra.

Terminology 2.1. Throughout this section, we work in a fixed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. The symmetry isomorphism will be denoted $\tau \colon A \otimes B \xrightarrow{\sim} B \otimes A$. An object A in $\mathcal{C}$ is *rigid* (a.k.a. *dualizable*) if there exists an object DA and morphisms $\eta \colon \mathbb{1} \to DA \otimes A$ and $\epsilon \colon A \otimes DA \to \mathbb{1}$ such that the composites

$$A \xrightarrow{1 \otimes \eta} A \otimes DA \otimes A \xrightarrow{\epsilon \otimes 1} A \quad \text{and} \quad DA \xrightarrow{\eta \otimes 1} DA \otimes A \otimes DA \xrightarrow{1 \otimes \epsilon} DA$$

are the identity morphisms. It follows that the functor $DA \otimes - \colon \mathcal{C} \to \mathcal{C}$ is right adjoint to $A \otimes - \colon \mathcal{C} \to \mathcal{C}$. An *algebra* A is an associative unital monoid in $\mathcal{C}$. The multiplication and unit maps will be denoted $\mu \colon A \otimes A \to A$ and $u \colon \mathbb{1} \to A$.

Definition 2.2. An algebra (A, μ, u) is *separable* if there exists a map $\sigma \colon A \to A \otimes A$ such that

(σ1) $\mu \circ \sigma = \mathrm{id}_A$ and
(σ2) $(1 \otimes \mu) \circ (\sigma \otimes 1) = \sigma \circ \mu = (\mu \otimes 1) \circ (1 \otimes \sigma)$ as maps $A \otimes A \to A \otimes A$.

In other words, A is separable if the multiplication map $\mu \colon A \otimes A \to A$ admits an (A, A)-bilinear section.

Remark 2.3. If we precompose such a section σ with the unit $u \colon \mathbb{1} \to A$, we obtain a map

$$\kappa := \sigma \circ u \colon \mathbb{1} \to A \otimes A$$

which satisfies

(κ1) $\mu \circ \kappa = u$, and
(κ2) $(1 \otimes \mu) \circ (\kappa \otimes 1) = (\mu \otimes 1) \circ (1 \otimes \kappa)$ as maps $A \to A \otimes A$.

Conversely, given such a κ, the map $A \to A \otimes A$ displayed in (κ2) satisfies axioms (σ1) and (σ2). Thus, an algebra A is separable if and only if it admits a map $\kappa \colon \mathbb{1} \to A \otimes A$ satisfying (κ1) and (κ2). Such a map κ is called a *separability idempotent*.

Remark 2.4. We refer the reader to [AG60, CHR65, KO74], [Pie82, Chapter 10], and [For17] for further information about separable algebras and their role in classical representation theory and algebraic geometry. The following notion of a *strongly* separable algebra was originally studied by Kanzaki and Hattori [Hat65, Kan62]:
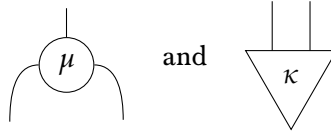
Definition 2.5. An algebra A is *strongly separable* if there exists a map $\kappa\colon \mathbb{1} \to A \otimes A$ satisfying $(\kappa 1)$, $(\kappa 2)$, and

($\kappa 3$) $\kappa = \tau \circ \kappa$.

In other words, A is strongly separable if it admits a *symmetric* separability idempotent.

Remark 2.6. For classical algebras over a field, [Agu00] provides several equivalent characterizations of strongly separable algebras. Our present goal is to clarify the extent to which these characterizations hold in an arbitrary symmetric monoidal category. For this purpose, the graphical calculus of string diagrams will be very convenient. We refer the reader to [Sel11, Sections 3–4] and [PS13, Section 2] for more information concerning these diagrams and suffice ourselves to remark that it is a routine exercise to convert a proof involving string diagrams into a detailed proof using commutative diagrams.

Notation 2.7. We will read our string diagrams from bottom to top. The multiplication map $\mu\colon A \otimes A \to A$ and the map $\kappa\colon \mathbb{1} \to A \otimes A$ will be represented by

while the unit $u\colon \mathbb{1} \to A$ and the identity $\mathrm{id}\colon A \to A$ will be represented by
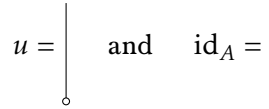
Thus, for example, axiom $(\kappa 2)$ reads

(2.8)

Proposition 2.9. *An algebra (A, μ, u) is strongly separable if and only if there exists a morphism $\kappa\colon \mathbb{1} \to A \otimes A$ satisfying $(\kappa 2)$ and*

($\kappa 4$) $\mu \circ \tau \circ \kappa = u$.

Proof. By definition, an algebra is strongly separable if it admits a morphism κ satisfying $(\kappa 1)$, $(\kappa 2)$, and $(\kappa 3)$. It is immediate that $(\kappa 1)$ and $(\kappa 3)$ together imply $(\kappa 4)$. It is also immediate that $(\kappa 3)$ and $(\kappa 4)$ together imply $(\kappa 1)$. Thus, the claim will be established if we can prove that $(\kappa 2)$ and $(\kappa 4)$ together imply $(\kappa 3)$. Using Notation 2.7, observe:

(2.10)

We can then rearrange the last diagram by pulling the left-hand multiplication to the right-hand side and continue:

(2.11)

This establishes $\kappa = \tau \circ \kappa$, which is axiom ($\kappa 3$). $\qquad\square$

Corollary 2.12. *Any commutative separable algebra is strongly separable.*

Proof. This follows from Proposition 2.9 since axiom ($\kappa 4$) coincides with axiom ($\kappa 1$) when the algebra is commutative. $\qquad\square$

Remark 2.13. For string diagrams involving a rigid object A, we will use the direction of a string to indicate whether it represents A or its dual DA. For example, the unit $\mathbb{1} \to DA \otimes A$ and counit $A \otimes DA \to \mathbb{1}$ are represented by

 and
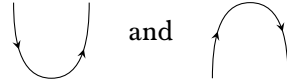
respectively, and the unit-counit relations are given by

 and
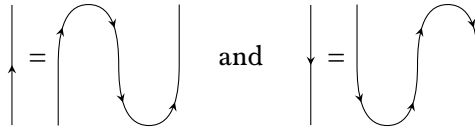
Definition 2.14. Let A be a rigid algebra in the symmetric monoidal category $\mathcal{C}$. Its *trace map* $\mathrm{tr}\colon A \to \mathbb{1}$ is given by

(2.15) $$A \simeq A \otimes \mathbb{1} \xrightarrow{1\otimes\eta} A \otimes DA \otimes A \xrightarrow{1\otimes\tau} A \otimes A \otimes DA \xrightarrow{\mu\otimes1} A \otimes DA \xrightarrow{\epsilon} \mathbb{1}.$$

Remark 2.16. To explain this definition, recall that every endomorphism $f\colon A \to A$ of the rigid object A has an associated "trace" $\mathrm{Tr}(f)\colon \mathbb{1} \to \mathbb{1}$ given as

$$\mathbb{1} \xrightarrow{\eta} DA \otimes A \xrightarrow{1\otimes f} DA \otimes A \xrightarrow{\tau} A \otimes DA \xrightarrow{\epsilon} \mathbb{1}.$$

Moreover, post-composition by the map

(2.17) $$DA \otimes A \xrightarrow{\tau} A \otimes DA \xrightarrow{\epsilon} \mathbb{1}$$

induces a function

$$\mathcal{C}(A, A) \simeq \mathcal{C}(\mathbb{1}, DA \otimes A) \to \mathcal{C}(\mathbb{1}, \mathbb{1})$$

which sends f to $\mathrm{Tr}(f)$. On the other hand, the multiplication map $\mu\colon A \otimes A \to A$ corresponds (by pulling the second A to the other side) to a morphism $A \to DA \otimes A$ given by

(2.18) $$A \simeq \mathbb{1} \otimes A \xrightarrow{\eta\otimes1} DA \otimes A \otimes A \xrightarrow{1\otimes\tau} DA \otimes A \otimes A \xrightarrow{1\otimes\mu} DA \otimes A.$$

Post-composition by this map provides a function

$$\mathcal{C}(\mathbb{1}, A) \to \mathcal{C}(\mathbb{1}, DA \otimes A) \simeq \mathcal{C}(A, A)$$

which sends a morphism $a\colon \mathbb{1} \to A$ to "left multiplication by a":

$$L_a\colon A \simeq A \otimes \mathbb{1} \xrightarrow{1\otimes a} A \otimes A \xrightarrow{\tau} A \otimes A \xrightarrow{\mu} A.$$

The map (2.15) defining the trace map $\mathrm{tr}\colon A \to \mathbb{1}$ is readily checked to equal the composite of (2.18) and (2.17). Post-composition by the trace map thus provides the function

$$\mathcal{C}(\mathbb{1}, A) \xrightarrow{\mathrm{tr}_*} \mathcal{C}(\mathbb{1}, \mathbb{1})$$
$$a \longmapsto \mathrm{Tr}(L_a).$$

Thus $\mathrm{tr}\colon A \to \mathbb{1}$ is morally the map which sends an "element" of A to the trace of left multiplication by that element.

Definition 2.19. The *trace form* of a rigid algebra A is the map $t\colon A \otimes A \to \mathbb{1}$ defined as the composite

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\mathrm{tr}} \mathbb{1}.$$

Remark 2.20. The trace map and trace form of a rigid algebra are given by the following string diagrams:

Remark 2.21. A map $f\colon A \otimes A \to \mathbb{1}$ is said to be an "associative" form (also called an "invariant" form) if $f \circ (\mu \otimes 1) = f \circ (1 \otimes \mu)$. Note that any form which factors through μ (such as the trace form of a rigid algebra) is necessarily associative by the associativity of the multiplication. The converse is also true: A form $A \otimes A \to \mathbb{1}$ is associative if and only if it factors through μ. In fact, we obtain a bijection

$$\left\{\text{maps } A \to \mathbb{1}\right\} \xrightarrow{\sim} \left\{\text{associative forms } A \otimes A \to \mathbb{1}\right\}$$

given by $\theta \mapsto \theta \circ \mu$ with inverse $f \mapsto f \circ (u \otimes 1) = f \circ (1 \otimes u)$.

Remark 2.22. A map $f\colon A \otimes A \to \mathbb{1}$ is said to be a "symmetric" form if $f = f \circ \tau$. Note that if A is a commutative algebra, then every associative form is automatically symmetric.

Proposition 2.23. *The trace form of a rigid algebra is symmetric.*

Proof. First note that we can rewrite the trace form as

(2.24)

as has already been mentioned in Remark 2.16. Next we establish

(2.25)

which is an equality of morphisms $A \otimes A \to DA \otimes A$. By adjunction, this can be checked after applying $A \otimes -$ and post-composing with $A \otimes DA \otimes A \xrightarrow{\epsilon \otimes 1} A$:

Next note that

(2.26)
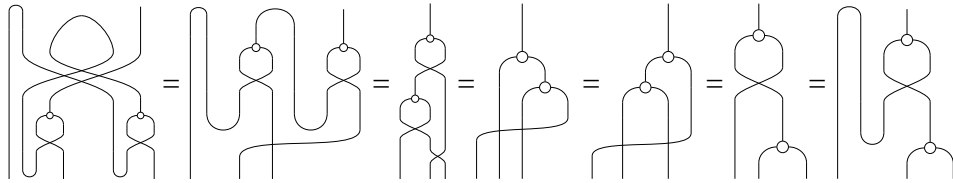
Then
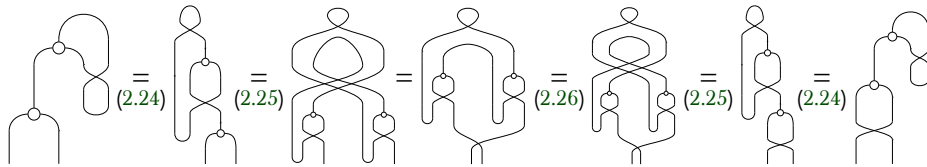
establishes that the trace form is symmetric. $\qquad\square$

Remark 2.27. Intuition for why the trace form is symmetric comes from the fact that for any two endomorphisms $f, g \colon A \to A$, we have $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$. Hence, at least morally, $t(a,b) = \mathrm{Tr}(L_{ab}) = \mathrm{Tr}(L_a \circ L_b) = \mathrm{Tr}(L_b \circ L_a) = \mathrm{Tr}(L_{ba}) = t(b,a)$.

Definition 2.28. If A is a rigid object in a symmetric monoidal category, then every map $f \colon A \otimes A \to \mathbb{1}$ gives rise to two morphisms $A \to DA$ by adjunction (moving each copy of A to the right-hand side). These two maps $A \to DA$ coincide when f is symmetric and are given by

(2.29) $$ f^* \colon A \simeq \mathbb{1} \otimes A \xrightarrow{\eta \otimes 1} DA \otimes A \otimes A \xrightarrow{1 \otimes f} DA \otimes \mathbb{1} \simeq DA. $$

We say that a symmetric form $f \colon A \otimes A \to \mathbb{1}$ is *nondegenerate* if $f^* \colon A \to DA$ is an isomorphism.

Proposition 2.30. *The trace form of a strongly separable rigid algebra is nondegenerate. Moreover, a strongly separable rigid algebra has a unique symmetric separability idempotent, which is given by*

(2.31) $$ \mathbb{1} \xrightarrow{\eta} DA \otimes A \xrightarrow{(t^*)^{-1} \otimes 1} A \otimes A. $$

Proof. Let κ be a symmetric separability idempotent. We will start by showing that the composite

(2.32) $$ A \simeq \mathbb{1} \otimes A \xrightarrow{\kappa \otimes 1} A \otimes A \otimes A \xrightarrow{1 \otimes t} A \otimes \mathbb{1} \simeq A $$

is the identity, where t denotes the trace form (Definition 2.19). First note:

(2.33)

Then observe that
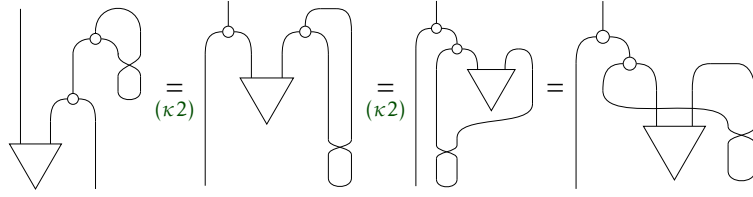
and
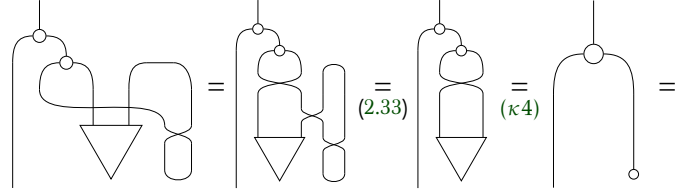
which shows that (2.32) is the identity map. Now κ is symmetric by assumption and the trace form t is symmetric by Proposition 2.23. Hence
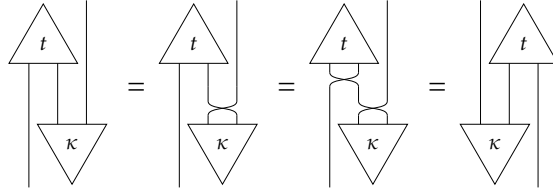
In other words, the composite (2.32) coincides with the other composite

$$A \simeq A \otimes \mathbb{1} \xrightarrow{1 \otimes \kappa} A \otimes A \otimes A \xrightarrow{t \otimes 1} \mathbb{1} \otimes A \simeq A.$$

It follows that the map $\kappa^* \colon DA \to A$ given by

$$DA \simeq \mathbb{1} \otimes DA \xrightarrow{\kappa \otimes 1} A \otimes A \otimes DA \xrightarrow{1 \otimes \epsilon} A \otimes \mathbb{1} \simeq A$$

is an inverse to $t^* \colon A \to DA$. Indeed:
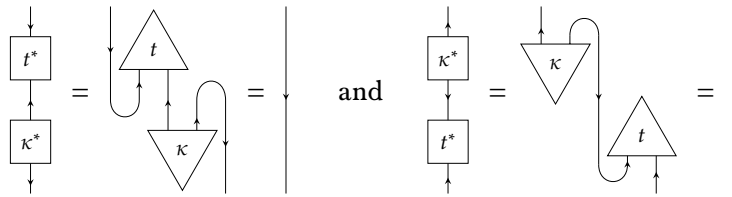
In particular, the trace form is nondegenerate. Moreover, one can readily check that $(t^* \otimes 1) \circ \kappa = \eta$, from which it follows that κ is given by (2.31). □

Theorem 2.34. *A rigid algebra is strongly separable if and only if its trace form is nondegenerate.*

Proof. The "only if" part is provided by Proposition 2.30. Conversely, suppose A is a rigid algebra whose trace form $t \colon A \otimes A \to \mathbb{1}$ is nondegenerate. Write $\theta \colon A \xrightarrow{\sim} DA$ for the associated isomorphism (that is, $\theta = t^*$ in the notation of Definition 2.28), and define $\kappa \colon \mathbb{1} \to A \otimes A$ by

$$(2.35) \qquad\qquad \mathbb{1} \xrightarrow{\eta} DA \otimes A \xrightarrow{\theta^{-1} \otimes 1} A \otimes A.$$

First we check that t and κ form a self-duality in the sense that

(2.36)

and

(2.37)

Armed with this relationship between t and κ, the fact that t is symmetric (by Proposition 2.23) implies that κ is also symmetric:

This establishes axiom (κ3). Next we establish (κ2), visualized in string diagrams in (2.8). It suffices to check equality after post-composition by the isomorphism $\theta \otimes \mathrm{id}_A$. Then by adjunction, it suffices to check equality after applying $A \otimes -$ and post-composing by $A \otimes DA \otimes A \xrightarrow{\epsilon \otimes 1} A$. Indeed,

where the equality (†) is the fact that the trace form is an associative form (Remark 2.21). Finally, we establish ($\kappa 1$). Observe that

and note that we showed (‡) was a consequence of ($\kappa 2$) in the proof of Proposition 2.30. Precomposing with the unit, we obtain

which is ($\kappa 1$). □

Corollary 2.38. *A rigid commutative algebra is separable if and only if it is strongly separable if and only if its trace form is nondegenerate.*

Proof. Every commutative separable algebra is strongly separable (Corollary 2.12); hence the claim follows from Theorem 2.34. □

Remark 2.39. A rigid strongly separable algebra A is automatically self-dual since the nondegeneracy of the trace form provides an isomorphism $A \cong DA$.

Example 2.40. Consider the case where $\mathcal{C} = R\text{-}\mathrm{Mod}$ is the category of R-modules for R a commutative ring. An R-algebra A is rigid precisely when it is finitely generated and projective (equivalently, finitely presented and flat) as an R-module. The trace map $A \to R$ is $a \mapsto \mathrm{Tr}(L_a)$, where $L_a \colon A \to A$ denotes left multiplication by a, and the trace form $t \colon A \otimes A \to R$ is given by $t(a \otimes b) = \mathrm{Tr}(L_{ab})$. In this example, the argument in Remark 2.27 shows immediately that the trace form is symmetric. It turns out that over a field $R = k$, a separable algebra is automatically rigid (that is, finite-dimensional), as shown by [VZ66, Proposition 1.1]. It was partly to clarify such finiteness assumptions that led the author to write this section on strongly separable algebras in arbitrary symmetric monoidal categories.

Example 2.41. An *idempotent algebra* in a symmetric monoidal category is an algebra (A, μ, u) whose multiplication map $\mu \colon A \otimes A \to A$ is an isomorphism. This is equivalent to the equality $u \otimes A = A \otimes u$ of morphisms $A \to A \otimes A$ (which then serve as an inverse to μ). It is also equivalent to the switch map $\tau \colon A \otimes A \to A \otimes A$ being equal to the identity map $A \otimes A \to A \otimes A$. Idempotent algebras are thus examples of commutative (strongly) separable algebras. They have a (unique) separability idempotent given by $\mu^{-1} \circ u \colon \mathbb{1} \to A \otimes A$. However, they are usually not rigid. For example, take $\mathcal{C} = R\text{-}\mathrm{Mod}$ for R a commutative ring. The idempotent R-algebra $R[1/s]$ is rarely finitely generated as an R-module. Indeed, this would imply that the principal open $D(s) \subset \mathrm{Spec}(R)$ is both an open and closed subset of $\mathrm{Spec}(R)$; see the argument in [San19, Example 7.4], for example.

Remark 2.42. A discussion of separable algebras would not be complete without saying something about their relationship with Frobenius algebras:

Definition 2.43. A *Frobenius algebra* in a symmetric monoidal category is an object A equipped with both an algebra structure (A, μ, u) and a coalgebra structure (A, Δ, c) such that the Frobenius law holds:

$$(1 \otimes \mu) \circ (\Delta \otimes 1) = \Delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \Delta).$$

See, for example, [Koc04, 3.6.8]. We say that A is a *symmetric* Frobenius algebra if the associative form

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{c} \mathbb{1}$$

is symmetric. Thus, every commutative Frobenius algebra is symmetric. A *special* Frobenius algebra is a Frobenius algebra such that $\mu \circ \Delta = \mathrm{id}_A$.

Remark 2.44. If (A, μ, u, Δ, c) is a Frobenius algebra, then the underlying object A is necessarily self-dual (*cf.* Remark 2.39). Indeed, the two maps $c \circ \mu \colon A \otimes A \to \mathbb{1}$ and $\Delta \circ u \colon \mathbb{1} \to A \otimes A$ provide a self-duality.

Remark 2.45. The following relationship between strongly separable algebras and special symmetric Frobenius algebras is well-known classically; we include a proof for precision and completeness. The interested reader will find more concerning these ideas in [LP07, Section 2.5], [FRS02, Section 3.3], and [Fau13], among other sources.

Proposition 2.46. *An algebra admits the structure of a special symmetric Frobenius algebra if and only if it is rigid and strongly separable. In this case, the special symmetric Frobenius structure is unique: The counit $A \to \mathbb{1}$ is the trace map (Definition 2.14), and the comultiplication $A \to A \otimes A$ is the map corresponding (Remark 2.3) to the unique symmetric separability idempotent (Proposition 2.30).*

Proof. If A is a strongly separable rigid algebra with (unique) symmetric separability idempotent $\kappa \colon \mathbb{1} \to A \otimes A$, then the corresponding map $A \to A \otimes A$ is coassociative. Indeed, using both descriptions provided by ($\kappa 2$), we have

This provides A with the structure of a coalgebra with counit $A \to \mathbb{1}$ given by the trace map. For the counital axiom, just observe that

where the last equalities (†) were established in the proof of Proposition 2.30. Alternatively, one can use the description (2.31) of the unique separability idempotent and check the counital diagrams after post-composition by the isomorphism $t^* \colon A \xrightarrow{\sim} DA$. This establishes that a strongly separable rigid algebra admits the structure of a special symmetric Frobenius algebra.

Now suppose that (A, μ, u, Δ, c) is a special symmetric Frobenius algebra. Every Frobenius algebra is self-dual (Remark 2.44), and the comultiplication $\Delta \colon A \to A \otimes A$ satisfies ($\sigma 2$). In our case, it also satisfies ($\sigma 1$) since A is assumed to be special. Symmetry of the associated separability idempotent $\Delta \circ u \colon \mathbb{1} \to A \otimes A$ then follows from the assumed symmetry of $c \circ \mu \colon A \otimes A \to \mathbb{1}$ via the self-duality (as in the beginning of the proof of Theorem 2.34). Thus A is strongly separable with symmetric separability idempotent $\Delta \circ u$.

To establish uniqueness, first observe that the commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\;u \otimes 1\;} & A \otimes A & \xrightarrow{\;\Delta \otimes 1\;} & A \otimes A \otimes A \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1 \otimes \Delta} & & \downarrow{\scriptstyle 1 \otimes \mu} \\
A \otimes A & \xrightarrow{u \otimes 1 \otimes 1} & A \otimes A \otimes A & \xrightarrow{\;\mu \otimes 1\;} & A \otimes A
\end{array}
$$
$$\text{id}$$

shows that the comultiplication Δ of a Frobenius algebra is determined by $\Delta \circ u$ and μ. If $(A, \mu, u, \Delta_1, c_1)$ and $(A, \mu, u, \Delta_2, c_2)$ are two special symmetric Frobenius structures on the (rigid strongly separable) algebra (A, μ, u), then $\Delta_1 \circ u = \Delta_2 \circ u$ by the uniqueness of symmetric separability idempotents (Proposition 2.30),

and hence $\Delta_1 = \Delta_2$. Moreover, since $c_i \circ \mu \colon A \otimes A \to \mathbb{1}$ and $\Delta_i \circ u \colon \mathbb{1} \to A \otimes A$ form a self-duality for each $i = 1, 2$, we have:

That is, $c_1 \circ \mu = c_2 \circ \mu$. Precomposing by the unit, we conclude that $c_1 = c_2$. This establishes that an algebra admits at most one special symmetric Frobenius structure. Finally, we have already proved that if A admits a special symmetric Frobenius structure then it is rigid and strongly separable and consequently the trace map and the symmetric separability idempotent provide it with a special symmetric Frobenius structure. These thus provide the unique such structure. $\square$

3. Separable algebras and triangulated categories

In this section, we recall the relationship between separable algebras and tensor-triangulated categories established in [Bal11].

Remark 3.1. Recall from [Bal11, Section 5] that for each $2 \le N \le \infty$, there is the notion of an N-triangulated category (or triangulated category of order N) which includes as part of the structure a distinguished class of n-triangles for each $n \le N$ which are required to satisfy suitable higher octahedral axioms. A 2-triangulated category is precisely the same thing as a pre-triangulated category, while the usual notion of triangulated category (in the sense of Verdier) lies between the notion of 2-triangulated and 3-triangulated. An N-triangulated functor is a functor which commutes with the suspension and preserves distinguished N-triangles (equivalently, preserves distinguished n-triangles for all $n \le N$).

Example 3.2. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a stable ∞-category has the structure of an ∞-triangulated category.

Remark 3.3. A tensor-triangulated category is a triangulated category equipped with a closed symmetric monoidal structure which is compatible with the triangulation in the sense of [HPS97, Definition A.2.1]. For $2 \le N \le \infty$, we similarly have the notion of an N-tensor-triangulated category by replacing all instances of "triangulated" in the definition with "N-triangulated." By an (N-)tensor-triangulated functor, we mean an (N-)triangulated functor which is also a strong symmetric monoidal functor.

Example 3.4. The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a presentably symmetric monoidal [NS17, Definition 2.1] stable ∞-category is an ∞-tensor-triangulated category.

Example 3.5. If A is a commutative separable algebra in an N-tensor-triangulated category $\mathcal{T}$ ($2 \le N \le \infty$), then the Eilenberg–Moore category $A\text{-}\mathrm{Mod}_{\mathcal{T}}$ inherits the structure of an N-tensor-triangulated category such that the extension-of-scalars functor $F_A \colon \mathcal{T} \to A\text{-}\mathrm{Mod}_{\mathcal{T}}$ is an N-tensor-triangulated functor. The distinguished n-triangles in $A\text{-}\mathrm{Mod}_{\mathcal{T}}$ ($n \le N$) are precisely those which are created by the forgetful functor $U_A \colon A\text{-}\mathrm{Mod}_{\mathcal{T}} \to \mathcal{T}$. This is established by [Bal11, Theorem 5.17] and [Bal14, Section 1].

Remark 3.6. The main theorem of [DS18] states that if $\mathcal{T}$ is an idempotent complete triangulated category, then any triangulated adjunction $F \colon \mathcal{T} \rightleftarrows \mathcal{S} \colon G$ is essentially monadic (that is, monadic up to idempotent completion and killing the kernel of G) whenever the Eilenberg–Moore category inherits a triangulation from $\mathcal{T}$:

$$(3.7) \qquad\qquad (\mathcal{S}/\ker G)^{\natural} \cong GF\text{-}\mathrm{Mod}_{\mathcal{T}}.$$

This theorem also holds (with the same proof) in the 2-category of N-triangulated categories for any $2 \le N \le \infty$. In this case, the equivalence (3.7) is an equivalence of N-triangulated categories. To be clear,

this is under the hypothesis that the Eilenberg–Moore category GF-$\mathrm{Mod}_{\mathcal{T}}$ inherits an N-triangulation from the N-triangulation of $\mathcal{T}$ (see [DS18, Remark 1.8]). This is a strong hypothesis on the adjunction but, as established by Balmer (Example 3.5), does hold in the separable case. The following proposition clarifies the situation with the tensor:

Proposition 3.8. *Let* $F\colon \mathcal{T} \to \mathcal{S}$ *be an* (N-)*tensor-triangulated functor with* $\mathcal{T}$ *idempotent complete. Suppose* F *admits a right adjoint* G *whose kernel is closed under the tensor product, and that the* $F \dashv G$ *adjunction satisfies the right projection formula* [BDS15, Definition 2.7]. *If the commutative algebra* $G(\mathbb{1}) \in \mathcal{T}$ *is separable then we have an induced equivalence*

$$(\mathcal{S}/\ker G)^{\natural} \cong G(\mathbb{1})\text{-}\mathrm{Mod}_{\mathcal{T}}$$

of (N-)*tensor-triangulated categories.*

Proof. By [BDS15, Lemma 2.8], the projection formula implies that the monad of the adjunction is the monad associated to the algebra $G(\mathbb{1})$. Since $G(\mathbb{1})$ is separable, the Eilenberg–Moore category inherits a triangulation from $\mathcal{T}$ (Example 3.5); hence by [DS18] and Remark 3.6, we have equivalences

$$(3.9) \qquad (G(\mathbb{1})\text{-}\mathrm{Free}_{\mathcal{T}})^{\natural} \xrightarrow{\sim} (\mathcal{S}/\ker G)^{\natural} \xrightarrow{\sim} G(\mathbb{1})\text{-}\mathrm{Mod}_{\mathcal{T}}.$$

We next prove that $\ker G$ is a tensor-ideal. To this end, let $q\colon \mathcal{S} \to \mathcal{S}/\ker G$ denote the Verdier quotient and $\overline{G}\colon \mathcal{S}/\ker G \to \mathcal{T}$ the induced functor. The induced adjunction $q \circ F \dashv \overline{G}$ realizes the same monad as the $F \dashv G$ adjunction. Since the comparison functor $K\colon \mathcal{S}/\ker G \to G(\mathbb{1})\text{-}\mathrm{Mod}_{\mathcal{T}}$ is fully faithful (hence separable), the functor $\overline{G} = U \circ K$ is separable, being the composite of such (*cf.* [Che15, Proposition 3.5]). Hence, the counit of the $q \circ F \dashv \overline{G}$ adjunction splits. Thus, for any $s \in \mathcal{S}$, the morphism α in the exact triangle $FGs \xrightarrow{\epsilon_s} s \xrightarrow{\alpha} x \to \Sigma FGs$ satisfies $q(\alpha) = 0$. This means that $\beta \circ \alpha = 0$ for some morphism $\beta\colon x \to y$ with $\mathrm{cone}(\beta) \in \ker G$. It follows that $s \in \mathrm{thick}\langle FGs, \ker G \rangle$. We conclude that $\ker G$ is a tensor-ideal since the thick subcategory $\{a \in \mathcal{S} \mid a \otimes \ker G \subseteq \ker G\}$ contains $\ker G$ by the hypothesis that $\ker G$ is closed under the tensor product, and it contains the image of F by the projection formula.

We have established that the thick subcategory $\ker G$ is a tensor-ideal. Thus $\mathcal{S}/\ker G$ and its idempotent completion $(\mathcal{S}/\ker G)^{\natural}$ inherit tensor structures from $\mathcal{S}$. On the other hand, the Kleisli category $G(\mathbb{1})\text{-}\mathrm{Free}_{\mathcal{T}}$ inherits a tensor structure from $\mathcal{T}$ such that the canonical functor $\mathcal{T} \to G(\mathbb{1})\text{-}\mathrm{Free}_{\mathcal{T}}$ is a strict symmetric monoidal functor. The canonical functor $G(\mathbb{1})\text{-}\mathrm{Free}_{\mathcal{T}} \to \mathcal{S}$ then inherits the structure of a strong symmetric monoidal functor from the corresponding structure on F. The first functor in (3.9) is a strong symmetric monoidal equivalence. It follows that the second functor is also a symmetric monoidal equivalence since the tensor structure on $G(\mathbb{1})\text{-}\mathrm{Mod}_{\mathcal{T}} \cong (G(\mathbb{1})\text{-}\mathrm{Free}_{\mathcal{T}})^{\natural}$ is the idempotent completion of the tensor structure on the Kleisli category (see [Pau15, Section 1.1] and [Bal14, Section 1]). $\qquad\square$

Remark 3.10. As an application of the proposition, we can provide a proof of Theorem 1.1 stated in the Introduction which characterizes smashing localizations of rigidly-compactly generated categories.

Proof of Theorem 1.1. The ($\Rightarrow$) direction is well-known: Any smashing localization of a rigidly-compactly generated category is a geometric functor to a rigidly-compactly generated category (whose right adjoint is fully faithful); see [HPS97, Section 3.3]. For the ($\Leftarrow$) direction, recall that smashing localizations are nothing but extension-of-scalars with respect to idempotent algebras. Suppose $f^*\colon \mathcal{D} \to \mathcal{C}$ is a geometric functor whose right adjoint f_* is fully faithful. The multiplication map $f_*(\mathbb{1}_{\mathcal{C}}) \otimes f_*(\mathbb{1}_{\mathcal{C}}) \to f_*(\mathbb{1}_{\mathcal{C}})$ becomes, under the projection formula, the counit $f_*(\epsilon)\colon f_* f^* f_*(\mathbb{1}_{\mathcal{C}}) \simeq f_*(\mathbb{1}_{\mathcal{D}} \otimes f^* f_*(\mathbb{1}_{\mathcal{C}})) \simeq f_*(\mathbb{1}_{\mathcal{C}}) \otimes f_*(\mathbb{1}_{\mathcal{C}}) \to f_*(\mathbb{1}_{\mathcal{C}})$. This is an isomorphism since f_* is fully faithful. Thus $f_*(\mathbb{1}_{\mathcal{C}})$ is an idempotent algebra. Idempotent algebras are separable, so Proposition 3.8 gives the result. $\qquad\square$

Remark 3.11. Note that we assume in Proposition 3.8 that the kernel of the right adjoint G is closed under the tensor product and show, under the other hypotheses, that it is then necessarily a tensor-ideal. The next example clarifies that this hypothesis on the kernel of G is not forced by the other assumptions.

Example 3.12. Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category. Then $\mathcal{L} := \mathrm{Loc}(\mathbb{1})$ is a tensor-triangulated subcategory of $\mathcal{T}$. It is also rigidly-compactly generated, and the inclusion functor $F: \mathcal{L} \hookrightarrow \mathcal{T}$ is a coproduct-preserving tensor-triangulated functor. Hence it has a right adjoint G and the projection formula holds. Moreover, since the left adjoint F is fully faithful, $G(\mathbb{1}) \simeq \mathbb{1}$ is just the trivial ring, which is certainly separable. Since the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a right adjoint, $\mathcal{L}$ is the kernel of a Bousfield localization on $\mathcal{T}$. The image of this Bousfield localization is $\mathcal{L}^{\perp} = \ker G$. Hence $\mathcal{L} = {}^{\perp}(\mathcal{L}^{\perp}) = {}^{\perp}(\ker G)$. If $\ker G$ were a tensor-ideal, then $\mathcal{L}$ would be forced to be a tensor-ideal. Indeed, the localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ is a tensor-ideal if and only if it is closed under tensoring with compact(=rigid) objects, and the left orthogonal of a tensor-ideal is certainly closed under tensoring with rigid objects. Thus, if $\ker G$ is a tensor-ideal, then $\mathrm{Loc}(\mathbb{1})$ is a tensor-ideal, and this is the case if and only if $\mathrm{Loc}(\mathbb{1}) = \mathcal{T}$. Thus, if we take $\mathcal{T}$ to be any non-monogenic rigidly-compactly generated tensor-triangulated category, we conclude that $\ker G$ is not a tensor-ideal, and in fact not closed under the tensor product.

4. Finite étale morphisms

The idea that extension-of-scalars with respect to a commutative separable algebra provides tensor triangular geometry with an analogue of an étale extension goes back to the work of Balmer [Bal15, Bal16a, Bal16b]. Here we focus on *finite* étale extensions of rigidly-compactly generated categories.

Terminology 4.1. A coproduct-preserving $(N\text{-})$tensor-triangulated functor between rigidly-compactly generated $(N\text{-})$tensor-triangulated categories is called a *geometric functor*.

Definition 4.2. A geometric functor $f^*: \mathcal{D} \to \mathcal{C}$ between rigidly-compactly generated $(N\text{-})$tensor-triangulated categories is *finite étale* if there exists a compact commutative separable algebra A in $\mathcal{D}$ and an $(N\text{-})$tensor-triangulated equivalence[1] $\mathcal{C} \cong A\text{-}\mathrm{Mod}_{\mathcal{D}}$ such that the functor f^* becomes isomorphic to the extension-of-scalars functor $F_A: \mathcal{D} \to A\text{-}\mathrm{Mod}_{\mathcal{D}}$.

Remark 4.3. It follows from [Bal16a, Theorem 4.2] that if $\mathcal{D}$ is rigidly-compactly generated, then $A\text{-}\mathrm{Mod}_{\mathcal{D}}$ is also rigidly-compactly generated (for A a commutative separable algebra in $\mathcal{D}$). Thus, there is no loss of generality in considering only geometric functors between rigidly-compactly generated categories.

Remark 4.4. Recall from [BDS16] that a geometric functor $f^*: \mathcal{D} \to \mathcal{C}$ between rigidly-compactly generated tensor-triangulated categories has a right adjoint $f_*: \mathcal{C} \to \mathcal{D}$ which itself has a right adjoint $f^!: \mathcal{D} \to \mathcal{C}$. The *relative dualizing object* of f^* is the object $\omega_f := f^!(\mathbb{1}_{\mathcal{D}}) \in \mathcal{C}$. Recall that f^* is said to satisfy *Grothendieck–Neeman duality* if the right adjoint f_* preserves compact objects. (A number of equivalent definitions are provided by [BDS16, Theorem 3.3].) In this case, the commutative algebra $f_*(\mathbb{1}_{\mathcal{C}})$ is compact(=rigid). Hence it has a trace map $f_*(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ (Definition 2.14), which corresponds to a map $\mathbb{1}_{\mathcal{C}} \to \omega_f$.

Remark 4.5. In general, morphisms $\mathbb{1}_{\mathcal{C}} \to \omega_f$ can be identified with morphisms $f_*(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ by adjunction, and these can be identified as in Remark 2.21 with the associative forms on the algebra $f_*(\mathbb{1}_{\mathcal{C}})$:

$$\left\{ \mathbb{1}_{\mathcal{C}} \to \omega_f \right\} \xrightarrow{\sim} \left\{ f_*(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}} \right\} \xrightarrow{\sim} \left\{ \text{associative forms } f_*(\mathbb{1}_{\mathcal{C}}) \otimes f_*(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}} \right\}.$$

Also recall (Definition 2.28) that an associative form $f_*(\mathbb{1}_{\mathcal{C}}) \otimes f_*(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is nondegenerate if the adjoint morphism $f_*(\mathbb{1}_{\mathcal{C}}) \to D f_*(\mathbb{1}_{\mathcal{C}})$ is an isomorphism. (Note that these associative forms are automatically symmetric since the algebra $f_*(\mathbb{1}_{\mathcal{C}})$ is commutative.) On the other hand, recall from [BDS16, (2.18)] that we have an isomorphism $f_*(\omega_f) \simeq D f_*(\mathbb{1}_{\mathcal{C}})$.

---

[1] For tensor-triangulated categories in the usual sense of Verdier, the category of modules $A\text{-}\mathrm{Mod}_{\mathcal{D}}$ is *a priori* only a pre-tensor-triangulated category, but this does not cause any trouble for the definition. Since $\mathcal{C}$ is tensor-triangulated by assumption, the equivalence $\mathcal{C} \cong A\text{-}\mathrm{Mod}_{\mathcal{D}}$ just forces $A\text{-}\mathrm{Mod}_{\mathcal{D}}$ to be tensor-triangulated as well. This technicality doesn't arise when working in the 2-category of N-tensor-triangulated categories for any $2 \leq N \leq \infty$.

Lemma 4.6. *Let $\theta \colon \mathbb{1}_{\mathcal{C}} \to \omega_f$ be any morphism. The map $f_*(\mathbb{1}_{\mathcal{C}}) \to D f_*(\mathbb{1}_{\mathcal{C}})$ which is adjoint to the associative form on $f_*(\mathbb{1}_{\mathcal{C}})$ corresponding to θ coincides with the map*

$$(4.7) \qquad f_*(\mathbb{1}_{\mathcal{C}}) \xrightarrow{f_*(\theta)} f_*(\omega_f) \simeq D f_*(\mathbb{1}_{\mathcal{C}}).$$

Consequently, the associative form associated to θ is nondegenerate if and only if $f_*(\theta)$ is an isomorphism.

Proof. This is a straightforward verification. From the definition of the isomorphism $f_*(\omega_f) \simeq D f_*(\mathbb{1}_{\mathcal{C}}) = [f_*(\mathbb{1}_{\mathcal{C}}), \mathbb{1}_{\mathcal{D}}]$ in [BDS16, (2.18)], one sees that the morphism (4.7) is obtained by going along the top of the following commutative diagram

$$
\begin{array}{ccccccc}
f_*(\omega_f) & \xrightarrow{\text{coev}} & [f_*(\mathbb{1}_{\mathcal{C}}), f_*(\omega_f) \otimes f_*(\mathbb{1}_{\mathcal{C}})] & \xrightarrow{[1,\text{lax}]} & [f_*(\mathbb{1}_{\mathcal{C}}), f_*(\omega_f)] & \xrightarrow{[1,\epsilon]} & [f_*(\mathbb{1}_{\mathcal{C}}), \mathbb{1}_{\mathcal{D}}] \\
f_*(\theta) \big\uparrow & & & & \big\uparrow {\scriptstyle [1, f_*(\theta)]} & & \\
f_*(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\text{coev}} & [f_*(\mathbb{1}_{\mathcal{C}}), f_*(\mathbb{1}_{\mathcal{C}}) \otimes f_*(\mathbb{1}_{\mathcal{C}})] & \xrightarrow{[1,\text{lax}]} & [f_*(\mathbb{1}_{\mathcal{C}}), f_*(\mathbb{1}_{\mathcal{C}})] & &
\end{array}
$$

while the adjoint of the corresponding associative form is obtained by going along the bottom. $\square$

Theorem 4.8. *Let $f^* \colon \mathcal{D} \to \mathcal{C}$ be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Then f^* is a finite étale morphism (Definition 4.2) if and only if the following three conditions hold:*

- *(a) f^* satisfies Grothendieck–Neeman duality;*
- *(b) the right adjoint f_* is conservative;*
- *(c) the map $\mathbb{1}_{\mathcal{C}} \to \omega_f$ adjoint to the trace map is an isomorphism.*

Proof. ($\Rightarrow$) If f^* is finite étale, then it is extension-of-scalars with respect to the compact separable commutative algebra $f_*(\mathbb{1})$. By the separable Neeman–Thomason Localization Theorem established by Balmer [Bal16a, Theorem 4.2], the compact objects in $\mathcal{C}$ are precisely the thick subcategory generated by the image $f^*(\mathcal{D}^c)$ of the compact objects in $\mathcal{D}$. Thus, by the projection formula, the fact that $f_*(\mathbb{1})$ is compact ensures that $f_*(c)$ is compact for all $c \in \mathcal{C}^c$. Thus, f^* satisfies Grothendieck–Neeman duality. The right adjoint f_* is certainly conservative (in fact faithful). Moreover, by Corollary 2.38, the commutative rigid separable algebra $f_*(\mathbb{1})$ is strongly separable, so that its trace form is nondegenerate. By Lemma 4.6, this means that the canonical map $\mathbb{1}_{\mathcal{C}} \to \omega_f$ becomes an isomorphism after applying f_*. But this means the canonical map is an isomorphism since f_* is conservative.

($\Leftarrow$) If f^* satisfies Grothendieck–Neeman duality, then the commutative algebra $f_*(\mathbb{1}_{\mathcal{C}})$ is rigid, hence has a trace map (so that part (c) makes sense). Moreover, by Lemma 4.6, if the map $\mathbb{1} \to \omega_f$ adjoint to the trace map is an isomorphism, then the trace form is nondegenerate; hence by Corollary 2.38, $f_*(\mathbb{1}_{\mathcal{C}})$ is a (strongly) separable algebra. By Proposition 3.8, we have a tensor-triangulated equivalence $\mathcal{C} \to f_*(\mathbb{1})\text{-}\mathrm{Mod}_{\mathcal{D}}$ compatible with the two adjunctions. Here we use the assumption that f_* is conservative and the fact that $\mathcal{C}$ is idempotent complete (since it has small coproducts). Therefore, f^* is finite étale. $\square$

Remark 4.9. Although in part (b) of Theorem 4.8 we just assume f_* is conservative, it follows from the other hypotheses that it is actually faithful. It also follows from (a) and (c) that f^* has the full Wirthmüller isomorphism of [BDS16, Theorem 1.9].

Remark 4.10. The conservativity condition (b) of Theorem 4.8 can be removed if we assume instead an additional hypothesis on the category $\mathcal{C}$. The remainder of this section is devoted to explaining this modification; see Corollary 4.20 below.

Definition 4.11. We say that a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ is *monogenic* if it is generated by its unit: $\mathcal{T} = \mathrm{Loc}\langle \mathbb{1} \rangle$. We say that $\mathcal{T}$ is *locally monogenic* if the local category $\mathcal{T}_{\mathcal{P}} := \mathcal{T}/\mathrm{Loc}\langle \mathcal{P} \rangle$ is monogenic for each $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$.

Example 4.12. The derived category $\mathrm{D}(A)$ of any commutative ring A is monogenic. For any quasi-compact and quasi-separated scheme X, the derived category $\mathrm{D}_{\mathrm{qc}}(X)$ is locally monogenic. Indeed, under the identification $\mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c) \cong X$, prime ideals $\mathcal{P} \in \mathrm{Spc}(\mathrm{D}_{\mathrm{qc}}(X)^c)$ correspond to points $x \in X$, and $\mathrm{D}_{\mathrm{qc}}(X)_{\mathcal{P}} \cong \mathrm{D}(\mathcal{O}_{X,x})$ is the derived category of the local ring at x. (See [Bal20, Section 4] for further discussion.)

Remark 4.13. A geometric functor $f^*: \mathcal{D} \to \mathcal{C}$ is fully faithful if and only if the unit map $\mathbb{1}_{\mathcal{D}} \to f_*(\mathbb{1}_{\mathcal{C}})$ is an isomorphism. Indeed, a left adjoint is fully faithful if and only if the unit of the adjunction is a natural isomorphism, and one readily checks that the unit $\eta_d: d \to f_* f^*(d)$ coincides with the composite

$$d \simeq d \otimes \mathbb{1}_{\mathcal{D}} \to d \otimes f_*(\mathbb{1}_{\mathcal{C}}) \simeq f_*(f^*(d) \otimes \mathbb{1}_{\mathcal{C}}) \simeq f_* f^*(d).$$

Moreover, note that in this case an object $c \in \mathcal{C}$ is in the essential image of f^* if and only if the counit $\epsilon_c: f^* f_*(c) \to c$ is an isomorphism.

Lemma 4.14. *If $f^*: \mathcal{D} \to \mathcal{C}$ is a fully faithful geometric functor, then the map $\theta: \mathbb{1}_{\mathcal{C}} \to \omega_f$ adjoint to the trace map is an isomorphism if and only if the counit $\epsilon_{\omega_f}: f^* f_*(\omega_f) \to \omega_f$ is an isomorphism.*

Proof. This follows from the commutative diagram

$$
\begin{array}{ccc}
f^* f_*(\mathbb{1}_{\mathcal{C}}) & \xrightarrow[\simeq]{f^* f_*(\theta)} & f^* f_*(\omega_f) \\
{\scriptstyle \simeq} \downarrow {\scriptstyle \epsilon} & & \downarrow {\scriptstyle \epsilon} \\
\mathbb{1}_{\mathcal{C}} & \xrightarrow{\theta} & \omega_f.
\end{array}
$$

The left-hand counit is an isomorphism since $\mathbb{1}_{\mathcal{C}} \simeq f^*(\mathbb{1}_{\mathcal{D}})$ is in the essential image of f^* and $f_*(\theta)$ is an isomorphism (by Lemma 4.6 and Corollary 2.38) since $f_*(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\mathcal{D}}$ is a rigid separable commutative algebra. $\square$

Proposition 4.15. *Let $f^*: \mathcal{D} \to \mathcal{C}$ be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Suppose that the following conditions hold:*

(a) the unit map $\mathbb{1}_{\mathcal{D}} \to f_*(\mathbb{1}_{\mathcal{C}})$ is an isomorphism;

(b) f^* satisfies Grothendieck–Neeman duality;

(c) the map $\mathbb{1}_{\mathcal{C}} \to \omega_f$ adjoint to the trace map is an isomorphism.

If $\mathcal{C}$ is locally monogenic (Definition 4.11), then f^* is an equivalence.

Proof. For any $t \in \mathcal{C}$, consider an exact triangle

(4.16) $$f^* f_*(t) \xrightarrow{\epsilon_t} t \to \mathrm{cone}(\epsilon_t) \to \Sigma f^* f_*(t).$$

Hypothesis (a) asserts that f^* is fully faithful (Remark 4.13). Hence $f_*(\mathrm{cone}(\epsilon_t)) = 0$. Hypotheses (b) and (c) imply (by [BDS16, Theorem 3.3]) that there is a natural isomorphism $f^* \simeq f^!$ where $f^!$ denotes the right adjoint to f_*. Hence the last map in (4.16) vanishes and we have a splitting

(4.17) $$t \simeq f^* f_*(t) \oplus \mathrm{cone}(\epsilon_t)$$

for any $t \in \mathcal{C}$. Now let $\mathcal{P} \in \mathrm{Spc}(\mathcal{C}^c)$ and let $L_{\mathcal{P}}: \mathcal{C} \to \mathcal{C}_{\mathcal{P}}$ denote the localization to the local category at $\mathcal{P}$ (see, *e.g.*, [BHS21, Remark 1.22–Definition 1.25]). This localization has an associated idempotent triangle

$$e_{\mathcal{P}} \to \mathbb{1}_{\mathcal{C}} \to f_{\mathcal{P}} \to \Sigma e_{\mathcal{P}}$$

in $\mathcal{C}$. The kernel of f_* does not contain any nonzero ring object (since if $f_*(R) = 0$ then the unit $\mathbb{1}_{\mathcal{C}} \to R$ would vanish by an argument similar to the proof of Lemma 4.14). Since the right idempotent $f_{\mathcal{P}}$ is a ring object, we conclude that $f^* f_*(f_{\mathcal{P}}) \neq 0$. Now the local category $\mathcal{C}_{\mathcal{P}}$ is local in the sense of [BHS21, Terminology 1.11]. Hence by [Bal10, Theorem 4.5], the endomorphism ring $\mathrm{End}_{\mathcal{C}_{\mathcal{P}}}(\mathbb{1}_{\mathcal{C}_{\mathcal{P}}})$ is local. The identifications $\mathrm{End}_{\mathcal{C}_{\mathcal{P}}}(\mathbb{1}_{\mathcal{C}_{\mathcal{P}}}) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, f_{\mathcal{P}}) \simeq \mathrm{End}_{\mathcal{C}}(f_{\mathcal{P}})$ are isomorphisms of rings, and we conclude that

$\operatorname{End}_{\mathcal{C}}(f_{\mathcal{P}})$ has no nontrivial idempotents. Hence the fact that $f^*f_*(f_{\mathcal{P}}) \neq 0$ implies that the splitting (4.17) is trivial: $f_{\mathcal{P}} \simeq f^*f_*(f_{\mathcal{P}})$.

We have thus established that $f_{\mathcal{P}}$ (and all its shifts $\Sigma^*f_{\mathcal{P}}$) is contained in the essential image of f^*. By invoking $f^* \simeq f^!$ again, we conclude that if $f_*(t) = 0$ then $\operatorname{Hom}_{\mathcal{C}}^*(t, f_{\mathcal{P}}) = 0$ for all $\mathcal{P} \in \operatorname{Spc}(\mathcal{C}^c)$. In particular, if $x \in \mathcal{C}^c$ is a compact(=rigid) object with $f_*(x) = 0$ then

$$(4.18) \qquad\qquad \operatorname{Hom}_{\mathcal{C}_{\mathcal{P}}}^*(L_{\mathcal{P}}(x), \mathbb{1}_{\mathcal{C}_{\mathcal{P}}}) \simeq \operatorname{Hom}_{\mathcal{C}}^*(x, f_{\mathcal{P}}) = 0.$$

Since $\mathcal{C}_{\mathcal{P}}$ is monogenic (by hypothesis) and $L_{\mathcal{P}}(x)$ is compact, (4.18) implies that $L_{\mathcal{P}}(x) = 0$, so that $x \in \mathcal{P}$. This is true for all $\mathcal{P}$ (*i.e.* $\operatorname{supp}(x) = \varnothing$), so $x = 0$. Thus, the kernel of f_* does not contain any nonzero compact objects. Hence (4.16) shows that $f^*f_*(x) \simeq x$ for every compact $x \in \mathcal{C}^c$. It follows that f_* is conservative. Indeed, if $f_*(t) = 0$, then $\operatorname{Hom}_{\mathcal{C}}(x, t) \simeq \operatorname{Hom}_{\mathcal{C}}(f^*f_*(x), t) \simeq \operatorname{Hom}_{\mathcal{D}}(f_*(x), f_*(t)) = 0$ for every compact $x \in \mathcal{C}^c$ and hence $t = 0$. Thus, since the kernel of f_* is trivial, the exact triangle (4.16) shows that $f^*f_*(t) \simeq t$ for every $t \in \mathcal{C}$. Hence f^* is essentially surjective, and the proof is complete. $\qquad\square$

Example 4.19. Hypothesis (c) in Proposition 4.15 does not follow from the other hypotheses. Indeed, consider the projective line $\mathbb{P}^1$ over the field k. The structure map $f: \mathbb{P}^1 \to \operatorname{Spec}(k)$ induces a fully faithful geometric functor $f^*: \operatorname{D}(k) \to \operatorname{D}_{\operatorname{qc}}(\mathbb{P}^1)$ which satisfies Grothendieck–Neeman duality (see [BDS16, Example 6.14] or [LN07]). Moreover, the target category is locally monogenic (see Example 4.12). However, $\mathbb{1} \not\simeq \omega_f$. Indeed, $\omega_f \simeq \Sigma \mathcal{O}_{\mathbb{P}^1}(-2)$ but

$$\operatorname{Hom}(\Sigma \mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \Sigma^{-1}\mathcal{O}_{\mathbb{P}^1}(2)) = H^{-1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = 0$$

while $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = k$. On the other hand, since $f_*(\mathbb{1}) \simeq H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k = \mathbb{1}$ is a rigid separable commutative algebra, we know from Lemma 4.6 that $f_*(\mathbb{1}) \simeq f_*(\omega_f)$. Indeed, $f_*(\omega_f) \simeq H^{*+1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = k$ (see, *e.g.*, [Har77, Theorem 5.1]). Note that this example is the $\mathcal{T} = \operatorname{D}_{\operatorname{qc}}(\mathbb{P}^1)$ case of Example 3.12.

Corollary 4.20. *Let $f^*: \mathcal{D} \to \mathcal{C}$ be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Suppose $\mathcal{C}$ is locally monogenic (Definition 4.11). Then f^* is a finite étale morphism if and only if the following two conditions hold:*

  (a) f^* satisfies Grothendieck–Neeman duality;
  (b) the map $\mathbb{1}_{\mathcal{C}} \to \omega_f$ adjoint to the trace map is an isomorphism.

Proof. The ($\Rightarrow$) direction is provided by Theorem 4.8. We need to prove that a geometric functor $f^*: \mathcal{D} \to \mathcal{C}$ satisfying (a) and (b) is finite étale provided that $\mathcal{C}$ is locally monogenic. As explained in the proof of Theorem 4.8, hypotheses (a) and (b) imply that $f_*(\mathbb{1}_{\mathcal{C}})$ is a rigid separable commutative algebra. Thus $\mathcal{D}' := (f_*(\mathbb{1}_{\mathcal{C}})\text{-}\operatorname{Free}_{\mathcal{D}})^{\natural} \cong f_*(\mathbb{1}_{\mathcal{C}})\text{-}\operatorname{Mod}_{\mathcal{D}}$ is a rigidly-compactly generated tensor-triangulated category (Example 3.5 and Remark 4.3), and we can factor f^* as a composite

$$\mathcal{D} \xrightarrow{g^*} \mathcal{D}' \xrightarrow{h^*} \mathcal{C},$$

where g^* is finite étale and h^* is a geometric functor with the property that $\mathbb{1}_{\mathcal{D}'} \to h_*(\mathbb{1}_{\mathcal{C}})$ is an isomorphism. (One can verify that the functor h^* preserves coproducts in a routine manner using the basic properties of the Kleisli adjunction and idempotent completion. That these are symmetric monoidal functors is explained in the proof of Proposition 3.8.) Now consider a compact object $x \in \mathcal{C}^c$. Since g^* is finite étale, the counit of the $g^* \dashv g_*$ adjunction has a section (*cf.* [Bal11, Proposition 3.11] and [Raf90, Theorem 1.2]). Thus $h_*(x)$ is a direct summand of $g^*g_*h_*(x) = g^*f_*(x)$ and hence is compact since f_* and g^* preserve compactness. Thus, h^* satisfies Grothendieck–Neeman duality. Moreover,

$$\omega_h = h^!(\mathbb{1}_{\mathcal{D}'}) \simeq h^!(\omega_g) = h^!(g^!(\mathbb{1}_{\mathcal{D}})) \simeq f^!(\mathbb{1}_{\mathcal{D}}) = \omega_f \simeq \mathbb{1}_{\mathcal{C}} \simeq h^*(\mathbb{1}_{\mathcal{D}'}),$$

so Lemma 4.14 implies that the map $\mathbb{1}_{\mathcal{C}} \to \omega_h$ adjoint to the trace form on $h_*(\mathbb{1}_{\mathcal{C}})$ is an isomorphism. Thus h^* satisfies all the hypotheses of Proposition 4.15. This establishes that h^* is an equivalence, and the proof is complete. $\qquad\square$

Remark 4.21. It follows from Example 5.12 below that if $f^* \colon \mathcal{D} \to \mathcal{C}$ is finite étale and $\mathcal{D}$ is locally monogenic, then $\mathcal{C}$ is locally monogenic; see Corollary 5.13. Thus, when studying the finite étale extensions of a locally monogenic category $\mathcal{D}$, there is no loss of generality in using the criterion provided by Corollary 4.20.

Remark 4.22. We now provide an example which shows that Proposition 4.15 and Corollary 4.20 do not hold (in general) without the locally monogenic hypothesis.

Example 4.23. Let $\mathcal{T}$ be a nonzero rigidly-compactly generated tensor-triangulated category. The product category $\mathcal{T} \times \mathcal{T}$ can be triangulated by defining the suspension and exact triangles coordinate-wise, and we have fully faithful triangulated functors

$$\mathcal{T} \hookrightarrow \mathcal{T} \times \mathcal{T} \hookleftarrow \mathcal{T}$$

given by $a \mapsto (a, 0)$ and $b \mapsto (0, b)$, respectively. We can turn $\mathcal{T} \times \mathcal{T}$ into a tensor-triangulated category by defining the tensor product in a $\mathbb{Z}/2$-graded fashion,

$$(a_0, a_1) \otimes (b_0, b_1) := ((a_0 \otimes b_0) \oplus (a_1 \otimes b_1), (a_0 \otimes b_1) \oplus (a_1 \otimes b_0)),$$

which can be interpreted as a Day convolution on the functor category $\mathcal{T} \times \mathcal{T} = \mathrm{Fun}(\mathbb{Z}/2, \mathcal{T})$. If $\mathcal{G}$ is a set of rigid-compact generators of $\mathcal{T}$, then $(\mathcal{G} \times 0) \cup (0 \times \mathcal{G})$ is a set of rigid-compact generators for $\mathcal{T} \times \mathcal{T}$. Moreover, the unit $(\mathbb{1}, 0)$ of $\mathcal{T} \times \mathcal{T}$ is compact. In summary, $\mathcal{T} \times \mathcal{T}$ is a rigidly-compactly generated tensor-triangulated category, and the inclusion $a \mapsto (a, 0)$ is a fully faithful coproduct-preserving tensor-triangulated functor $f^* \colon \mathcal{T} \hookrightarrow \mathcal{T} \times \mathcal{T}$. The projection $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$ onto the 0th coordinate is both left and right adjoint to f^*. It follows that f^* is a geometric functor which is not an equivalence and yet satisfies the hypotheses (a), (b), and (c) of Proposition 4.15. Also note that the right adjoint f_* is not conservative (so f^* is not finite étale) and yet f^* satisfies hypotheses (a) and (b) of Corollary 4.20.

Remark 4.24. The product category $\mathcal{T} \times \mathcal{T}$ from Example 4.23 is never locally monogenic (for $\mathcal{T}$ nonzero). This follows from our abstract theorems as noted above, but of course can be seen directly. Choose any prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$. Then $\mathcal{P} \times \mathcal{P} \in \mathrm{Spc}((\mathcal{T} \times \mathcal{T})^c)$. If the local category $(\mathcal{T} \times \mathcal{T})_{\mathcal{P} \times \mathcal{P}}$ at $\mathcal{P} \times \mathcal{P}$ were monogenic, then the composite

$$
\begin{array}{ccc}
\mathcal{T} \ \lhook\joinrel\longrightarrow \mathcal{T} \times \mathcal{T} \longrightarrow & (\mathcal{T} \times \mathcal{T})_{\mathcal{P} \times \mathcal{P}} \\
\downarrow & \downarrow{\scriptstyle \cong} \\
\mathcal{T}_{\mathcal{P}} \ \lhook\joinrel\longrightarrow & (\mathcal{T}_{\mathcal{P}}) \times (\mathcal{T}_{\mathcal{P}})
\end{array}
$$

would have a conservative right adjoint, which is a contradiction since $\mathcal{T}_{\mathcal{P}} \neq 0$.

Remark 4.25. Example 4.23 also demonstrates that hypothesis (b) of Theorem 4.8 does not follow (in general) from the other two hypotheses (a) and (c).

5. Examples

We will now discuss some examples of finite étale morphisms with an eye to future applications.

Example 5.1. Let G be a compact Lie group. It was proved in [BDS15, Theorem 1.1] that for any finite index subgroup $H \leq G$, the restriction functor $\mathrm{SH}(G) \to \mathrm{SH}(H)$ between equivariant stable homotopy categories is finite étale. We can use Theorem 4.8 to improve this to an "if and only if" statement:

Theorem 5.2. *Let G be a compact Lie group and let $H \leq G$ be a closed subgroup. The restriction functor* $\mathrm{res}_H^G \colon \mathrm{SH}(G) \to \mathrm{SH}(H)$ *is finite étale if and only if H has finite index in G.*

Proof. As already mentioned, the "if" part is [BDS15, Theorem 1.1]. For the "only if" part, recall that the relative dualizing object for res_H^G is the representation sphere $S^{L(H;G)}$ for the tangent H-representation at the coset $eH \in G/H$ (see [May03] and [San19, Remark 2.16]). By Theorem 4.8, if res_H^G is finite étale, the

canonical morphism $\mathbb{1}_{\mathrm{SH}(H)} \to S^{L(H;G)}$ is an isomorphism. Restricting to the trivial subgroup, we obtain an isomorphism $S^0 \to S^{\dim(G/H)}$ in the nonequivariant stable homotopy category SH. The dimension of (the suspension spectrum of) a sphere is recovered by rational cohomology. Hence $\dim(G/H) = 0$. The compact 0-dimensional manifold G/H is just a finite collection of points. That is, H has finite index in G. $\qquad\square$

Example 5.3. Let $p_n \colon S^1 \to S^1$ denote the degree n map $z \mapsto z^n$ on the unit circle. The induced functor $p_n^* \colon \mathrm{SH}(S^1) \to \mathrm{SH}(S^1)$ is *not* finite étale (for $n \geq 2$). Indeed, this amounts to the question of whether the quotient $S^1 \to S^1/C_n$ by the subgroup of nth roots of unity induces a finite étale morphism $\mathrm{SH}(S^1/C_n) \to \mathrm{SH}(S^1)$. But [San19, Proposition 3.2] establishes that inflation $\mathrm{infl}_{G/N}^G \colon \mathrm{SH}(G/N) \to \mathrm{SH}(G)$ never satisfies Grothendieck–Neeman duality except when $N = 1$ is the trivial subgroup.

Remark 5.4. Another way of appreciating why Example 5.3 is not finite étale is to look at its behaviour on the Balmer spectrum, which we know due to [BGH20, BS17]. The points of $\mathrm{Spc}(\mathrm{SH}(S^1)^c)$ are of the form $\mathcal{P}(H, \mathcal{C})$ for H a closed subgroup of S^1 and $\mathcal{C} \in \mathrm{Spc}(\mathrm{SH}^c)$. The closed subgroups of S^1 are, in addition to S^1 itself, the finite cyclic groups C_m ($m \geq 1$) realized as the roots of unity in S^1. Consider the map on the Balmer spectrum

$$\varphi := \mathrm{Spc}(p_n^*) \colon \mathrm{Spc}(\mathrm{SH}(S^1)^c) \to \mathrm{Spc}(\mathrm{SH}(S^1)^c)$$

induced by the degree n map $p_n \colon S^1 \to S^1$. One can show that $\varphi(\mathcal{P}(C_m, \mathcal{C})) = \mathcal{P}(C_{\mathrm{lcm}(m,n)/n}, \mathcal{C})$. For example, taking $n = 2$ and fixing the nonequivariant prime $\mathcal{C}$, it maps

$$m \mapsto \begin{cases} m/2 & \text{if } 2 \mid m, \\ m & \text{if } 2 \nmid m. \end{cases}$$

In particular, we find that the fibers have cardinality

$$|\varphi^{-1}(\{\mathcal{P}(C_N, \mathcal{C})\})| = \begin{cases} 1 & \text{if } 2 \mid N, \\ 2 & \text{if } 2 \nmid N. \end{cases}$$

For example, the fiber over $\mathcal{P}(C_1, \mathcal{C})$ consists of two points: $\{\mathcal{P}(C_1, \mathcal{C}), \mathcal{P}(C_2, \mathcal{C})\}$. Moreover, if the nonequivariant prime $\mathcal{C} := \mathcal{C}_{2,\infty}$ is the 2-local prime at chromatic height ∞, then $\mathcal{P}(C_1, \mathcal{C}_{2,\infty}) \subseteq \mathcal{P}(C_2, \mathcal{C}_{2,\infty})$ is a nontrivial inclusion in the fiber over $\mathcal{P}(C_1, \mathcal{C}_{2,\infty})$. This implies that the basic theorems of Balmer [Bal16b, Theorem 1.5] on the behaviour of finite étale morphisms do not hold for the morphisms $p_n^* \colon \mathrm{SH}(S^1) \to \mathrm{SH}(S^1)$.

Lemma 5.5. *Consider a diagram of coproduct-preserving (N-)tensor-triangulated functors between rigidly-compactly generated (N-)tensor-triangulated categories*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{g^*} & \mathcal{D} \\
{\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle k^*} \\
\mathcal{C}' & \xrightarrow{f^*} & \mathcal{D}'
\end{array}
$$

which commutes up to natural isomorphism of symmetric monoidal functors. Denote the right adjoints by $f^* \dashv f_*$ and $g^* \dashv g_*$, and suppose that the Beck–Chevalley comparison map

$$h^* g_* \to f_* k^*$$

is a natural isomorphism. If g^* is finite étale and f_* is conservative, then f^* is finite étale.

Proof. The Beck–Chevalley comparison map $h^* g_* \to f_* k^*$ is a monoidal natural transformation between lax symmetric monoidal functors; hence the natural isomorphism $h^* g_* \xrightarrow{\sim} f_* k^*$ provides an isomorphism of commutative algebras $h^* g_*(\mathbb{1}_{\mathcal{D}}) \simeq f_*(\mathbb{1}_{\mathcal{D}'})$. By assumption, $g_*(\mathbb{1}_{\mathcal{D}})$ is a compact commutative separable algebra in $\mathcal{C}$; hence $f_*(\mathbb{1}_{\mathcal{D}'})$ is a compact commutative separable algebra in $\mathcal{C}'$. The $f^* \dashv f_*$ adjunction satisfies the projection formula (see [BDS16, Proposition 2.15]), and f_* is conservative by hypothesis. Hence, Proposition 3.8 provides the result. $\qquad\square$

Example 5.6. If $\mathcal{C}$ is a presentably symmetric monoidal stable ∞-category and $A \in \mathrm{CAlg}(\mathcal{C})$ is a commutative algebra in $\mathcal{C}$, then we can consider the presentably symmetric monoidal stable ∞-category $A\text{-}\mathrm{Mod}_{\mathcal{C}}$ of A-modules. If $\mathcal{C}$ is rigidly-compactly generated, then so is $A\text{-}\mathrm{Mod}_{\mathcal{C}}$ (see [PSW22, Remark 3.11], for example). At the level of homotopy categories, the extension-of-scalars $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(A\text{-}\mathrm{Mod}_{\mathcal{C}})$ is then a geometric functor of rigidly-compactly generated ∞-tensor-triangulated categories whose right adjoint is conservative.

Example 5.7. Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category, and let $A, B \in \mathrm{CAlg}(\mathcal{C})$ be commutative algebras in $\mathcal{C}$. We then have

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & B\text{-}\mathrm{Mod}_{\mathcal{C}} \\
\downarrow & & \downarrow \\
A\text{-}\mathrm{Mod}_{\mathcal{C}} & \longrightarrow & (A \otimes B)\text{-}\mathrm{Mod}_{\mathcal{C}},
\end{array}
$$

where all four functors are extension-of-scalars. This is an example where the Beck–Chevalley property holds (at the level of the underlying stable ∞-categories). In particular, the induced diagram of ∞-tensor-triangulated categories

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathcal{C}) & \longrightarrow & \mathrm{Ho}(B\text{-}\mathrm{Mod}_{\mathcal{C}}) \\
\downarrow & & \downarrow \\
\mathrm{Ho}(A\text{-}\mathrm{Mod}_{\mathcal{C}}) & \longrightarrow & \mathrm{Ho}((A \otimes B)\text{-}\mathrm{Mod}_{\mathcal{C}})
\end{array}
$$

satisfies the first hypothesis of Lemma 5.5. Moreover, the right adjoints are all conservative (Example 5.6). Thus, if the top horizontal functor is finite étale (*i.e.* if B is a compact separable commutative algebra in $\mathrm{Ho}(\mathcal{C})$), then the bottom horizontal functor is also finite étale.

Example 5.8. Let G be a compact Lie group and let Sp_G denote the symmetric monoidal stable ∞-category of G-spectra (see [GM20, Appendix C]). Let $\mathrm{triv}_G \colon \mathrm{Sp} \to \mathrm{Sp}_G$ denote the unique colimit-preserving symmetric monoidal functor from the ∞-category of spectra. Since $\mathrm{res}_H^G \circ \mathrm{triv}_G \simeq \mathrm{triv}_H$ for any $H \leq G$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathrm{Sp}_G) & \longrightarrow & \mathrm{Ho}(\mathrm{Sp}_H) \\
\downarrow & & \downarrow \\
\mathrm{Ho}(\mathrm{triv}_G(\mathbb{E})\text{-}\mathrm{Mod}_{\mathrm{Sp}_G}) & \longrightarrow & \mathrm{Ho}(\mathrm{triv}_H(\mathbb{E})\text{-}\mathrm{Mod}_{\mathrm{Sp}_H})
\end{array}
$$

for any $\mathbb{E} \in \mathrm{CAlg}(\mathrm{Sp})$. If $H \leq G$ has finite index, then the top horizontal functor is finite étale (Example 5.1), and hence the bottom horizontal functor is finite étale. Taking $\mathbb{E} = \mathrm{H}\mathbb{Z}$, we obtain that the restriction functor

$$
\mathrm{D}(\mathrm{H}\mathbb{Z}_G) \to \mathrm{D}(\mathrm{H}\mathbb{Z}_H)
$$

between categories of derived Mackey functors studied in [PSW22] is finite étale. This will be utilized in the forthcoming [BHS21] which will classify the localizing tensor-ideals of these categories.

Example 5.9. A version of Example 5.7 holds purely at the level of triangulated categories if one assumes the two algebras are separable. More precisely, let A and B be two commutative separable algebras in a rigidly-compactly generated N-tensor-triangulated category $\mathcal{T}$. Iterated extension-of-scalars behaves as one expects (see [Pau17, Proposition 1.14]) and we have a diagram of rigidly-compactly generated N-tensor-triangulated categories

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\;F_A\;} & A\text{-}\mathrm{Mod}_{\mathcal{T}} \\
{\scriptstyle F_B}\downarrow & & \downarrow \\
B\text{-}\mathrm{Mod}_{\mathcal{T}} & \longrightarrow & (A \otimes B)\text{-}\mathrm{Mod}_{\mathcal{T}}
\end{array}
$$

which commutes up to isomorphism. Lemma 5.5 implies that if the top functor is finite étale, then so is the bottom functor.

Remark 5.10. Let $F\colon \mathcal{D} \to \mathcal{C}$ be a geometric functor of rigidly-compactly generated tensor-triangulated categories and let $\varphi\colon \operatorname{Spc}(\mathcal{C}^c) \to \operatorname{Spc}(\mathcal{D}^c)$ be the induced map on spectra. For any Thomason subset $Y \subseteq \operatorname{Spc}(\mathcal{D}^c)$ with $V := \operatorname{Spc}(\mathcal{D}^c) \setminus Y$, we have an induced functor $F|_V\colon \mathcal{D}(V) \to \mathcal{C}(\varphi^{-1}(V))$ on finite localizations such that

$$
(5.11) \qquad
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\ F\ } & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{D}(V) & \xrightarrow{\ F|_V\ } & \mathcal{C}(\varphi^{-1}(V))
\end{array}
$$

commutes up to isomorphism. Moreover, on spectra,

$$
\varphi^{-1}(V) \cong \operatorname{Spc}(\mathcal{C}(\varphi^{-1}(V))) \xrightarrow{\operatorname{Spc}(F|_V)} \operatorname{Spc}(\mathcal{D}(V)) \cong V
$$

is just the restriction $\varphi|_V\colon \varphi^{-1}(V) \to V$.

Example 5.12 (Restriction in the target). If $F\colon \mathcal{D} \to \mathcal{C}$ is finite étale, then the induced "restriction" functor

$$
F|_V\colon \mathcal{D}(V) \to \mathcal{C}(\varphi^{-1}(V))
$$

of Remark 5.10 is also finite étale. Here $V \subseteq \operatorname{Spc}(\mathcal{D}^c)$ is the complement of a Thomason subset. For example, V could be a quasi-compact open subset. Indeed, this is just a special case of Example 5.9 with $B = f_{V^c}$ the idempotent algebra for the finite localization $\mathcal{D} \to \mathcal{D}(V)$.

Corollary 5.13. *Let $F\colon \mathcal{D} \to \mathcal{C}$ be finite étale. If $\mathcal{D}$ is locally monogenic, then $\mathcal{C}$ is locally monogenic.*

Proof. We use the notation of Remark 5.10. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{C}^c)$ and consider its image $\varphi(\mathcal{P}) \in \operatorname{Spc}(\mathcal{D}^c)$. Let $V := \operatorname{gen}(\varphi(\mathcal{P}))$ be the subset of $\operatorname{Spc}(\mathcal{D}^c)$ consisting of all generalizations of the point $\varphi(\mathcal{P})$. It is the complement of a Thomason subset, and restriction to V is localization at the point $\varphi(\mathcal{P})$; see [BHS21, Remark 1.21 and Definition 1.25]. Note that $\operatorname{gen}(\mathcal{P}) \subseteq \varphi^{-1}(V)$. Then consider the composition

$$
\mathcal{D}_{\varphi(\mathcal{P})} = \mathcal{D}(V) \to \mathcal{C}(\varphi^{-1}(V)) \to \mathcal{C}(\operatorname{gen}(\mathcal{P})) = \mathcal{C}_{\mathcal{P}},
$$

where the first functor is finite étale (Example 5.12) and the second functor is a localization. Since the right adjoints are conservative, any set of compact generators of $\mathcal{D}_{\varphi(\mathcal{P})}$ is mapped to a set of compact generators of $\mathcal{C}_{\mathcal{P}}$. Thus, $\mathcal{D}_{\varphi(\mathcal{P})}$ monogenic implies $\mathcal{C}_{\mathcal{P}}$ monogenic. $\qquad\square$

Remark 5.14. Additional equivariant examples are featured in the work of Balmer and Dell'Ambrogio on Mackey 2-motives [BD20, Del21]. On the other hand, the following basic example relates the tensor-triangular notion of finite étale with the ordinary scheme-theoretic notion:

Theorem 5.15 (Balmer). *If $f\colon X \to Y$ is a finite étale morphism of quasi-compact and quasi-separated schemes, then the derived functor $\mathbb{L}f^*\colon \operatorname{D_{qc}}(Y) \to \operatorname{D_{qc}}(X)$ is a finite étale morphism in the sense of Definition 4.2.*

Proof. This is provided by [Bal16a, Theorem 3.5]; see also [Nee18, Example 0.3]. $\qquad\square$

Remark 5.16. The proof of the above theorem works verbatim for other tensor-triangulated categories $\mathcal{T}(X)$ fibered over a category of schemes, provided the pseudofunctor $X \mapsto \mathcal{T}(X)$ satisfies flat base change. Many motivic examples of such pseudofunctors are discussed in [CD19]. We just mention:

Example 5.17. Let $\mathbb{L}/\mathbb{K}$ be a finite separable extension of fields whose characteristic (if positive) is invertible in the ring R. The induced functor $\operatorname{SH}(\mathbb{K}; R) \to \operatorname{SH}(\mathbb{L}; R)$ between motivic stable homotopy categories (with coefficients in R) is a finite étale morphism in the sense of Definition 4.2. The same is true of the induced functor $\operatorname{DM}(\mathbb{K}; R) \to \operatorname{DM}(\mathbb{L}; R)$ between derived categories of motives. See [CD19, Ayo07a, Ayo07b, Tot18]

for more information about these categories. The assumption on the characteristic ensures that these categories are rigidly-compactly generated. To see this, first recall that $\mathrm{SH}(\mathbb{K}; R)$ is compactly generated by the twists of smooth $\mathbb{K}$-schemes of finite type (see [Rio05, Corollary 1.3] and [Ayo07b, Theorem 4.5.67]) and, similarly, $\mathrm{DM}(\mathbb{K}; R)$ is compactly generated by the twists of smooth separated $\mathbb{K}$-schemes of finite type (see [CD19, Section 11.1] and [Tot16, Lemma 5.4]). It is nontrivial that these generators are dualizable. For the motivic stable homotopy category, see [EK20, Theorem 3.2.1], which builds on [LYZ$^+$19, Appendix B]; for the derived category of motives, see [Tot16, Lemma 5.5], which extends [Kel12, Theorem 5.5.14] and [Voe00, Theorem. 4.3.7]. Since the unit object is compact and there is a generating set of dualizable objects, it follows that the compact and dualizable objects coincide. This follows from [HPS97, Theorem A.2.5.(a)] and [Nee92, Lemma 2.2] (which is also proved in [Ayo07a, Proposition 2.1.24]).

Remark 5.18. The author thinks it is interesting to have an "intrinsic" characterization of finite étale morphisms in tensor triangular geometry as expressed in Theorem 4.8. Nevertheless, actually classifying the finite étale extensions of a given category $\mathcal{T}$ amounts to classifying the rigid (strongly) separable commutative algebras in $\mathcal{T}$. For the equivariant stable homotopy category $\mathcal{T} = \mathrm{SH}(G)$, this classification will be studied in forthcoming work with Balmer. The analogous problem for the stable module category $\mathcal{T} = \mathrm{StMod}(kG)$ has been studied in [BC18] and is surprisingly subtle. It is currently only understood when G is cyclic.

Remark 5.19. For the derived category $\mathcal{T} = \mathrm{D}_{\mathrm{qc}}(X)$ of a noetherian scheme, Neeman [Nee18] has obtained a very satisfactory classification of the (not necessarily compact) commutative separable algebras. His work shows that the tensor-triangular analogue of étale morphism (a.k.a. extension by a commutative separable algebra) lies somewhere between the classical étale morphisms of schemes and the pro-étale morphisms of Bhatt–Scholze [BS15]. His results also show that there are no exotic étale extensions of derived categories of schemes: An étale extension of a derived category of a scheme is another derived category of a scheme. We will state this result precisely in the case of finite étale extensions:

Theorem 5.20 (Neeman). *Let X be a noetherian scheme. If $F\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathcal{S}$ is a finite étale morphism (Definition 4.2), then there exist a finite étale morphism of schemes $f\colon U \to X$ and a tensor-triangulated equivalence $\mathcal{S} \cong \mathrm{D}_{\mathrm{qc}}(U)$. With this identification, F is naturally isomorphic to $\mathbb{L}f^*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(U)$.*

Proof. Let G denote the right adjoint of F. By definition, F is extension-of-scalars with respect to the compact commutative separable algebra $G(\mathbb{1}) \in \mathrm{D}_{\mathrm{qc}}(X)$. Neeman [Nee18, Theorem 7.10] establishes that there is a separated finite-type étale map of schemes $g\colon V \to X$ and a generalization-closed subset $U \subset V$ such that $G(\mathbb{1}) \cong \mathbb{R}f_*(\mathcal{O}_U)$, where $f\colon U \to X$ denotes the composite $U \hookrightarrow V \xrightarrow{g} X$. It then follows from Proposition 3.8 that $\mathcal{S} \cong \mathrm{D}_{\mathrm{qc}}(U)$ with $F \cong \mathbb{L}f^*$. Now, since $G(\mathbb{1})$ is compact, the argument in [Nee18, Remark 0.6] shows that $U \subset V$ is actually an open subset. (Take $L := 0$, $\widetilde{K} := f_* f^*(K)$ and the identity map $\widetilde{K} \to f_* f^* K$ in *loc. cit.*) Thus, $f\colon U \to X$ is a separated finite-type étale map. It is also proper since $\mathbb{L}f^* \cong F$ satisfies GN-duality (by [LN07]; see also [San19, Section 7] and [Lip09, Section 4.3]). This completes the proof since an étale map is proper if and only if it is finite. $\qquad\square$

Remark 5.21. For the purpose of classifying the *finite* étale extensions of a given tensor-triangulated category, the results of Section 2 are worth keeping in mind. They clarify that the compact/rigid commutative separable algebras that provide finite étale extensions are necessarily self-dual. This puts limits on the role finite étale morphisms can play in equivariant contexts over non-finite groups. Stated differently, Theorem 4.8 shows that the relative dualizing object ω_f for a finite étale morphism f^* must be trivial. It is natural to wonder if there is a reasonable generalization of "finite étale" in tensor triangular geometry which shares some of its good properties (*e.g.*, the results of [Bal16a, Bal16b]) and yet covers examples having non-trivial dualizing objects (*e.g.*, the examples which arise in [Rog08]).

References

[Agu00]   M. Aguiar, *A note on strongly separable algebras*, Bol. Acad. Nac. Cienc. (Córdoba) **65** (2000), 51–60.

[AG60]   M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[Ayo07a]   J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I*, Astérisque **314** (2007).

[Ayo07b]   ———, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II*, Astérisque **315** (2007).

[Bal10]   P. Balmer, *Spectra, spectra, spectra – tensor triangular spectra versus Zariski spectra of endomorphism rings*, Algebr. Geom. Topol. **10** (2010), no. 3, 1521–1563.

[Bal11]   ———, *Separability and triangulated categories*, Adv. Math. **226** (2011), no. 5, 4352–4372.

[Bal14]   ———, *Splitting tower and degree of tt-rings*, Algebra Number Theory **8** (2014), no. 3, 767–779.

[Bal15]   ———, *Stacks of group representations*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 1, 189–228.

[Bal16a]   ———, *The derived category of an étale extension and the separable Neeman-Thomason theorem*, J. Inst. Math. Jussieu **15** (2016), no. 3, 613–623.

[Bal16b]   ———, *Separable extensions in tensor-triangular geometry and generalized Quillen stratification*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 907–925.

[Bal20]   ———, *A guide to tensor-triangular classification*, In: *Handbook of Homotopy Theory*, pp. 145–162, CRC Press, Boca Raton, FL, 2020.

[BC18]   P. Balmer and J. F. Carlson, *Separable commutative rings in the stable module category of cyclic groups*, Algebr. Represent. Theory **21** (2018), no. 2, 399–417.

[BD20]   P. Balmer and I. Dell'Ambrogio, *Mackey 2-functors and Mackey 2-motives*, EMS Monogr. Math., Euro. Math. Soc., Zürich, 2020.

[BDS15]   P. Balmer, I. Dell'Ambrogio, and B. Sanders, *Restriction to finite-index subgroups as étale extensions in topology, KK-theory and geometry*, Algebr. Geom. Topol. **15** (2015), no. 5, 3025–3047.

[BDS16]   ———, *Grothendieck–Neeman duality and the Wirthmüller isomorphism*, Compos. Math. **152** (2016), no. 8, 1740–1776.

[BS17]   P. Balmer and B. Sanders, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. **208** (2017), no. 1, 283–326.

[BGH20]   T. Barthel, J. P. C. Greenlees, and M. Hausmann, *On the Balmer spectrum for compact Lie groups*, Compos. Math. **156** (2020), no. 1, 39–76.

[BHS21]   T. Barthel, D. Heard, and B. Sanders, *Stratification in tensor triangular geometry with applications to spectral Mackey functors*, preprint arXiv:2106.15540 (2021).

[BS15]   B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2015), 99–201.

[CHR65]   S. U. Chase, D. K. Harrison, and A. Rosenberg, *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. No. **52** (1965), 15–33.

[Che15]   X.-W. Chen, *A note on separable functors and monads with an application to equivariant derived categories*, Abh. Math. Semin. Univ. Hambg. **85** (2015), no. 1, 43–52.

[CD19]   D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, Springer Monogr. Math., Springer, Cham, 2019.

[Del21]   I. Dell'Ambrogio, *Green 2-functors*, Trans. Amer. Math. Soc. **375** (2022), no. 11, 7783–7829.

[DS18]   I. Dell'Ambrogio and B. Sanders, *A note on triangulated monads and categories of module spectra*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 8, 839–842.

[EK20]   E. Elmanto and A. A. Khan, *Perfection in motivic homotopy theory*, Proc. Lond. Math. Soc. (3) **120** (2020), no. 1, 28–38.

[Fau13]   B. Fauser, *Some graphical aspects of Frobenius algebras*, In: *Quantum Physics and Linguistics: A Compositional, Diagrammatic Discourse*, pp. 22–49, Oxford Univ. Press, Oxford, 2013.

[For17]   T. J. Ford, *Separable algebras*, Grad. Stud. Math., vol. 183, Amer. Math. Soc., Providence, RI, 2017.

[FRS02]   J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators. I. Partition functions*, Nuclear Phys. B **646** (2002), no. 3, 353–497.

[GM20]   D. Gepner and L. Meier, *On equivariant topological modular forms*, preprint arXiv:2004.10254 (2020).

[Har77]   R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52, Springer-Verlag, New York, 1977.

[Hat65]   A. Hattori, *On strongly separable algebras*, Osaka Math. J. **2** (1965), 369–372.

[HPS97]   M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.

[Kan62]   T. Kanzaki, *A type of separable algebras*, J. Math. Osaka City Univ. **13** (1962), 39–43.

[Kel12]   S. Kelly, *Triangulated categories of motives in positive characteristic*, Ph.D. thesis, Université Paris 13, 2012. Available at https://arxiv.org/abs/1305.5349.

[KO74]   M.-A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math., vol. 389, Springer-Verlag, Berlin, 1974.

[Koc04]   J. Kock, *Frobenius algebras and 2D topological quantum field theories*, London Math. Soc. Stud. Texts, vol. 59, Cambridge Univ. Press, Cambridge, 2004.

[LP07]   A. D. Lauda and H. Pfeiffer, *State sum construction of two-dimensional open-closed topological quantum field theories*, J. Knot Theory Ramifications **16** (2007), no. 9, 1121–1163.

[LYZ+19] M. Levine, Y. Yang, G. Zhao, and J. Riou, *Algebraic elliptic cohomology theory and flops I*, Math. Ann. **375** (2019), no. 3-4, 1823–1855.

[Lip09]   J. Lipman, *Notes on derived functors and Grothendieck duality*, In: *Foundations of Grothendieck duality for diagrams of schemes*, pp. 1–259, Lecture Notes in Math., vol. 1960, Springer, Berlin, 2009.

[LN07]   J. Lipman and A. Neeman, *Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), no. 1, 209–236.

[May03]   J. P. May, *The Wirthmüller isomorphism revisited*, Theory Appl. Categ. **11** (2003), No. 5, 132–142.

[Nee92]   A. Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[Nee18]   ———, *Separable monoids in* $\mathbf{D_{qc}}(X)$, J. reine angew. Math. **738** (2018), 237–280.

[NS17]   T. Nikolaus and S. Sagave, *Presentably symmetric monoidal ∞-categories are represented by symmetric monoidal model categories*, Algebr. Geom. Topol. **17** (2017), no. 5, 3189–3212.

[PSW22]   I. Patchkoria, B. Sanders, and C. Wimmer, *The spectrum of derived Mackey functors*, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4057–4105.

[Pau15]   B. Pauwels, *Quasi-Galois theory in tensor-triangulated categories*, Ph.D. thesis, University of California, Los Angeles, 2015. Available at https://escholarship.org/uc/item/65q0q1gv.

[Pau17]   ———, *Quasi-Galois theory in symmetric monoidal categories*, Algebra Number Theory **11** (2017), no. 8, 1891–1920.

[Pie82]   R. S. Pierce, *Associative algebras*, Grad. Texts in Math., vol. 88, Springer-Verlag, New York-Berlin, 1982.

[PS13]   K. Ponto and M. Shulman, *Shadows and traces in bicategories*, J. Homotopy Relat. Struct. **8** (2013), no. 2, 151–200.

[Raf90]   M. D. Rafael, *Separable functors revisited*, Comm. Algebra **18** (1990), no. 5, 1445–1459.

[Rio05]   J. Riou, *Dualité de Spanier-Whitehead en géométrie algébrique*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 6, 431–436.

[Rog08]   J. Rognes, *Galois extensions of structured ring spectra. Stably dualizable groups*, Mem. Amer. Math. Soc. **192** (2008), no. 898.

[San19]   B. Sanders, *The compactness locus of a geometric functor and the formal construction of the Adams isomorphism*, J. Topol. **12** (2019), no. 2, 287–327.

[Sel11]   P. Selinger, *A survey of graphical languages for monoidal categories*, In: *New structures for physics*, pp. 289–355, Lecture Notes in Phys., vol. 813, Springer, Heidelberg, 2011.

[Tot16]   B. Totaro, *The motive of a classifying space*, Geom. Topol. **20** (2016), no. 4, 2079–2133.

[Tot18]   ———, *Adjoint functors on the derived category of motives*, J. Inst. Math. Jussieu **17** (2018), no. 3, 489–507.

[VZ66]   O. E. Villamayor and D. Zelinsky, *Galois theory for rings with finitely many idempotents*, Nagoya Math. J. **27** (1966), 721–731.

[Voe00]   V. Voevodsky, *Triangulated categories of motives over a field*, In: *Cycles, transfers, and motivic homology theories*, pp. 188–238, Ann. of Math. Stud., vol. 143, Princeton Univ. Press, Princeton, NJ, 2000.