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Divisorial contractions to codimension three orbits

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Abstract. Let G be a connected algebraic group. We study G-equivariant extremal contractions whose centre is a codimension three G-simply connected orbit. In the spirit of an important result by Kawakita in 2001, we prove that those contractions are weighted blow-ups.

Keywords. Divisorial contraction; weighted blow-ups; equivariant morphism; algebraic groups

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[Français]

Contractions divisorielles vers des orbites de codimension 3

Résumé. Soit G un groupe algébrique connexe. Nous étudions les contractions extrémales G-équivariantes dont le centre est une orbite G-simplement connexe de dimension 3. Dans l'esprit d'un résultat important obtenu par Kawakita en 2001, nous démontrons que ces contractions sont des éclatements à poids.

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Contents

1. Introduction

Let Y be a smooth complex projective variety. The determinant K_Y of its tangent vector bundle is a line bundle canonically attached to Y and called the *canonical divisor*. The so-called *numerical properties* of K_Y determine the geometry of Y. For example, it is well-known that if Y is a smooth surface, then either Y is covered by rational curves which have negative intersection with the canonical divisor or, by the Castelnuovo theorem, the curves having negative intersection with the canonical divisor are rational and can be contracted, giving a birational morphism:

$$Y \longrightarrow X,$$

such that the canonical divisor of X has non-negative intersection with every curve.

The minimal model program (MMP) aims to achieve a similar description for higher-dimensional varieties. In the development of the theory, it has become necessary to take singular varieties into consideration. If Y is normal, its canonical divisor can be defined in the following way. Denote by Y^{sm} the non-singular locus of Y. Writing $K_{Y^{\mathrm{sm}}} = \sum_i a_i D_i$, where D_i are prime divisors on Y^{sm} and $a_i \in \mathbb{Z}$, one defines $K_Y := \sum_i a_i \overline{D}_i$, where $\overline{D}_i$ is the Zariski closure of D_i in Y. Since the singular locus Y^{sing} of Y has codimension at least two in Y, the divisor K_Y is the unique divisor extending $K_{Y^{\mathrm{sm}}}$. In order to be able to compute intersection numbers, we need the divisor K_Y, which is a priori only a Weil divisor, to be a $\mathbb{Q}$-Cartier divisor. Moreover, we always assume that Y has at worst *terminal* singularities (see §3).

The first step of the MMP consists in looking at the curves which have negative intersection with K_Y. This is achieved by the *cone and contraction theorem* [KM98, Theorem 3.7], which describes the cone of numerical equivalence classes of curves in Y. It states that if R is an extremal ray of the cone, having negative intersection with the canonical divisor, then there is a morphism $f\colon Y \to X$, called an *extremal contraction*, contracting exactly those curves whose class belongs to R. There are three possibilities for the morphism f (see [KM98, Proposition 2.5]):

- *Divisorial contraction*: $\dim Y = \dim X$ and the exceptional locus of f has codimension 1;
- *Small contraction*: $\dim Y = \dim X$ and the exceptional locus of f has codimension ≥ 2;
- *Mori fibre space*: $\dim Y > \dim X$.

Those morphisms are the elementary bricks of the minimal model program and of the Sarkisov program. The first conjecturally associates to a variety a simpler model (either a minimal model or a Mori fibre space, see [KM98, §2.1]) and the second describes the relation between two different Mori fibre spaces associated to the same variety (see [Cor95, HM13]).

In this note, we focus on divisorial contractions. It is indeed useful for many applications to know what a divisorial contraction from a certain variety Y looks like, as it gives information on the possible outcomes after performing an MMP on Y. Divisorial contractions from a smooth variety Y have been studied for instance in [And85, Wiś91] and extremal contractions from mildly singular varieties in [And95, And18, AT14, AT16]. Here we focus our attention on the singularities of X, rather than on the singularities of Y. If X is a smooth surface and if the centre $Z := f(\mathrm{Exc}(f))$ of f is a point, then by the Castelnuovo theorem, f is a smooth blow-up. In dimension three, Kawakita proves the following result:

Theorem 1.1 (Kawakita, [Kaw01, Theorem 1.1]). *Let $f\colon Y \to X$ be 3-dimensional divisorial contraction, which contracts its exceptional divisor to a smooth point. Then f is a weighted blow-up.*

This result is particularly interesting if X has a Mori fibre space structure $X \to B$ and if one wants to study in detail the Sarkisov program starting from X/B. Indeed, for that, it is necessary to know the contractions *from* and *to* X.

If G is a connected algebraic group acting on X, then the Sarkisov program can be used to determine whether G is a maximal subgroup of the group $\mathrm{Bir}(X)$ of birational automorphisms of X (see [BFT19, Flo20]). Motivated by the study of the G-equivariant Sarkisov program, we turn our attention to G-equivariant divisorial contractions to a codimension three centre contained in the smooth locus and we prove the following result for any $d \geq 3$.

Theorem 1.2. *Every d-dimensional G-equivariant divisorial contraction to a G-simply connected G-orbit of dimension $d-3$, contained in the smooth locus of X is a G-equivariant weighted blow-up.*

The strategy of proof of Theorem 1.2 is based on a construction of Kawakita [Kaw01] and uses an induction argument to reduce the statement to Kawakita's theorem 1.1. Given a G-equivariant divisorial contraction $f\colon Y \to X$ with exceptional divisor E, satisfying our assumptions (see §2.3), the starting point in §5 is the iterative construction of a second birational model of X, which ends up to a G-equivariant birational morphism $h\colon X_n \to X$ contracting an irreducible exceptional divisor E_n. The G-equivariant iterative process provides the needed information on the valuation defined by E_n, which corresponds to a weighted blow-up, and by an induction argument in §6, using iterated hyperplane sections, we show that E defines the same valuation on X as E_n.

Finally, we present in §7 an example showing that the hypothesis that Z is an orbit is essential. We describe a terminal extraction with one-dimensional center which is not a weighted blow-up.

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2. The setup

2.1. Convention

We work over the field $\mathbb{C}$ of complex numbers. A *variety* is an integral, separated scheme of finite type over $\mathbb{C}$. A *divisor* is either Cartier, $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier, depending on the context. We denote by $\sim$ (*resp.* $\sim_{\mathbb{Q}}$, $\sim_{\mathbb{R}}$) the linear (*resp.* $\mathbb{Q}$-linear, $\mathbb{R}$-linear) equivalence relation of divisors. Unless explicitely mentioned, all varieties are assumed to be projective.

If $f\colon X \to Y$ is a morphism between two varieties X and Y, and D_1, D_2 are divisors on X, we write $D_1 \sim_f D_2$ (*resp.* $\sim_{\mathbb{Q},f}$, $\sim_{\mathbb{R},f}$) if there is a Cartier (*resp.* $\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier) divisor δ on Y such that $D_1 \sim D_2 + f^*\delta$ (*resp.* $\sim_{\mathbb{Q}f}$, $\sim_{\mathbb{R}f}$). We say that a divisor D is f-ample (*resp.* f-antiample, f-effective) if there exists an ample (*resp.* antiample, effective) divisor D' such that $D \sim_f D'$. A ***pair*** (X, Δ) is the data of a normal projective variety X and a $\mathbb{Q}$-divisor Δ.

2.2. Divisorial contractions

Definition 2.1. A morphism $f\colon Y \to X$ with connected fibers, between normal projective varieties Y and X is called a ***divisorial contraction*** if it satisfies all the following conditions:

(1) Y is locally $\mathbb{Q}$-factorial with terminal singularities;

(2) the morphism f is birational and its exceptional locus E is a prime divisor;

(3) the canonical divisor K_Y is f-antiample;

(4) the morphism f has relative Picard number one.

Throughout this paper, we consider a d-dimensional divisorial contraction:

$$f\colon Y \to X,$$

with $d \geq 3$, which contracts its exceptional divisor E to its ***centre*** Z, that is assumed to be a smooth subvariety contained in the smooth locus of X: for short we call Z a ***smooth centre***. Recall that X is also locally $\mathbb{Q}$-factorial with terminal singularities [KM98, Proposition 3.36 & Corollary 3.43(3)] and therefore the singular locus X^{sing} of X has codimension at least three [BS95, Lemma 1.3.1]. We have a $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier divisors:

$$K_Y \sim_{\mathbb{Q}} f^*K_X + aE,$$

where the positive rational number $a := \mathrm{a}(E, X)$ is the discrepancy of E with respect to X.

2.3. Equivariant divisorial contractions

Definition 2.2. Let G be a connected algebraic group. A divisorial contraction $f\colon Y \to X$ is called G-***equivariant*** if it satisfies the following conditions:

(1) X and Y are endowed with a regular action of G;

(2) the contraction f is G-equivariant.

We still denote by E the exceptional divisor of the G-equivariant contraction $f\colon Y \to X$ and by Z its centre. The action of G on X induces a regular action on Z. The variety Z is said G-***simply connected*** if its G-equivariant fundamental group $\pi_1^G(Z)$ is trivial (see for instance [Hui01, Loo08]). This implies that every connected, étale, G-equivariant morphism with target Z is an isomorphism.

Remark 2.3.

(1) If the action of G on Z is transitive, then Z is reduced and smooth, and every finite G-equivariant morphism is étale.

(2) If Z is a Fano manifold, it is automatically G-simply connected since every étale cover of a Fano variety is an isomorphism (see [Deb02, Corollary 4.18(b)]). Indeed, take an étale cover $\eta\colon Z' \to Z$. The Euler characteristic of Z' is given by:

$$\chi(Z') = (\deg \eta)\chi(Z),$$

and Z' is also a Fano manifold since $K_{Z'} = \eta^* K_Z$. By the Kodaira vanishing theorem, the Euler characteristic of a Fano manifold is one, so η is an isomorphism.

(3) If G is a semi-simple linear group, then its closed orbits are Fano manifolds by [Sha99, Corollary 2.1.7].

To compare with Theorem 1.2, which deals with codimension 3 orbits, the understanding of codimension 2 orbits is an easy consequence of a result due to Ando [And85]:

Proposition 2.4. *Every d-dimensional G-equivariant divisorial contraction to a $(d-2)$-dimensional G-orbit contained in the smooth locus of X is a blow-up.*

Proof. Let $f\colon Y \to X$ be a G-equivariant divisorial contraction with centre Z and exceptional divisor E. We first show that the singular locus Y^{sing} of Y does not meet E. Otherwise, Y would have a singularity over Z, but f is G-equivariant and the action of G preserves the singularities, so we would get $f(Y^{\mathrm{sing}}) \supseteq Z$. Since Z has codimension 2, this would imply that Y^{sing} has a component of codimension 2: this is impossible since Y has terminal singularities. So Y is smooth in a neighbourhood of E and $f_{|E}\colon E \to Z$ is equidimensional since f is G-equivariant. We can thus apply [And85, Theorem 2.3], showing that f is a blow-up. □

Our assumption of a G-action might look strong, however we believe that it is necessary. In Section 7 we provide an example of an extremal contraction in dimension four, non equivariant, which is not a weighted blow-up.

3. Preliminaries on the MMP with scaling

We recall for further use some notions on the Minimal Model Program (MMP), following the terminology and notation of Kollár–Mori [KM98].

Definition 3.1. A pair (X,Δ) is called *klt* if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, $\lfloor\Delta\rfloor = 0$ and there exists a log resolution $\mu\colon \hat{X} \to X$ (see [KM98, Notation 0.4 (10)]) such that:

$$K_{\hat{X}} + \mu_*^{-1}(\Delta) \sim_{\mathbb{Q}} \mu^*(K_X+\Delta) + \sum_E a_E E, \text{ with } a_E > -1 \text{ for all } E,$$

where the sum runs over the exceptional divisors of μ. A $\mathbb{Q}$-factorial variety X is called *terminal* if there exists a log resolution $\mu\colon \hat{X} \to X$ such that:

$$K_{\hat{X}} \sim_{\mathbb{Q}} \mu^* K_X + \sum_E a_E E, \text{ with } a_E > 0 \text{ for all } E.$$

where the sum runs over the exceptional prime divisors of μ.

Definition 3.2. Let $\widetilde{W}$, W be two varieties together with a projective morphism $f\colon \widetilde{W} \to W$. A $\mathbb{Q}$-divisor D on $\widetilde{W}$ is *movable* over W if there is a positive integer m such that the intersection of the base locus of mD with every fibre of f has codimension at least two in the fibre. An $\mathbb{R}$-divisor D is movable if there are $r_1,\dots,r_k \in \mathbb{R}_{\geq 0}$ and movable $\mathbb{Q}$-divisors $D_1,\dots,D_k$ such that $D \sim_{\mathbb{R}} \sum_{i=1}^k r_i D_i$.

The sum of two movable $\mathbb{Q}$-divisors is movable, and therefore numerical equivalence classes of movable divisors form a cone $\mathrm{Mov}(\widetilde{W}/W) \subseteq \mathrm{N}^1(\widetilde{W}/W)$, which is in general neither open nor closed.

Definition 3.3. The *movable cone* $\overline{\mathrm{Mov}}(\widetilde{W}/W)$ of $\widetilde{W}$ over W is the closure of $\mathrm{Mov}(\widetilde{W}/W)$ in $\mathrm{N}^1(\widetilde{W}/W)$ with respect to the euclidean topology.

Remark 3.4. If $D \in \overline{\mathrm{Mov}}(\widetilde{W}/W)$, then for every family of curves $\{\Gamma_t\}_t$ such that $\cup_t \Gamma_t$ has codimension one in a fibre of f, we have $D \cdot \Gamma_t \geq 0$ for all t.

We recall the procedure called the *Minimal Model Program with scaling of an ample divisor.* The hypotheses here are stronger than the usual ones, but they are exactly what we need in the sequel.

Construction 3.5 (Minimal Model Program with scaling). (See [BCHM10, §3.10].) Let $(\widetilde{W},\Delta)$ be a *klt* pair such that Δ is a $\mathbb{Q}$-divisor, and let $f\colon \widetilde{W}\to W$ be a proper birational morphism such that $K_{\widetilde{W}}+\Delta$ is f-antiample. Let A be an ample $\mathbb{Q}$-divisor on $\widetilde{W}$ such that $(\widetilde{W},\Delta+A)$ is *klt* and $K_{\widetilde{W}}+\Delta+A$ is f-nef. We set:

$$\lambda_0 := \inf\{t\in\mathbb{R}_{\geq 0} \mid K_{\widetilde{W}}+\Delta+tA \text{ is nef over } W\}.$$

Then either $\lambda_0=0$ and $K_{\widetilde{W}}+\Delta$ is nef, or there is an extremal ray $R_0\in\overline{\mathrm{NE}}(\widetilde{W}/W)$ such that:

$$(K_{\widetilde{W}}+\Delta)\cdot R_0<0 \text{ and } (K_{\widetilde{W}}+\Delta+\lambda_0 A)\cdot R_0=0.$$

If $K_{\widetilde{W}}+\Delta$ is f-nef, or if R_0 defines a Mori fibre space, we stop. Otherwise R_0 gives a divisorial contraction or a flip $\varphi_0\colon \widetilde{W}\dashrightarrow W_1$. We recall that flips exist by [HM10, Corollary 1.4.1]. Let Δ_1 and A_1 be the strict transforms of Δ and A.

We prove that $K_{W_1}+\Delta_1+\lambda_0 A_1$ is f_1-nef. Let $(p,q)\colon \widehat{W}\to\widetilde{W}\times W_1$ be a resolution of the indeterminacies of φ_0 (if φ_0 is a morphism, then p is the identity). By the negativity lemma [KM98, Lemma 3.39] we have:

$$p^*(K_{\widetilde{W}}+\Delta+\lambda_0 A)=q^*(K_{W_1}+\Delta_1+\lambda_0 A_1),$$

so the result follows from the projection formula.

We further continue with $K_{W_1}+\Delta_1$ and A_1. We finally obtain a sequence of divisorial contractions and flips:

$$\begin{array}{ccccccccc}
(\widetilde{W},\Delta)=:(W_0,\Delta_0) & \overset{\varphi_0}{\dashrightarrow} & (W_1,\Delta_1) & \overset{\varphi_1}{\dashrightarrow} & \cdots & \overset{\varphi_{i-2}}{\dashrightarrow} & (W_{i-1},\Delta_{i-1}) & \overset{\varphi_{i-1}}{\dashrightarrow} & \cdots \\
 & {\scriptstyle f=:f_0}\searrow & & {\scriptstyle f_1}\searrow & W & \swarrow{\scriptstyle f_{i-1}} & & &
\end{array}$$

At each step i, we set:

$$\lambda_i := \inf\{t\in\mathbb{R}_{\geq 0} \mid K_{W_i}+\Delta_i+tA_i \text{ is nef over } W\},$$

where A_i (*resp.* Δ_i) is the push-forward of A (*resp.* Δ) on W_i. It follows from the definition that $0\leq\lambda_i\leq 1$ for all i and that the sequence $(\lambda_i)_i$ is decreasing. We call this contruction the $(K_{\widetilde{W}}+\Delta)$-***MMP with scaling of*** A ***over*** W.

Remark 3.6.

(1) In the above construction, if there is a flip, then the strict transform of A is not ample anymore. Indeed, assume for instance that φ_0 is a flip and let $\phi\colon W_0\to Y$ and $\phi^+\colon W_1\to Y$ be the two small contractions involved. Then we know that:
- $K_{W_0}+\Delta+\lambda_0 A$ is ϕ-trivial and $K_{W_1}+\Delta_1+\lambda_0 A_1$ is ϕ^+-trivial;
- $K_{W_0}+\Delta$ is ϕ-antiample and $K_{W_1}+\Delta_1$ is ϕ^+-ample,

therefore A_1 is ϕ^+-antiample.

(2) Isomorphisms in codimension one preserve the movable cone. Indeed, let $\varphi\colon W_0\dashrightarrow W_1$ be an isomorphism in codimension one. It is enough to prove that for any movable $\mathbb{Q}$-divisor D, its push-forward φ_*D is a movable $\mathbb{Q}$-divisor. Let $(p,q)\colon \widehat{W}\to\widetilde{W}\times W$ be a resolution of the indeterminacies. Since φ is an isomorphism in codimension one, we have $\mathrm{Exc}(p)=\mathrm{Exc}(q)$ and for any m such that mD is Cartier we have:

$$\mathrm{Bs}(|mp^*D|)\subseteq p^{-1}\,\mathrm{Bs}(|mD|).$$

By definition, $\varphi_*D=q_*p^*D$ and $\mathrm{Bs}(|mq_*p^*D|)=\mathrm{Bs}(q_*|mp^*D|)$. An irreducible component of $p^{-1}\,\mathrm{Bs}(|mD|)$ has either codimension at least 2 or is contained in the exceptional locus of q. Then:

$$\mathrm{Bs}(|mq_*p^*D|)=\mathrm{Bs}(q_*|mp^*D|)=q(\mathrm{Bs}(|mp^*D|))\subseteq q(p^{-1}\,\mathrm{Bs}(|mD|)),$$

and this last set has codimension two in W_1.

The following result is an easy termination lemma known to experts. We present here a proof for the reader's convenience, following closely [Fuj11, Theorem 2.3] (see also [BCHM10, Corollary 1.4.2]).

Lemma 3.7. *In the MMP with scaling of an ample divisor as above, there is no infinite sequence of flips.*

Proof. We prove the statement by contradiction. Assume that n is an integer such that φ_i is a flip for every $i \geq n$.

Step 1. We prove first that $K_{W_i} + \Delta_i \notin \overline{\mathrm{Mov}}(W_i/W)$ for every i. Indeed $-(K_{W_i} + \Delta_i)$ is the pushforward of $-(K_W + \Delta)$. This divisor is ample and we can choose an effectif divisor D such that $-(K_W + \Delta) \sim_{\mathbb{R},f} D$ and such that the support of D is not contained in the exceptional locus of $\varphi_{i-1} \circ \cdots \circ \varphi_1$. Denote by D_i the pushforward of D on W_i. Then $-(K_{W_i} + \Delta_i) \sim_{\mathbb{R},f} D_i$. The divisor D_i is effective and non-zero. Thus, for any family of curves $\{\Gamma_t\}_t$ such that $\cup_t \Gamma_t$ has codimension one in a fibre of f_i and which is not contained in D_i, we have $D_i \cdot \Gamma_t > 0$, and by Remark 3.4 we have $K_{W_i} + \Delta_i \notin \overline{\mathrm{Mov}}(W_i/W)$. In particular, $K_{W_n} + \Delta_n \notin \overline{\mathrm{Mov}}(W_n/W)$.

Step 2. Let $\lambda := \lim_{i\to\infty} \lambda_i$. We prove that we can assume $\lambda = 0$. Indeed, if $\lambda > 0$, we pick $\Delta' \sim_{\mathbb{R}} \lambda A$ such that the pair $(\widetilde{W}, \Delta + \Delta')$ is *klt* and we run the $(K_{\widetilde{W}} + \Delta + \Delta')$-MMP with scaling of A over W. We notice that each step of the $(K_{\widetilde{W}} + \Delta + \Delta')$-MMP with scaling of A is a step of the $(K_{\widetilde{W}} + \Delta)$-MMP with scaling of A over W. Moreover:

$$\mu_i := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_{W_i} + \Delta_i + \Delta'_i + tA_i \text{ is nef over } W\} = \lambda_i - \lambda,$$

so $\lim_{i\to\infty} \mu_i = 0$. we can thus assume that $\lambda = 0$.

Step 3. Take for any $i \geq n$ an f_i-ample $\mathbb{Q}$-divisor G_i on W_i such that:

$$\lim_{i\to\infty} G_{i,n} = 0 \in \mathrm{N}^1(W_n/W),$$

where $G_{i,n}$ is the strict transform of G_i on W_n. For this, fix an euclidean norm $\|\cdot\|$ on $\mathrm{N}^1(W_n/W)$ and let A_i be an f_i-ample $\mathbb{Q}$-divisor on W_i and $A_{i,n}$ its strict transform. Then put $G_{i,n} = \frac{A_{i,n}}{i\|A_{i,n}\|}$. We note that the divisor:

$$K_{W_i} + \Delta_i + \lambda_i A_i + G_i$$

is ample over W for every i, since by construction $K_{W_i} + \Delta_i + \lambda_i A_i$ is nef. By Remark 3.6(2), the strict transform:

$$K_{W_n} + \Delta_n + \lambda_i A_n + G_{i,n}$$

is movable on W_n for every i. Thus $K_{W_n} + \Delta_n$ is a limit of movable $\mathbb{R}$-divisors in $\mathrm{N}^1(W_n/W)$ and therefore $K_{W_n} + \Delta_n \in \overline{\mathrm{Mov}}(W_n/W)$: this contradicts Step 1, so there is no infinite sequence of flips. □

Lemma 3.7 says that, in the setup of Construction 3.5, every MMP with scaling of an ample divisor terminates and, in particular, minimal models exist.

4. Preliminaries on equivariant birational transformations

We gather some classical constructions on G-equivariant birational transformations that will be needed in the sequel. We consider a variety X and a connected algebraic subgroup G of $\mathrm{Aut}^0(X)$.

4.1. Equivariant blow-ups

Let Z be a G-invariant subvariety of X. Then G acts on the ideal sheaf I_Z of Z, so there is a natural action of G on the blow-up $\mathrm{Bl}_Z X := \mathrm{Proj}\left(\oplus_{k\geq 0} I_Z^k\right)$ of Z on X and the natural morphism $\mathrm{Bl}_Z X \to X$ is G-equivariant.

4.2. Equivariant weighted blow-ups

Definition 4.1. The weighted blow-up of X of weights $\omega = (\omega_1, \dots, \omega_r)$, where the positive integers ω_i have no common divisor, with smooth centre Z of codimension r, contained in the smooth locus of X is the normalisation of the projectivisation:

$$\mathrm{Bl}_Z^{\omega} X := \mathrm{Proj}\left(\mathcal{O}_X \oplus \bigoplus_{k\geq 1} \mathcal{I}_k\right) \to X,$$

where the sheaves $\mathcal{I}_k$ are ideal sheaves on X such that, locally analytically on X, there are smooth coordinate functions $x_1, \dots, x_r$ defining Z, and for every k, the sheaf $\mathcal{I}_k$ is generated by the monomials $x_1^{a_1} \cdots x_r^{a_r}$ with $\omega_1 a_1 + \cdots + \omega_r a_r \geq k$. A weighted blow-up is called G-equivariant if $g(\mathcal{I}_k) = \mathcal{I}_k$ for every $g \in G$ and $k \geq 1$. This means that $g^* x_1, \dots, g^* x_r$ are smooth coordinate functions defining locally Z, for any $g \in G$. In this case, the morphism $\mathrm{Bl}_Z^{\omega} X \to X$ is G-equivariant.

In the sequel, we consider the weighted blow-up of a d-dimensional variety X along a codimension $r = 3$ smooth centre Z contained in the smooth locus of X, with weights $\omega = (n, m, 1)$ such that n and m are coprime. From now on, we assume that $X = \mathrm{Spec}\,\mathbb{C}[x_1, \dots, x_d]$ and that Z is given by the equations $x_1 = x_2 = x_3 = 0$. Then $\mathrm{Bl}_Z^{\omega} X = \mathrm{Proj}\left(\bigoplus_{k\geq 0} I_k\right)$, where $I_k \subset \mathbb{C}[x_1, \dots, x_d]$ is the ideal generated by the monomials $x_1^{a_1} x_2^{a_2} x_3^{a_3}$ with $na_1 + ma_2 + a_3 \geq k$. We can define a surjective map of graded rings:

$$\Phi \colon \mathbb{C}[x_1, \dots, x_d][u, v, w] \to \bigoplus_{k\geq 0} I_k, \quad u \mapsto x_1, \quad v \mapsto x_2, \quad w \mapsto x_3,$$

where, on the left hand side, the degrees are $\deg(x_i) = 0$ for all i, $\deg(u) = n$, $\deg(v) = m$ and $\deg(w) = 1$. Then $\mathrm{Bl}_Z^{\omega} X$ is isomorphic to the proper closed subscheme of $\mathbb{C}^d \times \mathbb{P}(n, m, 1)$ defined by the weighted homogeneous polynomials which generate the kernel of Φ, that is:

$$x_1^m v^n = x_2^n u^m, \quad x_1 w^n = x_3^n u, \quad x_2 w^m = x_3^m v. \tag{4.1}$$

This shows that the weighted blow-up may be equally defined as the normalization of the closure of the image of the map:

$$\left(\mathbb{C}^d \setminus Z\right) \to \left(\mathbb{C}^d \setminus Z\right) \times \mathbb{P}(n, m, 1), \quad (x_1, \dots, x_d) \mapsto (x_1, \dots, x_d, [x_1 : x_2 : x_3]).$$

The exceptional divisor E of the weighted blow-up is isomorphic to the weighted projective space $\mathbb{P}(n, m, 1)$. The valuation $v_\omega \colon \mathbb{C}(x_1, \dots, x_d) \setminus \{0\} \to \mathbb{Z}$ associated to this weighted blow-up is characterized by:

$$v_\omega(x_1) = n, \quad v_\omega(x_2) = m, \quad v_\omega(x_3) = 1, \quad v_\omega(x_i) = 0 \quad \forall i \geq 4.$$

4.3. Equivariant resolution of singularities

The singular locus of X is a G-invariant subvariety of X. By [Kol07, Proposition 3.9.1, Theorem 3.35 & Theorem 3.36] there is a smooth variety $\widetilde{X}$ together with a regular action of G on $\tilde{X}$ and a G-equivariant birational morphism $\widetilde{X} \to X$.

4.4. Equivariant MMP

Any MMP on X is automatically G-equivariant. To see it, consider the first step $X \dashrightarrow X_1$ of an MMP:

- If it is an extremal contraction, this is a consequence of the Blanchard lemma [BSU13, Proposition 4.2.1]. We present here an alternative proof. The group G acts trivially on the extremal rays contained in the K_X-negative part of the Mori cone, since these rays are discrete. Then the extremal ray corresponding to the contraction $X \to X_1$ is G-invariant and so is the locus spanned by it, so X_1 inherites an action of G making the contraction G-equivariant.

- If it is a flip, given by the composition of two small contractions $\mu\colon X \to Y$ and $\mu^+\colon X_1 \to Y$, by the discussion above, there is a G-action on Y such that μ is G-equivariant. Moreover we have:

$$X_1 \cong \operatorname{Proj}_Y\left(\bigoplus_m \mu_*\mathcal{O}_X(m(K_X))\right).$$

  Since K_X is G-invariant, the group G acts on $\mathcal{O}_X(mK_X)$ for any m such that mK_X is Cartier, and subsequently on X_1.

5. The tower

Let us recall the setup. We consider a d-dimensional G-equivariant divisorial contraction $f\colon Y \to X$ with exceptional divisor E, as in Definitions 2.1 & 2.2. We further assume that the center $Z \subset X$ of the contraction is smooth and contained in the smooth locus of X.

5.1. Construction of the tower

Starting from $X_0 := X$ and $Z_0 := Z$, we construct inductively the following objects: G-equivariant birational morphisms $g_i\colon X_i \to X_{i-1}$, integral closed subschemes $Z_i \subset X_i$ and prime divisors E_i on X_i. This construction follows the same lines as those of Kawakita [Kaw01, Construction 3.1], we recall it for the reader's convenience.

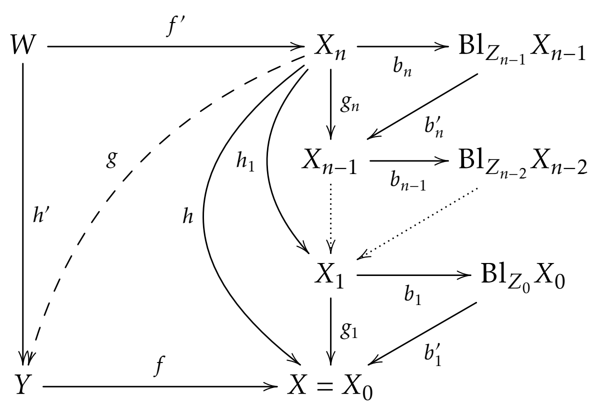
(1) We consider the blow-up $b_i'\colon \mathrm{Bl}_{Z_{i-1}}X_{i-1} \to X_{i-1}$ and we take $b_i\colon X_i \to \mathrm{Bl}_{Z_{i-1}}X_{i-1}$ a resolution of singularities. We put $g_i := b_i' \circ b_i$. Since Z_{i-1} is generically smooth, the exceptional divisor E_i' of b_i' is smooth over the generic point of Z_{i-1}, and so is its blow-up $\mathrm{Bl}_{Z_{i-1}}X_{i-1}$. In particular, we can find b_i such that none of the b_i-exceptional divisors surjects onto Z_{i-1}.

(2) We define Z_i as the centre of E in X_i, that is the closure of the image of E under the birational map $g_i^{-1} \circ f$. Note that Z_i is reduced and generically smooth.

(3) We denote by $E_i \subset X_i$ the strict transform of E_i'. Since Z_i surjects onto Z_{i-1}, the divisor E_i is generically smooth along Z_i, which is contained in E_i but in none of the b_i-exceptional divisors. Thus E_i is the only prime exceptional divisor of g_i which contains Z_i, and furthermore Z_i has multiplicity one along E_i.

The construction is summarized in Diagram 5.1 and is illustrated in Figure 1 (see Appendix).

(5.1)



The process ends up at some n-th step, when $Z_n = E_n$ is a prime divisor. Let us explain why it does so. At step $i \geq 1$, denote by $\mathrm{a}(E_i, X_{i-1}) \geq 1$ the discrepancy of E_i with respect to X_{i-1}. Since Z_i has multiplicity one along E_i, we have:

$$g_{i+1}^* E_i \equiv (g_{i+1}^{-1})_* E_i + E_{i+1} + \text{other components}.$$

It is then easy to compute that:

$$\mathrm{a}(E_{i+1}, X_{i-1}) = \mathrm{a}(E_{i+1}, X_i) + \mathrm{a}(E_i, X_{i-1}),$$

and similarly, going down by induction, for any $1 \leq j \leq i$:

$$\mathrm{a}(E_{i+1}, X_{i-j}) \geq \mathrm{a}(E_{i+1}, X_i) + \mathrm{a}(E_i, X_{i-j}), \tag{5.2}$$

where the inequality comes from the possible contribution of E_{i+1} to the total transforms of the birational images of the divisors $E_{i-1}, \dots, E_1$, for instance:

$$g_{i+1}^*(g_i^{-1})_*E_{i-1} \equiv (g_{i+1}^{-1})_*(g_i^{-1})_*E_{i-1} + \mathrm{mult}_{Z_i}((g_i^{-1})_*E_{i-1})E_{i+1}. \tag{5.3}$$

In particular, $\mathrm{a}(E_i, X_{i-j}) < \mathrm{a}(E_{i+1}, X_{i-j})$ for any j. Moreover the first blow-up is the one of a centre of codimension three, so $\mathrm{a}(E_1, X) = 2$ and:

$$2 = \mathrm{a}(E_1, X) < \cdots < \mathrm{a}(E_i, X) < \cdots < \mathrm{a}(E_n, X) = \mathrm{a}(E, X) = a.$$

This shows that the process ends up after at most $a-1$ steps. We finally take an elimination of indeterminacies W of the birational map $f^{-1} \circ h$.

We put:

$$\begin{aligned} h &:= g_1 \circ \cdots \circ g_n, \\ h_i &:= g_{i+1} \circ \cdots \circ g_n, \quad \forall i = 1, \dots, n-1, \\ g_{i,j} &:= g_{j+1} \circ \cdots \circ g_i \colon X_i \to X_j, \quad \forall i > j \geq 0. \end{aligned}$$

For simplicity, in the sequel we still denote by E_j the divisor $(g_{i,j}^{-1})_*E_j$ on X_i. Recall that we denote by $n \leq a-1$ the number of steps of the construction, which stops when $Z_n = E_n$ is a prime divisor. We also denote by $m \leq n$ the largest integer such that $\dim Z_{m-1} = \dim Z$. Using the results recalled in §4, we see that we may assume that the tower construction (5.1) is G-equivariant.

Remark 5.1. Since E and E_n define the same valuation on X, we have:

$$f_*\mathcal{O}_Y(-iE) \cong h_*\mathcal{O}_{X_n}(-iE_n) \quad \forall i$$

(see also [Kaw01, Remark 3.4]). By the same argument as above, it is easy to see that for any i, the possible contribution of E_{i+1} to the total transforms of the birational images of the divisors $E_{i-1}, \dots, E_1$, introduced in Equation (5.3) have multiplicity at most one. Proposition 5.3 below says in particular that there is no such contribution during the process exactly when $f_*\mathcal{O}_Y(-2E) \neq \mathcal{I}_Z$, or equivalently $h_*\mathcal{O}_{X_n}(-2E_n) \neq \mathcal{I}_Z$.

5.2. Characterisation of the tower as a weighted blow-up

Recall that if $f\colon Y \to X$ is a divisorial contraction with exceptional divisor E, the local ring $\mathcal{O}_{Y,E}$ is a discrete valuation ring. The basic observation to interpret the tower construction as a weighted blow-up is the comparison of valuations (see §4.2).

Lemma 5.2 ([Kaw01, Lemma 3.4]). *Two divisorial contractions over X whose exceptional divisors define the same valuation are isomorphic.*

This means that, to prove that the divisorial contraction $f\colon Y \to X$ of Theorem 1.2 is a G-equivariant weighted blow-up, it is enough to show that the exceptional divisor E_n obtained by the tower construction in §5.1 defines the same valuation as those of a G-equivariant weighted blow-up over X. To do this, one of the key results in Kawakita [Kaw01] is a characterisation of weighted blow-ups of smooth points in a threefold. We prove a similar result as [Kaw01, Proposition 3.6] in our setup.

Proposition 5.3. *Let $f : Y \to X$ be a d-dimensional G-equivariant divisorial contraction to a $(d-3)$-dimensional G-orbit contained in the smooth locus of X. Keeping the same notation as above, the divisor E_n defines the same valuation as those of an exceptional divisor obtained by a G-equivariant weighted blow-up of weights $(n, m, 1)$ if and only if the following conditions hold for every analytic open set U in X:*

(1) $f_*\mathcal{O}_Y(-2E)|_U \neq \mathcal{I}_Z|_U$, *that is* $h_*\mathcal{O}_{X_n}(-2E_n)|_U \neq \mathcal{I}_Z|_U$;

(2) $f_*\mathcal{O}_Y(-nE)|_U \not\subseteq \mathcal{I}_Z^2|_U$, *that is* $h_*\mathcal{O}_{X_n}(-nE_n)|_U \not\subseteq \mathcal{I}_Z^2|_U$.

Proof. Assume first that f is a G-equivariant weighted blow-up of Z with weights $(n, m, 1)$. Let U be an analytic open set of X, meeting Z. Since Z is smooth and contained in the smooth locus of X, shrinking U if necessary, we may assume that there are local coordinates $x_1, \ldots, x_d$ on U such that $Z \cap U$ is given by the equations:

$$x_1 = x_2 = x_3 = 0,$$

with the property that for any $k \geq 0$ we have:

$$f_*\mathcal{O}_Y(-kE)|_U = \mathcal{I}_k|_U, \tag{5.4}$$

where $\mathcal{I}_k$ is the ideal sheaf generated over U by the monomials $x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with $na_1 + ma_2 + a_3 \geq k$ (see §4.2). Taking $k = 2$, we see that $x_3 \notin \mathcal{I}_2|_U$, so $\mathcal{I}_2|_U \neq \mathcal{I}_Z|_U$: this is Condition (1). Taking $k = n$, we see that $x_1 \in \mathcal{I}_n|_U$, but $x_1 \notin \mathcal{I}_Z^2|_U$, so $\mathcal{I}_n|_U \not\subseteq \mathcal{I}_Z^2|_U$: this is Condition (2).

Conversely, assume now that Conditions (1) and (2) are satisfied. Let U be an analytic open subset of X, meeting Z. We may assume that U is nonsingular. We have $f_*\mathcal{O}_Y(-2E)|_U \subseteq \mathcal{I}_Z|_U$. During the tower construction, if there were a contribution of some divisor E_{i+1} to the total transforms of the birational images of the divisors $E_{i-1}, \ldots, E_1$, that is, if there were an index $j \leq i$ such that $Z_j \subseteq E_{i+1}$, then at the end of the process, the multiplicity of E_n in $h_1^*E_1$ would be greater of equal than two (see Equation (5.3)). This would mean that for any $\varphi \in \mathcal{I}_Z|_U$, the function $\varphi \circ h$ vanishes at least twice along E_n, so:

$$\mathcal{I}_Z|_U \subset h_*\mathcal{O}_{X_n}(-2E_n)|_U = f_*\mathcal{O}_Y(-2E)|_U.$$

This would contradict Condition (1), so we deduce that there is no such contribution and we have the formulas:

$$g_{i,j}^*E_j = \sum_{k=j}^{i} E_k + \text{other components}, \quad 0 \leq j < i \leq n. \tag{5.5}$$

The sheaf $\left(\mathcal{I}_Z/\mathcal{I}_Z^2\right)|_U$ is locally free of rank 3 on $U \cap Z$ and is locally generated by three coordinate functions of U (see [Har77, Theorem 8.17]). We have:

$$h_*\mathcal{O}_{X_n}(-nE_n)|_U \subseteq h_*\mathcal{O}_{X_n}(-E_n)|_U \subseteq \mathcal{I}_Z|_U,$$

and by Condition (2), we can choose an element $x_1 \in \left(\mathcal{I}_Z \setminus \mathcal{I}_Z\right)|_U$ such that $h_n^* \operatorname{div}(x_1) \geq nE_n$. The function x_1 has multiplicity one along $Z \cap U$, this means that, shrinking U if necessary, we may assume that x_1 is part of a coordinate system of $Z \cap U$. At step n of the tower construction, the process ends up when the centre Z_n is the prime divisor E_n in X_n. We take a coordinate $x_3 \in \left(\mathcal{I}_Z/\mathcal{I}_Z^2\right)|_U$ such that $x_3 \circ h$ vanishes with multiplicity one along each E_i, shrinking again U if necessary. At step m of the construction, the centre Z_m has codimension at most two, so, shrinking once more U we choose a third coordinate function x_2 such that until step m, the local equations of Z_i are the strict transforms of x_1 and x_2 and a local equation of E_i, that is:

$$\sum_{i=1}^{n} E_i = \operatorname{div}(x_3 \circ h),$$

$$Z_i = E_i \cap \operatorname{div}(x_1) \cap \operatorname{div}(x_2), \quad \forall i = 1, \ldots, m-1,$$

$$Z_i = E_i \cap \operatorname{div}(x_1), \quad \forall i = m, \ldots, n-1,$$

where, as explained above, we still denote by D, instead of $(g_{i,0}^{-1})_*D$, the strict transform in X_j of a divisor D in X_0. We finally take complementary coordinate functions $x_4,\dots,x_d$ on U. Denote by v_{E_n} the valuation associated to the discrete valuation ring $\mathcal{O}_{X_n,E_n}$. Locally on U, it is characterized by its value on the coordinate functions $x_1,\dots,x_d$. Since x_1 vanishes along Z_i for all $i=1,\dots,n$, using Equation (5.5) we have by induction:

$$g_{i,0}^*\operatorname{div}(x_1)=\operatorname{div}(x_1)+\sum_{j=1}^{i} jE_j+\text{other components},\quad \forall i=1,\dots,n.$$

In particular, for $i=n$ we get $v_{E_n}(x_1)=n$. Similarly, since x_2 vanishes along Z_i only for $i=1,\dots,m-1$, we compute:

$$\begin{aligned}
g_{m,0}^*\operatorname{div}(x_2)&=\operatorname{div}(x_2)+\sum_{j=1}^{m} jE_j+\text{other components},\\
g_{m+1,0}^*\operatorname{div}(x_2)&=\operatorname{div}(x_2)+\sum_{j=1}^{m} jE_j+mE_{m+1}+\text{other components},\\
g_{n,0}^*\operatorname{div}(x_2)&=\operatorname{div}(x_2)+\sum_{j=1}^{m} jE_j+m(E_{m+1}+\cdots+E_n)+\text{other components}.
\end{aligned}$$

This shows that $v_{E_n}(x_2)=m$. Clearly $v_{E_n}(x_3)=1$ and $v_{E_n}(x_i)=0$ for $i\geq 4$, so locally over U, E_n defines the same valuation as a weighted blow-up with weights $(n,m,1)$. Since the divisorial contraction f is G-equivariant, and since Z is a G-orbit, a local analysis at another analytic neighbourhood of Z can be performed by translating the above local coordinate functions by the action of G, so the weighted blow-up is given by the same weights everywhere and f is a G-equivariant weighted blow-up with weights $(n,m,1)$. □

Remark 5.4. If there is no contribution of E_{i+1} to the total transform of the birational images of the divisors $E_{i-1},\dots,E_1$ in Equation (5.3), for any i, then Equation (5.2) shows that the discrepancy function $i\mapsto \mathrm{a}(E_i,X)$ increases by 2 until $i=m$ and by 1 after, so:

$$a=\mathrm{a}(E,X)=2m+(n-m)=n+m.$$

This can be also directly computed from Equations (4.1) of a weighted blow-up.

6. Proof of Theorem 1.2

The proof of Theorem 1.2 uses an induction argument by considering a general hyperplane section of a locally $\mathbb{Q}$-factorial variety. The next three lemmata contain the necessary ingredients for this induction. For any variety X, we denote by $\mathrm{Cl}(X)$ the group of linear equivalence classes $[D]$ of Weil divisors D on X.

Lemma 6.1. *Let X be a normal, locally $\mathbb{Q}$-factorial and projective variety of dimension at least four, with terminal singularities. Let L be a base point free line bundle on X.*

(1) *A general element $H\in|L|$ is normal with terminal singularities.*
(2) *If moreover L is ample, then H is also locally $\mathbb{Q}$-factorial.*

Proof. By the Bertini theorem for irreductibility [Jou83], the Seidenberg theorem [Sei50] and the generalized Seidenberg theorem [BS95, Theorem 1.7.1] and its proof, a general element $H\in|L|$ is an irreducible Cartier divisor of X, which is normal with terminal singularities in codimension at least three, proving (1). As for (2), by the Grothendieck–Lefschetz theorem for normal varieties, proven by Ravindra–Srinivas [RS06, Theorem 1], for H general the restriction map $\mathrm{Cl}(X)\to\mathrm{Cl}(H)$ defined by $[D]\mapsto[D\cap H]$ is an isomorphism, so H is locally $\mathbb{Q}$-factorial. □

The following lemma is a consequence of [RS06, Theorem 2].

Lemma 6.2. *Let W be a normal, irreducible, projective variety of dimension greater or equal to four, L a big line bundle on W and $V \subseteq \mathrm{H}^0(W,L)$ linear subspace giving a base point free linear system $|V|$ on W. Let $\varphi\colon W \to \mathbb{P}(V^*)$ be the morphism determined by V and consider the Stein factorization of φ:*

$$\varphi\colon W \xrightarrow{\pi} W' \longrightarrow \mathbb{P}(V^*).$$

Then there exists a Zariski open subset of divisors $Z \in |V|$ such that the cokernel of the restriction map:

$$\rho\colon \operatorname{Cl}(W) \to \operatorname{Cl}(Z),$$

is generated by the classes of $\mathcal{O}_Z(E)$, where E is supported on $\operatorname{Exc}(\pi) \cap Z$.

Proof. Let $\varepsilon\colon \widehat{W} \to W$ be a resolution of singularities given by a sequence of blow-ups $\varepsilon = \varepsilon_s \circ \cdots \circ \varepsilon_1$. Since Z is general, the morphism ε induces a resolution of singularities $\varepsilon|_{\widehat{Z}}\colon \widehat{Z} \to Z$ such that no centre of a blow-up ε_i is contained in the strict transform of Z. Let $[F] \in \operatorname{Cl}(Z)$ and let $\widehat{F}$ be its strict transform in $\widehat{Z}$. By [RS06, Theorem 2] there are a divisor $\widehat{D}$ on $\widehat{W}$ and divisors F_i contained in $\operatorname{Exc}(\pi \circ \varepsilon|_{\widehat{Z}})$ such that:

$$[\widehat{F}] = [\widehat{D}|_{\widehat{Z}}] + \sum_i m_i [F_i].$$

Then $[F] = \varepsilon_*[\widehat{F}] = \varepsilon_*[\widehat{D}]|_Z + \sum_i m_i \varepsilon_*[F_i]$, so the cokernel of the restriction map ρ is generated by the classes of exceptional divisors of π. □

Lemma 6.3. *Let $f\colon Y \to X$ be a G-equivariant divisorial contraction to a smooth G-simply connected G-orbit Z. Then general fibers of f over Z are irreducible.*

Proof. Denote by $\nu\colon \widetilde{E} \to E$ the normalization of E. The morphism ν is birational and finite and the action of G on E naturally extends to $\widetilde{E}$, so ν is G-equivariant. Consider the restricted morphism $f_{|E}\colon E \to Z$ and the Stein factorization of the morphism $f_{|E} \circ \nu\colon \widetilde{E} \to Z$:

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\;f_{|E}\circ\nu\;} & Z \\ & {\scriptstyle \nu'} \searrow \quad \nearrow {\scriptstyle \eta} & \\ & Z' & \end{array}$$

By construction, the fibration ν' and the finite morphism η are G-equivariant. Since η is finite, surjective and G-equivariant, it is étale, otherwise it would be ramified everywhere since G acts transitively on Z. Since Z is G-simply connected, η is an isomorphism, so $f_{|E} \circ \nu = \nu'$ has connected fibers. Since $\widetilde{E}$ is normal, by the generalized Seidenberg theorem [BS95, Theorem 1.7.1], a general fiber of $f_{|E} \circ \nu = \nu'$ is normal, hence irreducible since it is connected. It follows that a general fiber of $f_{|E}$ is irreducible. □

Proof of Theorem 1.2. Let $f\colon Y \to X$ be a G-equivariant divisorial contraction. We assume that its exceptional prime divisor is contracted to a smooth $(d-3)$-dimensional centre Z contained in the smooth locus of X, and that Z is a G-simply connected G-orbit. If $d = 3$, the result follows from Kawakita's theorem (see Theorem 1.1) so we assume that $d \geq 4$. By Lemma 6.1(1), a general complete intersection of $(d-3)$ hyperplanes of X, that we denote by:

$$H := H_1 \cap \cdots \cap H_{d-3}$$

is such that H and $\widetilde{H} := f^{-1}(H)$ are reduced, irreducible, normal with terminal singularities. Moreover, by Lemma 6.1(2) the variety H is locally $\mathbb{Q}$-factorial. We claim:

Claim 6.4. *The variety $\widetilde{H}$ is locally $\mathbb{Q}$-factorial.*

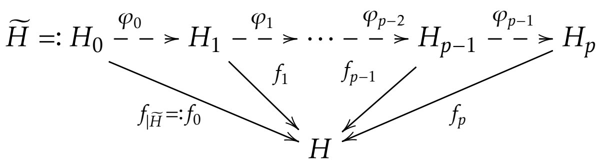
Assuming the claim, we finish the proof. Since the centre Z of f has codimension three in X, for H general the intersection $H \cap Z$ is reduced and consists of a finite number of points, so the restricted morphism $f_{|\widetilde{H}}\colon \widetilde{H} \to H$ is birational and its exceptional locus is $E \cap \widetilde{H}$. By Lemma 6.3, the fibers of $f_{|\widetilde{H}}$ over Z are irreducible. Notice that $f^*H_k = \widetilde{H}_k$ for all k where $\widetilde{H}_k$ is the strict transform of H_k in Y.

By the adjunction formula we have:

$$K_{\widetilde{H}} \equiv \left(K_Y + \widetilde{H}_1 + \cdots + \widetilde{H}_{d-3}\right)\Big|_{\widetilde{H}} \equiv (K_Y + f^*H_1 + \cdots + f^*H_{d-3})|_{\widetilde{H}}.$$

Since K_Y is f-antiample by assumption, we deduce that $K_{\widetilde{H}}$ is $f_{|\widetilde{H}}$-antiample.

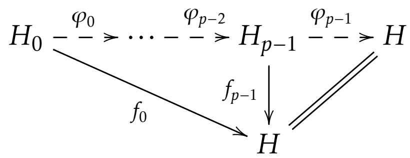
To show that $f_{|\widetilde{H}}$ is a divisorial contraction over every point of $H \cap Z$, we run on $\widetilde{H}$ a $K_{\widetilde{H}}$-MMP with scaling of an ample divisor over H. This MMP terminates by Lemma 3.7 applied to $(\widetilde{H}, 0)$:



In this diagram, each dotted arrow φ_i is either a divisorial contraction or a flip. Since f_0 is birational, all structural morphisms f_i are birational too. The variety H_p is a minimal model relatively to H, so it is normal, locally $\mathbb{Q}$-factorial, with terminal singularities, and its canonical divisor is f_p-nef. Assume that f_p is not an isomorphism. Since H has terminal singularities, there exist effective divisors F_i and positive rational numbers b_i such that:

$$K_{H_p} \equiv f_p^* K_H + \sum_i b_i F_i.$$

putting $D := \sum_i b_i F_i$, we see that D is f_p-nef, so by the negativity lemma [KM98, Lemma 3.39], $(-D)$ is effective, this is a contradiction. So f_p is an isomorphism and we may assume that the last step of the MMP is:



It follows that the birational map φ_{p-1} is everywhere defined, so it is a divisorial contraction, with exceptional divisor denoted $D_{p-1} \subset H_{p-1}$. Its strict transform $\widetilde{D}_{p-1} \subset \widetilde{H}$ is an irreducible component of the exceptional locus $E \cap \widetilde{H}$, so by Lemma 6.3 it is one of its connected components. This means that, locally over H in a neighbourhood of $\varphi_{p-1}(D_{p-1})$ for the Zariski topology, the morphism $f_{|\widetilde{H}}$ looks like a divisorial contraction. More precisely, let j be the maximum integer such that the exceptional locus of φ_j^{-1} is not disjoint from D_{p-1}. Then φ_j cannot be a flip, otherwise D_{p-1} would contain a $K_{H_{p-1}}$-positive curve, which is impossible since $K_{H_{p-1}}$ is f_{p-1}-negative. Moreover φ_j cannot be a divisorial contraction, otherwise there would be at least two irreducible components in $f_{|\widetilde{H}}^{-1}(\varphi_{p-1}(D_{p-1}))$. Therefore the exceptional locus of φ_j^{-1} is disjoint from D_{p-1} for all $j \leq p-2$. In particular, f_{p-1} commutes with φ_j for all j and we can repeat the argument to show that all φ_i are divisorial contractions, whose exceptional divisors are the connected components of $E \cap \widetilde{H}$. This shows that $f_{|\widetilde{H}}\colon \widetilde{H} \to H$ is a composition of divisorial contractions with disjoint exceptional loci.

To conclude, we prove that f is a G-equivariant weighted blow-up by proving that it satisfies Conditions (1) and (2) of Proposition 5.3. Indeed, assume by contradiction that (1) or (2) is not satisfied. Then there is a point $p \in Z$ such that:

$$f_*\mathcal{O}_Y(-2E) \otimes \mathcal{O}_p = \mathcal{I}_Z \otimes \mathcal{O}_p, \tag{6.1}$$

$$f_*\mathcal{O}_Y(-nE) \otimes \mathcal{O}_p \subseteq \mathcal{I}_Z^2 \otimes \mathcal{O}_p. \tag{6.2}$$

(6.1) either; (6.2) or.

Translating H using the action of G if necessary, we may assume that $p \in H$. Then locally around p, by Theorem 1.1, the divisorial contraction $f|_{\widetilde{H}}$ is a weighted blow-up. So by Proposition 5.3 applied to $f|_{\widetilde{H}}$, we have:

$$\left(f|_{\widetilde{H}}\right)_* \mathcal{O}_{\widetilde{H}}\left(-2(E \cap \widetilde{H})\right) \neq \mathcal{I}_p,$$
$$\text{and } \left(f|_{\widetilde{H}}\right)_* \mathcal{O}_{\widetilde{H}}\left(-n(E \cap \widetilde{H})\right) \not\subseteq \mathcal{I}_p^2.$$

Tensoring Formulas (6.1) and (6.2) by $\mathcal{O}_H$, we get:

$$\text{either } \left(f|_{\widetilde{H}}\right)_* \mathcal{O}_{\widetilde{H}}\left(-2(E \cap \widetilde{H})\right) = \mathcal{I}_Z \otimes \mathcal{O}_p \otimes \mathcal{O}_H,$$
$$\text{or } \left(f|_{\widetilde{H}}\right)_* \mathcal{O}_{\widetilde{H}}\left(-n(E \cap \widetilde{H})\right) \subseteq \mathcal{I}_Z^2 \otimes \mathcal{O}_p \otimes \mathcal{O}_H.$$

But by generality of H, we have $\mathcal{I}_Z \otimes \mathcal{O}_p \otimes \mathcal{O}_H = \mathcal{I}_p$: this gives a contradiction, so f is a G-equivariant weighted blow-up. We are left with the proof of Claim 6.4. We prove by induction on k that $\widetilde{H}_1 \cap \cdots \cap \widetilde{H}_k$ is locally $\mathbb{Q}$-factorial for any $k \geq 0$. For $k = 0$ there is nothing to prove, so assume that $k > 0$. The divisor:

$$L := \widetilde{H}_k|_{\widetilde{H}_1 \cap \cdots \cap \widetilde{H}_{k-1}}$$

is big and base point free. We apply Lemma 6.2 with:

$$W := \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_{k-1}, \qquad |V| := |L|,$$
$$Z := \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_k, \qquad \pi = f|_W \colon W \to f(W).$$

We get that for any $[D] \in \mathrm{Cl}(Z)$, either $[D]$ is in the image of the restriction map $\rho\colon \mathrm{Cl}(W) \to \mathrm{Cl}(Z)$, and therefore is $\mathbb{Q}$-Cartier by induction, or, modulo an element of the image of the restriction map, we can assume that D is supported on the exceptional locus of the restriction of π. So we have to prove that the irreducible components of the exceptional locus of π are $\mathbb{Q}$-Cartier divisors. Denote by $(E_i)_{i=1,\dots,s}$ the irreducible components of $E \cap \widetilde{Z}$. By Lemma 6.3, these are also the connected components of $E \cap Z$ (the only case where the exceptional locus is not connected in general is for $k = d - 3$). We thus have a decomposition into connected irreducible components:

$$E \cap Z = E \cap \widetilde{H}_1 \cap \cdots \cap \widetilde{H}_k = E_1 \cup \cdots \cup E_s.$$

Then each E_i is $\mathbb{Q}$-Cartier, since it is the intersection of the $\mathbb{Q}$-Cartier divisor E with:

$$Z \setminus (E_1 \cup \cdots \cup E_{i-1} \cup E_{i+1} \cup \cdots \cup E_s).$$

□

Remark 6.5. ***A posteriori***, our induction argument, reducing to Kawakita's theorem 1.1 shows that the arithmetic between the integer parameters in issue still holds in the setup of Theorem 1.2:

(1) The integers n and m are coprime (see [Kaw01, Theorem 3.5]);

(2) The integers a is prime to the index of K_Y (see [Kaw01, Lemma 4.3]);

(3) $a \geq 2$ (see [Kaw01, §5, p.116]).

7. A counter-example

In this section, to emphasize the importance of the group action in the statement of Theorem 1.2, we construct a non equivariant 4-dimensional divisorial contraction which is not a weighted blow-up. We use similar notation as in the tower construction in Section 5.

Let $X_0 := X$ be a smooth projective variety of dimension 4 and $Z_0 := Z \subset X$ be a smooth curve.

(1) We denote by $g_1\colon X_1 \to X$ the blow-up of Z, with exceptional divisor E_1. Let $Z_1 \subset X_1$ be a section of the $\mathbb{P}^2$-bundle $g_1|_{E_1} : E_1 \to Z$.

(2) We denote by $g_2\colon X_2 \to X_1$ the blow-up of Z_1, with exceptional divisor E_2. Again $g_1 \circ g_2|_{E_2} : E_2 \to Z$ is a $\mathbb{P}^2$-bundle. We denote by E_1^2 the strict transform of E_1 in X_2. Observe that $g_1 \circ g_2|_{E_1^2} : E_1^2 \to Z$ is an $\mathbb{F}_1$-bundle. The natural fibration $\mathbb{F}_1 \to \mathbb{P}^1$ defines a $\mathbb{P}^1$-bundle structure $\pi\colon E_1^2 \to S$ over a surface S.

(3) Let $Z_2 \subsetneq E_2$ be a surface such that the intersection of Z_2 with a general fibre of $g_1 \circ g_2|_{E_2} : E_2 \to Z$ is a line and the intersection of Z_2 with E_1^2 is of the form $s_2 \cup h_1 \cup \ldots \cup h_k$ where s_2 is a section of $g_1 \circ g_2|_{E_1^2} : E_1^2 \to Z$, and $h_i = (g_1 \circ g_2)^{-1}(p_i) \cap E_1^2 \cap E_2$ for some point $p_i \in Z$. We can further assume that h_i and s_2 meet transversally for all i. We denote by $g_3\colon X_3 \to X_2$ the blow up of Z_2, with exceptional divisor E_3. We denote by E_1^3 the strict transform of E_1^2 in X_3.

The construction is summarized in Diagram 7.1 and is illustrated in Figure 2 (Appendix). When there is no risk of confusion, we simply denote by E_j, instead of E_j^i, the strict transform of E_j in X_i for $i = 2, 3$ and $j = 1, 2$.

Proposition 7.1. *The pair (X_3, E_1) admits a log resolution $\mu\colon \widetilde{X} \to X_3$ such that:*

$$K_{\widetilde{X}} + (\mu^{-1})_* E_1 = \mu^*(K_{X_3} + E_1) + \sum_i F_i,$$

with $\cup_i F_i = \mathrm{Exc}(\mu)$. *The pair (X_3, tE_1) is klt for any $t \in \,]0,1[$.*

Proof. The singularities of E_1^3 lie over the singular points of $s_2 \cup h_1 \cup \ldots \cup h_k$. To understand them, we can choose local analytic coordinates (x, y, z, w) on X_2 such that Z_2 is locally given by $\{w = z = 0\}$ and E_1^2 by $\{xy - w = 0\}$. The blow up of X_2 along Z_2 is then locally given by:

$$\{((x, y, z, w), [s : t]) \in \mathbb{C}^4 \times \mathbb{P}^1 \mid zt - ws = 0\}.$$

We compute that the variety E_1^3 has one singular point in the chart $s \neq 0$, with equation $zv - xy$, where $v = t/s$. It is smooth on the other chart. A desingularization of E_1^3 is given by the blow up $\mu\colon \widetilde{X} \to X_3$ of the singular points of E_1^3. Since E_1^3 has isolated singularities of type A_1, a local computation shows that the exceptional divisors F_i of μ meet the strict transform $(\mu^{-1})_* E_1^3$ transversally, so μ is a log resolution and we have:

$$\mu^* E_1^3 = (\mu^{-1})_* E_1^3 + 2 \sum_i F_i,$$

$$K_{\widetilde{X}} = \mu^* K_{X_3} + 3 \sum_i F_i.$$

It follows that:

$$K_{\widetilde{X}} + (\mu^{-1})_* E_1^3 = \mu^*(K_{X_3} + E_1^3) + \sum_i F_i,$$

and that for any $t \in \,]0,1[$ we have:

$$K_{\widetilde{X}} + (\mu^{-1})_*(tE_1^3) = \mu^*(K_{X_3} + tE_1^3) + (3 - 2t) \sum_i F_i,$$

proving that the pair (X_3, E_1) is *klt* . □

Over a general point of Z, the fiber of E_1^3 is isomorphic to the blow-up of $\mathbb{F}_1$ at a point not belonging to a (-1)-curve. Over the points p_i, the fiber of E_1^3 is the union of the blow-up of $\mathbb{F}_1$ at a point and a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Consider the composition $\pi'\colon E_1^3 \to E_1^2 \xrightarrow{\pi} S$. We denote by ℓ_1' the strict transform of a fibre of π passing through s_2 and by ℓ_1'' the strict transform by π' of a fibre of π passing through h_i.

Lemma 7.2. *The curves ℓ_1' and ℓ_1'' are numerically equivalent and generate an extremal ray of the cone $\overline{\mathrm{NE}}(X_3/X)$. More precisely, we have:*

$$\begin{aligned} \ell_1' \cdot E_1 &= \ell_1'' \cdot E_1 = -2, \\ \ell_1' \cdot E_2 &= \ell_1'' \cdot E_2 = 0, \\ \ell_1' \cdot E_3 &= \ell_1'' \cdot E_3 = 1, \\ \ell_1' \cdot K_{X_3} &= \ell_1'' \cdot K_{X_3} = 1. \end{aligned}$$

Proof. To prove the numerical equivalence, it is enough to prove that:

$$\ell_1' \cdot E_i = \ell_1'' \cdot E_i \text{ for } i = 1, 2, 3.$$

- We have $\ell_1' \cdot E_2 = \ell_1'' \cdot E_2 = 0$ as both ℓ_1' and ℓ_1'' are disjoint from E_2.
- We have $\ell_1' \cdot E_3 = \ell_1'' \cdot E_3 = 1$ as both ℓ_1' and ℓ_1'' meet E_3 transversally.
- We have $g_2^*E_1 = E_1^2 + E_2$, $g_3^*E_1^2 = E_1^3$, and $g_3^*E_2^2 = E_2^3 + E_3$. Set $\bar{\ell}_1' = g_{3*}\ell_1'$ and $\bar{\ell}_1'' = g_{3*}\ell_1''$. The curves $\bar{\ell}_1'$ and $\bar{\ell}_1''$ are numerically equivalent as they are both fibres of $\pi\colon E_1^2 \to S$. We get the numerical equivalence by computing as follows:
$$E_1^3 \cdot \ell_1' = g_3^*E_1^2 \cdot \ell_1' = E_1^2 \cdot \bar{\ell}_1' = E_1^2 \cdot \bar{\ell}_1'' = g_3^*E_1^2 \cdot \ell_1'' = E_1^3 \cdot \ell_1''.$$
Moreover, $E_1^2 \cdot \bar{\ell}_1' = (\varepsilon_2^*E_1 - E_2) \cdot \bar{\ell}_1' = -2$. Using the formulas:
$$K_{X_1} = g_1^*K_X + 2E_1, \quad K_{X_2} = g_2^*K_{X_1} + 2E_2, \quad K_{X_3} = g_3^*K_{X_2} + E_3,$$
we get $K_{X_3} = (g_1g_2g_3)^*K_X + 2E_1 + 4E_2 + 5E_3$, hence $K_{X_3} \cdot \ell_1' = 1$.

Let us show that the ray $\mathbb{R}_+[\ell_1']$ is extremal. Assume $\ell_1' = C_1 + C_2$ in $\overline{\mathrm{NE}}(X_3/Z)$. We have to show that $C_1, C_2 \in \mathbb{R}_+[\ell_1']$. As $\ell_1' \cdot E_1 = -2$, we can assume that every component of C_2 is linearly equivalent to a component not in E_1 and all the components of C_1 are in E_1. We have

$$\overline{\mathrm{NE}}(E_1/Z) = \mathbb{R}_+[\ell_1'] + \mathbb{R}_+[\ell_1''] + \mathbb{R}_+[\ell_2] + \mathbb{R}_+[\ell_3],$$

therefore:

$$\begin{aligned} C_1 &\equiv a\ell_1' + b\ell_1'' + c\ell_2 + d\ell_3 \quad \text{and} \\ \ell_1' &\equiv a\ell_1' + b\ell_1'' + c\ell_2 + d\ell_3 + C_2, \end{aligned}$$

where the first numerical equivalence is in E_1 and the second in X_3. We already proved that $\ell_1' \equiv \ell_1''$, hence we get:

$$(1 - a - b)\ell_1' \equiv c\ell_2 + d\ell_3 + C_2.$$

As the right hand side is an effective curve, we have $1 - a - b \geq 0$. By taking the intersection product with E_1, we get $-2(1 - a - b) = c + C_2 \cdot E_1 \geq 0$. Therefore $a + b = 1$ and $c\ell_2 + d\ell_3 + C_2 \equiv 0$. This proves that $\mathbb{R}_+[\ell_1']$ is an extremal ray. The proof for $\mathbb{R}_+[\ell_1'']$ is similar. □

Lemma 7.3. *The antiflip $\phi\colon X_3 \dashrightarrow Y_3$ of the ray $\mathbb{R}_+[\ell_1']$ exists. The variety Y_3 has terminal singularities.*

Proof. In order to prove that the antiflip exists, it is enough to find a *klt* boundary Δ such that $(K_{X_3}+\Delta)\cdot\ell_1' < 0$. We set $\Delta = tE_1$ with $1/2 < t < 1$. The pair (X_3, Δ) is *klt* by Proposition 7.1, and using Lemma 7.2 we get:

$$(K_{X_3} + tE_1) \cdot \ell_1' = 1 - 2t < 0.$$

Then the $(K_{X_3} + tE_1)$-flip of ℓ_1' exists by [HM10, Corollary 1.4.1], we denote it by $\phi\colon X_3 \dashrightarrow Y_3$. By Proposition 7.1 the pair (X_3, Δ) is terminal in the sense of [KM98, Definition 2.34], so by [KM98, Corollary 3.42] the pair (Y_3, Δ^+) is also terminal. By [KM98, Corollary 2.35] the variety Y_3 has terminal singularities. □

We notice that ϕ is an isomorphism in a neighborhood of E_2. We denote by $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$ the strict transforms of E_1, E_2, E_3 in Y_3. In the next lemmata, we construct extremal contractions $Y_3 \to Y_2 \to Y_1$.

Lemma 7.4. *Let ℓ_2 be the strict transform in $\overline{E}_2$ of a line in a fibre of $E_2 \to Z$. The extremal ray $\mathbb{R}_+[\ell_2]$ is K_{Y_3}-negative in $\overline{\mathrm{NE}}(Y_3/X)$. We denote by $\eta_3\colon Y_3 \to Y_2$ the contraction of $\mathbb{R}_+[\ell_2]$. We have:*

$$K_{Y_3} = \eta_3^* K_{Y_2} + \frac{1}{2}\overline{E}_2.$$

Proof. As E_2 is disjoint from the indeterminacy locus of ϕ, we have $K_{Y_3} \cdot \ell_2 = K_{X_3} \cdot \ell_2 = -1$. Writing $K_{Y_3} = \eta_3^* K_{Y_2} + a\overline{E}_2$, we have $a = \frac{1}{2}$ since:

$$-1 = K_{Y_3} \cdot \ell_2 = \eta_3^* K_{Y_2} \cdot \ell_2 + a\overline{E}_2 \cdot \ell_2 = -2a.$$

□

Lemma 7.5. *Let $\ell_1 \subset Y_2$ be the strict transform in $\overline{E}_1$ of a fibre of $\pi\colon E_1 \to Z$ not passing through $s_2 \cup h_1 \cup \ldots \cup h_k$. Then $\mathbb{R}_+[\ell_1]$ is a K_{Y_2}-negative extremal ray in $\overline{\mathrm{NE}}(Y_2/X)$.*

Proof. By abuse of notation we still denote by ℓ_1 the strict transform of ℓ_1 in Y_3. By Lemma 7.4, we obtain:

$$K_{Y_2} \cdot \ell_1 = K_{Y_3} \cdot \ell_1 - \frac{1}{2}\overline{E}_2 \cdot \ell_1 = K_{X_3} \cdot \ell_1 - \frac{1}{2}E_2 \cdot \ell_1 = -\frac{1}{2}.$$

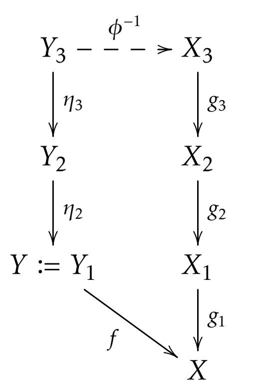
□

We denote by $\eta_2\colon Y_2 \to Y_1 =: Y$ the contraction of $\mathbb{R}_+[\ell_1]$

Proposition 7.6. *There is a divisorial contraction $f\colon \overline{X} \to X$ with codimension three center contained in the smooth locus. The divisorial contraction f is not a weighted blow-up.*

Proof. Starting from the variety Y_3 obtained in Lemma 7.3 by flipping over X the extremal ray $\mathbb{R}_+[l_1']$, we contract the divisor $\overline{E}_2$ by Lemma 7.4, obtaining the variety Y_2. By Lemma 7.5 we can contract the divisor $\overline{E}_1$ over X, obtaining a variety Y with a birational morphism $f\colon Y \to X$. The variety Y has terminal singularities by Lemma 7.3 and because it is obtained by contracting K-negative rays of the Mori cone. Furthermore $\rho(Y) = \rho(X_3) - 2 = \rho(X) + 1$ so f is a divisorial contraction. We set E the exceptional divisor of f, which is the strict transform of $\overline{E}_3$. The construction is summarized in Diagram 7.1.

(7.1)



We prove now that f is not a weighted blow-up. As g_1 and g_2 blow-up curves and g_3 a surface, by the characterization of the tower construction given in Section 5 the fibre of f over a general point of Z is the weighted projective plane $\mathbb{P}(3,2,1)$, which has three singular points.

We prove that over the special points there is either a curve of singularities or a unique singular point. From this it follows that f cannot be a weighted blow-up, otherwise all the fibres of the restriction $f: E \to Z$ would be isomorphic.

Let $p \in Z$ be a special point, that is, one of the images of the singularities of $s_2 \cup h_1 \cup \ldots \cup h_k$. Let $H \subseteq X$ be a smooth hyperplane through p and $S \subseteq H$ a smooth surface through p. By choosing H, S general enough, we can assume that the centre Z_i is not contained in the strict transforms of S in X_i and that the strict transform of S in X_3 is smooth. We can find such S as X is smooth. Let S_3 be the strict transform of S in X_3. Let $\bar{g}$ be the restriction of $g_3 \circ g_2 \circ g_1$ to S_3. Then $g^{-1}(p)$ is a chain of curves e_1, e_2, e_3 where e_3 meets e_2 and e_1, E_2 marks on S_3 the (-2)-curve e_2, E_3 the (-1)-curve e_3 and E_1 the union $e_3 \cup e_1$ and e_1 is a (-3)-curve.

Let $\overline{S}_3$ be the strict transform of S_3 via the antiflip and $\lambda\colon S_3 \to \overline{S}$ and $\mu\colon \overline{S}_3 \to \overline{S}$ be the restriction of the antiflip to S_3. Then $\overline{E}_1 \cup \overline{E}_2 \cup \overline{E}_3$ marks on $\overline{S}_3$ a chain of curves $\overline{e}_2, \overline{e}_3, \overline{\ell}_3$ where $\overline{e}_3$ meets $\overline{e}_2$ and $\overline{\ell}_3$, and $\overline{e}_3$ and $\overline{e}_2$ are the strict transforms of e_3 and e_2. From the adjunction formula it follows that $\overline{\ell}_3$ has positive intersection with the canonical bundle of $\overline{S}_3$. Indeed, if we let $\overline{H}_3$ be the strict transform of H in Y_3 by our hypothesis $\overline{H}_3 = \eta_3^* \eta_2^* f^* H$, and we have $K_{\overline{H}_3} = K_{Y_3} + \overline{H}_3$. Moreover, again by the generality of S, $\overline{S}_3$ is the pullback of S via the restriction of $f \circ \eta_3 \circ \eta_2$. By the adjunction formula we have

$$K_{\overline{S}_3} \cdot \overline{\ell}_3 = (K_{\overline{H}_3} + \overline{S}_3) \cdot \overline{\ell}_3 = K_{\overline{H}_3} \cdot \overline{\ell}_3 = K_{Y_3} \cdot \overline{\ell}_3 > 0.$$

Therefore there is a singular point for $\overline{S}_3$ in $\overline{\ell}_3$: otherwise $\overline{S}_3$ would be the minimal desingularisation of the surface obtained by contracting e_1, but this surface is isomorphic to S_3 and the exceptional curve is not K-negative.

Two cases may appear. Either for S general the singular point lies away from $\overline{e}_3$, in which case the fibre of $f: E \to Z$ over p is singular along a curve, or the singular point is the intersection of $\overline{e}_3$ and $\overline{\ell}_3$, and this curve gets contracted to a point via the MMP, in which case the fibre of $f: E \to Z$ over p has just one singular point. □

Appendix. Graphical illustrations of the key geometric constructions of the paper

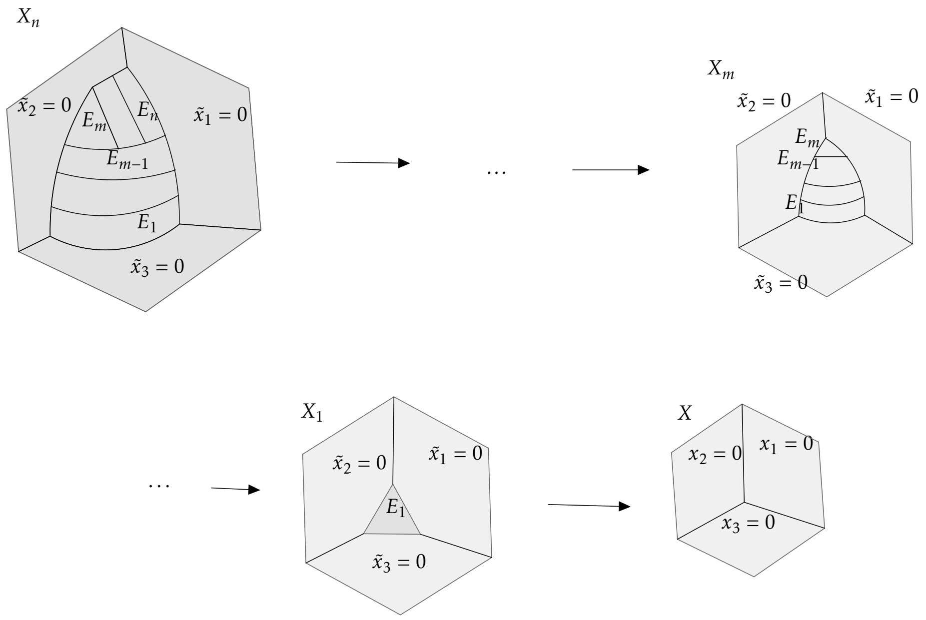


Figure 1. The tower construction

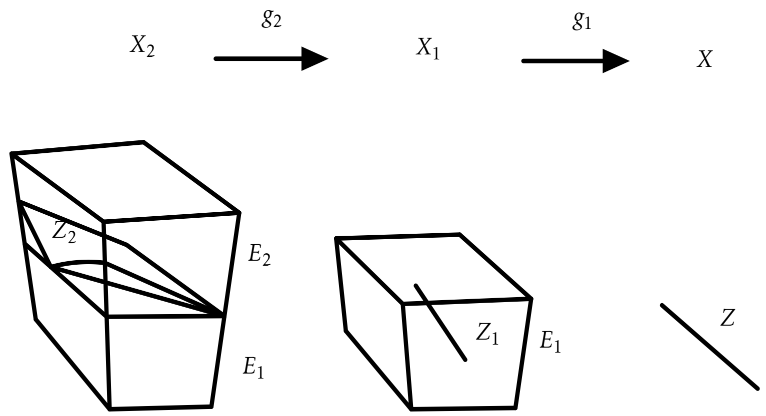


Figure 2. The counter-example

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