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Big Picard theorems and algebraic hyperbolicity for varieties admitting a variation of Hodge structures

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Abstract. In this paper, we study various hyperbolicity properties for a quasi-compact Kähler manifold U which admits a complex polarized variation of Hodge structures so that each fiber of the period map is zero-dimensional. In the first part, we prove that U is algebraically hyperbolic and that the generalized big Picard theorem holds for U. In the second part, we prove that there is a finite étale cover $\tilde{U}$ of U from a quasi-projective manifold $\tilde{U}$ such that any projective compactification X of $\tilde{U}$ is Picard hyperbolic modulo the boundary $X - \tilde{U}$, and any irreducible subvariety of X not contained in $X - \tilde{U}$ is of general type. This result coarsely incorporates previous works by Nadel, Rousseau, Brunebarbe and Cadorel on the hyperbolicity of compactifications of quotients of bounded symmetric domains by torsion-free lattices.

Keywords. Picard hyperbolicity, algebraic hyperbolicity, complex variation of Hodge structures, system of (log) Hodge bundles, Finsler metric, quotients of bounded symmetric domains by torsion-free lattices

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1. Introduction

1.1. Background

The classical big Picard theorem says that any holomorphic map from the punctured disk Δ^* into $\mathbb{P}^1$ which omits three points can be extended to a holomorphic map $\Delta \to \mathbb{P}^1$, where Δ denotes the unit disk. Therefore, we introduce a new notion of hyperbolicity which generalizes the big Picard theorem. We say a complex manifold U is *quasi-compact Kähler* if it is a Zariski open subset of a compact Kähler manifold Y. Such a Y will be called a *smooth Kähler compactification* of U.

Definition 1.1 (Picard hyperbolicity). A quasi-compact Kähler manifold U is called *Picard hyperbolic* if there is a smooth Kähler compactification Y of U such that any holomorphic map $f: \Delta^* \to U$ extends to a holomorphic map $\bar{f}: \Delta \to Y$.

We will prove in Lemma 5.3 that this definition does not depend on the compactification of U. Picard hyperbolic varieties first attracted the author's interest because of the recent interesting work [JK20] by Javanpeykar–Kucharczyk on the *algebraicity of analytic maps*. In [JK20, Definition 1.1], they introduce a new notion of hyperbolicity: a reduced quasi-projective variety U is *Borel hyperbolic* if any holomorphic map from a quasi-projective variety to U is *algebraic*. In [JK20, Corollary 3.11], they prove that a Picard hyperbolic variety is Borel hyperbolic. We refer the readers to [JK20, Section 1] for their motivation on the Borel hyperbolicity. Picard hyperbolic varieties fascinate us further when we realize in Proposition 5.2 that a more general extension theorem is also valid for them: any holomorphic map from $\Delta^p \times (\Delta^*)^q$ to the manifold U in Definition 1.1 extends to a *meromorphic map* from Δ^{p+q} to Y.

By Borel [Bor72] and Kobayashi–Ochiai [KO71], it has long been known to us that the quotients of bounded symmetric domains by torsion-free arithmetic lattices are hyperbolically embedded into their Baily–Borel compactification, and thus they are Picard hyperbolic (see [Kob98, Theorem 6.1.3]). Period domains, first introduced by Griffiths [Gri68a] and later systematically studied by him in the seminal work [Gri68b, Gri70a, Gri70b], are classifying spaces of polarized Hodge structures and are a generalization of bounded symmetric domains. In [JK20, Section 1.1], Javanpeykar–Kucharczyk conjectured that an algebraic variety U which admits an *integral variation of Hodge structures* ($\mathbb{Z}$-VHS for short) with quasi-finite period map is Borel hyperbolic. Their conjecture was recently proved in a remarkable work of Bakker–Brunebarbe–Tsimerman [BBT23]. The proof is based on the tame geometry for quotients $\mathcal{D}/\Gamma$ of period domains $\mathcal{D}$ by arithmetic groups Γ containing the monodromy group of the $\mathbb{Z}$-VHS. However, when one studies Picard hyperbolicity (or Borel hyperbolicity) for varieties admitting the more general *complex polarized variation of Hodge structures* ($\mathbb{C}$-PVHS for short), there are several problems which seem difficult to tackle if one uses o-minimal geometry in [BBT23]:

- The monodromy group Γ might act non-discretely on the period domain $\mathcal{D}$, and thus the quotient of the period domain by the monodromy group $\mathcal{D}/\Gamma$ is even non-Hausdorff.
- The local monodromies at infinity might not be quasi-unipotent, though it always is for a $\mathbb{Z}$-VHS by the theorem of Borel.

Therefore, the great differences between a $\mathbb{Z}$-VHS and a $\mathbb{C}$-PVHS require new ideas if one would like to extend the theorem by Bakker–Brunebarbe–Tsimerman to the context of $\mathbb{C}$-PVHSs. This is one of the main goals in this paper. In the first part, we prove the Picard hyperbolicity of quasi-compact Kähler manifolds admitting a $\mathbb{C}$-PVHS using techniques in complex-analytic geometry and Hodge theory.

1.2. Big Picard theorem and algebraic hyperbolicity

Theorem A. *Let U be a quasi-compact Kähler manifold. Assume that there is a $\mathbb{C}$-PVHS over U so that each fiber of the period map $U_{\rm uni} \to \mathcal{D}$ is zero-dimensional, where $U_{\rm uni}$ is the universal cover of U and $\mathcal{D}$ is the period domain associated to the above $\mathbb{C}$-PVHS. Then U is both algebraically hyperbolic and Picard hyperbolic. In particular, U is Borel hyperbolic.*

Note that we make no assumptions on the local monodromies of the $\mathbb{C}$-PVHS at infinity (which can be non–quasi-unipotent) or on its global monodromy group (thus it might act non-discretely on U). Let us mention that when the $\mathbb{C}$-PVHS over U in Theorem A is moreover a $\mathbb{Z}$-PVHS, the Borel hyperbolicity of U in Theorem A has been proven in [BBT23, Corollary 7.1], and the algebraic hyperbolicity of U is implicitly shown by Javanpeykar–Litt in [JL19, Theorem 4.2] if the local monodromies of the $\mathbb{Z}$-VHS at infinity are unipotent (see Remark 5.5). Our proof of Theorem A is based on complex-analytic and Hodge-theoretic methods, and it does not use the delicate o-minimal geometry in [PS08, PS09, BKT20, BBT23]. Indeed, we are not sure that the tame geometry for period domains of $\mathbb{Z}$-PVHSs in [BKT20, BBT23] can be applied to prove Theorem A.

We can even generalize Theorem A to higher-dimensional domain spaces.

Corollary 1.2 (= Theorem A + Proposition 5.2). *Let U be the quasi-compact Kähler manifold in Theorem A, and let Y be a smooth Kähler compactification of U. Then any holomorphic map $f\colon \Delta^p \times (\Delta^*)^q \to U$ extends to a meromorphic map $\overline{f}\colon \Delta^{p+q} \dashrightarrow Y$. In particular, if W is a Zariski open subset of a compact complex manifold X, then any holomorphic map $g\colon W \to U$ extends to a meromorphic map $\overline{g}\colon X \dashrightarrow Y$.*

1.3. Hyperbolicity for the compactification after a finite étale cover

The second main result of this paper is on the hyperbolicity for the compactification of some finite étale cover of the quasi-compact Kähler manifold U in Theorem A. Let us first introduce several definitions of hyperbolicity. We refer the readers to the recent survey by Javanpeykar [Jav20, Section 8] for more conjectural relations among them.

Definition 1.3 (Notions of hyperbolicity). Let (X, ω) be a compact Kähler manifold, and let $Z \subsetneq X$ be a closed subset of X.

(i) The manifold X is *Kobayashi hyperbolic modulo Z* if the Kobayashi pseudo distance satisfies $d_X(x,y) > 0$ for any distinct points $x, y \in X$ not both contained in Z.
(ii) The manifold X is *Picard hyperbolic modulo Z* if any holomorphic map $\gamma\colon \Delta^* \to X$ whose image is not contained in Z extends across the origin.
(iii) The manifold X is *Brody hyperbolic modulo Z* if any entire curve $\gamma\colon \mathbb{C} \to X$ is contained in Z.

(iv) The manifold X is *algebraically hyperbolic modulo* Z if there is an $\varepsilon > 0$ so that for any compact irreducible curve $C \subset X$ not contained in Z, one has

$$2g(\tilde{C}) - 2 \geq \varepsilon \deg_\omega C,$$

where $g(\tilde{C})$ is the genus of the normalization $\tilde{C}$ of C, and $\deg_\omega C := \int_C \omega$.

Note that Definition 1.3(iv) was first introduced in [JX22]. It is easy to show that if X is Kobayashi hyperbolic modulo Z (resp. Picard hyperbolic modulo Z), then X is Brody hyperbolic modulo Z.

The second main result of the paper is the following theorem.

Theorem B. *Let U be a quasi-compact Kähler manifold. Assume that there is a $\mathbb{C}$-PVHS over U so that each fiber of the period map is zero-dimensional. Then there are a quasi-projective manifold $\tilde{U}$ and a finite étale cover $\tilde{U} \to U$ such that for any projective compactification X of $\tilde{U}$,*

(i) *an irreducible Zariski closed subvariety of X not contained in $\tilde{D} := X - \tilde{U}$ is of general type;*
(ii) *the variety X is Picard hyperbolic modulo $\tilde{D}$;*
(iii) *the variety X is Brody hyperbolic modulo $\tilde{D}$;*
(iv) *the variety X is algebraically hyperbolic modulo $\tilde{D}$.*

By the work of Deligne (see [Mil13, Theorem 7.10]), the quotient of any bounded symmetric domain by a torsion-free lattice always admits a $\mathbb{C}$-PVHS whose period map is *immersive everywhere*. Theorem B then yields the following.

Corollary C. *Let U be the quotient of a bounded symmetric domain by a torsion-free lattice. Then there is a finite étale cover $\tilde{U} \to U$ from a quasi-projective manifold $\tilde{U}$ with any projective compactification X of $\tilde{U}$ Picard and algebraically hyperbolic modulo $X - \tilde{U}$.*

Let us stress here that Nadel [Nad89] and Rousseau [Rou16] proved that the variety X in Corollary C is Brody and Kobayashi hyperbolic modulo $X - \tilde{U}$, and Brunebarbe [Bru20a] and Cadorel [Cad21, Cad22] proved that any Zariski closed subvariety not contained in $X - \tilde{U}$ is of general type. Theorem B thus incorporates their results, but at the cost of loss of effectivity for the level structures (see Remark 6.4) due to the generality of our result in Theorem B.

1.4. Main strategy

1.4.1. Negatively curved Finsler metric.— Let Y be a compact Kähler manifold, and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS over $U := Y - D$. In [Gri70a], Griffiths constructed a metrized line bundle on U whose curvature is semipositive and strictly positive at the points where the period map is immersive. Based on the work by Simpson and Mochizuki, in Proposition 3.5, we can extend this Griffiths line bundle over Y to obtain a more refined positivity result.(1) We then construct a special system of log Hodge bundles $(E, \theta) = (\oplus_{p+q=m} E^{p,q}, \oplus_{p+q=m} \theta_{p,q})$ on the log pair (Y, D) so that some higher-stage E^{p_0,q_0} contains a big line bundle which admits enough *local positivity along* D. Inspired by our previous work [Den22a] on the proof of Viehweg–Zuo's conjecture on Brody hyperbolicity of moduli of polarized manifolds, in Theorem 3.9 we show that (E, θ) still enjoys a "partially" infinitesimal Torelli property. These results enable us to construct a negatively curved and generically positive definite Finsler metric on $T_Y(-\log D)$ in a similar vein as [Den22a].

Theorem 1.4 (= Theorem 3.6 + Theorem 4.6). *Let Y be a compact Kähler manifold, and let D be a simple normal crossing divisor on Y. Assume that there is a $\mathbb{C}$-PVHS over $U := Y - D$ whose period map is immersive at one point. Then there are a* Finsler metric h (*see Definition 4.1*) *on $T_Y(-\log D)$ which is positive definite on a*

(1)If the local monodromy around D is unipotent, this is well known.

dense Zariski open subset U° of $Y-D$ and a smooth Kähler form ω on Y such that for any holomorphic map $\gamma\colon C\to U$ from an open set $C\subset\mathbb{C}$ to U, one has

$$\mathrm{dd^c}\log|\gamma'(t)|_h^2\geq\gamma^*\omega \tag{1.1}$$

when $\gamma(C)\cap U^\circ\neq\varnothing$.

Let us mention that, though we only construct the (possibly degenerate) Finsler metric over $T_Y(-\log D)$, it follows from (1.1) that we know exactly the behavior of its curvature near the boundary D since ω is a smooth Kähler form over Y. The proof of Theorem A is then based on Theorem 1.4 and some criteria for the big Picard theorem established in [DLS+19] (see Theorem 5.4). Let us also mention that the Finsler metric constructed in Theorem 1.4 is also crucially used in the proof of Theorem B.

1.4.2. On the hyperbolicity of the compactification.— The proof of Theorem B is based on Theorem 6.1, whose proof is technically involved. It is worthwhile to mention that our proof is quite different from those in [Nad89, Rou16, Bru20a, Cad22]. All these proofs relied heavily on the special property of quotients of bounded symmetric domains by torsion-free lattices. They all applied the toroidal compactifications by Mumford to find the desired finite étale cover $\tilde{U}\to U$ when U is a quotient of a bounded symmetric domain by a torsion-free lattice. We construct the cover $\tilde{U}\to U$ in Theorem B in a subtle way using the residual finiteness of the global monodromy group. We refer the readers to the beginning of Section 6 for the general strategy.

1.5. Some new developments

Shortly after this paper appeared on arXiv, Brunebarbe–Brotbek [BB20, Theorem 1.5] proved the Borel hyperbolicity of U in Theorem A under the additional assumption that the local monodromy of the $\mathbb{C}$-PVHS at infinity is unipotent. Moreover, they also obtained a weaker result than Theorem B(ii) in [BB20, Theorem 1.7], in which they showed that for a quasi-projective manifold U admitting a $\mathbb{Z}$-PVHS with quasi-finite period map, there is a finite étale cover $\tilde{U}\to U$ so that the projective compactification X of $\tilde{U}$ is *Borel hyperbolic* modulo $X-\tilde{U}$. Our proofs are indeed quite different: Brotbek–Brunebarbe's proof is based on their Second Main Theorem using the *Griffiths–Schmid metric*, which coincides with the curvature form of the Griffiths line bundle. Let us also mention that result similar to Theorem B(i) is also obtained by Brunebarbe in [Bru20b, Theorem 1.1] when the underlying local system of the $\mathbb{C}$-PVHS is defined over $\mathbb{Z}$.

In [CD21] Cadorel and the author generalized Theorems A and B and Corollary C in this paper to quasi-compact Kähler manifolds admitting *nilpotent Higgs bundles.* More recently, in [CDY22, Theorem 0.1] Cadorel, Yamanoi and the author proved that for any complex quasi-projective normal variety X, if there is a big representation $\rho\colon\pi_1(X)\to GL_N(\mathbb{C})$ such that the Zariski closure of $\rho(\pi_1(X))$ is a semisimple algebraic group, then there is a proper Zariski closed subset $Z\subsetneq X$ such that

- any closed subvariety of X not contained in X is log general type;
- X is Picard hyperbolic modulo Z.

We stress here that Theorem A in this paper is applied in [CDY22].

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Notation and Conventions

- A *log pair* (Y,D) consists of a (possibly non-compact) complex manifold and a simple normal crossing divisor D. It will be called a *compact Kähler log pair* (resp. *projective log pair*) if Y is a compact Kähler (resp. projective) manifold.
- A complex manifold is called *quasi-compact Kähler* if it is a Zariski open subset of a compact Kähler manifold.
- A *log morphism* $f:(X,\tilde{D})\to(Y,D)$ between log pairs is a morphism $f:X\to Y$ with $\tilde{D}\subset f^{-1}(D)$.
- For a big line bundle L on a projective manifold, $\mathbf{B}_+(L)$ denotes its *augmented base locus* (see [Laz04, Definition 10.3.2]).

2. Preliminaries on Hodge theory

2.1. Systems of Hodge bundles

Following Simpson [Sim88], a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS) is equivalent to a system of Hodge bundles. Let us recall the definition in this subsection.

Definition 2.1 (Higgs bundle). A *Higgs bundle* on a complex manifold Y is a pair (E,θ) consisting of a holomorphic vector bundle E on Y and an $\mathcal{O}_Y$-linear map

$$\theta: E\longrightarrow E\otimes\Omega_Y^1$$

so that $\theta\wedge\theta=0$. Such a map θ is called the *Higgs field*.

Definition 2.2 (Harmonic bundle). A *harmonic bundle* (E,θ,h) consists of a Higgs bundle (E,θ) and a hermitian metric h for E such that

$$D:=\partial_h+\bar{\partial}_E+\theta+\theta_h^*$$

is flat. Here $\partial_h+\bar{\partial}_E$ is the Chern connection, and $\theta_h^*\in C^\infty(Y,\mathrm{End}(E)\otimes\Omega_Y^{0,1})$ is the adjoint of θ with respect to h.

Definition 2.3 (System of Hodge bundles). A *system of Hodge bundles* of weight m is a harmonic bundle (E,θ,h) satisfying the following:

- The vector bundle $E=\oplus_{p+q=m}E^{p,q}$ is a direct sum of holomorphic vector bundles $E^{p,q}$.
- The map θ restricts to
$$\theta|_{E^{p,q}}: E^{p,q}\longrightarrow E^{p-1,q+1}\otimes\Omega_Y^1.$$
- The splitting $E=\oplus_{p+q=m}E^{p,q}$ is orthogonal with respect to h.

We write $h_{p,q}=h|_{E^{p,q}}$ and $\theta_{p,q}=\theta|_{E^{p,q}}$. This harmonic metric h will be called the *Hodge metric*.

Throughout this paper, we observe the convention that $0\le p,q\le m$ for the decomposition $E=\oplus_{p+q=m}E^{p,q}$. This can always be achieved if we make a Tate twist (k,k) to increase the weight by $2k$ when $k\in\mathbb{Z}_{>0}$ is large enough.

2.2. Filtered bundles and parabolic Higgs bundles

In this section, we recall the notions of filtered bundles and parabolic Higgs bundles from [Sim88, Moc07]. Let $(Y,D=\sum_{i=1}^c D_i)$ be a log pair.

Definition 2.4. A filtered bundle $(E,\mathcal{P}_{\boldsymbol{a}}E)$ on (Y,D) is a locally free sheaf E on $U:=Y-D$, together with an $\mathbb{R}^c$-indexed filtration $\mathcal{P}_{\boldsymbol{a}}E$ by locally free sheaves on Y such that

(i) $\boldsymbol{a}\in\mathbb{R}^c$ and $\mathcal{P}_{\boldsymbol{a}}E|_U=E$;
(ii) $\mathcal{P}_{\boldsymbol{a}}E\subset\mathcal{P}_{\boldsymbol{b}}E$ for $\mathbf{a}\le\mathbf{b}$ (*i.e.*, $a_i\le b_i$ for all i);

(iii) $\mathcal{P}_{\boldsymbol{a}}E \otimes \mathcal{O}_Y(D_i) = \mathcal{P}_{\mathbf{a}+1_i}E$ with $1_i = (0,\ldots,1,\ldots,0)$ with 1 in the i^{th} component;
(iv) $\mathcal{P}_{\boldsymbol{a}+\boldsymbol{\epsilon}}E = \mathcal{P}_{\boldsymbol{a}}E$ for any vector $\boldsymbol{\epsilon} = (\epsilon,\ldots,\epsilon)$ with $0 < \epsilon \ll 1$;
(v) write $\mathcal{P}_{<\boldsymbol{a}}E = \cup_{\boldsymbol{b}<\boldsymbol{a}}\mathcal{P}_{\boldsymbol{b}}E$; the set of ***weights*** $\boldsymbol{a}$ such that $\mathcal{P}_{\boldsymbol{a}}E/\mathcal{P}_{<\boldsymbol{a}}E \neq 0$ is discrete in $\mathbb{R}^c$.

A weight is normalized if it lies in $(-1,0]^c$. Denote $\mathcal{P}_{\mathbf{0}}E$ by $^\diamond E$, where $\mathbf{0} = (0,\ldots,0)$. Note that the set of weights of $(E,\mathcal{P}_{\boldsymbol{a}}E)$ is uniquely determined by the weights lying in $(-1,0]^c$.

Definition 2.5. A *parabolic Higgs bundle* on (Y,D) is a filtered bundle $(E,\mathcal{P}_{\boldsymbol{a}}E)$ together with $\mathcal{O}_Y$ linear map

$$\theta\colon {}^\diamond E \longrightarrow \Omega^1_Y(\log D) \otimes {}^\diamond E$$

such that

$$\theta \wedge \theta = 0 \quad \text{and} \quad \theta(\mathcal{P}_{\boldsymbol{a}}E) \subseteq \Omega^1_Y(\log D) \otimes \mathcal{P}_{\boldsymbol{a}}E \ \text{ for } \boldsymbol{a} \in [-1,0)^c.$$

A natural class of filtered bundles comes from extensions of systems of Hodge bundles, which will be discussed in Section 2.4.

2.3. Admissible coordinates

Definition 2.6 (Admissible coordinate). Let $(Y, D = \sum_{i=1}^c D_i)$ be log pair. Let p be a point of Y, and let $\{D_j\}_{j=1,\ldots,\ell}$ be the components of D containing p. An *admissible coordinate* around p is a tuple $(\mathcal{U}; z_1,\ldots,z_n;\varphi)$ (or simply $(\mathcal{U}; z_1,\ldots,z_n)$ if no confusion arises) where

- $\mathcal{U}$ is an open subset of Y containing p;
- there is a holomorphic isomorphism $\varphi\colon \mathcal{U} \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1,\ldots,\ell$.

We shall write $\mathcal{U}^* := \mathcal{U} - D$.

Recall that the complete Poincaré metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ is described as

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{k=\ell+1}^{n} \sqrt{-1}\frac{dz_k \wedge d\bar{z}_k}{(1-|z_k|^2)^2} \tag{2.1}$$

Note that $\omega_P = dd^c\varphi$ with

$$\varphi := -\log\left(\prod_{j=1}^{\ell}\left(-\log|z_j|^2\right)\cdot\left(\prod_{k=\ell+1}^{n}(1-|z_k|)^2\right)\right). \tag{2.2}$$

Remark 2.7 (Global Kähler metric with Poincaré growth). Let (Y,ω) be a compact Kähler manifold, and let $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor on Y. Let σ_i be the section $H^0(Y,\mathcal{O}_Y(D_i))$ defining D_i, and pick any smooth metric h_i for the line bundle $\mathcal{O}_Y(D_i)$. One can prove that when $\varepsilon > 0$ is small enough, the closed $(1,1)$-current

$$T := \omega - dd^c\log\left(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|^2_{h_i}\right) \tag{2.3}$$

is a Kähler current (*i.e.*, $T \geq \delta\omega$ for some $\delta > 0$), and on any admissible coordinate $(\mathcal{U}; z_1,\ldots,z_n)$, $T|_{\mathcal{U}-D}$ is mutually bounded with ω_P.

2.4. Extension of systems of Hodge bundles

Let $(Y, D = \sum_{i=1}^{\ell} D_i)$ be log pair. Let (E,h) be a hermitian bundle on $Y - D$. For any $\mathbf{a} = (a_1,\ldots,a_\ell) \in \mathbb{R}^\ell$, we can prolong E over Y by $\mathcal{P}^h_{\boldsymbol{a}}E$ as follows:

$$\mathcal{P}^h_{\boldsymbol{a}}E(\mathcal{U}) = \left\{\sigma \in \Gamma(\mathcal{U}-D, E|_{\mathcal{U}-D}) \mid |\sigma|_h \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{a_i+\varepsilon}} \ \forall\varepsilon > 0\right\}, \tag{2.4}$$

where $(\mathcal{U}; z_1, \ldots, z_n)$ is any admissible coordinate. We still use the notation ${}^\diamond E$ in the case $\boldsymbol{a} = (0, \ldots, 0)$. In general, $\mathcal{P}^h_{\boldsymbol{a}} E$ is not coherent. However, by the deep work of Simpson [Sim88, Theorem 3] and Mochizuki [Moc07], this is the case for systems of Hodge bundles.

Theorem 2.8 (Simpson, Mochizuki). *If $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ is a system of Hodge bundles on $Y - D$, then $(E, \mathcal{P}^h_{\boldsymbol{a}} E, \theta)$ is a parabolic Higgs bundle on (Y, D).*

In this case, we write $\mathcal{P}_{\boldsymbol{a}} E$ for $\mathcal{P}^h_{\boldsymbol{a}} E$ to lighten the notation, and denote by

$$\theta \colon \mathcal{P}_{\boldsymbol{a}} E \longrightarrow \mathcal{P}_{\boldsymbol{a}} E \otimes \Omega^1_Y(\log D)$$

the prolonged Higgs field by abuse of notation. From Theorem 2.8, one can easily deduce the following.

Lemma 2.9. *Let $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ be as above.*

(i) *We have $\mathcal{P}_{\boldsymbol{a}} E = \oplus_{p+q=m} \mathcal{P}_{\boldsymbol{a}} E^{p,q}$. Here $\mathcal{P}_{\boldsymbol{a}} E^{p,q}$ is the extension of $(E^{p,q}, h_{p,q})$.*

(ii) *The map θ restricts to*

$$\theta|_{\mathcal{P}_{\boldsymbol{a}} E^{p,q}} \colon \mathcal{P}_{\boldsymbol{a}} E^{p,q} \longrightarrow \mathcal{P}_{\boldsymbol{a}} E^{p-1,q+1} \otimes \Omega^1_Y(\log D).$$

Remark 2.10. If $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ is a system of Hodge bundles, $\mathcal{P}_{\boldsymbol{a}} E$ coincides with the Deligne extension with real part of the eigenvalue in $[\boldsymbol{a}, \boldsymbol{a} + \mathbf{1})$. See the table in [Sim90, p. 746].

Definition 2.11. Let (Y, D) be a log pair. Let $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ be a system of Hodge bundles defined over $Y - D$. The extension $({}^\diamond E = \oplus_{p+q=m} {}^\diamond E^{p,q}, \theta)$ is called the *canonical extension* of $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$.

Lemma 2.9 inspires us to introduce the definition of systems of log Hodge bundles.

Definition 2.12 (System of log Hodge bundles). Let (Y, D) be a log pair. A *system of log Hodge bundles* of weight m over (Y, D) consists of a pair $(E = \oplus_{p+q=m} E^{p,q}, \theta = \oplus_{p+q=m} \theta_{p,q})$, where

- $E = \oplus_{p+q=m} E^{p,q}$ is a direct sum of holomorphic vector bundles $E^{p,q}$ on Y;
- θ is a direct sum of

$$\theta_{p,q} \colon E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega^1_Y(\log D)$$

  with $\theta \wedge \theta = 0$.

3. Construction of a special system of log Hodge bundles

In this section, we first study the refined positivity for the Griffiths line bundle associated to a system of Hodge bundles. This positivity is well known when the corresponding $\mathbb{C}$-PVHS has unipotent monodromies near the boundary. We then construct a special system of log Hodge bundles (see Theorem 3.6) over the log pair (Y, D) in Theorem 1.4. Such a system of Hodge bundles will be used to construct a negatively curved Finsler metric in Section 4.

3.1. Refined positivity for Griffiths line bundles

For a system of Hodge bundles $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ over a complex manifold U, in [Gri70a] Griffiths constructed a line bundle $\mathfrak{L}$ on U, which can be endowed with a natural metric with semipositive curvature. Precisely, one has

$$\mathfrak{L} := \left(\det E^{m,0}\right)^{\otimes m} \otimes \left(\det E^{m-1,1}\right)^{\otimes (m-1)} \otimes \cdots \otimes \det E^{1,m-1}. \tag{3.1}$$

Here $\theta^*_{p,q}$ is the adjoint of $\theta_{p,q}$ with respect to $h_{p,q}$. The Hodge metric h then induces a metric $h_{\mathfrak{L}}$ on $\mathfrak{L}$ whose curvature is

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) = -\mathrm{tr}\left(\sum_{q=0}^{m-1} \theta^*_{m-q,q} \wedge \theta_{m-q,q}\right) \geq 0. \tag{3.2}$$

One can see that $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) > 0$ at the point y where $\theta\colon T_{Y,y} \to \mathrm{End}(E_y)$ is injective. Note that θ is the differential of the period map (see, *e.g.*, [KKM11, p. 429] for the proof). This means that $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is strictly positive at the point where the period map is immersive.

Now assume $U = Y - D$, where (Y, D) is a compact Kähler log pair. Let T be the Kähler current on Y defined in Remark 2.7. Then $\omega_U := T|_U$ is a complete Kähler metric with Poincaré type near D. We recall the following theorem by Simpson [Sim88, Lemma 10.1] and Mochizuki [Moc07].

Theorem 3.1 (Simpson, Mochizuki). *Let $(E = \oplus_{p+q=m} E^{p,q}, \theta, h)$ be a system of Hodge bundles on $U = Y - D$. Then*

$$|\theta|_{h,\omega_U} \le C \tag{3.3}$$

for some constant $C > 0$.

Lemma 3.2. *In the notation above, $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is less singular than ω_U, which we denote by*

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L}) \lesssim \omega_U.$$

Proof. By Theorem 3.1, one has $|\theta_{p,q}|_{h,\omega_U} \le C$. Then $|\theta^*_{p,q}|_{h,\omega_U} \le C$. Hence

$$\left|\theta^*_{p,q} \wedge \theta_{p,q}\right|_{h,\omega_U} \le C^2.$$

It follows from (3.2) that

$$\left|\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})\right| \le C'$$

for some constant $C' > 0$. The lemma follows directly from the above inequality. □

By Lemma 3.2, the mass of $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is bounded near D, and one can thus apply the Skoda extension theorem (see [Dem97, Theorem 2.3]) so that the trivial extension of $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ over Y is a positive closed $(1,1)$-current, which is denoted by S. The current S is therefore less singular than the current T defined in Remark 2.7, which we denote by $S \lesssim T$.

Let us consider the extension $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$ of $(\mathfrak{L}, h_{\mathfrak{L}})$ defined in (2.4), where $\mathbf{1} = (1, \ldots, 1)$. Then $h_{\mathfrak{L}}$ can be seen as the *singular hermitian metric* for $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$; this can be seen explicitly from the proof of the next lemma.

Lemma 3.3. *The curvature $\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathcal{P}_{\mathbf{1}}\mathfrak{L})$ is a closed positive $(1,1)$-current. In particular, $\mathcal{P}_{\mathbf{1}}\mathfrak{L}$ is a pseudo effective line bundle on Y.*

Proof. Pick any $p \in Y$. We take an admissible coordinate $(W; z_1, \ldots, z_n)$ around p as in Definition 2.6. Since S is a closed positive current on Y, over W there is a plurisubharmonic function ψ so that $S = \mathrm{dd}^c\psi$. Note that $S \lesssim T$. One thus has $\varphi \lesssim \psi$, where φ is defined in (2.2). For the new metric $\tilde{h}_{\mathfrak{L}} := h_{\mathfrak{L}} \cdot e^{\psi}$ of $\mathfrak{L}$, one has $\Theta_{\tilde{h}_{\mathfrak{L}}}(\mathfrak{L}) = 0$ over $W - D \simeq (\Delta^*)^\ell \times \Delta^{n-\ell}$.

Let ∇ be the corresponding Chern connection of $(\mathfrak{L}, \tilde{h}_{\mathfrak{L}})$, which is flat by the relation $\Theta_{\tilde{h}_{\mathfrak{L}}}(\mathfrak{L}) = 0$. It corresponds to a unitary representation $\rho\colon \mathbb{Z}^\ell \simeq \pi_1(W - D) \to U(1)$. Let γ_i be a clockwise loop around the origin in the i^{th} factor $(\Delta^*)^\ell \times \Delta^{n-\ell} \simeq W - D$. Let $T_i = \rho([\gamma_i]) \in U(1)$ be the monodromy corresponding to the loop γ_i.

Consider the universal covering map

$$\begin{aligned} \pi\colon \mathbb{H}^\ell \times \Delta^{n-\ell} &\longrightarrow (\Delta^*)^\ell \times \Delta^{n-\ell} \\ (t_1, \ldots, t_\ell, z_{\ell+1}, \ldots, z_n) &\longmapsto \left(e^{2\pi\sqrt{-1}t_1}, \ldots, e^{2\pi\sqrt{-1}t_\ell}, z_{\ell+1}, \ldots, z_n\right), \end{aligned}$$

where $\mathbb{H} = \{t \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Choose a flat section Φ of the flat line bundle $\pi^*(\mathfrak{L}, \nabla)$. Since $(\mathfrak{L}, \tilde{h}_{\mathfrak{L}})$ is unitary flat, $|\Phi|_{\tilde{h}_{\mathfrak{L}}}$ is constant, and we may assume that $|\Phi|_{\tilde{h}_{\mathfrak{L}}} \equiv 1$. Recall that $T_i = \rho([\gamma_i]) \in U(1)$ is the monodromy corresponding to the loop γ_i; one has

$$T_i \cdot \Phi(t_1, \ldots, t_\ell, z_{\ell+1}, \ldots, z_n) = \Phi(t_1, \ldots, t_i + 1, \ldots, t_\ell, z_{\ell+1}, \ldots, z_n). \tag{3.4}$$

Write $T_i = e^{2\pi\sqrt{-1}b_i}$ for some $0 < b_i \leq 1$. Define a new section of $\pi^*\mathfrak{L}$ by

$$\Psi(t_1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n) := \Phi(t_1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n)e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i}.$$

By (3.4), one has

$$\Psi(t_1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n) := \Psi(t_1,\ldots,t_i+1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n)$$

for any $i = 1,\ldots,\ell$. It thus descends to a section $\sigma(z)$ of $\mathfrak{L}|_{W-D}$; *i.e.*,

$$\sigma(\pi(t_1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n)) = \Psi(t_1,\ldots,t_\ell,z_{\ell+1},\ldots,z_n).$$

Note that $\nabla(\Phi) = 0$; one has

$$\nabla(\Psi) = \Phi\cdot e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i}\left(-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i\cdot dt_i\right) = \Psi\cdot\left(-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i\cdot dt_i\right).$$

Hence

$$\nabla(\sigma(z)) = -\sum_{i=1}^{\ell} b_i d\log z_i\cdot\sigma(z).$$

Therefore, $\sigma(z)$ is a holomorphic section trivializing $\mathfrak{L}|_{W-D}$. Note that

$$|\Psi|_{\tilde{h}_{\mathfrak{L}}} = \left|\Phi\cdot e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i}\right| = \left|e^{-2\pi\sqrt{-1}\sum_{i=1}^{\ell} b_i t_i}\right|,$$

where the second equality follows from the fact that $|\Phi|_{\tilde{h}_{\mathfrak{L}}} \equiv 1$. It follows that $|\sigma(z)|_{\tilde{h}_{\mathfrak{L}}} = \prod_{i=1}^{\ell}|z_i|^{-b_i}$, and thus

$$|\sigma(z)|_{h_{\mathfrak{L}}} = \prod_{i=1}^{\ell}|z_i|^{-2b_i}\cdot e^{-\psi} \tag{3.5}$$

by the relation $\tilde{h}_{\mathfrak{L}} := h_{\mathfrak{L}}\cdot e^{\psi}$. Since $\varphi \lesssim \psi$, one has

$$1 \lesssim e^{-\psi} \lesssim e^{-\varphi} \lesssim \left(\prod_{j=1}^{\ell}\left(-\log|z_j|^2\right)\right)^N$$

for some $N > 0$, where the last inequality follows from (2.2). Therefore,

$$\prod_{i=1}^{\ell}\frac{1}{|z_i|^{b_i-\varepsilon}} \lesssim |\sigma(z)|_{h_{\mathfrak{L}}} \lesssim \prod_{i=1}^{\ell}\frac{1}{|z_i|^{b_i+\varepsilon}} \tag{3.6}$$

for any $\varepsilon > 0$. Since $0 < b_i \leq 1$, one has $\sigma \in \mathcal{P}_1\mathfrak{L}|_W$ by (2.4). Let us show that σ is a generator of $\mathcal{P}_1\mathfrak{L}|_W$.

For any section $s \in \mathcal{P}_1\mathfrak{L}(W)$, there is a holomorphic function $f \in \mathcal{O}(W-D)$ so that $s = f\cdot\sigma$. By (2.4) again,

$$|f|\cdot|\sigma|_{h_{\mathfrak{L}}} = |s|_{h_{\mathfrak{L}}} \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1+\varepsilon}}$$

for all $\varepsilon > 0$. By (3.6), one has

$$|f| \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1-b_i+\varepsilon}}$$

for all $\varepsilon > 0$. Pick $\varepsilon \ll 1$ with $1-b_i+\varepsilon < 1$ for all i. The above inequality shows that f extends to a holomorphic function over W. Hence σ is a generator of $\mathcal{P}_1\mathfrak{L}|_W$.

By (3.5), one has

$$\sqrt{-1}\Theta_{h_{\mathfrak{L}}}(\mathcal{P}_1\mathfrak{L}) = dd^c\log|\sigma|_{h_{\mathfrak{L}}} = S + \sum_{i=1}^{\ell} b_i[D_i], \tag{3.7}$$

where $[D_i]$ is the current of integration associated to D_i. This finishes the proof of the theorem. $\square$

The following lemma is a consequence of the above proof.

Lemma 3.4. *For any $N \in \mathbb{Z}_{>0}$, let $\mathcal{P}_1(\mathfrak{L}^{\otimes N})$ be the extension of $(\mathfrak{L}^{\otimes N}, h_{\mathfrak{L}}^{\otimes N})$ defined in* (2.4)*. Then*

$$\mathcal{P}_1\left(\mathfrak{L}^{\otimes N}\right) = (\mathcal{P}_1\mathfrak{L})^{\otimes N} \otimes \mathcal{O}\left(-\sum_{i=1}^{\ell}(\lceil Nb_i\rceil - 1)D_i\right).$$

Proof. We use the same notation as that in the proof of Lemma 3.3. Consider the section σ^N, which is a generator of $(\mathcal{P}_1\mathfrak{L})^{\otimes N}|_W$. For any section $s \in \mathcal{P}_1(\mathfrak{L}^{\otimes N})(W)$, there is a holomorphic function $f \in \mathcal{O}(W^*)$ so that $s = f\cdot\sigma^N$, where $W^* := W - D$. By (2.4) again, one has

$$|f|\cdot|\sigma^N|_{h_{\mathfrak{L}}^{\otimes N}} = |s|_{h_{\mathfrak{L}}^{\otimes N}} \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1+\varepsilon}}$$

for all $\varepsilon > 0$. By (3.6), one has

$$|f| \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1-Nb_i+\varepsilon}}$$

for all $\varepsilon > 0$. This shows that $f \in \mathcal{O}_Y(-\sum_{i=1}^{\ell}(\lceil Nb_i\rceil - 1)D_i)$.

On the other hand, if $g \in \mathcal{O}_Y(-\sum_{i=1}^{\ell}(\lceil Nb_i\rceil - 1)D_i)$, then by (3.6), one has

$$|g\cdot\sigma^N| \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1-\lceil Nb_i\rceil+Nb_i+\varepsilon}} \lesssim \frac{1}{\prod_{i=1}^{\ell}|z_i|^{1+\varepsilon}}$$

for any $\varepsilon > 0$. This yields the lemma. □

In summary, we have the following positivity result for Griffiths line bundles.

Proposition 3.5. *Let (Y, D) be a compact Kähler log pair. Let $(E = \oplus_{p+q=m}E^{p,q}, \theta, h)$ be a system of Hodge bundles over $Y - D$. Assume that its period map is immersive at one point. Then ${}^{\diamond}(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D)$ is a big line bundle on Y for $N \gg 1$. In particular, Y is projective.*

Proof. Recall that the closed positive current S is the trivial extension of the semipositive $(1,1)$-form $\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ over Y. By (3.7), one has

$$\{c_1(\mathcal{P}_1\mathfrak{L})\} = \{S\} + \sum_{i=1}^{\ell} b_i\{D_i\}.$$

Lemma 3.4 then yields

$$\left\{c_1\left(\mathcal{P}_1\left(\mathfrak{L}^{\otimes N}\right)\right)\right\} = N\{S\} + \sum_{i=1}^{\ell}(Nb_i - \lceil Nb_i\rceil + 1)\{D_i\}.$$

Note that

$$\mathcal{P}_1\left(\mathfrak{L}^{\otimes N}\right) = {}^{\diamond}\left(\mathfrak{L}^{\otimes N}\right) \otimes \mathcal{O}_Y(D).$$

Therefore,

$$c_1\left({}^{\diamond}\left(\mathfrak{L}^{\otimes N}\right) \otimes \mathcal{O}_Y(-D)\right) = N\{S\} + \sum_{i=1}^{\ell}(-1 + Nb_i - \lceil Nb_i\rceil)\{D_i\}.$$

By the discussion at the beginning of this subsection, the semipositive $(1,1)$-form $\Theta_{h_{\mathfrak{L}}}(\mathfrak{L})$ is strictly positive at the point where the period map is immersive. By Boucksom's criterion [Bou02], the cohomology class $\{S\}$ is a big $(1,1)$-class. Therefore, $N\{S\} - 2D$ is big for $N \gg 1$. Note that

$$1 + Nb_i - \lceil Nb_i\rceil \geq 0.$$

Since the sum of a big class with an effective class is still big, we conclude that $c_1({}^{\diamond}(\mathfrak{L}^{\otimes N}) \otimes \mathcal{O}_Y(-D))$ is big. This proves the lemma. □

3.2. Special system of Hodge bundles

Let (Y,D) be a compact Kähler log pair. Let $(F=\oplus_{p+q=m}F^{p,q},\eta,h_F)$ be a system of Hodge bundle over $U:=Y-D$ whose period map is immersive at one point. Let us write $r_p:=\operatorname{rank}F^{p,q}$. Recall that the Griffiths line bundle for $(F=\oplus_{p+q=m}F^{p,q},\eta,h_F)$ is

$$\mathfrak{L}:=\left(\det F^{m,0}\right)^{\otimes m}\otimes\left(\det F^{m-1,1}\right)^{\otimes(m-1)}\otimes\cdots\otimes\det F^{1,m-1}.$$

By Proposition 3.5, ${}^{\diamond}(\mathfrak{L}^{\otimes N})\otimes\mathcal{O}_Y(-D)$ is a big line bundle for some $N\gg 1$. Let us write

$$r:=N\Big(mr_m+(m-1)r_{m-1}+\cdots+r_1\Big).$$

We define a new system of Hodge bundle $(E=\oplus_{P+Q=rm}E^{P,Q},\theta,h)$ on $U=Y-D$ by setting $(E,\theta,h):=(F,\eta,h_F)^{\otimes r}$. Precisely, $E:=F^{\otimes r}$, and

$$\theta:=\eta\otimes\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{(r-1)-\text{tuple}}+\mathbb{1}\otimes\eta\otimes\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{(r-2)-\text{tuple}}+\cdots+\underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{(r-1)-\text{tuple}}\otimes\eta.$$

Define

$$E^{P,Q}:=\oplus_{p_1+\cdots+p_r=P;q_1+\cdots+q_r=Q}F^{p_1,q_1}\otimes\cdots\otimes F^{p_r,q_r}. \tag{3.8}$$

Then we have

$$\theta\colon E^{P,Q}\longrightarrow E^{P-1,Q+1}\otimes\Omega_U^1,$$

and one can easily check that $h=h_F^{\otimes r}$ is the Hodge metric for $(E=\oplus_{P+Q=rm}E^{P,Q},\theta)$.

Note that $\det F^{p,q}=\wedge^{r_p}F^{p,q}\subset(F^{p,q})^{\otimes r_p}\subset F^{\otimes r_p}$. Hence

$$\begin{aligned}\mathfrak{L}^{\otimes N}&=\left(\det F^{m,0}\right)^{\otimes Nm}\otimes\left(\det F^{m-1,1}\right)^{\otimes(N(m-1))}\otimes\cdots\otimes\left(\det F^{1,m-1}\right)^{\otimes N}\\&\subset\left(F^{m,0}\right)^{\otimes Nmr_m}\otimes\cdots\otimes\left(F^{1,m-1}\right)^{\otimes Nr_1}\subset E^{P_0,Q_0},\end{aligned}$$

where $P_0=N(r_mm^2+r_{m-1}(m-1)^2+\cdots+r_1)$ and $Q_0=rm-P_0$. In other words, $\mathfrak{L}^{\otimes N}$ is a subbundle of E^{P_0,Q_0}. Moreover, their hermitian metrics are compatible in the following sense: $h_{\mathfrak{L}}^{\otimes N}=h|_{\mathfrak{L}}$. By the very definition of the extension (2.4), one has

$${}^{\diamond}\left(\mathfrak{L}^{\otimes N}\right)\subset{}^{\diamond}E^{P_0,Q_0}.$$

In summary, we construct a *special system of log Hodge bundles* on (Y,D) as follows (we change the notation for brevity's sake).

Theorem 3.6. *Let (Y,D) be a compact Kähler log pair. Let $(F=\oplus_{p+q=m}F^{p,q},\eta,h_F)$ be a system of Hodge bundles over $Y-D$ whose period map is immersive at one point. Then there is a system of log Hodge bundles $(E=\oplus_{p+q=\ell}E^{p,q},\theta=\oplus_{p+q=\ell}\theta_{p,q})$ on (Y,D) satisfying the following properties:*

- (i) *The pair (E,θ) is the canonical extension (in the sense of Definition 2.11) of some system of Hodge bundles $(\tilde{E},\tilde{\theta},h_{\rm hod})$ defined over $Y-D$.*
- (ii) *There is a big line bundle L over Y such that $L\subset E^{p_0,q_0}$ for some $p_0+q_0=\ell$, and $L\otimes\mathcal{O}_Y(-D)$ is still big.*
- (iii) *If the period map moreover has zero-dimensional fibers, then the augmented base locus satisfies $\mathbf{B}_+(L)\subset D$.*

Remark 3.7. The interested readers can compare the Higgs bundle in Theorem 3.6 with the Viehweg–Zuo Higgs bundle in [VZ02, VZ03] (see also [PTW19]). Loosely speaking, a *Viehweg–Zuo Higgs bundle* for a log pair (Y,D) is a Higgs bundle $(E=\oplus_{p+q=m}E^{p,q},\theta)$ over $(Y,D+S)$ induced by some (geometric) $\mathbb{Z}$-PVHS defined over a Zariski open subset of $Y-(D\cup S)$, where S is another divisor on Y so that $D+S$ is simple normal crossing. The extra data is that there is a sub-Higgs sheaf $(F=\oplus_{p+q=m}F^{p,q},\eta)\subset(E,\theta)$ such that the *first stage* $F^{n,0}$ is a big line bundle, and that we have

$$\eta\colon F^{p,q}\longrightarrow F^{p-1,q+1}\otimes\Omega_Y^1(\log D).$$

As we explained in Section 1.4.1, the positivity $F^{n,0}$ comes in a sophisticated way from Kawamata's big line bundle $\det f_*(mK_{X/Y})$, where $f\colon X\to Y$ is some algebraic fiber space between projective manifolds. For our Higgs bundle $(E=\oplus_{p+q=m}E^{p,q},\theta)$ over the log pair (Y,D) in Theorem 3.6, the global positivity is the Griffiths line bundle which is contained in some *intermediate stage* E^{p_0,q_0} of $(E=\oplus_{p+q=m}E^{p,q},\theta)$.

3.3. Iterating Higgs fields

Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be the system of log Hodge bundles on a compact Kähler log pair (Y,D) satisfying the two conditions in Theorem 3.6. We apply ideas by Viehweg–Zuo [VZ02, VZ03] to iterate Higgs fields.

Since we have $\theta\colon E^{p,q}\to E^{p-1,q+1}\otimes\Omega_Y^1(\log D)$, one can iterate θ k times to obtain

$$E^{p_0,q_0}\longrightarrow E^{p_0-1,q_0+1}\otimes\Omega_Y^1(\log D)\longrightarrow\cdots\longrightarrow E^{p_0-k,q_0+k}\otimes\otimes^k\Omega_Y^1(\log D).$$

Since $\theta\wedge\theta=0$, the above morphism factors through

$$E^{p_0,q_0}\longrightarrow E^{p_0-k,q_0+k}\otimes\mathrm{Sym}^k\Omega_Y^1(\log D).\tag{3.9}$$

Since L is a subsheaf of E^{p_0,q_0}, it induces

$$L\longrightarrow E^{p_0-k,q_0+k}\otimes\mathrm{Sym}^k\Omega_Y^1(\log D),$$

which is equivalent to a morphism

$$\tau_k\colon \mathrm{Sym}^kT_Y(-\log D)\longrightarrow L^{-1}\otimes E^{p_0-k,q_0+k}.\tag{3.10}$$

The readers might be worried that all τ_k might be trivial, so that the above construction will be meaningless. In the next subsection, we will show that this cannot happen.

3.4. An infinitesimal Torelli-type theorem

We begin with the following technical lemma.

Proposition 3.8. *Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y,D) satisfying the two conditions in Theorem 3.6. Then there is a singular hermitian metric h_L with analytic singularities for L such that*

(i) *the curvature current satisfies*
$$\sqrt{-1}\Theta_{h_L}(L)\geq T,$$
where T is the Kähler current on Y defined in Remark 2.7;

(ii) *the singular hermitian metric $h:=h_L^{-1}\otimes h_{\rm hod}$ on $L^{-1}\otimes E$ is locally bounded on Y and smooth outside $D\cup\mathbf{B}_+(L-D)$, where $h_{\rm hod}$ is the Hodge metric for the system of Hodge bundles $(E=\oplus_{p+q=\ell}E^{p,q},\theta)|_U$. Moreover, $h\cdot\prod_{i=1}^{\ell}|\sigma_i|_{h_i}^{-\varepsilon}$ vanishes on $D\cup\mathbf{B}_+(L-D)$ for $\varepsilon>0$ small enough. Here σ_i is the canonical section in $H^0(Y,\mathcal{O}_Y(D_i))$ defining D_i, and h_i is a smooth metric for the line bundle $\mathcal{O}_Y(D_i)$.*

Proof. By Theorem 3.6(ii), the line bundle $L\otimes\mathcal{O}_Y(-D)$ is big, and thus by [Bou04, Theorem 3.17], we can put a singular hermitian metric g_0 on it with analytic singularities for $L\otimes\mathcal{O}_Y(-D)$ such that g_0 is smooth on $Y\setminus\mathbf{B}_+(L\otimes\mathcal{O}_Y(-D))$, where $\mathbf{B}_+(L\otimes\mathcal{O}_Y(-D))$ is the augmented base locus of $L\otimes\mathcal{O}_Y(-D)$, and the curvature current satisfies $\sqrt{-1}\Theta_{g_0}(L-D)\geqslant\omega$ for some smooth Kähler form ω on Y. Take $g:=g_0(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|_{h_i}^2)$. Then

$$\sqrt{-1}\Theta_g(L-D)\geqslant T:=\omega-\mathrm{dd^c}\log\left(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|_{h_i}^2\right).$$

Note that T is a Kähler current when $0<\varepsilon\ll1$.

Let h_D be the canonical singular hermitian metric for D so that $\sqrt{-1}\Theta_{h_D}(\mathcal{O}_Y(D))=[D]$. We define a singular hermitian metric on L as follows:

$$h_L:=g\cdot h_D.$$

Then

$$\sqrt{-1}\Theta_{h_L}(L)=\sqrt{-1}\Theta_g(L\otimes\mathcal{O}_X(-D))+[D]\geq T.$$

The first condition is verified.

Note that g^{-1} vanishes on $\mathbf{B}_+(L\otimes\mathcal{O}_Y(-D))$, and h_D^{-1} vanishes on D. Since $h_{\rm hod}$ is smooth over $Y-D$, we have $\mathbf{B}_+(L)\subset\mathbf{B}_+(L\otimes\mathcal{O}_Y(-D))$, so $h:=h_{\rm hod}\cdot h_L^{-1}$ vanishes on $\mathbf{B}_+(L)-D$. For any point $y\in D$, we pick an admissible coordinate $(W;z_1,\ldots,z_n)$ and a frame $(e_1,\ldots,e_r)$ for $E|_W$. Since (E,θ) is the canonical extension of a system of Hodge bundles $(\tilde{E},\tilde{\theta},h_{\rm hod})$, by (2.4) one has

$$|e_i|_h\lesssim\frac{1}{\prod_{i=1}^{\ell}|z_i|^{\varepsilon}}$$

for all $\varepsilon>0$. Pick a section $e\in L(W)$ which trivializes $L|_W$. By the definition of h_L, one has

$$|e|^2_{h_L}\gtrsim\frac{1}{\prod_{i=1}^{\ell}|z_i|}.$$

Hence for the frame $(e_1\otimes e^{-1},\ldots,e_r\otimes e^{-1})$ trivializing $E\otimes L^{-1}|_W$, one has

$$|e_i\otimes e^{-1}|_h\lesssim\prod_{i=1}^{\ell}|z_i|^{1-\varepsilon}$$

for any $\varepsilon>0$. This shows that $h\cdot\prod_{i=1}^{\ell}|\sigma_i|^{-\varepsilon}_{h_i}$ vanishes on D when $\varepsilon>0$ is small enough. The proposition is proved. □

Theorem 3.9 (Infinitesimal Torelli-type property). *The morphism $\tau_1: T_Y(-\log D)\to L^{-1}\otimes E^{p_0-1,q_0+1}$ defined in (3.10) is generically injective.*

The proof is almost the same at that of [Den22a, Theorem D]. We provide it here for the sake of completeness.

Proof. The inclusion $L\subset E^{p_0,q_0}$ induces a global section $s\in H^0(Y,L^{-1}\otimes E^{p_0,q_0})$ by Theorem 3.6(ii); this section is *generically* non-vanishing over $U=Y-D$. Set

$$U_1:=\{y\in Y-(D\cup\mathbf{B}_+(L-D))\mid s(y)\neq 0\},\tag{3.11}$$

which is a non-empty Zariski open subset of U. Since the Hodge metric $h_{\rm hod}$ is a direct sum of metrics h_p on $E^{p,q}$, the metric h for $L^{-1}\otimes E$ is a direct sum of metrics $h_L^{-1}\cdot h_p$ on $L^{-1}\otimes E^{p,q}$, which is smooth over $U_0:=Y-(D\cup\mathbf{B}_+(L-D))$. Let D' be the $(1,0)$-part of its Chern connection over U_1 and Θ to be its curvature form. Then over U_0, we have

$$\begin{aligned}\Theta&=-\Theta_{L,h_L}\otimes\mathbb{1}+\mathbb{1}\otimes\Theta_{h_{p_0}}(E^{p_0,q_0})\\&=-\Theta_{L,h_L}\otimes\mathbb{1}-\mathbb{1}\otimes\left(\theta^*_{p_0,q_0}\wedge\theta_{p_0,q_0}\right)-\mathbb{1}\otimes\left(\theta_{p_0+1,q_0-1}\wedge\theta^*_{p_0+1,q_0-1}\right)\\&=-\Theta_{L,h_L}\otimes\mathbb{1}-\tilde{\theta}^*_{p_0,q_0}\wedge\tilde{\theta}_{p_0,q_0}-\tilde{\theta}_{p_0+1,q_0-1}\wedge\tilde{\theta}^*_{p_0+1,q_0-1},\end{aligned}\tag{3.12}$$

where we set

$$\theta_{p,q}=\theta|_{E^{p,q}}: E^{p,q}\longrightarrow E^{p-1,q+1}\otimes\Omega^1_Y(\log D)$$

and

$$\tilde{\theta}_{p,q}=\mathbb{1}\otimes\theta_{p,q}: L^{-1}\otimes E^{p,q}\longrightarrow L^{-1}\otimes E^{p-1,q+1}\otimes\Omega^1_Y(\log D)$$

and define $\tilde{\theta}^*_{p,q}$ to be the adjoint of $\tilde{\theta}_{p,q}$ with respect to the metric $h_L^{-1}\cdot h$. Hence over U_1, one has

$$-\mathrm{dd^c}\log|s|_h^2 = \frac{\left\{\sqrt{-1}\Theta(s),s\right\}_h}{|s|_h^2}+\frac{\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h}{|s|_h^4}-\frac{\sqrt{-1}\{D's,D's\}_h}{|s|_h^2}$$

$$\leqslant \frac{\left\{\sqrt{-1}\Theta(s),s\right\}_h}{|s|_h^2} \tag{3.13}$$

thanks to Cauchy–Schwarz inequality

$$\sqrt{-1}|s|_h^2\cdot\{D's,D's\}_h\geqslant\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h.$$

Substituting (3.12) into (3.13), over U_1, one has

$$\sqrt{-1}\Theta_{L,h_L}-\mathrm{dd^c}\log|s|_h^2\leqslant-\frac{\left\{\sqrt{-1}\tilde{\theta}^*_{p_0,q_0}\wedge\tilde{\theta}_{p_0,q_0}(s),s\right\}_h}{|s|_h^2}-\frac{\left\{\sqrt{-1}\tilde{\theta}_{p_0+1,q_0-1}\wedge\tilde{\theta}^*_{p_0+1,q_0-1}(s),s\right\}_h}{|s|_h^2}$$

$$=\frac{\sqrt{-1}\left\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\right\}_h}{|s|_h^2}+\frac{\left\{\tilde{\theta}^*_{p_0+1,q_0-1}(s),\tilde{\theta}^*_{p_0+1,q_0-1}(s)\right\}_h}{|s|_h^2}$$

$$\leq\frac{\sqrt{-1}\left\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\right\}_h}{|s|_h^2}, \tag{3.14}$$

where $\tilde{\theta}_{p_0,q_0}(s)\in H^0\big(Y,L^{-1}\otimes E^{p_0-1,q_0+1}\otimes\Omega_Y^1(\log D)\big)$. By Proposition 3.8(ii), one has $|s|_h^2(y)=0$ for any $y\in D\cup\mathbf{B}_+(L-D)$. Therefore, there exists a $y_0\in U_0$ so that $|s|_h^2(y_0)\geqslant|s|_h^2(y)$ for any $y\in U_0$. Hence $|s|_h^2(y_0)>0$, and by (3.11), $y_0\in U_1$. Since $|s|_h^2$ is smooth over U_0, $\mathrm{dd^c}\log|s|_h^2$ is seminegative at y_0 by the maximal principle. By Proposition 3.8(i), $\sqrt{-1}\Theta_{L,h_L}$ is strictly positive at y_0. By (3.14) and the relation $|s|_h^2(y_0)>0$, we conclude that $\sqrt{-1}\left\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\right\}_h$ is strictly positive at y_0. In particular, for any non-zero $\xi\in T_{Y,y_0}$, one has $\tilde{\theta}_{p_0,q_0}(s)(\xi)\neq0$. For $k=1$, we write τ_k in (3.10) as

$$\tau_1:\,T_Y(-\log D)\longrightarrow L^{-1}\otimes E^{p_0-1,q_0+1}.$$

Then over U, it is defined by $\tau_1(\xi):=\tilde{\theta}_{p_0,q_0}(s)(\xi)$ and is thus ***injective at*** $y_0\in U_1$. Hence τ_1 is ***generically injective***. The theorem is thus proved. □

4. Construction of a negatively curved Finsler metric

The aim of this technical section is to prove Theorem 1.4 based on Theorem 3.6. We first give the definition of a Finsler metric.

Definition 4.1 (Finsler metric). Let E be a holomorphic vector bundle on a complex manifold X. A *Finsler metric* on E is a real non-negative *continuous* function $h\colon E\to[0,+\infty[$ such that

$$h(av)=|a|h(v)$$

for any $a\in\mathbb{C}$ and $v\in E$. The metric h is ***positive definite*** on a subset $U\subset X$ if $h(v)>0$ for any non-zero $v\in E_x$ and any $x\in U$.

We mention that our definition is a bit different from that in [Kob98, Section 2.3], which requires *convexity*, and the Finsler metric therein can be upper-semicontinuous.

Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y,D) satisfying the two conditions in Theorem 3.6. We adopt the same notation as that in Theorem 3.6 and Section 3.4 throughout this section. Let us denote by n the largest non-negative number for k so that τ_k in (3.10) is not trivial. By Theorem 3.9, $n > 0$. Following [Den22a, Section 2.3], we construct Finsler metrics $F_1, \ldots, F_n$ on $T_Y(-\log D)$ as follows. By (3.10), for each $k = 1, \ldots, n$, there exists a

$$\tau_k \colon \operatorname{Sym}^k T_Y(-\log D) \longrightarrow L^{-1} \otimes E^{p_0-k, q_0+k}.$$

Then it follows from Proposition 3.8(ii) that the (Finsler) metric h on $L^{-1} \otimes E^{p_0-k,q_0+k}$ induces a Finsler metric F_k on $T_Y(-\log D)$ defined as follows: for any $e \in T_Y(-\log D)_y$,

$$F_k(e) := h\left(\tau_k\left(e^{\otimes k}\right)\right)^{\frac{1}{k}}. \tag{4.1}$$

Let $C \subset \mathbb{C}$ be any open subset of $\mathbb{C}$. For any holomorphic map $\gamma : C \to U := Y - D$, one has

$$d\gamma \colon T_C \longrightarrow \gamma^* T_U = \gamma^* T_Y(-\log D). \tag{4.2}$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector field in $C \subset \mathbb{C}$, and by $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. The Finsler metric F_k induces a continuous hermitian pseudo metric on C, defined by

$$\gamma^* F_k^2 = \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \tag{4.3}$$

Hence $G_k(t) = |\tau_k\big(d\gamma(\partial_t)^{\otimes k}\big)|_h^{2/k}$, where τ_k is defined in (3.10).

By Theorem 3.9, there is a Zariski open subset U° of U such that $U^\circ \cap \mathbf{B}_+(L) = \varnothing$ and τ_1 is injective at any point of U°. We now fix any holomorphic map $\gamma \colon C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. By Proposition 3.8(ii), the metric h for $L^{-1} \otimes E$ is smooth and positive definite over $U - \mathbf{B}_+(L)$. Hence $G_1(t) \not\equiv 0$. Let C° be a (non-empty) open subset of C whose complement $C \setminus C^\circ$ is a *discrete set* so that

- $\gamma(C^\circ) \subset U^\circ$;
- for every $k = 1, \ldots, n$, either $G_k(t) \equiv 0$ on C°, or $G_k(t) > 0$ for every $t \in C^\circ$;
- $\gamma'(t) \neq 0$ for any $t \in C^\circ$; namely $\gamma|_{C^\circ} \colon C^\circ \to U^0$ is immersive everywhere.

By the definition of $G_k(t)$, if $G_k(t) \equiv 0$ for some $k > 1$, then $\tau_k(\partial_t^{\otimes k}) \equiv 0$, where τ_k is defined in (3.10). Note that one has $\tau_{k+1}(\partial_t^{\otimes(k+1)}) = \tilde{\theta}(\tau_k(\partial_t^{\otimes k}))(\partial_t)$, where

$$\tilde{\theta} = \mathbb{1}_{L^{-1}} \otimes \theta \colon L^{-1} \otimes E \longrightarrow L^{-1} \otimes E \otimes \Omega_Y^1(\log D).$$

We thus conclude that $G_{k+1}(t) \equiv 0$. Hence there exists an m with $1 \leq m \leq n$ so that the set $\{k \mid G_k(t) > 0 \text{ over } C^\circ\} = \{1, \ldots, m\}$ and $G_\ell(t) \equiv 0$ for all $\ell = m+1, \ldots, n$. From now on, *all the computations* are made over C° if not specified.

Using the same computations as those in the proof of [Den22a, Proposition 2.10], we have the following curvature formula.

Theorem 4.2. *For $k = 1, \ldots, m$, over C°, one has*

$$\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} \geq \Theta_{L,h_L}\left(\partial_t, \bar{\partial}_t\right) - \frac{G_2^2}{G_1} \qquad \text{if } k = 1, \tag{4.4}$$

$$\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \geq \frac{1}{k}\left(\Theta_{L,h_L}\left(\partial_t, \bar{\partial}_t\right) + \frac{G_k^k}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^k}\right) \qquad \text{if } k > 1. \tag{4.5}$$

Here we make the convention that $G_{m+1} \equiv 0$ and $\frac{0}{0} = 0$. We also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively, where $d\gamma$ is defined in (4.2).

Let us mention that in [Den22a, Equation (2.2.11)], we dropped the term $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$ in (4.5), though it can be easily seen from the proof of [Den22a, Lemma 2.7].

We will follow ideas in [Den22a, Section 2.3] (inspired by [TY15, BPW22, Sch18]) to introduce a new Finsler metric F on $T_Y(-\log D)$ by taking a convex sum of the form

$$F := \sqrt{\sum_{k=1}^{n} k\alpha_k F_k^2}, \tag{4.6}$$

where $\alpha_1,\ldots,\alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later.

For the above, for $\gamma\colon C \to U$ with $\gamma(C)\cap U^\circ \neq \varnothing$, we write

$$\gamma^* F^2 = \sqrt{-1}H(t)dt\wedge d\bar{t}.$$

Then

$$H(t) = \sum_{k=1}^{n} k\alpha_k G_k(t), \tag{4.7}$$

where G_k is defined in (4.3). Recall that for $k=1,\ldots,m$, $G_k(t)>0$ for any $t\in C^\circ$.

We first recall a computational lemma by Schumacher.

Lemma 4.3 ([Sch18, Lemma 17]). *Let α_j and G_j be positive real numbers for $j=1,\ldots,n$. Then*

$$\sum_{j=2}^{n}\left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^{j}}{G_{j-1}^{j-2}}\right) \geqslant \frac{1}{2}\left(-\frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2 + \sum_{j=2}^{n-1}\left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\right)G_j^2\right) \tag{4.8}$$

Now we are ready to compute the curvature of the Finsler metric F based on Theorem 4.2.

Theorem 4.4. *Fix a smooth Kähler metric ω on Y. There exist* universal *constants $0<\alpha_1<\ldots<\alpha_n$ and $\delta>0$ such that for any holomorphic map $\gamma\colon C\to U = Y-D$ with C an open subset of $\mathbb{C}$ and $\gamma(C)\cap U^\circ\neq\varnothing$, one has*

$$\mathrm{dd^c}\log|\gamma'(t)|_F^2 \geq \delta\gamma^*\omega. \tag{4.9}$$

Proof. By Theorem 3.9 and the assumption that $\gamma(C)\cap U^\circ\neq\varnothing$, we have $G_1(t)\not\equiv 0$.

We first recall a result in [Den22a, Lemma 2.9]; we write its proof here as it is crucial in what follows.

Claim 4.5. *There is a universal constant $c_0>0$ (i.e., it does not depend on γ) so that $\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\geq c_0 G_1(t)$ for all $t\in C$.*

Proof of Claim 4.5. Indeed, by Proposition 3.8(i), it suffices to prove that

$$\frac{|\partial_t|^2_{\gamma^*(T)}}{\left|\tau_1(d\gamma(\partial_t))\right|_h^2} \geqslant c_0 \tag{4.10}$$

for some $c_0>0$, where T is a Kähler current on Y, which is a smooth complete metric over $Y-D$ of Poincaré type. It can be seen as a singular hermitian metric for $T_Y(-\log D)$. Hence for any admissible coordinate $(\mathcal{U};z_1,\ldots,z_n)$, one has

$$\left|z_i\frac{\partial}{\partial z_i}\right|_T \sim (-\log|z_i|)^{-1}.$$

On the other hand, by Proposition 3.8(ii), one has

$$\left|\tau_1\left(z_i\frac{\partial}{\partial z_i}\right)\right|_h \lesssim C\cdot\prod_{i=1}^{\ell}|z_i|^{\varepsilon}$$

for some constant $\varepsilon>0$. Hence one has $\tau_1^*h\lesssim T$. Since Y is compact, there exists a constant $c_0>0$ such that $T\geqslant c_0\tau_1^*h$. Therefore,

$$\frac{|\partial_t|^2_{\gamma^*T}}{\left|\tau_1(d\gamma(\partial_t))\right|_h^2} = \frac{|\partial_t|^2_{\gamma^*T}}{|\partial_t|^2_{\gamma^*\tau_1^*h}} \geq c_0.$$

Hence (4.10) holds for any $\gamma\colon C\to U$ with $\gamma(C)\cap U^\circ\neq\varnothing$. The claim is proved. □

By [Sch12, Lemma 8],

$$\sqrt{-1}\partial\bar{\partial}\log\left(\sum_{j=1}^{n} j\alpha_j G_j\right) \geqslant \frac{\sum_{j=1}^{n} j\alpha_j G_j \sqrt{-1}\partial\bar{\partial}\log G_j}{\sum_{i=1}^{n} j\alpha_j G_i}. \tag{4.11}$$

Substituting (4.4) and (4.5) into (4.11), and observing the convention that $\frac{0}{0} = 0$, we obtain

$$\begin{aligned}
\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} &\geq \frac{1}{H}\left(-\alpha_1 G_2^2 + \sum_{k=2}^{n} \alpha_k \left(\frac{G_k^{k+1}}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^{k-1}}\right)\right) + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&= \frac{1}{H}\left(\sum_{j=2}^{n}\left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\right)\right) + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&\overset{(4.8)}{\geq} \frac{1}{H}\left(-\frac{1}{2}\frac{\alpha_1^3}{\alpha_2^2}G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\right)G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\right) \\
&\quad + \frac{\sum_{k=1}^{n} \alpha_k G_k}{H}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \\
&\overset{\text{Claim } 4.5}{\geq} \frac{1}{H}\left(\frac{\alpha_1}{2}\left(c_0 - \frac{\alpha_1^2}{\alpha_2^2}\right)G_1^2 + \frac{1}{2}\sum_{j=2}^{n-1}\left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\right)G_j^2 + \frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\right) \\
&\quad + \frac{1}{H}\left(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n}\alpha_k G_k\right)\Theta_{L,h_L}(\partial_t, \bar{\partial}_t).
\end{aligned}$$

One can take $\alpha_1 = 1$ and choose the further $\alpha_j > \alpha_{j-1}$ inductively so that

$$c_0 - \frac{\alpha_1^2}{\alpha_2^2} > 0, \quad \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} > 0 \quad \forall\ j = 2, \ldots, n-1. \tag{4.12}$$

Hence

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq \frac{1}{H}\left(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n}\alpha_k G_k\right)\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \overset{(4.7)}{\geq} \frac{1}{n}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$$

over C°. By Proposition 3.8(i), this implies that

$$\mathrm{dd^c}\log|\gamma'|_F^2 = \mathrm{dd^c}\log H(t) \geq \frac{1}{n}\gamma^*\sqrt{-1}\Theta_{L,h_L} \geq \delta\gamma^*\omega \tag{4.13}$$

over C° for some positive constant δ which does not depend on γ. Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C, by the extension theorem of subharmonic function, (4.13) holds over the whole C. Since $c_0 > 0$ is a constant which does not depend on γ, so are $\alpha_1, \ldots, \alpha_n$ by (4.12). The theorem is thus proved. □

As a summary of the results in this subsection, we obtain the following theorem.

Theorem 4.6. *Let $(E = \oplus_{p+q=\ell}E^{p,q}, \theta)$ be a system of log Hodge bundles on a compact Kähler log pair (Y, D) satisfying the two conditions in Theorem 3.6. Then there are a Finsler metric h on $T_Y(-\log D)$ which is* positive definite *on a dense Zariski open subset U° of $U := Y - D$ and a smooth Kähler form ω on Y such that for any holomorphic map $\gamma\colon C \to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\mathrm{dd^c}\log|\gamma'|_h^2 \geq \gamma^*\omega. \tag{4.14}$$

Proof of Theorem 1.4. Theorem 3.6 together with Theorem 4.6 imply Theorem 1.4. □

5. Big Picard theorem and algebraic hyperbolicity

5.1. Algebraic and Picard hyperbolicity

In Definition 1.3, we have seen the definition of *algebraic hyperbolicity* for a compact complex manifold X, which was introduced by Demailly in [Dem97, Definition 2.2]. He proved in [Dem97, Theorem 2.1] that X is algebraically hyperbolic if it is Kobayashi hyperbolic. The notion of algebraic hyperbolicity was generalized to log pairs by Chen [Che04].

Definition 5.1 (Algebraic hyperbolicity). Let (X,D) be a compact Kähler log pair. For any reduced irreducible curve $C \subset X$ such that $C \not\subset D$, we denote by $i_X(C,D)$ the number of distinct points in the set $\nu^{-1}(D)$, where $\nu: \tilde{C} \to C$ is the normalization of C. The log pair (X,D) is *algebraically hyperbolic* if there is a smooth Kähler metric ω on X such that

$$2g(\tilde{C}) - 2 + i(C,D) \geq \deg_\omega C := \int_C \omega$$

for all curves $C \subset X$ as above.

Note that $2g(\tilde{C}) - 2 + i(C,D)$ depends only on the complement $C - D$. Hence the above notion of hyperbolicity also makes sense for quasi-projective manifolds: we say that a quasi-projective manifold U is algebraically hyperbolic if it has a log compactification (X,D) which is algebraically hyperbolic.

However, unlike Demailly's theorem, it is unclear to us that Kobayashi hyperbolicity or Picard hyperbolicity of $X - D$ will imply algebraic hyperbolicity of (X,D). In [PR07], Pacienza–Rousseau proved that if $X - D$ is hyperbolically embedded into X, the log pair (X,D) (and thus $X - D$) is algebraically hyperbolic.

Before we prove that Definition 1.1 does not depend on the compactification of U, we will need the following proposition, which is a consequence of the deep extension theorem of meromorphic maps by Siu [Siu75]. The *meromorphic map* in this paper is defined in the sense of Remmert, and we refer the reader to [FG02, p. 243] for the precise definition.

Proposition 5.2. *Let Y° be a Zariski open subset of a compact Kähler manifold Y. Assume that Y° is Picard hyperbolic. Then any holomorphic map $f: \Delta^p \times (\Delta^*)^q \to Y^\circ$ extends to a meromorphic map $\overline{f}: \Delta^{p+q} \dashrightarrow Y$. In particular, any holomorphic map g from a Zariski open subset X° of a compact complex manifold X to Y° extends to a meromorphic map from X to Y.*

Proof. By [Siu75, Theorem 1], any meromorphic map from a Zariski open subset Z° of a complex manifold Z to a compact Kähler manifold Y extends to a meromorphic map from Z to Y provided that the codimension of $Z - Z^\circ$ is at least 2. The complement $\Delta^p \times (\Delta^*)^q$ in Δ^{p+q} is a simple normal crossing divisor D. We remove a subvariety $Z \subset \Delta^{p+q}$ of codimension at least 2 with $D - Z$ smooth. Then any point $x \in D - Z$ has an open neighborhood $\Omega_x \subset \Delta^{p+q} - Z$ which is isomorphic to $\Delta^{p+q-1} \times \Delta^*$. It then suffices to prove the extension theorem for any holomorphic map $f: \Delta^r \times \Delta^* \to Y^\circ$.

By the assumption that Y° is Picard hyperbolic, for any $z \in \Delta^r$, the holomorphic map $f|_{\{z\}\times\Delta^*}: \{z\} \times \Delta^* \to Y^\circ$ can be extended to a holomorphic map from $\{z\} \times \Delta$ to Y. It then follows from [Siu75, p.442, (*)] that f extends to a meromorphic map $\overline{f}: \Delta^{r+1} \dashrightarrow Y$. This proves the first part of the proposition. To prove the second part, we first apply the Hironaka theorem on resolution of singularities to assume that $X - X^\circ$ is a simple normal crossing divisor on X. Then any point $x \in X - X^\circ$ has an open neighborhood Ω_x which is isomorphic to Δ^{p+q} so that $X^\circ \simeq \Delta^p \times (\Delta^*)^q$ under this isomorphism. The above arguments show that $g|_{\Omega_x \cap X^\circ}$ extends to a meromorphic map from Ω_x to Y, and thus g can be extended to a meromorphic map from X to Y. The proposition is proved. □

Let us prove that Definition 1.1 does not depend on the compactification of U. This independence also implies the following result.

Lemma 5.3. *Let U be a Zariski open subset of a compact Kähler manifold Y. If any holomorphic map $f: \Delta^* \to U$ extends to $\bar{f}: \Delta \to Y$, then Y is bimeromorphic to any other compact Kähler manifold Y' which contains U as a Zariski open set. In particular, $f: \Delta^* \to U$ also extends to a holomorphic map $\Delta \to Y'$.*

Proof. By blowing up $Y - U$ and $Y' - U$, we can assume that both $Y - U$ and $Y' - U$ are simple normal crossing divisors. By the same arguments as those in the proof of Proposition 5.2, the identity map of U extends to meromorphic maps $a: Y \dashrightarrow Y'$ and $b: Y' \dashrightarrow Y$. Note that $a \circ b|_U$ and $b \circ a|_U$ are identity maps. Hence Y and Y' are bimeromorphic. Composing b with $\bar{f}$, one obtains the desired extension $\Delta \to Y'$ of $f: \Delta^* \to U$ in Y'. □

By Chow's theorem, Proposition 5.2 in particular gives an alternative proof of the fact that a Picard hyperbolic variety is moreover Borel hyperbolic, proven in [JK20, Corollary 3.11].

5.2. Proof of Theorem A

This subsection is devoted to the proof of Theorem A. We first recall the following criteria for Picard hyperbolicity established in [DLS+19], whose proof is Nevanlinna-theoretic.

Theorem 5.4 ([DLS+19, Theorem A]). *Let Y be a projective manifold, and let D be a simple normal crossing divisor on Y. Let $f: \Delta^* \to Y - D$ be a holomorphic map. Assume that there is a (possibly degenerate) Finsler metric h of $T_Y(-\log D)$ such that $|f'(t)|_h^2 \not\equiv 0$ and*

$$\mathrm{dd^c} \log |f'(t)|_h^2 \geq f^*\omega \tag{5.1}$$

for some smooth Kähler metric ω on Y. Then f extends to a holomorphic map $\overline{f}: \Delta \to Y$.

We will combine Theorem 5.4 with Theorem 1.4 to prove Theorem A.

Proof of Theorem A. By Theorem 1.4, there exist finitely many compact Kähler log pairs $\{(X_i, D_i)\}_{i=0,\ldots,N}$ so that the following hold:

1. There are morphisms $\mu_i: X_i \to Y$ with $\mu_i^{-1}(D) = D_i$ so that each $\mu_i: X_i \to \mu_i(X_i)$ is a birational morphism and $X_0 = Y$ with $\mu_0 = \mathbb{1}$.
2. There are smooth Finsler metrics h_i for $T_{X_i}(-\log D_i)$ which is positive definite over a Zariski open subset U_i° of $U_i := X_i - D_i$.
3. The restriction $\mu_i|_{U_i^\circ}: U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
4. There are smooth Kähler metrics ω_i on X_i such that for any holomorphic map $\gamma: C \to U_i$ with C an open subset of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, one has
$$\mathrm{dd^c} \log |\gamma'|_{h_i}^2 \geq \gamma^*\omega_i. \tag{5.2}$$
5. For any $i \in \{0, \ldots, N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero-dimensional, or there exists an $I \subset \{0, \ldots, N\}$ so that
$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j \in I} \mu_j(X_j).$$

Let us explain how to construct these log pairs. By the assumption, there is a $\mathbb{C}$-PVHS on $Y - D$ so that each fiber of the period map is zero-dimensional. In particular, the period map is generically immersive. We then apply Theorem 1.4 to construct a Finsler metric on $T_Y(-\log D)$ which is positive definite over some Zariski open subset U° of $U = Y - D$ with the desired curvature property (4.14). Set $X_0 = Y$, $\mu_0 = \mathbb{1}$ and $U_0^\circ = U^\circ$. Let $Z_1, \ldots, Z_m$ be all irreducible subvarieties of $Y - U^\circ$ which are not components of D. Then $Z_1 \cup \ldots \cup Z_m \supset U \setminus U^\circ$. For each i, we take a desingularization $\mu_i: X_i \to Z_i$ so that $D_i := \mu_i^{-1}(D)$ is a simple normal crossing divisor in X_i. We pull back the $\mathbb{C}$-PVHS to $U_i = X_i - D_i$ via μ_i. Then its period map is still generically immersive. We then apply Theorem 1.4 to construct the desired Finsler metrics in item (4) for $T_{X_i}(-\log D_i)$. We iterate this construction, and since at each step the dimension of X_i is strictly decreasing, this algorithm stops after finitely many steps.

(i) We will first prove that U is Picard hyperbolic. Fix any holomorphic map $f: \Delta^* \to U$. If $f(\Delta^*) \cap U_0^\circ \neq \varnothing$, then $|f'(t)|_{h_0} \not\equiv 0$ by item (2). By item (4), there is a smooth Kähler metric ω_0 on X_0 such that

$$\mathrm{dd^c} \log |f'(t)|_{h_0}^2 \geq f^*\omega_0.$$

We now apply Theorem 5.4 to conclude that f extends to a holomorphic map $\overline{f}: \Delta \to X_0 = Y$.

Now assume $f(\Delta^*) \cap \mu_0(U_0^\circ) = \varnothing$. By item (5), there exists an $I_0 \subset \{0, \ldots, N\}$ so that

$$f(\Delta^*) \subset \mu_0(U_0) - \mu_0(U_0^\circ) \subset \cup_{j \in I_0} \mu_j(X_j).$$

Since the $\mu_j(X_j)$ are all irreducible, there exists a $k \in I_0$ so that $f(\Delta^*) \subset \mu_k(X_k)$. Note that $U_k := \mu_k^{-1}(U)$. Hence $f(\Delta^*) \subset \mu_k(U_k)$. If $f(\Delta^*) \cap \mu_k(U_k^\circ) \neq \varnothing$, then by item (3), $f(\Delta^*)$ is not contained in the exceptional set of μ_k. Hence f can be lifted to $f_k: \Delta^* \to U_k$, so that $\mu_k \circ f_k = f$ and $f_k(\Delta^*) \cap U_k^\circ \neq \varnothing$. By Theorem 5.4 and item (4) again, we conclude that f_k extends to a holomorphic map $\overline{f}_k: \Delta \to X_k$. Hence $\mu_k \circ \overline{f}_k$ extends f. If $f(\Delta^*) \cap \mu_k(U_k^\circ) = \varnothing$, we apply item (5) to iterate the above arguments, and after finitely many steps, there exists an X_i so that $f(\Delta^*) \subset \mu_i(U_i)$ and $f(\Delta^*) \cap \mu_i(U_i^\circ) \neq \varnothing$. By item (3), f can be lifted to $f_i: \Delta^* \to U_i$ so that $\mu_i \circ f_i = f$ and $f_i(\Delta^*) \cap U_i^\circ \neq \varnothing$. By Theorem 5.4 and item (4) again, f_i extends to the origin, and so does f. This proves the Picard hyperbolicity of $U = Y - D$.

(ii) Let us prove the algebraic hyperbolicity of U in as similar vein as [DLS+19, Proof of Theorem D]. Fix any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$. By the above arguments, there exists an $i \in \{0, \ldots, N\}$ so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. Let $C_i \subset X_i$ be the strict transform of C under μ_i. By item (3), $h_i|_{C_i}$ is not identically equal to zero.

Denote by $\nu_i: \tilde{C}_i \to C_i \subset X_i$ the normalization of C_i, and set $P_i := (\mu_i \circ \nu_i)^{-1}(D) = \nu_i^{-1}(D_i)$. Then

$$d\nu_i: T_{\tilde{C}_i}(-\log P_i) \longrightarrow \nu_i^* T_{X_i}(-\log D_i)$$

induces a (non-trivial) hermitian pseudo metric $\tilde{h}_i := \nu_i^* h_i$ over $T_{\tilde{C}_i}(-\log P_i)$. By (5.2), the *curvature current* of $\tilde{h}_i^{-1}$ on $\tilde{C}_i$ satisfies

$$\frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \nu_i^* \omega_i.$$

Hence

$$2g(\tilde{C}_i) - 2 + i(C, D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi} \Theta_{\tilde{h}_i^{-1}}\left(K_{\tilde{C}_i}(\log P_i)\right) \geq \int_{\tilde{C}_i} \nu_i^* \omega_i.$$

Fix a Kähler metric Ω_Y^1 on Y. Then there is a constant $\varepsilon_i > 0$ so that $\omega_i \geq \varepsilon_i \mu_i^* \Omega_Y^1$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C, D) \geq \varepsilon_i \int_{\tilde{C}_i} (\mu_i \circ \nu_i)^* \Omega_Y^1 = \varepsilon_i \deg_{\Omega_Y^1} C$$

for $\mu_i \circ \nu_i: \tilde{C}_i \to C$ the normalization of C. Set $\varepsilon := \inf_{i=0,\ldots,N} \varepsilon_i$. Then we conclude that for any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\Omega_Y^1} C,$$

where $\tilde{C} \to C$ is its normalization. This shows the algebraic hyperbolicity of U. The proof of the theorem is accomplished. □

Let us mention that the idea of using Finsler metrics to prove the hyperbolicity in the above theorem was inspired by the work of To–Yeung in [TY15].

Remark 5.5. Let U be a quasi-projective manifold admitting an integral variation of Hodge structures whose period map is quasi-finite. In [JL19, Theorem 4.2], Javanpeykar–Litt proved that U is *weakly bounded* in the sense of Kovács–Lieblich [KL10, Definition 2.4] (which is weaker than algebraic hyperbolicity). Though not mentioned explicitly, their proof of [JL19, Theorem 4.2] implicitly shows that such a U is also algebraically hyperbolic when the local monodromies of the $\mathbb{C}$-PVHS at infinity are unipotent. Their proof is based on the work [BBT23] as well as the Arakelov-type inequality for Hodge bundles by Peters [Pet00].

We end this section with the following remark.

Remark 5.6. Let (E,θ) be the system of log Hodge bundles on a log pair (Y,D) as that in Theorem 4.6. One can also use the idea by Viehweg–Zuo [VZ02] in constructing their *Viehweg–Zuo* sheaf (based on the negativity of kernels of Higgs fields by Zuo [Zuo00]) to prove a weaker result than Theorem 4.6: for any holomorphic map $\gamma\colon C\to U$ from any open subset C of $\mathbb{C}$ with $\gamma(C)\cap U^{\circ}\neq\varnothing$, there exist a Finsler metric h_C of $T_Y(-\log D)$ (depending on C) and a Kähler metric ω_C for Y (also depending on C) so that $|\gamma'(t)|_h^2\not\equiv 0$ and

$$\mathrm{dd}^{\mathrm{c}}\log|\gamma'|^2_{h_C}\geq\gamma^*\omega_C.$$

It follows from our proof of Theorem A that one can also combine Theorem 5.4 with this result, which is only weaker in appearance, to prove Theorem A. The more general result Theorem 1.4 will be used to prove Theorem 6.1(ii) in the next section.

6. Hyperbolicity for the compactification after a finite étale cover

In this section, we will prove Theorem B and Corollary C. We first prove the following theorem.

Theorem 6.1. *Let U be a quasi-compact Kähler manifold. Assume that there is a $\mathbb{C}$-PVHS over U whose period map is immersive at one point. Then there are a finite étale cover $\tilde{U}\to U$ together with a compact Kähler compactification X of $\tilde{U}$ and a proper Zariski closed subvariety $Z\subsetneq X$ so that*

(i) *the variety X is of general type;*
(ii) *the variety X is Kobayashi hyperbolic modulo Z;*
(iii) *the variety X is Picard hyperbolic modulo Z;*
(iv) *the variety X is algebraically hyperbolic modulo Z.*

Let us briefly explain the idea of the proof of Theorem 6.1 and some related results. Let Y be a compact Kähler manifold compactifying U with $D:=Y-U$ a simple normal crossing divisor. By Theorem 3.6, there is a special system of log Hodge bundles $(E,\theta):=(\oplus_{p+q=\ell}E^{p,q},\oplus_{p+q=\ell}\theta_{p,q})$ on (Y,D) satisfying the properties therein. We divide the proof into four steps.

(1) The first step is devoted to constructing a compact Kähler log pair $(X,\tilde{D})$ and a generically finite surjective log morphism $\mu\colon(X,\tilde{D})\to(Y,D)$ which is étale over U so that for each irreducible component $\tilde{D}_i$ of $\tilde{D}$,
   - either $\mathrm{ord}_{\tilde{D}_i}(\mu^*D)\gg 1$,
   - or the local monodromy of the pull-back $\mathbb{C}$-PVHS over $\tilde{U}$ around $\tilde{D}_i$ is trivial.

   To find this μ, we apply the well-known result that that monodromy group of a $\mathbb{C}$-PVHS is residually finite and use the Cauchy argument principle to show the high ramification over irreducible components of $\tilde{D}$ around which the local monodromies are not trivial. Let us mention that this step is quite different from those in [Nad89, Rou16, Bru20a, Cad22] for the hyperbolicity of compactifications of quotients of bounded symmetric domains by a torsion-free lattice, as they all applied Mumford's work on toroidal compactifications of quotients of bounded symmetric domains [Mum77] so that $\mathrm{ord}_{\tilde{D}_i}(\mu^*D)\gg 1$ for all $\tilde{D}_i$. In general, we are not sure that such a covering can be found in our case.
(2) The second step is to construct a new system of log Hodge bundles $(G=\oplus_{p+q=\ell}G^{p,q},\eta)$ over $(X,\tilde{D})$ which is the canonical extension of the pull-back of the $\mathbb{C}$-PVHS via μ. This system of log Hodge bundles $(G=\oplus_{p+q=\ell}G^{p,q},\eta)$ on $(X,\tilde{D})$ satisfies the two conditions in Theorem 3.6. Moreover, some $G^{p_0,\ell-p_0}$ contains $\tilde{L}\otimes\mathcal{O}_X(\ell D_X)$ with $\tilde{L}$ a big line bundle. Here D_X is the sum of irreducible

components of $\tilde{D}$ around which the local monodromies of the pull-back $\mathbb{C}$-PVHS are not trivial (hence μ is highly ramified over D_X). Note that (G,η) has singularities along D_X instead of $\tilde{D}$ since the pull-back $\mathbb{C}$-PVHS extends across the components where the local monodromies are trivial (see (6.4).)

(3) The third step is to prove Theorem 6.1(i). We start with $G^{p_0,\ell-p_0}$ and iterate the Higgs field η, ending at finitely many steps. By the negativity of the kernel of $\tilde{\theta}$, $\tilde{L}\otimes\mathcal{O}(\ell D_X)\subset G^{p_0,\ell-p_0}$, and (6.4), we can construct an ample sheaf contained in some symmetric differential $\mathrm{Sym}^\beta\Omega_X^1$ (rather than on $\mathrm{Sym}^\beta\Omega_X^1(\log\tilde{D})$!). It follows from a celebrated theorem of Campana–Păun [CP19] that X is of general type. Let us mention that this idea of iterating Higgs fields to their kernels, originally due to Viehweg–Zuo [VZ02], has been used by Brunebarbe in [Bru20a], in which he proved similar results for quotients of bounded symmetric domains by arithmetic groups. There are also some similar results for *quotients of bounded domains* by Boucksom–Diverio [BD21] and Cadorel–Diverio–Guenancia [CDG19].

(4) The last step is to prove Theorem 6.1(ii)–Theorem 6.1(iv). We use the above system of log Hodge bundles (G,η) and ideas in Section 4 to construct a Finsler metric F on T_X (rather than $T_X(-\log D)$!) due to the extra positivity $\tilde{L}\otimes\mathcal{O}(\ell D_X)\subset G^{p_0,\ell-p_0}$. Such a metric F is generically positive and has holomorphic sectional curvature bounded from above by a negative constant by Theorem 4.4. By the Ahlfors–Schwarz lemma, we conclude that X is Kobayashi hyperbolic modulo a proper closed subvariety, and by Theorem 5.4, the Picard hyperbolicity modulo a proper subset of X follows. Let us mention that Rousseau [Rou16] has proved a similar result for hermitian symmetric spaces, which was later refined by Cadorel [Cad22]. Their methods use Bergman metrics for bounded symmetric domains instead of Hodge theory.

Now we start the detailed proof of Theorem 6.1.

Proof of Theorem 6.1. By Theorem 3.6, there is a system of log Hodge bundles $(E,\theta)=(\oplus_{p+q=\ell}E^{p,q},\oplus_{p+q=\ell}\theta_{p,q})$ over (Y,D) satisfying the two conditions therein. In particular, there are a big line bundle L on Y and an inclusion $L\subset E^{p_0,\ell-p_0}$ for some $0\leq p_0\leq\ell$. Pick $m\gg 1$ so that $L-\frac{\ell+1}{m}D$ is a big $\mathbb{Q}$-line bundle.

Step 1a. Fix a base point $y\in U:=Y-D$. Let us denote by $\rho\colon\pi_1(U,y)\to GL(r,\mathbb{C})$ the monodromy representation of the corresponding $\mathbb{C}$-PVHS and denote by $\Gamma:=\rho(\pi_1(U,y))$ its monodromy group, which is a finitely generated linear group, hence residually finite by a theorem of Malcev [Mal40]. Let us cover Y by finitely many admissible coordinate systems

$$\left\{\left(\mathcal{U}_\alpha;z_1^{(\alpha)},\ldots,z_d^{(\alpha)}\right)\right\}_{\alpha\in S},$$

where S is a finite set, so that

$$D\cap\mathcal{U}_\alpha=\left(z_1^{(\alpha)}\cdots z_{k_\alpha}^{(\alpha)}=0\right).$$

Write $\mathcal{U}_\alpha^*:=\mathcal{U}_\alpha-D$. The fundamental group $\pi_1(\mathcal{U}_\alpha^*,y_\alpha)\simeq\pi_1((\Delta^*)^{k_\alpha}\times\Delta^{d-k_\alpha},y_\alpha)\simeq\mathbb{Z}^{k_\alpha}$ is abelian. Pick a base point $y_\alpha\in\mathcal{U}_\alpha^*$. Let $e_1^{(\alpha)},\ldots,e_{k_\alpha}^{(\alpha)}$ be the generators of $\pi_1((\Delta^*)^{k_\alpha}\times\Delta^{d-k_\alpha},y_\alpha)$; namely, $e_i^{(\alpha)}$ is a clockwise loop around the origin in the i^{th} factor Δ^*. Pick a path $h_\alpha\colon[0,1]\to Y-D$ connecting y_α with y, and denote by $\gamma_i^{(\alpha)}\in\pi_1(Y-D,y)$ the equivalent class of the loop $h_\alpha^{-1}\cdot e_i^{(\alpha)}\cdot h_\alpha$. Set $T_i^{(\alpha)}:=\rho(\gamma_i^{(\alpha)})$. Clearly, $T_1^{(\alpha)},\ldots,T_{\alpha_k}^{(\alpha)}$ commute pairwise.

Let $\mathfrak{S}\subset\Gamma$ be the finite subset defined by

$$\left\{\left(T_1^{(\alpha)}\right)^{\ell_1}\cdots\left(T_{k_\alpha}^{(\alpha)}\right)^{\ell_{k_\alpha}}\mid\alpha\in S,0\leq\ell_i<m\right\},\tag{6.1}$$

where m is the integer chosen at the beginning. It follows from the definition of a residually finite group that there is a normal subgroup $\tilde{\Gamma}$ of Γ with finite index so that

$$\mathfrak{S}\cap\tilde{\Gamma}=\{0\}.\tag{6.2}$$

Then $\rho^{-1}(\tilde{\Gamma})$ is a normal subgroup of $\pi_1(U,y)$ with finite index. Let $\nu\colon \tilde{U} \to U$ be the finite étale cover of U so that for the induced map of the fundamental group $\nu_*\colon \pi_1(\tilde{U},x) \to \pi_1(U,y)$, its image is $\rho^{-1}(\tilde{\Gamma})$. Here $x \in \tilde{U}$ with $\mu(x) = y$. We consider $\pi_1(\tilde{U},x)$ as a subgroup of $\pi_1(U,y)$ of finite index. Since the monodromy representation of the pull-back of the $\mathbb{C}$-PVHS on $\tilde{U}$ is the restriction

$$\rho|_{\pi_1(\tilde{U},x)}\colon \pi_1(\tilde{U},x) \longrightarrow GL(r,\mathbb{C}),$$

its monodromy group is thus $\tilde{\Gamma}$.

Step 1b. Note that U is quasi-projective. Hence $\tilde{U}$ is also quasi-projective. Let us take a smooth projective compactification X of $\tilde{U}$ with $\tilde{D} := X - \tilde{U}$ simple normal crossing so that $\nu\colon \tilde{U} \to U$ extends to a log morphism $\mu\colon (X,\tilde{D}) \to (Y,D)$. Write $\tilde{D} = \sum_{j=1}^{n} \tilde{D}_j$, where the $\tilde{D}_j$ are irreducible components of $\tilde{D}$.

Claim 6.2. *For each $j = 1,\ldots,n$, one has*

- *either* $\mathrm{ord}_{\tilde{D}_j}(\mu^* D) \geq m$,
- *or the local monodromy group of the pull-back $\mathbb{C}$-PVHS around $\tilde{D}_j$ is trivial.*

Proof of Claim 6.2. Since $\{\mathcal{U}^{(\alpha)}\}_{\alpha\in S}$ covers D, there is an $\alpha \in S$ so that for the admissible coordinate system $(\mathcal{U}^{(\alpha)}; z_1^{(\alpha)},\ldots,z_d^{(\alpha)})$, one has $\mu^{-1}(\mathcal{U}^{(\alpha)}) \cap \tilde{D}_j \neq \varnothing$. We will write $(\mathcal{U}; z_1,\ldots,z_d)$ instead of $(\mathcal{U}^{(\alpha)}; z_1^{(\alpha)},\ldots,z_d^{(\alpha)})$ and k instead of k_α to lighten the notation. Namely, $\mathcal{U} \cap D = (z_1 \cdots z_k = 0)$. Note that $k \geq 1$.

Pick a point $x \in \tilde{D}_j - \cup_{i\neq j}\tilde{D}_i$ so that there is an admissible coordinate system $(\mathcal{W}; x_1,\ldots,x_n)$ with $\mu(\mathcal{W}) \subset \mathcal{U}$ and $\mathcal{W} \cap \tilde{D} = (x_1 = 0)$. Denote by $(\mu_1(x),\ldots,\mu_d(x))$ the expression of μ within these coordinates. Then

$$(\mu_1(x),\ldots,\mu_d(x)) = \left(x_1^{n_1}\nu_1(x),\ldots,x_1^{n_k}\nu_k(x),\mu_{k+1}(x),\ldots,\mu_d(x)\right),$$

where $\nu_1(x),\ldots,\nu_k(x)$ are holomorphic functions defined on $\mathcal{W}$ so that none of them is identically equal to zero on $(x_1 = 0)$, and $n_p \geq 0$ for $p = 1,\ldots,k$.

We thus can choose a slice $S := \{(x_1,\ldots,x_d) \mid \{|x_1| \leq \varepsilon, x_2 = \zeta_2,\ldots,x_d = \zeta_d\} \subset \mathcal{W}$ so that $\nu_i(x) \neq 0$ for any $x \in S$ and any $i = 1,\ldots,k$. Let us define a loop $e(\theta)\colon [0,1] \to \mathcal{W}^* := \mathcal{W} - \tilde{D}$ by $e(\theta) := (\varepsilon e^{2\pi i\theta},\zeta_2,\ldots,\zeta_d)$ which is the generator of $\pi_1(\mathcal{W}^*,x_0)$, where $x_0 \in \mathcal{W}^*$ is a point with $\mu(x_0) = y_\alpha \in \mathcal{U}^*$. By *Cauchy's argument principle*, the winding number of $\mu_p \circ e(\theta)$ around 0 is n_p for $p = 1,\ldots,k$. Hence by the diagram

$$\begin{tikzcd}
\pi_1(\mathcal{W}^*,x_0) \arrow[r, "\nu_*"] \arrow[d, "\simeq"] & \pi_1(\mathcal{U}^*,y_\alpha) \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}^k,
\end{tikzcd}$$

one has $\nu_*(1) = (n_1,\ldots,n_p)$.

Pick a path $\tilde{h}\colon [0,1] \to \tilde{U}$ connecting x and x_0, which lifts the above path $h_\alpha\colon [0,1] \to U$. Set $\tilde{\gamma}_0 \in \pi_1(\tilde{U},x)$ to be the equivalence class of the loop $\tilde{h}^{-1}\cdot e \cdot \tilde{h}$. Then

$$\nu_*(\tilde{\gamma}_0) = \left[h_\alpha^{-1}\cdot\left(e_1^{(\alpha)}\right)^{n_1}\cdots\left(e_k^{(\alpha)}\right)^{n_k}\cdot h_\alpha\right] = \left(\gamma_1^{(\alpha)}\right)^{n_1}\cdots\left(\gamma_k^{(\alpha)}\right)^{n_k}$$

Therefore,

$$\left(T_1^{(\alpha)}\right)^{n_1}\cdots\left(T_k^{(\alpha)}\right)^{n_k} = \rho\left(\nu_*(\tilde{\gamma}_0)\right) \in \tilde{\Gamma}.$$

By (6.1) and (6.2), either $\rho\left(\nu_*(\tilde{\gamma}_0)\right) = 0$, or there is some $i \in \{1,\ldots,k\}$ so that $n_i \geq m$. The first case means that the local monodromy of the pull-back $\mathbb{C}$-PVHS around $\tilde{D}_j$ is trivial. In the latter case, one has

$$\mathrm{ord}_{\tilde{D}_j}(\mu^* D) = \sum_{i=1}^{k} n_i \geq m.$$

The claim is proved. □

Step 2. Set $D_X \subset D$ to be the sum of all $\tilde{D}_j$ so that the local monodromy group of the pull-back $\mathbb{C}$-PVHS around $\tilde{D}_j$ is ***not*** trivial. Then by the dichotomy in Claim 6.2, $\mu^*D - mD_X$ is an effective divisor, and the pull-back $\mathbb{C}$-PVHS on $\tilde{U}$ around $\tilde{D}_i$ with $\tilde{D}_i \not\subset D_X$ is trivial. Note that the pull-back $\mathbb{C}$-PVHS extends to a $\mathbb{C}$-PVHS defined over $X - D_X$.

By the second condition in Theorem 3.6(i), (E, θ) is the canonical extension (in the sense of Definition 2.11) of some system of Hodge bundles $(\tilde{E} = \oplus_{p+q=\ell}\tilde{E}^{p,q}, \tilde{\theta}, h_{\rm hod})$ defined over $Y - D$. Hence for any admissible coordinate $(\mathcal{U}; z_1, \ldots, z_d)$ and any holomorphic frame $(e_1, \ldots, e_{r'})|_U$ for $E^{p,q}$, one has

$$|e_j|_{\rm hod} \lesssim \frac{1}{\prod_{i=1}^k |z_i|^\varepsilon}$$

for all $\varepsilon > 0$. If we take an admissibe coordinate $(\mathcal{W}; x_1, \ldots, x_d)$ with $\mathcal{W} \cap \tilde{D} = (x_1 \cdots x_c = 0)$ and $\mu(\mathcal{W}) \subset \mathcal{U}$, one can see that

$$|\mu^* e_j|_{\mu^*{\rm hod}} \lesssim \frac{1}{\prod_{i=1}^c |x_i|^{\varepsilon \cdot n_i}}$$

for all $\varepsilon > 0$. Here $n_i := {\rm ord}_{(x_i=0)}(\mu^*(z_1 \cdots z_k))$. It then follows from the definition of the extension (2.4) that

$$\mu^* E^{p,q} \subset {}^\diamond(\mu^* \tilde{E}^{p,q}). \tag{6.3}$$

Note that $\mu^*(\tilde{E}, \tilde{\theta}, h_{\rm hod})$ is still a system of Hodge bundles over $\tilde{U}$, which corresponds to the pull-back of the given $\mathbb{C}$-PVHS on U. Recall that the pull-back $\mathbb{C}$-PVHS extends to a $\mathbb{C}$-PVHS defined over $X - D_X$. Hence $\mu^*(\tilde{E}, \tilde{\theta}, h_{\rm hod})$ extends to a system of Hodge bundles over $X - D_X$.

We denote by $(G = \oplus_{p+q=\ell} G^{p,q}, \eta = \oplus_{p+q=\ell}\eta_{p,q})$ the canonical extension (in the sense of Definition 2.11) of $\mu^*(\tilde{E}, \tilde{\theta}, h_{\rm hod})$ (which is defined over $X - D_X$) over the log pair (X, D_X), which is thus a system of log Hodge bundles on (X, D_X). In particular, one has

$$\eta_{p,q} \colon G^{p,q} \longrightarrow G^{p-1,q+1} \otimes \Omega_X^1(\log D_X). \tag{6.4}$$

By Lemma 2.9(i), one has

$$G^{p,q} = {}^\diamond(\mu^* \tilde{E}^{p,q}). \tag{6.5}$$

Since L is a subsheaf of $E^{p_0, \ell - p_0}$, by (6.3) and (6.5), one has

$$\mu^* L \subset \mu^* E^{p_0,q_0} \subset G^{p_0,q_0}.$$

Recall that $\mu^*D - mD_X$ is an effective divisor and $L - \frac{\ell+1}{m}D$ is a big $\mathbb{Q}$-line bundle. Write $\tilde{L} := \mu^*L - \ell D_X$. Then $\tilde{L}$ and $\tilde{L} - D_X$ are both big line bundles. The above inclusion yields

$$\tilde{L} \otimes \mathcal{O}_X(\ell D_X) \subset G^{p_0,q_0}. \tag{6.6}$$

Step 3. Now we iterate η k times as in Section 3.3 to obtain a morphism

$$G^{p_0, \ell - p_0} \longrightarrow G^{p_0 - k, \ell - p_0 + k} \otimes {\rm Sym}^k \Omega_X^1(\log D_X). \tag{6.7}$$

The inclusion (6.6) then induces a morphism

$$\kappa_k \colon \tilde{L} \otimes \mathcal{O}_X(\ell D_X) \longrightarrow G^{p_0 - k, \ell - p_0 + k} \otimes {\rm Sym}^k \Omega_X^1(\log D_X). \tag{6.8}$$

Write k_0 for the largest k so that κ_{k_0} is non-trivial. Then $0 \leq k_0 \leq p_0 \leq \ell$. Let us denote by N_p the kernel of $\theta_{p, \ell - p}$. Hence κ_{k_0} admits a factorization

$$\kappa_{k_0} \colon \tilde{L} \otimes \mathcal{O}_X(\ell D_X) \longrightarrow N_{p_0 - k_0} \otimes {\rm Sym}^{k_0} \Omega_X^1(\log D_X).$$

We first note that $k_0 > 0$; or else, there is a morphism from the big line bundle $\tilde{L} \otimes \mathcal{O}_X(\ell D_X)$ to N_{p_0}, whose dual $N_{p_0}^*$ is weakly positive in the sense of Viehweg by [Bru17] (see also [Den22b, Theorem 4.6]). Hence κ_{k_0} induces

$$\tilde{L} \longrightarrow N_{p_0 - k_0} \otimes {\rm Sym}^{k_0} \Omega_X^1(\log D_X) \otimes \mathcal{O}_X(-\ell D_X) \subset N_{p_0 - k_0} \otimes {\rm Sym}^{k_0} \Omega_X^1$$

due to $k_0 \le p_0 \le \ell$. In other words, there exists a non-trivial morphism

$$\tilde{L} \otimes N^*_{p_0-k_0} \longrightarrow \mathrm{Sym}^{k_0}\Omega^1_X.$$

Recall that $N^*_{p_0-k_0}$ is weakly positive. The torsion-free coherent sheaf $\tilde{L} \otimes N^*_{p_0-k_0}$ is big in the sense of Viehweg. Hence there is an $\alpha > 0$ so that

$$\mathrm{Sym}^{\alpha}\left(\tilde{L} \otimes N^*_{p_0-k_0}\right) \otimes \mathcal{O}_X(-A)$$

is generically globally generated for some ample divisor A. One thus has a non-trivial morphism

$$\mathcal{O}_X(A) \longrightarrow \mathrm{Sym}^{\alpha k_0}\Omega^1_X.$$

By a theorem of Campana–Păun [CP19, Corollary 8.7], X is of general type.

Step 4. Let us prove that X is both pseudo Picard and pseudo Kobayashi hyperbolic. Note that κ_k in (6.8) induces a morphism

$$\tau_k \colon \mathrm{Sym}^k T_X(-\log D_X) \longrightarrow G^{p_0-k,\ell-p_0+k} \otimes \tilde{L}^{-1} \otimes \mathcal{O}_X(-\ell D_X)$$

By Theorem 3.9, we know that τ_1 is injective on a Zariski open set $\tilde{U}' \subset \tilde{U}$. The morphism τ_k induces a morphism

$$\tilde{\tau}_k \colon \mathrm{Sym}^k T_X \longrightarrow \mathrm{Sym}^k T_X(-\log D_X) \otimes \mathcal{O}_X(\ell D_X) \longrightarrow G^{p_0-k,\ell-p_0+k} \otimes \tilde{L}^{-1}$$

which coincides with τ_k over $\tilde{U}$. Hence $\tilde{\tau}_1$ is also injective over $\tilde{U}'$. By Proposition 3.8, we can take a singular hermitian metric $h_{\tilde{L}}$ for $\tilde{L}$ so that $h := h^{-1}_{\tilde{L}} \otimes \tilde{h}_{\mathrm{hod}}$ on $G \otimes \tilde{L}^{-1}$ is locally bounded on Y and smooth outside $D_X \cup \mathbf{B}_+(\tilde{L} - D_X)$, where $\tilde{h}_{\mathrm{hod}}$ is the Hodge metric for the system of Hodge bundles $(G,\eta)|_{X-D_X}$. Moreover, h vanishes on $D_X \cup \mathbf{B}_+(\tilde{L} - D_X)$. This metric h on $G \otimes \tilde{L}^{-1}$ induces a Finsler metric F_k on T_X defined as follows: for any $e \in T_{X,x}$,

$$F_k(e) := h\left(\tilde{\tau}_k\left(e^{\otimes k}\right)\right)^{\frac{1}{k}}$$

We apply the same method as in Section 4 to construct a new Finsler metric F on T_X by taking a convex sum in the form

$$F := \sqrt{\sum_{i=1}^{k_0} \alpha_i F_i^2},$$

where $\alpha_1, \dots, \alpha_{k_0} \in \mathbb{R}^+$ are certain constants. This Finsler metric F on T_X is positive definite over $\tilde{U}^\circ := \tilde{U}' - \mathbf{B}_+(\tilde{L} - D_X)$ as $\tilde{\tau}_1$ is injective over $\tilde{U}'$ and h is smooth on $\tilde{U} - \mathbf{B}_+(\tilde{L} - D_X)$. Set $Z := X \setminus \tilde{U}^\circ$, which is a proper Zariski closed subvariety of X. By Theorem 4.4 one can choose $\alpha_1, \dots, \alpha_{k_0} \in \mathbb{R}^+$ properly so that for any $\gamma \colon C \to X$ with C an open subset of $\mathbb{C}$ and $\gamma(C) \cap \tilde{U}^\circ \neq \varnothing$, one has

$$\mathrm{dd^c} \log |\gamma'(t)|^2_F \ge \gamma^*\omega \tag{6.9}$$

for some fixed smooth Kähler form ω on X. Indeed, it follows from the proof of Theorem 4.4 that there is an open subset C° of C whose complement is a discrete set such that (6.9) holds over C°. By Definition 4.1, $|\gamma'(t)|^2_F$ is continuous and locally bounded from above over C, and by the extension theorem of subharmonic functions, (6.9) holds over the whole unit disk C. Applying Theorem 5.4 to (6.9), we conclude that X is Picard hyperbolic modulo Z. Hence Theorem 6.1(iii) follows.

Let C be an irreducible compact curve in X not contained in Z. Write $h_{\tilde{C}}$ the induced singular hermitian metric for $T_{\tilde{C}}$ by F, where $\tilde{C}$ is the normalization of C. Then by (6.9), one has

$$2g(\tilde{C}) - 2 = -\sqrt{-1}\Theta_{h_{\tilde{C}}}(T_{\tilde{C}}) \ge \deg_\omega(C).$$

This proves Theorem 6.1(iv).

By Definition 4.1 again, there is an $\varepsilon > 0$ so that $\omega \ge \varepsilon F^2$. Hence (6.9) implies that

$$\frac{\partial^2 \log |\gamma'(t)|^2_F}{\partial t \partial \bar{t}} \ge \varepsilon |\gamma'(t)|^2_F$$

for any $\gamma\colon \Delta \to X$ with $\gamma(\Delta)\cap \tilde{U}^\circ \neq \varnothing$. In other words, the holomorphic sectional curvature of F is bounded from above by the negative constant $-\varepsilon$ (see [Kob98, Theorem 2.3.5]). By the Ahlfors–Schwarz lemma, we conclude that X is Kobayashi hyperbolic modulo Z (see [Den22a, Lemma 2.4]). This proves Theorem 6.1(ii). The theorem is proved. □

During the above proof, we indeed obtained the following result.

Theorem 6.3. *Let (Y,D) be a compact Kähler log pair, and let $(E,\theta)=(\oplus_{p+q=\ell}E^{p,q},\oplus_{p+q=\ell}\theta_{p,q})$ be a system of log Hodge bundles on (Y,D) satisfying the following conditions:*

(1) *The pair (E,θ) is the canonical extension of some system of Hodge bundles over $Y-D$ of weight ℓ.*
(2) *There is a big line bundle L over Y such that $L\otimes\mathcal{O}_X(\ell D)\subset E^{p_0,q_0}$ for some $p_0+q_0=\ell$.*
(3) *The line bundle $L\otimes\mathcal{O}_X(-D)$ is still big.*

Then there is a proper Zariski closed subset $Z\subsetneq Y$ so that

(i) *Y is Kobayashi hyperbolic modulo Z;*
(ii) *Y is Picard hyperbolic modulo Z;*
(iii) *Y is algebraically hyperbolic modulo Z.*
(iv) *Y is of general type;*

Now we are able to prove Theorem B.

Proof of Theorem B. By Steps 1–3 in the proof of Theorem 6.1, we can construct a projective log pair (X,D) and a log morphism $\mu\colon (X,\tilde{D})\to(Y,D)$ which is a finite étale cover over U. Over (X,D), there is a system of log Hodge bundles $(G=\oplus_{p+q=\ell}G^{p,q},\eta=\oplus_{p+q=\ell}\eta_{p,q})$ satisfying the following properties:

(1) The pair (G,η) is the canonical extension of some system of Hodge bundle on $X-D_X$, where D_X is a reduced simple normal crossing divisor supported on D.
(2) There is a big line bundle $\tilde{L}$ on X so that $\tilde{L}\otimes\mathcal{O}(-D_X)$ is also big,
(3) There is an inclusion $\tilde{L}\otimes\mathcal{O}_X(\ell D_X)\subset G^{p_0,\ell-p_0}$ for some $p_0>0$.

Since the period map has zero-dimensional fibers, by Theorem 3.6(iii) and the construction of $\tilde{L}$ at the beginning of the proof of Theorem 6.1, we moreover have that

(4) the augmented base locus satisfies $\mathbf{B}_+(\tilde{L})\subset\tilde{D}$.

Let $\tilde{Z}$ be any irreducible Zariski closed subvariety of X of positive dimension which is not contained in $\tilde{D}$. Take a resolution of singularities $g\colon Z\to\tilde{Z}$ so that $D_Z:=\nu^{-1}(D_X)$ is simple normal crossing. Then $g\colon(Z,D_Z)\to(X,D_X)$ is a log morphism which is generically finite.

By item (4), we can see that $L_Z:=g^*\tilde{L}$ is big. Since $g^*D_X-D_Z$ is an effective divisor, $L_Z\otimes\mathcal{O}_Z(-D_Z)$ is also big. By item (3), one has

$$L_Z\otimes\mathcal{O}_Z(\ell D_Z)\subset g^*\left(\tilde{L}\otimes\mathcal{O}_X(\ell D_X)\right)\subset g^*G^{p_0,\ell-p_0}.$$

For the $\mathbb{C}$-PVHS corresponding to $(G,\eta)|_{X-D_X}$, we pull it back to $Z-D_Z$ via g and denote by $(\tilde{E},\tilde{\theta})$ the induced system of Hodge bundles on $Z-D_Z$. Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be the canonical extension of such a system of Hodge bundles. In the same vein as the proof of (6.3), one has

$$g^*G^{p_0,\ell-p_0}\subset E^{p_0,\ell-p_0}.$$

In summary, we construct a system of log Hodge bundles $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ on (Z,D_Z) satisfying the two conditions in Theorem 6.3. By Theorem 6.3, Z is of general type. We have proved Theorem B(i).

Let us prove Theorem B(ii). For any $\tilde{\gamma}\colon\Delta^*\to X$ whose image is not contained in $\tilde{D}$, let $\tilde{Z}$ be its Zariski closure. Take a desingularization $\nu\colon Z\to\tilde{Z}$ as above, and let $\gamma\colon\Delta^*\to Z$ be the lift of γ. By the above argument and Theorem 6.3, γ extends to a holomorphic map $\overline{\gamma}\colon\Delta\to Z$. Therefore, $\nu\circ\overline{\gamma}$ extends $\tilde{\gamma}$. We proved Theorem B(ii). It is easy to see that Theorem B(ii) implies Theorem B(iii).

The proof of Theorem B(iv) is exactly the same as that of Theorem A. We will not repeat the arguments and leave the proof to the interested readers. □

We now show how to deduce Corollary C from Theorem B.

Proof of Corollary C. By the work of Baily–Borel and Mok, we know that U is quasi-projective. By the work of Deligne, U admits a $\mathbb{C}$-PVHS whose period map is immersive everywhere (see, *e.g.*, [Mil13, Theorem 7.10]). The corollary immediately follows from Theorem B(ii). □

Remark 6.4. Corollary C unifies the previous result by Nadel who proved that X is Brody hyperbolic modulo $X - \tilde{U}$. Applying Theorem B(i), it also re-proves theorems by Brunebarbe [Bru20a] and Cadorel [Cad22]: any positive-dimensional irreducible subvariety of X not contained in $X - \tilde{U}$ is of general type. However, since our proof does not rely on special properties of bounded symmetric domains (we use neither Mumford's work on toroidal compactifications nor the existence of variations of Hodge structures of Calabi-Yau type over quotients of bounded symmetric domains by arithmetic groups), we certainly loose the effectivity result regarding the level structures of the étale coverings, which is also a main result in [Nad89, Bru20a, Cad22].

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