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A gluing construction of projective K3 surfaces

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Abstract. We construct a non-Kummer projective K3 surface X which admits compact Levi-flats by holomorphically patching two open complex surfaces obtained as the complements of tubular neighborhoods of elliptic curves embedded in blow-ups of the projective plane at nine general points.

Keywords. K3 surfaces; blow-up of the projective plane at nine general points; Levi-flat hypersurfaces

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Contents

1. Introduction

In the paper [KU19], we gave a method, the so-called *gluing method*, for constructing a family of K3 surfaces, that is, we constructed such a K3 surface by holomorphically gluing two open complex surfaces obtained as the complements of tubular neighborhoods of elliptic curves embedded in blow-ups of the projective planes at nine points. The family has complex dimension 19 and each K3 surface of the family admits compact Levi-flat hypersurfaces. In this paper, we will show that there are *projective* K3 surfaces among the family. One of the main results is given as follows:

Theorem 1.1. *There exists a deformation $\pi\colon \mathcal{X} \to B$ of projective K3 surfaces over an 18 dimensional complex manifold B with injective Kodaira-Spencer map such that each fiber $X_b := \pi^{-1}(b)$ admits a holomorphic immersion $F_b\colon \mathbb{C} \to X_b$ with the property that the Euclidean closure of the image $F_b(\mathbb{C})$ in X_b is a compact real analytic hypersurface C^ω-diffeomorphic to a real 3-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ which is Levi-flat. Especially, $F_b(\mathbb{C})$ is Zariski dense in X_b whereas it is not Euclidean dense. Moreover, X_b is non-Kummer for almost every $b \in B$ in the sense of the Lebesgue measure.*

In the construction of K3 surfaces given in the paper [KU19], we prepare two surfaces S^+ and S^- obtained from the blow-ups of the projective plane $\mathbb{P}^2$ at nine points $\{p_1^\pm, \dots, p_9^\pm\}$ with smooth elliptic curves $C^\pm \in |K_{S^\pm}^{-1}|$. Here we assume that $(S^\pm, C^\pm)$ satisfy the following two conditions:

(a) there exists an isomorphism $g : C^+ \to C^-$ such that $g^*N_- \cong N_+$, where $N_\pm := N_{C^\pm/S^\pm}$ are the normal bundles of $C^\pm$ in $S^\pm$, and
(b) the normal bundles $N_\pm \in \mathrm{Pic}^0(C^\pm)$ satisfy the Diophantine condition (see Definition 2.2).

Then Arnol'd's theorem [Arn77] guarantees that there exist *analytically linearizable neighborhoods* $W^\pm \subset S^\pm$ of $C^\pm$ in $S^\pm$, namely, $W^\pm$ are tubular neighborhoods of $C^\pm$ in $S^\pm$ which are biholomorphic to neighborhoods of the zero sections in $N_\pm$. In other words, there exist a pair $(p, q) \in \mathbb{R}^2$ that satisfies the Diophantine condition (see Definition 2.1) and a positive real number $R > 1$ such that $W^\pm$ are expressed as

$$W^\pm \cong \left\{(z^\pm, w^\pm) \in \mathbb{C}^2 \mid \left|w^\pm\right| < R\right\}/\sim_\pm, \tag{1.1}$$

where $\sim_\pm$ are the equivalence relations generated by

$$(z^\pm, w^\pm) \sim_\pm (z^\pm + 1,\ \exp(\pm p \cdot 2\pi\sqrt{-1}) \cdot w^\pm) \sim_\pm (z^\pm + \tau,\ \exp(\pm q \cdot 2\pi\sqrt{-1}) \cdot w^\pm)$$

with $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$ (here note that $C^+ \cong C^-$ via g). From now on, we fix (p, q), $(S^\pm, C^\pm)$, g, and isomorphisms (1.1).

In the present paper, we take an appropriate $\xi \in \mathbb{C}$ and consider $g_\xi := \ell_\xi \circ g$, where $\ell_\xi : C^- \cong \mathbb{C}/\langle 1, \tau\rangle \circlearrowleft$ is the translation induced from $\mathbb{C} \ni z \mapsto z + \xi \in \mathbb{C}$. Note that $g_\xi^* N_- \cong N_+$ remains true since $N_\pm \in \mathrm{Pic}^0(C^\pm)$. For each $s \in \Delta := \{s \in \mathbb{C} \mid |s| < 1\}$ with $s \neq 0$, we define open submanifolds $M_s^\pm$ of $S^\pm$ by

$$M_s^\pm := S^\pm \setminus \left\{[(z^\pm, w^\pm)] \in W^\pm \mid \left|w^\pm\right| \leq \sqrt{|s|}/R\right\},$$

which contain

$$V_s^\pm := \left\{[(z^\pm, w^\pm)] \in W^\pm \mid \sqrt{|s|}/R < |w^\pm| < \sqrt{|s|}R\right\}$$

as neighborhoods of boundaries of $M_s^\pm$, and a biholomorphism $f_s\colon V_s^+ \to V_s^-$ by

$$f_s([(z^+, w^+)]) = \left[(g_\xi(z^+), s/w^+)\right].$$

Then by identifying V_s^+ and V_s^- via the biholomorphic map f_s, we can patch M_s^+ and M_s^- to define a compact complex surface X_s. In the paper [KU19], we showed that X_s is a K3 surface and that the nowhere vanishing holomorphic 2-form σ_s on X_s satisfies

$$\sigma_s|_{V_s} = c \cdot \frac{dz \wedge dw}{w}$$

for some $c \in \mathbb{C}^*$, where $V_s \subset X_s$ is the open submanifold corresponding to $V_s^+ \cong V_s^-$ and (z, w) are the coordinates induced from (z^+, w^+).

For each ξ, these K3 surfaces X_s with $s \in \Delta \setminus \{0\}$ are the fibers of a proper holomorphic map

$$\mathcal{X} \to \Delta$$

from a smooth complex manifold $\mathcal{X}(= \mathcal{X}(\xi))$ such that

— each fiber over $s \in \Delta \setminus \{0\}$ coincides with the K3 surface X_s,

— the fiber X_0 over $0 \in \Delta$ is a compact complex variety with normal crossing singularities whose irreducible components are S^+ and S^- and whose singular part is the one obtained by identifying C^+ and C^- via g_ξ, and thus

— $\mathcal{X} \to \Delta$ is a type II degeneration of K3 surfaces (see Section 4.1).

We notice that $V_s \subset X_s$ is biholomorphic to a topologically trivial annulus bundle over the elliptic curve $C := C^+ \cong C^-$, and hence homotopic to $\mathbb{S}^1_\alpha \times \mathbb{S}^1_\beta \times \mathbb{S}^1_\gamma$, where $\mathbb{S}^1_\alpha$ and $\mathbb{S}^1_\beta$ are circles in V_s such that $\mathbb{S}^1_\alpha \times \mathbb{S}^1_\beta$ is a C^∞ section of the bundle, and $\mathbb{S}^1_\gamma$ is a circle in a fiber of the bundle which generates the fundamental group. Then we define the 2-cycles $A_{\alpha\beta}$, $A_{\beta\gamma}$, $A_{\gamma\alpha}$ by

$$A_{\alpha\beta} = \mathbb{S}^1_\alpha \times \mathbb{S}^1_\beta, \quad A_{\beta\gamma} = \mathbb{S}^1_\beta \times \mathbb{S}^1_\gamma, \quad \text{and} \quad A_{\gamma\alpha} = \mathbb{S}^1_\gamma \times \mathbb{S}^1_\alpha.$$

In addition to the 2-cycles $A_{\alpha\beta}$, $A_{\beta\gamma}$, $A_{\gamma\alpha}$, each K3 surface X_s admits a marking, which gives 22 generators of the second homology group $H_2(X_s, \mathbb{Z})$ denoted by

$$A_{\alpha\beta},\ A_{\beta\gamma},\ A_{\gamma\alpha},\ B_\alpha,\ B_\beta,\ B_\gamma,\ C^+_{12},\ C^+_{23},\ \dots, C^+_{78},\ C^+_{678},\ C^-_{12},\ C^-_{23},\ \dots,\ C^-_{78},\ C^-_{678}. \tag{1.2}$$

In §5, we will give the definitions of these generators.

Now let $L^\pm$ be holomorphic line bundles on $S^\pm$ with $(L^+ \cdot C^+) = (L^- \cdot C^-)$. Assume that there exists $\xi \in \mathbb{C}$ such that $g_\xi^*(L^-|_{C^-}) \cong L^+|_{C^+}$. Note that such a ξ always exists when $(L^+ \cdot C^+) = (L^- \cdot C^-) \neq 0$. We fix such a $\xi \in \mathbb{C}$, and consider the deformation family $\mathcal{X} \to \Delta$.

Theorem 1.2. *Under the above setting, we have the following.*

(i) For any $s \in \Delta$, the line bundles $L^+|_{M_s^+}$ and $L^-|_{M_s^-}$ glue to define a holomorphic line bundle $L_s = L^+ \vee L^-$ on X_s. Moreover there exists a holomorphic line bundle $\mathcal{L} \to \mathcal{X}$ such that $\mathcal{L}|_{X_s} = L_s$ for each $s \in \Delta$.

(ii) If $L^\pm$ are ample, then there exists $\varepsilon_0 > 0$ such that L_s is ample for any $s \in \Delta$ with $0 < |s| < \varepsilon_0$.

(iii) Let L be a holomorphic line bundle on X_s for some $s \in \Delta \setminus \{0\}$. Then the following are equivalent.

(a) There exist line bundles $L^\pm$ on $S^\pm$ with $(L^+ \cdot C^+) = (L^- \cdot C^-)$ such that $L = L^+ \vee L^-$.

(b) There exists a line bundle $\mathcal{L} \to \mathcal{X}$ such that $L = \mathcal{L}|_{X_s}$.

(c) $(L \cdot A_{\beta\gamma}) = (L \cdot A_{\gamma\alpha}) = 0$.

In our arguments it is important to describe the line bundles on $V_s^{\pm}$ and on $W^{\pm}$, which is given in Section 3 after preliminary studies in Section 2. Then we will prove the main theorems in Section 4. Moreover, we will determine the Chern class $c_1(L_s)$ of the line bundle L_s in terms of the marking (1.2) in Section 5.

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2. Preliminaries

2.1. Neighborhoods of elliptic curves

First we give the following definition.

Definition 2.1. Let $(p,q) \in \mathbb{R}^2$ be a pair of real numbers.

(1) (p,q) is called a ***torsion pair*** if $(p,q) \in \mathbb{Q}^2$. Otherwise, (p,q) is called a ***non-torsion pair***.
(2) (p,q) is said to satisfy the ***Diophantine condition*** if there exist $\alpha > 0$ and $A > 0$ such that

$$\min_{\mu,\nu\in\mathbb{Z}} \left| n(p+q\sqrt{-1}) - (\mu + \nu\sqrt{-1}) \right| \geq A \cdot n^{-\alpha}$$

for any $n \in \mathbb{Z}_{>0}$.

Of course, if (p,q) satisfies the Diophantine condition, then (p,q) is a non-torsion pair.

Let X be a complex manifold. Denote by $\mathrm{Pic}(X)$ the Picard group of X, the group of isomorphism classes of holomorphic line bundles on X, and by $\mathrm{Pic}^0(X)$ the subgroup of $\mathrm{Pic}(X)$ consisting of (isomorphism classes of) topologically trivial line bundles. Note that $L \in \mathrm{Pic}(X)$ is topologically trivial if and only if L satisfies $c_1(L) = 0 \in H^2(X,\mathbb{Z})$, where $c_1(L)$ stands for the first Chern class of $L \in \mathrm{Pic}(X)$. If $X = C$ is a smooth elliptic curve, then any topologically trivial line bundle $L \in \mathrm{Pic}^0(C)$ admits a structure of unitary flat line bundle (see [Ued83]). In particular, the monodromy of $L \in \mathrm{Pic}^0(C)$ along any loop in C is expressed as a complex number with modulus 1.

Definition 2.2. For $\tau \in \mathbb{H}$, let $C = \mathbb{C}/\langle 1, \tau\rangle$ be a smooth elliptic curve, and let α and β be the loops in C corresponding to the line segments $[0,1]$ and $[0,\tau]$, respectively. Then a topologically trivial line bundle $L \in \mathrm{Pic}^0(C)$ on C is said to satisfy the ***Diophantine condition*** if so does the pair $(p,q) \in \mathbb{R}^2$, where (p,q) is defined from L, that is, $\exp(p \cdot 2\pi\sqrt{-1})$ and $\exp(q \cdot 2\pi\sqrt{-1})$ are the monodromies of L along the loops α and β, respectively.

Now, assume $C_0 = \mathbb{C}/\langle 1, \tau\rangle \subset \mathbb{P}^2$ is a smooth elliptic curve embedded in the projective plane $\mathbb{P}^2$. Let $Z := \{p_1, \dots, p_9\} \subset C_0$ be nine points on C_0, and $S := \mathrm{Bl}_Z\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at Z with the strict transform C of C_0. In this case, the normal bundle $N_{C/S} \in \mathrm{Pic}(C)$ of C in S is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - \cdots - p_9) \in \mathrm{Pic}^0(C_0) \cong \mathrm{Pic}^0(C)$, and the pair $(p,q) \in \mathbb{R}^2$ defined from $L = N_{C/S}$ (see Definition 2.2) is given by

$$9p_0 - \sum_{j=1}^{9} p_j = q - p\cdot\tau \mod \langle 1, \tau\rangle,$$

where p_0 is an inflection point of C_0. Moreover, if $N_{C/S} \in \mathrm{Pic}^0(C)$ satisfies the Diophantine condition, then Arnol'd's theorem [Arn77] guarantees that there exists a analytically linearizable neighborhood of C in S, namely, a tubular neighborhood of C in S which is biholomorphic to a neighborhood of the zero section in $N_{C/S}$. In other words, there exists a neighborhood of C in S biholomorphic to

$$W := \left\{(z,w) \in \mathbb{C}^2 \mid |w| < R\right\}/\sim \tag{2.1}$$

for some $R > 1$, where $\sim$ is the equivalence relation generated by

$$(z,w) \sim (z+1,\ \exp(p \cdot 2\pi\sqrt{-1}) \cdot w) \sim (z+\tau,\ \exp(q \cdot 2\pi\sqrt{-1}) \cdot w). \tag{2.2}$$

With the neighborhood W at hand, we can construct a family of K3 surfaces as mentioned in the introduction.

Remark 2.3. For a given $w_0 \in \mathbb{C}$ with $0 < |w_0| < R$, let $F : \mathbb{C} \to W \subset S$ be a holomorphic map defined by $F(z) = [(z, w_0)]$. Since (p,q) satisfies the Diophantine condition, the Euclidean closure of $F(\mathbb{C})$ in S coincides with $\{[(z,w)] \mid |w| = |w_0|\} \subset W$, which is a real analytic hypersurface C^ω-diffeomorphic to a real 3-dimensional torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. The maps F_b in Theorem 1.1 can be constructed in this manner.

2.2. Holomorphic line bundles on toroidal groups

The neighborhood W given in (2.1) is closely related to the *toroidal group*. For $\tau \in \mathbb{H}$ and a non-torsion pair $(p,q) \in \mathbb{R}^2$, we consider

$$U = U_{\tau,(p,q)} := \mathbb{C}^2_{(z,\eta)}/\Lambda \quad \text{with} \quad \Lambda = \Lambda_{\tau,(p,q)} := \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} \tau \\ q \end{pmatrix} \right\rangle.$$

It is seen that U becomes a toroidal group (see *e.g.* [AK01]). On the toroidal group U, an important class of line bundles is the *theta line bundles*, given as follows. Let

$$H = \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \in \mathrm{M}_2(\mathbb{C})$$

be a Hermitian matrix satisfying the condition

$$\operatorname{Im} H(\lambda,\mu) \in \mathbb{Z} \quad (\lambda,\mu \in \Lambda), \tag{2.3}$$

where $H(x,y) = {}^t x H \overline{y}$ for $x, y \in \mathbb{C}^2$, and let $\rho : \Lambda \to U(1)$ be a *semi-character* of $\operatorname{Im} H$, that is, it satisfies

$$\rho(\lambda+\mu) = \rho(\lambda)\rho(\mu)\exp\left(\pi\sqrt{-1}\operatorname{Im} H(\lambda,\mu)\right) \quad (\lambda,\mu \in \Lambda).$$

Then we define the holomorphic function $\alpha_\lambda = \alpha_\lambda^{(H,\rho)} : \mathbb{C}^2_{(z,\eta)} \to \mathbb{C}$ by

$$\alpha_\lambda(x) := \rho(\lambda)\exp\left(\pi H(x,\lambda) + (\pi/2)H(\lambda,\lambda)\right), \quad \lambda \in \Lambda,\ x = {}^t(z,\eta) \in \mathbb{C}^2.$$

From (2.3), the function $\alpha_\lambda(x)$ satisfies the cocycle condition

$$\alpha_{\lambda+\mu}(x) = \alpha_\lambda(x+\mu)\alpha_\mu(x), \quad \lambda,\mu \in \Lambda,\ x \in \mathbb{C}^2,$$

and hence

$$L = L_{H,\rho} := (\mathbb{C}_\zeta \times \mathbb{C}^2)/\Lambda$$

with

$$\lambda \cdot (\zeta, x) := (\alpha_\lambda(x) \cdot \zeta, x+\lambda), \quad \lambda \in \Lambda,\ \zeta \in \mathbb{C}_\zeta,\ x \in \mathbb{C}^2$$

defines a line bundle on U, which is called a *theta line bundle* on U. In our setting, note that $\lambda_2 \in \mathbb{R}$ for any ${}^t(\lambda_1,\lambda_2) \in \Lambda$. Hence a nowhere vanishing holomorphic function $\beta : \mathbb{C}^2 \to \mathbb{C}^*$, given by

$$\beta(z,\eta) = \exp(-\pi c \eta^2/2),$$

satisfies

$$\alpha_\lambda^{(H_0,\rho)}(x) = \beta(x+\lambda)\alpha_\lambda^{(H,\rho)}(x)\beta(x)^{-1} \quad (\lambda \in \Lambda, x \in \mathbb{C}^2) \qquad \text{with} \quad H_0 = \begin{pmatrix} a & b \\ \overline{b} & 0 \end{pmatrix},$$

which means that $L_{H,\rho}$ is holomorphically isomorphic to $L_{H_0,\rho}$. Hereafter, we assume $c = 0$ and put

$$H = \begin{pmatrix} a & b \\ \overline{b} & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}). \tag{2.4}$$

On the line bundle $L_{H,\rho}$, there is a natural metric $h = h_H$, given by

$$|\zeta|^2_{h,x} := \exp(-\pi H(x,x))|\zeta|^2,$$

which is well-defined because

$$\begin{aligned}|\alpha_\lambda(x)\cdot\zeta|^2_{h,x+\lambda} &= |\alpha_\lambda(x)|^2\cdot\exp(-\pi H(x+\lambda,x+\lambda))|\zeta|^2\\ &= \exp(\mathrm{Re}(2\pi H(x,\lambda)+\pi H(\lambda,\lambda)))\cdot\exp(-\pi H(x+\lambda,x+\lambda))|\zeta|^2\\ &= \exp(\pi H(x,\lambda)+\pi H(\lambda,x)+\pi H(\lambda,\lambda))\\ &\quad\cdot\exp(-\pi H(x,x)-\pi H(x,\lambda)-\pi H(\lambda,x)-\pi H(\lambda,\lambda))|\zeta|^2\\ &= \exp(-\pi H(x,x))|\zeta|^2 = |\zeta|^2_{h,x}.\end{aligned}$$

In particular, the curvature form of h_H is given by

$$\Theta_{h_H} := -\partial\overline{\partial}\log h_H = \pi\cdot(a dz\wedge d\overline{z} + b dz\wedge d\overline{\eta} + \overline{b} d\eta\wedge d\overline{z})$$

with $x = {}^t(z,\eta)\in\mathbb{C}^2$, and $c_1(L_{H,\rho}) = [\sqrt{-1}\Theta_{h_H}/2\pi]$. Moreover the following result holds (see [AK01]).

Proposition 2.4. *Assume that (p,q) satisfies the Diophantine condition. Then any line bundle L on $U_{\tau,(p,q)}$ is holomorphically isomorphic to $L_{H,\rho}$ for some (H,ρ).*

2.3. Deformations of K3 surfaces and Picard numbers

The following results are taught by Dr. Takeru Fukuoka.

Proposition 2.5. *Let $P\colon \mathcal{X}\to T$ be a deformation family of K3 surfaces. Assume that the Kodaira–Spencer map $\rho_{\mathrm{KS},P}\colon T_T\to R^1P_*T_{\mathcal{X}/T}$ is injective. Then, for almost every $t\in T$, it holds that $\rho(X_t)\le 20-\dim(T)$, where $X_t := P^{-1}(t)$ and $\rho(X_t)$ is the Picard number of X_t.*

Proof. Take a base point $0\in T$ and denote by $L := \Pi_{3,19}$ the K3 lattice $H^2(X_0,\mathbb{Z})$. Fix a marking $R^2P_*\mathbb{C}_{\mathcal{X}}\cong(L_{\mathbb{C}})_T$, where $L_{\mathbb{C}} := L\otimes\mathbb{C}$. Consider the map $V_\bullet\colon T\to\mathbb{P}(L_{\mathbb{C}})$ defined by $t\mapsto V_t := H^0(X_t,K_{X_t})^\perp$ for each $t\in T$, where we are regarding $\mathbb{P}(L_{\mathbb{C}})$ as the set of hyperplanes of $L_{\mathbb{C}}$. It follows from Torelli's theorem that the map $V_\bullet$ is a locally closed embedding of T into $\mathbb{P}(L_{\mathbb{C}})$. Therefore Image $V_\bullet$ is a locally closed subvariety of $\mathbb{P}(L_{\mathbb{C}})$ of dimension $\dim(T)$. Define $r\colon\mathbb{P}(L_{\mathbb{C}})\to\mathbb{Z}$ by $r(V) := \mathrm{rank}(L\cap V)$. Note that $r(V_t) = \mathrm{rank}(H^2(X_t,\mathbb{Z})\cap(H^{1,1}(X_t,\mathbb{C})\oplus H^{0,2}(X_t,\mathbb{C}))) = \rho(X_t)+1$ holds for each $t\in T$. Therefore the set $\{t\in T\mid \rho(X_t)<21-\dim(T)\}$ can be rewritten as $V_\bullet^{-1}((\mathrm{Image}\,V_\bullet)\setminus\{V\in\mathbb{P}(L_{\mathbb{C}})\mid r(V)\ge 22-\dim(T)\})$. By Lemma 2.6 below, $\{V\in\mathbb{P}(L_{\mathbb{C}})\mid r(V)\ge 22-\dim(T)\}$ is a countable union of $(\dim(T)-1)$-dimensional linear subspaces of $\mathbb{P}(L_{\mathbb{C}})$. □

Lemma 2.6. *Let $r\colon\mathbb{P}(L_{\mathbb{C}})\to\mathbb{Z}$ be as in the proof of Proposition 2.5. Then $F_n := \{V\in\mathbb{P}(L_{\mathbb{C}})\mid r(V)\ge n\}$ is a countable union of $(21-n)$-dimensional linear subspaces of $\mathbb{P}(L_{\mathbb{C}})$ for each $n = 0,1,2,\dots,21$.*

Proof. Set $\Lambda := \{M\subset L\mid M : \text{sub module},\ \mathrm{rank}\,M = n\}$. For $M\in\Lambda$ and $W\in\mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})$, it clearly holds that $p_M^{-1}(W)\in F_n$, where $M_{\mathbb{C}} := M\otimes\mathbb{C}$ and $p_M\colon L_{\mathbb{C}}\to L_{\mathbb{C}}/M_{\mathbb{C}}$ is the natural projection. Conversely, for each $V\in F_n$ and a sublattice $M\subset V$ of rank n, we have $V = p_M^{-1}(W)$ by defining $W := V/M_{\mathbb{C}}\in\mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})$. Therefore we obtain the description

$$F_n = \bigcup_{M\in\Lambda}\left\{p_M^{-1}(W)\mid W\in\mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})\right\}.$$

As Λ is countable and the map $p_M^{-1}(-)\colon\mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})\ni W\mapsto p_M^{-1}(W)\in F_n\subset\mathbb{P}(L_{\mathbb{C}})$ is a linear embedding for each M, the lemma follows. □

3. Line bundles on W and V

For $\tau \in \mathbb{H}$, let $C = \mathbb{C}_z/\langle 1, \tau\rangle$ be a complex torus, and for a non-torsion pair $(p,q) \in \mathbb{R}^2$ and $0 \le r < R \le \infty$, let $W = W^R_{\tau,(p,q)}$ be defined in (2.1) and $V = V^{r,R}_{\tau,(p,q)}$ be defined by

$$V = V^{r,R}_{\tau,(p,q)} := \Big\{(z,w) \in \mathbb{C}^2 \mid r < |w| < R\Big\}/\sim,$$

where $\sim$ is given by (2.2). We notice that V is isomorphic to an open submanifold of the toroidal group $U = U_{\tau,(p,q)} = (\mathbb{C}_z \times \mathbb{C}_\eta)/\Lambda$, namely,

$$U \supset (\mathbb{C}_z \times \{-\log R < 2\pi \mathrm{Im}\,\eta < -\log r\})/\Lambda \ni [(z,\eta)] \overset{\cong}{\longmapsto} \Big[\Big(z, \exp(2\pi\sqrt{-1}\eta)\Big)\Big] \in V$$

with $U_{\tau,(p,q)} \cong V^{0,\infty}_{\tau,(p,q)}$, and W is obtained from $V^{0,R}_{\tau,(p,q)}$ by adding the complex torus C. Let $\pi : W \to C$ be the natural projection, given by $\pi([(z,w)]) = [z]$, and denote $\pi|_V : V \to C$ by $\pi : V \to C$ for simplicity.

Lemma 3.1. *Assume that (p,q) satisfies the Diophantine condition. Then for any $L \in \mathrm{Pic}^0(W)$, the equality $L = \pi^*(L|_C)$ holds.*

Proof. As the topologically trivial bundle L satisfies $c_1(L) = 0$, L can be represented by some $\alpha \in H^1(W, \mathcal{O}_W)$ from the exact sequence $H^1(W, \mathcal{O}_W) \to \mathrm{Pic}(W) \xrightarrow{c_1} H^2(W, \mathbb{Z})$. Hence it is enough to show that $\pi^*(\alpha|_C) = \alpha$.

Put $\alpha = \{(W_{jk}, f_{jk})\}$, where $W_{jk} = W_j \cap W_k$ and $W_j = \pi^{-1}(U_j) \cong U_j \times \Delta$ with a Stein open covering $\{U_j\}$ of C. Moreover f_{jk} can be expressed on W_j as a convergent power series

$$f_{jk}(z_j, w_j) = \sum_{n=0}^{\infty} f_{jk,n}(z_j) \cdot w_j^n,$$

where (z_j, w_j) are coordinates on W_j which come from (z,w). Then it is enough to show that there are holomorphic functions $g_j\colon W_j \to \mathbb{C}$ such that

$$\Big\{(W_{jk}, \widehat{f_{jk}})\Big\} = \delta\Big\{(W_j, g_j)\Big\} := \Big\{(W_{jk}, -g_j + g_k)\Big\},$$

where

$$\widehat{f_{jk}}(z_j, w_j) := f(z_j, w_j) - f(z_j, 0) = \sum_{n=1}^{\infty} f_{jk,n}(z_j) \cdot w_j^n.$$

Note that there exists a multiplicative 1-cocycle $\{t_{jk}\}$ with $t_{jk} \in U(1)$ representing $N_{C/W}$ such that $w_k = t_{kj} \cdot w_j$ for any j,k. Since $\Big\{(U_{jk}, f_{jk,n})\Big\} \in H^1\Big(\{U_j\}, N_{C/W}^{-n}\Big)$ and $N_{C/W}$ is non-torsion, the δ-equation

$$-g_{j,n} + t_{jk}^{-n} \cdot g_{k,n} = f_{jk,n}$$

has a unique solution $g_{j,n}\colon U_j \to \mathbb{C}$ for each $n > 0$. Furthermore the power series

$$g_j(z_j, w_j) = \sum_{n=1}^{\infty} g_{j,n}(z_j) \cdot w_j^n \tag{3.1}$$

converges. Indeed, Ueda's lemma (see [Ued83, Lemma 4]) says that there exists a constant $K > 0$ depending only on C and $\Big\{U_j\Big\}$ such that for any flat line bundle E over C and for any 0-cochain $\Big\{h_j\Big\} \in C^0\Big(\Big\{U_j\Big\}, \mathcal{O}(E)\Big)$, the inequality

$$d(\mathbb{I}_C, E) \cdot \Big\|\Big\{h_j\Big\}\Big\| \le K \cdot \Big\|\delta\Big\{h_j\Big\}\Big\|$$

holds, where $\mathbb{I}_C$ is the holomorphically trivial line bundle on C, $d(\mathbb{I}_C, E)$ is the Euclidean distance of $\mathrm{Pic}^0(C) \cong \mathbb{C}/\langle 1, \tau\rangle$, which clearly is an invariant distance, and

$$\Big\|\Big\{h_j\Big\}\Big\| := \max_j \sup_{z \in U_j} \Big|h_j(z)\Big| \quad \text{and} \quad \Big\|\delta\Big\{h_j\Big\}\Big\| := \max_{j,k} \sup_{z \in U_j \cap U_k} \Big|h_{jk}(z)\Big| \quad \text{with} \ \Big\{h_{jk}\Big\} := \delta\Big\{h_j\Big\}.$$

In our setting, since $N_{C/W}$ satisfies the Diophantine condition, there exist $A > 0$ and $\alpha > 0$ such that $d(\mathbb{I}_C, N^n_{C/W}) \geq A \cdot n^{-\alpha}$ holds for any $n \geq 1$. Cauchy's inequality shows that for any $\ell \in (0, R)$, there exists $M > 0$ such that $|f_{jk,n}(z_j)| \leq M/\ell^n$ for any $n \geq 1$ and $z_j \in U_j \cap U_k$. Hence we have

$$|g_{j,n}(z_j)| \leq \frac{K}{d(\mathbb{I}_C, N^n_{C/W})} \cdot \max_{j,k} \sup_{z_j \in U_j \cap U_k} |f_{jk}(z_j)| \leq \frac{K}{A \cdot n^{-\alpha}} \cdot \frac{M}{\ell^n} = \frac{KM}{A} \cdot \frac{n^\alpha}{\ell^n},$$

which means that the power series (3.1) indeed converges because $\ell \in (0, R)$ is chosen arbitrarily. Therefore we have $\pi^*(\alpha|_C) = \alpha$ in $H^1(W, \mathcal{O}_W)$. □

Remark 3.2. The following can be proved in a similar manner by replacing a Taylor power series with a Laurent power one: for any $L \in \mathrm{Pic}^0(V)$, there exists an $F \in \mathrm{Pic}^0(C)$ such that $L = \pi^* F$, which is proved in [AK01] for the case where $V = U$ is a toroidal group. Conversely, [AK01] also proves the statement that if a pair (p, q) *does not* satisfy the Diophantine condition, then there exists an $L \in \mathrm{Pic}^0(U)$ such that $L \neq \pi^* F$ for any $F \in \mathrm{Pic}^0(C)$.

Proposition 3.3. *Assume that* (p, q) *satisfies the Diophantine condition. Then* $L = \pi^*(L|_C)$ *holds for any* $L \in \mathrm{Pic}(W)$*. In particular, the restriction map* $\mathrm{Pic}(W) \to \mathrm{Pic}(C)$ *is an isomorphism.*

Proof. As C is a deformation retract of W, the restriction map $H^2(W, \mathbb{Z}) \to H^2(C, \mathbb{Z})$ is an isomorphism. Hence we have $c_1(L \otimes \pi^*(L^{-1}|_C)) = 0$ and $L \otimes \pi^*(L^{-1}|_C)$ is topologically trivial. Since $(L \otimes \pi^*(L^{-1}|_C))|_C$ is a trivial bundle on C, one has $L = \pi^*(L|_C)$ by Lemma 3.1. □

Now let us recall the three 2-cycles

$$A_{\alpha\beta} = \mathbb{S}^1_\alpha \times \mathbb{S}^1_\beta, \quad A_{\beta\gamma} = \mathbb{S}^1_\beta \times \mathbb{S}^1_\gamma, \quad \text{and} \quad A_{\gamma\alpha} = \mathbb{S}^1_\alpha \times \mathbb{S}^1_\gamma$$

on V, where, for a base point $[(0, w_0)] \in V$, $\mathbb{S}^1_\alpha, \mathbb{S}^1_\beta, \mathbb{S}^1_\gamma$ are circles given by the images of

- $i_\alpha \colon [0,1] \ni \alpha \mapsto \left[(\alpha, \exp(\alpha p \cdot 2\pi\sqrt{-1}) w_0)\right] \in V,$
- $i_\beta \colon [0,1] \ni \beta \mapsto \left[(\beta\tau, \exp(\beta q \cdot 2\pi\sqrt{-1}) w_0)\right] \in V,$
- $i_\gamma \colon [0,1] \ni \gamma \mapsto \left[(0, \exp(\gamma \cdot 2\pi\sqrt{-1}) w_0)\right] \in V,$

respectively. Here, the orientations of $A_{\alpha\beta}, A_{\beta\gamma}, A_{\gamma\alpha}$ are defined by $d\alpha \wedge d\beta$, $d\beta \wedge d\gamma$, $d\alpha \wedge d\gamma$, respectively.

Lemma 3.4. *For a Hermitian matrix* H *given in* (2.4) *satisfying condition* (2.3) *and a semi-character* ρ *of* $\mathrm{Im}\, H$*, we have*

(1) $(L_{H,\rho} \cdot A_{\alpha\beta}) = \mathrm{Im}\, H(x_\beta, x_\alpha) = a \cdot \mathrm{Im}\,\tau + p \cdot \mathrm{Im}(b\tau) - q \cdot \mathrm{Im}\, b,$

(2) $(L_{H,\rho} \cdot A_{\beta\gamma}) = \mathrm{Im}\, H(x_\gamma, x_\beta) = -\mathrm{Im}(b\tau),$

(3) $(L_{H,\rho} \cdot A_{\gamma\alpha}) = \mathrm{Im}\, H(x_\gamma, x_\alpha) = -\mathrm{Im}\, b,$

where $x_\alpha := {}^t(1, p)$, $x_\beta := {}^t(\tau, q)$, *and* $x_\gamma := {}^t(0, 1)$.

Proof. We will only prove the assertion (1) as the other cases can be treated in the same manner. Note that the class $c_1(L_{H,\rho})$ can be represented as

$$\frac{\sqrt{-1}}{2} \cdot (a dz \wedge d\overline{z} + b dz \wedge d\overline{\eta} + \overline{b} d\eta \wedge d\overline{z}),$$

where $w = \exp(\eta \cdot 2\pi\sqrt{-1})$. By the definition of $A_{\alpha\beta}$, put $z = \alpha + \tau\beta$ and $\eta = p\alpha + q\beta$. Since $p, q, \alpha, \beta \in \mathbb{R}$, we have

$$j^*_{\alpha\beta} dz \wedge d\overline{z} = d(\alpha + \tau\beta) \wedge d(\alpha + \overline{\tau}\beta) = (\overline{\tau} - \tau) d\alpha \wedge d\beta = -2\sqrt{-1}\,\mathrm{Im}\,\tau\, d\alpha \wedge d\beta,$$

where $j_{\alpha\beta}\colon A_{\alpha\beta} \to V$ is the embedding induced by i_α and i_β. In a similar manner, one has

$$j_{\alpha\beta}^* dz \wedge d\overline{\eta} = -(p\tau - q) d\alpha \wedge d\beta, \quad j_{\alpha\beta}^* d\eta \wedge d\overline{z} = \overline{(p\tau - q)} d\alpha \wedge d\beta,$$

and hence

$$j_{\alpha\beta}^* (b dz \wedge d\overline{\eta} + \overline{b} d\eta \wedge d\overline{z}) = -2\sqrt{-1}\,\mathrm{Im}\,(b(p\tau - q))\, d\alpha \wedge d\beta.$$

Therefore we have

$$(L_{H,\rho} \cdot A_{\alpha\beta}) = \int_{[0,1]\times[0,1]} (a \mathrm{Im}\,\tau + \mathrm{Im}(b(p\tau - q)))\, d\alpha \wedge d\beta = a \mathrm{Im}\,\tau + \mathrm{Im}(b(p\tau - q)).$$

□

Proposition 3.5. *Let $L \in \mathrm{Pic}(V)$ be a holomorphic line bundle on V. Assume that (p,q) satisfies the Diophantine condition. Then the following are equivalent.*

(1) There exists a holomorphic line bundle $G \in \mathrm{Pic}(W)$ on W such that $L = G|_V$.

(2) $(L \cdot A_{\beta\gamma}) = (L \cdot A_{\gamma\alpha}) = 0$.

(3) The equality $b = 0$ holds, where b is the $(1,2)$-element of the Hermitian matrix $H \in \mathrm{M}_2(\mathbb{C})$ as (2.4) satisfying the condition (2.3) and $L = L_{H,\rho}$ for a semi-character ρ of $\mathrm{Im}\,H$, whose existence is assured by Proposition 2.4.

Note that the Diophantine assumption on the pair (p,q) in this proposition can be dropped if one assumes that $L = L_{H,\rho}$ for some Hermitian matrix $H \in \mathrm{M}_2(\mathbb{C})$ satisfying condition (2.3) and ρ is a semi-character of $\mathrm{Im}\,H$.

Proof. The equivalence (2) $\Longleftrightarrow$ (3) follows from Lemma 3.4 and (1) $\Longrightarrow$ (2) holds since the circle $\mathbb{S}^1_\gamma$ is contractible in W. The implication (3) $\Longrightarrow$ (1) follows since the factor $\alpha_\lambda^{(H,\rho)}(z,\eta)$ depends only on z and thus L is expressed as $L = \pi^*(L_0)$ for some $L_0 \in \mathrm{Pic}(C)$. □

4. Proofs of main theorems

4.1. Proof of Theorem 1.2 (i)

It follows from Proposition 3.3 and the assumption $g_\xi^*(L^-|_{C^-}) \cong L^+|_{C^+}$ that the restrictions $L^\pm|_{V_s^\pm}$ of $L^\pm|_{W^\pm}$ are isomorphic via the biholomorphic map $f_s : V_s^+ \to V_s^-$. Thus, $\left(M_s^+, L^+|_{M_s^+}\right)$ and $\left(M_s^-, L^-|_{M_s^-}\right)$ are glued together to yield a holomorphic line bundle $L_s = L^+ \vee L^-$ on X_s.

In order to describe the holomorphic line bundle $\mathcal{L} \to \mathcal{X}$ on $\mathcal{X}$ via the isomorphisms (1.1), we define manifolds $\mathcal{M}^\pm$ and $\mathcal{V}$ by

$$\mathcal{M}^\pm := (S^\pm \times \Delta) \setminus \left\{ \left(\left[(z^\pm, w^\pm)\right], s\right) \in W^\pm \times \Delta \mid \left|w^\pm\right| \leq \sqrt{|s|}R \right\}$$

and

$$\mathcal{V} := \left\{ (z^+, w^+, w^-) \in \mathbb{C}^3 \mid \left|w^+\right| < R, |w^-| < R, \left|w^+ w^-\right| < 1 \right\} / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(z^+, w^+, w^-) \sim (z^+ + 1,\ e^{p\cdot 2\pi\sqrt{-1}} \cdot w^+,\ e^{-p\cdot 2\pi\sqrt{-1}} \cdot w^-) \sim (z^+ + \tau,\ e^{q\cdot 2\pi\sqrt{-1}} \cdot w^+,\ e^{-q\cdot 2\pi\sqrt{-1}} \cdot w^-).$$

Then $\mathcal{M}^\pm$ and $\mathcal{V}$ are glued together to yield the deformation family $\mathcal{X}$ via injective holomorphic maps $f_\pm : \mathcal{M}^\pm \supset \mathcal{V}^\pm \to \mathcal{V}$, where

$$\mathcal{V}^\pm := \left\{ ([(z^\pm, w^\pm)], s) \in W^\pm \times \Delta \mid \sqrt{|s|}R < \left|w^\pm\right| < R \right\} \subset \mathcal{M}^\pm$$

and

$$f_+\left(([(z^+, w^+)], s)\right) = [(z^+, w^+, s/w^+)], \quad f_-\left(([(z^-, w^-)], s)\right) = [(g_\xi^{-1}(z^-), s/w^-, w^-)].$$

The restriction of $\mathcal{X} \to \Delta$ on $\mathcal{M}^\pm$ is the natural projection $\mathcal{M}^\pm \to \Delta$, while that on $\mathcal{V}$ is given by $[(z^+, w^+, w^-)] \mapsto w^+ \cdot w^-$. Moreover, it should be noted that there are natural projections $\varphi_\pm : \mathcal{M}^\pm \to S^\pm$ and $\varphi : \mathcal{V} \to C^+$ given by $\varphi([(z^+, w^+, w^-)]) = [z^+]$. Then a holomorphic line bundle $\mathcal{L} \to \mathcal{X}$ is defined by the pullbacks $\varphi_\pm^*(L^\pm)$ on $\mathcal{M}^\pm$ and $\varphi^*(L^+|_{C^+})$ on $\mathcal{V}$. We notice that the line bundle $\mathcal{L} \to \mathcal{X}$ is well-defined since the line bundles $f_+^*\varphi^*(L^+|_{C^+})$ and $f_-^*\varphi^*(g_\xi^*(L^-|_{C^-}))$ are the same as the restrictions $\mathcal{L}|_{\mathcal{V}^+}$ and $\mathcal{L}|_{\mathcal{V}^-}$ respectively, by virtue of Proposition 3.3 and the assumption $g_\xi^*(L^-|_{C^-}) \cong L^+|_{C^+}$. □

4.2. Idea of proof of Theorem 1.2 (ii)

Let $X = X_s$ be a K3 surface obtained by gluing $M^+ = M_s^+$ and $M^- = M_s^-$, and $L^\pm$ be an ample line bundle on $S^\pm$. In order to show Theorem 1.2 (ii), we will construct a C^∞-Hermitian metric on $L := L_s = L^+ \vee L^-$ with positive curvature in the following manner for fixed $0 < R_1 < R_2 < R$:

Step 1: Construct a C^∞-Hermitian metric $h_\pm$ on $L^\pm$ such that:
- $h_\pm$ can be glued to define a C^∞-Hermitian metric h on L (if $0 < |s| < \varepsilon_0$),
- the Chern curvature of $h_\pm$ is semi-positive: $\sqrt{-1}\Theta_{h_\pm} \geq 0$,
- $\sqrt{-1}\Theta_{h_\pm} > 0$ holds on $S^\pm \setminus \{|w^\pm| \leq R_1\}$, and
- $\sqrt{-1}\Theta_{h_\pm}(\partial/\partial z^\pm, \partial/\partial z^\pm) > 0$ holds on $S^\pm$.

Step 2: Construct a C^∞ function $\psi^\pm$ on $S^\pm \setminus C^\pm$ such that:
- $\psi^\pm$ can be glued to define a C^∞ function ψ on X,
- $\psi^\pm$ is psh on $M^\pm \setminus \{R_2 \leq |w^\pm| \leq R\}$: $\sqrt{-1}\partial\bar{\partial}\psi^\pm|_{M^\pm \setminus \{R_2 \leq |w^\pm| \leq R\}} \geq 0$,
- $\psi^\pm|_{W^\pm}$ depends only on $|w^\pm|$, and
- $\sqrt{-1}\partial\bar{\partial}\psi^\pm(\partial/\partial w^\pm, \partial/\partial w^\pm) > 0$ holds on $\{|w^\pm| < R_2\}$.

Step 3: For $0 < c \ll 1$, $h \cdot e^{-c\psi}$ is a desired metric on L with positive Chern curvature $\sqrt{-1}\Theta_h + c\sqrt{-1}\partial\bar{\partial}\psi > 0$.

In our construction, $h_\pm \cdot e^{-c\psi^\pm}$ is a C^∞-Hermitian metric on $L^\pm|_{S^\pm \setminus C^\pm}$ with positive Chern curvature such that $h_\pm \cdot e^{-c\psi^\pm} \sim (\log|w^\pm|)^2$ as $w^\pm \to 0$. Moreover, $\omega^\pm := \sqrt{-1}\Theta_{h_\pm} + c\sqrt{-1}\partial\bar{\partial}\psi^\pm \in c_1\left(L^\pm|_{S^\pm \setminus C^\pm}\right)$ gives a complete Kähler metric on $S^\pm \setminus C^\pm$, and on a neighborhood $\left\{|w^\pm| < \sqrt{\varepsilon_0}R\right\}$ of $C^\pm$, the form $\omega^\pm$ is expressed as

$$\omega^\pm|_{\{|w^\pm| < \sqrt{\varepsilon_0}R\}} = \frac{\pi(L^\pm \cdot C^\pm)}{\operatorname{Im}\tau} \cdot \sqrt{-1}dz^\pm \wedge d\overline{z}^\pm + 2c \cdot \frac{\sqrt{-1}dw^\pm \wedge d\overline{w}^\pm}{|w^\pm|^2}.$$

4.3. Proof of Theorem 1.2 (ii)

Let S be the blow-up of $\mathbb{P}^2$ at nine points, and $C \subset S$ be an elliptic curve in $|K_S^{-1}|$ such that $N_{C/S} \in \operatorname{Pic}^0(C)$ satisfies the Diophantine condition. Then Arnol'd's theorem says that there is an analytically linearizable neighborhood $W \subset S$ of C. By shrinking W if necessary, we may assume that W is isomorphic to $W_{\tau,(p,q)}^R$ for some $R > 0$, $\tau \in \mathbb{H}$ and $(p, q) \in \mathbb{R}^2$ satisfying the Diophantine condition, and let $\pi : W \to C$ be the projection given in Section 3.

Let $L \in \operatorname{Pic}(S)$ be an ample line bundle, which implies that there exists $n \in \mathbb{N}$ such that $L^n \otimes [-C]$ is very ample, and let $g_1, g_2, \ldots, g_N$ be a basis of $H^0(S, L^n \otimes [-C])$, which are regarded as sections of L^n with zeros along C. Then the *singular* Hermitian metric h_L on L is defined by

$$\langle \xi, \eta \rangle_{h_L,x} := \frac{\xi \cdot \overline{\eta}}{\left(|g_1(x)|^2 + |g_2(x)|^2 + \cdots + |g_N(x)|^2\right)^{\frac{1}{n}}}, \quad \text{where} \quad \xi, \eta \in L|_x.$$

The metric h_L has a pole along C and its restriction $h_L|_{S \setminus C}$ induces a C^∞-metric on $S \setminus C$ with positive curvature form $\sqrt{-1}\Theta_{h_L}|_{S \setminus C} > 0$. Moreover let h_C be a C^∞-metric on $L|_W$ satisfying $\sqrt{-1}\Theta_{h_C} = b\sqrt{-1}dz \wedge d\overline{z}$ for $b := \pi(L \cdot C)/\operatorname{Im}\tau > 0$.

Fix $0 < R_1 < R_2 < R$. Then we define a metric h on L by

$$h^{-1} := \begin{cases} \mathrm{RegularizedMax}(h_L^{-1}, \varepsilon \cdot \pi^* h_C^{-1}) & \text{on } W \\ h_L^{-1} & \text{on } S \setminus \overline{W} \end{cases}$$

where $\varepsilon > 0$ and RegularizedMax: $\mathbb{R}^2 \to \mathbb{R}$ is the regularized maximum function (see [Dem12, Chapter I, Lemma 5.18]). Note that, by choosing $\varepsilon > 0$ sufficiently small, one may assume that $h = h_L$ holds on $\{[(z,w)] \in W \mid R_1 < |w|\}$, which ensures the smoothness of h. Then $\sqrt{-1}\Theta_h \geq 0$, since the local weight function φ of h satisfies

$$\varphi = \mathrm{RegularizedMax}(\varphi_L, \varphi_C - \log \varepsilon),$$

where φ_L and φ_C are the local weight functions of h_L and h_C, respectively. By the construction of h, there exists a positive constant ε_0 such that $h = \varepsilon^{-1} \cdot \pi^* h_C$ holds on $\left\{|w| < \sqrt{\varepsilon_0} R\right\}$. By shrinking ε_0 if necessary, we may assume $\sqrt{\varepsilon_0} R < R_1$. For $s \in \Delta$ with $|s| < \varepsilon_0$, let $\lambda = \lambda_s : \mathbb{R}_{>0} \to \mathbb{R}$ be a C^∞-function satisfying the conditions

$$\begin{cases} \lambda(t) = \left(\log(t^2/|s|)\right)^2 & \text{if } 0 < t < R_2, \\ \lambda(t) \equiv \text{constant} & \text{if } t \geq R, \end{cases}$$

and $\psi = \psi_s : S \setminus C \to \mathbb{R}$ be the C^∞-function defined by

$$\psi(p) := \begin{cases} \lambda(|w|) & \forall\, p = (z,w) \in W \setminus C \\ \lambda(R) & \forall\, p \notin W. \end{cases}$$

It is easy to see that $\partial\overline{\partial}\psi = 0$ outside $\{|w| \leq R\}$ and $\partial\overline{\partial}\psi = 2 \cdot dw \wedge d\overline{w}/|w|^2$ on $\{0 < |w| < R_2\}$. Finally, we choose $c > 0$ so that

$$\sqrt{-1}\Theta_{h_L} + c\sqrt{-1}\partial\overline{\partial}\psi > 0$$

on the compact subset $\{R_2 \leq |w| \leq R\}$. Here note that such a $c > 0$ exists since $\sqrt{-1}\Theta_{h_L}$ is strictly positive on $\{R_2 \leq |w| \leq R\} \subset S \setminus C$.

We consider the metric $h \cdot e^{-c\psi}$ on $S \setminus C$. Our assumption on $c > 0$ says that

$$\sqrt{-1}\Theta_{h \cdot e^{-c\psi}} = \sqrt{-1}\Theta_{h_L} + c\sqrt{-1}\partial\overline{\partial}\psi > 0$$

outside $\{|w| < R_2\}$. Moreover, $h \cdot e^{-c\psi}$ has positive curvature also on $\{0 < |w| < R_2\}$, since it holds

$$\sqrt{-1}\Theta_{h_L \cdot e^{-c\psi}} > \sqrt{-1}\Theta_{h_L} > 0, \quad \sqrt{-1}\Theta_{\varepsilon^{-1} \cdot \pi^* h_C \cdot e^{-c\psi}} = b\sqrt{-1} dz \wedge \overline{z} + c\frac{\sqrt{-1} dw \wedge d\overline{w}}{|w|^2} > 0$$

and

$$(h \cdot e^{-c\psi})^{-1} = \mathrm{RegularizedMax}\left((h_L \cdot e^{-c\psi})^{-1}, (\varepsilon^{-1} \cdot \pi^* h_C \cdot e^{-c\psi})^{-1}\right)$$

on $\{0 < |w| < R_2\}$ (see [Dem12, Chapter I, Lemma 5.18(e)]). Therefore the curvature of $h \cdot e^{-c\psi}$ is positive on $S \setminus C$.

Now we consider two pairs $(S^\pm, C^\pm)$ of surfaces $S^\pm$ and curves $C^\pm \subset S^\pm$ given in the introduction, which admit analytically linearizable neighborhoods $W^\pm \subset S^\pm$ of $C^\pm$, and assume that $W^\pm$ are regarded as subspaces $\{[(z^\pm, w^\pm)] \mid |w^\pm| < R\}$ of toroidal groups. Moreover let $L^\pm$ be ample line bundles with $(L^+ \cdot C^+) = (L^- \cdot C^-)$ and $g_\xi : C^+ \to C^-$ be an isomorphism with $g_\xi^*(L^-|_{C^-}) = L^+|_{C^+}$. In what follows we abuse the notation to denote g_ξ simply by g. Then the above argument shows that there exist C^∞-metrics $h_\pm \cdot e^{-c\psi^\pm}$ on $S^\pm \setminus C^\pm$ such that $\sqrt{-1}\Theta_{h_\pm \cdot e^{-c\psi^\pm}} > 0$ on $S^\pm \setminus C^\pm$ and

$$h_\pm = \varepsilon^{-1} \cdot \pi_\pm^* h_{C_\pm}, \quad \psi^\pm(z^\pm, w^\pm) = \left(\log \frac{|w^\pm|^2}{|s|}\right)^2$$

on $\left\{0 < |w^\pm| < \sqrt{|s|}R\ (< \sqrt{\varepsilon_0}R < R_1)\right\}$. As our K3 surface X_s is given by gluing two surfaces

$$M_s^\pm = S^\pm \setminus \left\{\left|w^\pm\right| \le \sqrt{|s|}/R\right\}$$

via the map $(z^+, w^+) \mapsto (z^-, w^-) = (g(z^+), s/w^+)$, it follows from Proposition 3.3 that $h_\pm$ can be glued together and become a global C^∞-Hermitian metric on $L_s = L^+ \vee L^-$. Moreover, on $\left\{\sqrt{|s|}/R < |w^+| < \sqrt{|s|}R\right\}$, we have $\psi^+(z^+, w^+) = \left(\log |w^+|^2/|s|\right)^2$ and

$$\psi^-\left(g(z^+), \frac{s}{w^+}\right) = \left(\log \frac{|s/w^+|^2}{|s|}\right)^2 = \left(-\log \frac{|w^+|^2}{|s|}\right)^2 = \psi^+(z^+, w^+),$$

which means that $\psi^\pm$ can be glued together and become a global C^∞-function ψ on X_s. Therefore $h_\pm \cdot e^{-c\psi^\pm}$ yield a C^∞-metric on X_s with positive definite curvature form. □

4.4. Proof of Theorem 1.2 (iii)

The equivalence (a) $\Longleftrightarrow$ (c) follows from Proposition 3.5, and the implications (a) $\Longrightarrow$ (b) follows from Theorem 1.2 (i). In what follows we show (b) $\Longrightarrow$ (c). Take a line bundle $\mathcal{L} \to \mathcal{X}$ as in (b) and consider the function $h\colon \Delta \to \mathbb{Z}$ defined by

$$h(t) := \left(\mathcal{L}|_{M_t^+} \cdot A_{\beta\gamma}\right),$$

where we are regarding $A_{\beta\gamma}$ as a cycle of M_t^+. As $\mathcal{M}^+ \to \Delta$ is a submersion, h is continuous. Thus h is a constant function. Therefore, in order to show that $(L \cdot A_{\beta\gamma})(= h(s))$ is equal to zero, it is sufficient to show that $h(0) = 0$, which follows from Proposition 3.5 since $\mathcal{L}|_{M_0^+}$ coincides with the restriction of the line bundle $(\mathcal{L}|_{X_0})|_{S^+}$ to M_0^+. The equation $(L \cdot A_{\gamma\alpha}) = 0$ can be shown in the same manner. □

4.5. Proof of Theorem 1.1

Our construction of K3 surfaces has 19 complex dimensional degrees of freedom if we allow the variation of ξ [KU19]. Indeed, for a fixed pair $(p, q) \in \mathbb{R}^2$ satisfying the Diophantine condition, we have the following parameters:

(I) 1 parameter $\tau \in \mathbb{H}$ determining the elliptic curve $C^+ \cong C^-$,
(II) 16 parameters $\{p_1^\pm, \dots, p_8^\pm\}$ determining the centers of the blow-ups $\pi^\pm$ (here p_9^+ and p_9^- are fixed from the conditions (a) and (b) in the introduction),
(III) 1 parameter $\xi \in \mathbb{C}$ determining the isomorphism $g_\xi : C^+ \to C^-$, and
(IV) 1 parameter $s \in \Delta \setminus \{0\}$ determining the gluing function $f_s : V_s^+ \to V_s^-$.

Note that there always exist ample line bundles $L^\pm \to S^\pm$ with $(L^+ \cdot C^+) = (L^- \cdot C^-)$. If such ample line bundles $L^\pm$ are fixed, then ξ is determined uniquely up to modulo $\langle 1, \tau\rangle$ from the condition $g_\xi^*(L^-|_{C^-}) \cong L^+|_{C^+}$, and depends holomorphically on the parameters given in (I) and (II) (see also the relation (5.5)). Moreover, for any $s \in \Delta \setminus \{0\}$ with sufficiently small $|s| \ll 1$, the K3 surface X_s admits an ample line bundle $L_s = L^+ \vee L^-$ by Theorem 1.2 (ii). Hence we have an 18 dimensional family of projective K3 surfaces, whose Kodaira-Spencer map is injective by [KU19, Theorem 1.1]. Moreover it follows from [KU19] that there exists a holomorphic immersion $F_b\colon \mathbb{C} \to X_b$ mentioned in Theorem 1.1 (see also Remark 2.3). Finally among the family, almost every fiber is a non-Kummer K3 surface since if follows from Proposition 2.5 that almost every fiber X_s has the Picard number $\rho(X_s) \le 2$. □

5. Calculation of the Chern class $c_1(L)$

Let $S^\pm$ be surfaces obtained from the blow-ups $\pi^\pm : S^\pm \to \mathbb{P}^2$ of the projective plane $\mathbb{P}^2$ at nine points $\{p_1^\pm, \dots, p_9^\pm\}$ with smooth elliptic curves $C^\pm \in \left|K_{S^\pm}^{-1}\right|$. In our assumption $(S^\pm, C^\pm)$ satisfy Conditions (a) and (b) given in the introduction. Moreover let $L^\pm$ be holomorphic line bundles on $S^\pm$. In this section, we compute the Chern class $c_1(L)$ in the lattice $H^2(X,\mathbb{Z}) \cong H_2(X,\mathbb{Z})$, where X is a K3 surface given by the gluing construction and $L = L^+ \vee L^-$ is the line bundle on X (see the introduction).

First we notice that the second homology group of $S^\pm$ is expressed as

$$H_2(S^\pm,\mathbb{Z}) \cong H^2(S^\pm,\mathbb{Z}) \cong \mathrm{Pic}(S^\pm) = \langle H^\pm, E_1^\pm, \dots, E_9^\pm \rangle,$$

where $E_\nu^\pm$ is (the class of) the exceptional divisor in $S^\pm$ which is the preimage of $p_\nu^\pm$ for $\nu = 1,2,\dots,9$, and $H^\pm$ is (the class of) the preimage of a line in $\mathbb{P}^2$ by the blow-up $\pi^\pm \colon S^\pm \to \mathbb{P}^2$. In the homology group $H_2(S^\pm,\mathbb{Z})$, the elliptic curve $C^\pm$ is expressed as

$$C^\pm = 3H^\pm - \sum_{j=1}^{9} E_j^\pm.$$

We also notice that the points $p_1^\pm, \dots, p_9^\pm$ lie in the elliptic curve $C_0^\pm := \pi^\pm(C^\pm)$. Then fix isomorphisms

$$C_0^+ \cong C_0^- \cong \mathbb{C}/\langle 1, \tau \rangle$$

and also fix an inflection point $p_0^\pm$ so that

$$9p_0^\pm - \sum_{j=1}^{9} p_j^\pm = \pm\mu \mod \langle 1, \tau \rangle,$$

where $\mu := q - p \cdot \tau$ and the points $p_j^\pm \in \mathbb{C}$ $(j = 0,\dots,9)$ are regarded as complex numbers (see Subsection 2.1). By choosing the complex number corresponding to the point $p_0^\pm$ appropriately, we may assume that

$$9p_0^\pm - \sum_{j=1}^{9} p_j^\pm = \pm\mu, \tag{5.1}$$

actually holds. In what follows we assume that $g(p_0^+) = p_0^-$ by changing g if necessary. For $j \neq k \in \{0,1,\dots,9\}$, let $\Gamma_{jk}^\pm \subset C^\pm$ be the lift of an arc in $C_0^\pm$ connecting $p_j^\pm$ and $p_k^\pm$.

Now we give the definitions of the generators (1.2) (see also [KU19]). The 2-cycles $A_{\alpha\beta}$, $A_{\beta\gamma}$, $A_{\gamma\alpha}$ are already defined in the introduction. In order to define the 2-cycle $B_\bullet$ for $\bullet \in \{\alpha,\beta,\gamma\}$, we first notice that $M_s^\pm$ are simply connected. Thus, there exist topological discs $D_\bullet^\pm \subset M_s^\pm$ such that $\partial D_\bullet^\pm = \pm\mathbb{S}_\bullet^1$ hold, where $\mathbb{S}_\bullet^1 \subset V_s$, which are given in the introduction, are regarded as 1-cycles of $V_s^\pm \subset M_s^\pm$. Then $B_\bullet$ is defined by $B_\bullet = D_\bullet^+ \cup_{\mathbb{S}_\bullet^1} (-D_\bullet^-)$, that is, the patch of $D_\bullet^+$ and $-D_\bullet^-$ through $\mathbb{S}_\bullet^1$. In order to define the 2-cycles $C_\bullet^\pm$, we prepare the tube $T_{jk}^\pm$ given by $T_{jk}^\pm := \mathrm{pr}_\pm^{-1}(\Gamma_{jk}^\pm) \subset \left\{|w^\pm| = \sqrt{|s|}\right\}$, where

$$\mathrm{pr}_\pm : \left\{[(z^\pm, w^\pm)] \in W^\pm \mid \left|w^\pm\right| = \sqrt{|s|}\right\} \ni [(z^\pm, w^\pm)] \longmapsto [z^\pm] \in C^\pm$$

is a natural projection. Then for $\nu = 1,\dots,7$, the 2-cycle $C_{\nu,\nu+1}^\pm$ is defined by the connected sum $(\pm E_\nu^\pm)\#(\mp E_{\nu+1}^\pm)$ of $\pm E_\nu^\pm$ and $\mp E_{\nu+1}^\pm$ given by connecting them through the tube $T_{\nu,\nu+1}^\pm$. In a similar manner, the 2-cycle $C_{678}^\pm$ is defined by the connected sum

$$C_{678}^\pm := (\mp H^\pm)\#(\pm E_6^\pm)\#(\pm E_7^\pm)\#(\pm E_8^\pm)$$

of $\mp H^\pm$, $\pm E_6^\pm$, $\pm E_7^\pm$, $\pm E_8^\pm$ given by connecting them through the tubes $T_{06}^\pm$, $T_{07}^\pm$, $T_{08}^\pm$. In particular, $C_\bullet^\pm$ is represented as

$$C_{12}^\pm = \pm(E_1^\pm - E_2^\pm),\ \dots,\ C_{78}^\pm = \pm(E_7^\pm - E_8^\pm),\ C_{678}^\pm = \pm(-H^\pm + E_6^\pm + E_7^\pm + E_8^\pm)$$

in $H_2(S^\pm,\mathbb{Z})$. It should be noted that $C_\bullet^\pm$ lies in $M_s^\pm$. Moreover, $H_2(S^\pm,\mathbb{Z})$ admits an orthogonal decomposition

$$H_2(S^\pm,\mathbb{C}) = \langle C^\pm, E_9^\pm \rangle \oplus \mathcal{C}^\pm$$

with respect to the intersection product, where $\mathcal{C}^\pm$ is given by $\mathcal{C}^\pm := \langle C^\pm_{12}, C^\pm_{23}, \dots, C^\pm_{78}, C^\pm_{678}\rangle$, and any element $q^\pm \in H_2(S^\pm, \mathbb{C}) = H_2(S^\pm, \mathbb{Z}) \otimes \mathbb{C}$ admits an expression

$$q^\pm = q^\pm_0 H^\pm - \sum_{j=1}^{9} q^\pm_j E^\pm_j = \left(3q^\pm_0 - \sum_{j=1}^{8} q^\pm_j\right) C^\pm + \left(3q^\pm_0 - \sum_{j=1}^{9} q^\pm_j\right) E^\pm_9 + q^\pm|_{\mathcal{C}^\pm}. \tag{5.2}$$

Next let us consider a K3 surface $X = X_s$ given by the gluing construction. It is seen (*cf.* [KU19]) that the second homology group of X is given by the orthogonal decomposition

$$H_2(X, \mathbb{Z}) = \Pi_{3,19} \cong \langle A_{\alpha\beta}, B_\gamma\rangle \oplus \langle A_{\beta\gamma}, B_\alpha\rangle \oplus \langle A_{\gamma\alpha}, B_\beta\rangle \oplus \mathcal{C}^+ \oplus \mathcal{C}^-.$$

with respect to the intersection product. Here note that

$$\begin{aligned}
&(A_{\alpha\beta} \cdot A_{\alpha\beta}) = (A_{\beta\gamma} \cdot A_{\beta\gamma}) = (A_{\gamma\alpha} \cdot A_{\gamma\alpha}) = 0,\\
&(B_\gamma \cdot B_\gamma) = (B_\alpha \cdot B_\alpha) = (B_\beta \cdot B_\beta) = -2,\\
&(A_{\alpha\beta} \cdot B_\gamma) = (A_{\beta\gamma} \cdot B_\alpha) = (A_{\gamma\alpha} \cdot B_\beta) = 1.
\end{aligned}$$

The K3 surface X admits a nowhere vanishing holomorphic 2-form σ, which is expressed as

$$\sigma = (2\mu + c^-_9) A_{\alpha\beta} + \mu B_\gamma + x A_{\beta\gamma} + \tau B_\alpha + y A_{\gamma\alpha} + B_\beta + \sum c^+_\bullet C^+_\bullet + \sum c^-_\bullet C^-_\bullet$$

in $H_2(X, \mathbb{C})$ by multiplying a constant to σ if necessary, where $x = x(s)$ and $y = y(s)$ are constants and $c^\pm_\bullet$ is given by

$$c^\pm_\bullet = \int_{\Gamma^\pm_\bullet} dz^\pm$$

with $\Gamma^-_9 \subset C^-$ being the lift of an arc in C^-_0 connecting p^-_9 and $g_\xi(p^+_9)$. Hence, one has

$$c^\pm_{12} = \pm(p^\pm_1 - p^\pm_2),\ \dots,\ c^\pm_{78} = \pm(p^\pm_7 - p^\pm_8),\ c^\pm_{678} = \pm(-3p^\pm_0 + p^\pm_6 + p^\pm_7 + p^\pm_8), \quad \text{and} \quad c^-_9 = g_\xi(p^+_9) - p^-_9 \tag{5.3}$$

if one selects the arcs appropriately.

Proposition 5.1. *The Chern class $c_1(L)$ in $H^2(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ is expressed as*

$$c_1(L) = (2b + n^+_9 + n^-_9) A_{\alpha\beta} + b B_\gamma + L^+|_{\mathcal{C}^+} + L^-|_{\mathcal{C}^-},$$

where $b := (L^+ \cdot C^+) = (L^- \cdot C^-)$ and $n^\pm_9 := (L^\pm \cdot E^\pm_9)$.

Proof. We put

$$c_1(L) = \widehat{a}_{\alpha\beta} A_{\alpha\beta} + \widehat{b}_\gamma B_\gamma + \widehat{a}_{\beta\gamma} A_{\beta\gamma} + \widehat{b}_\alpha B_\alpha + \widehat{a}_{\gamma\alpha} A_{\gamma\alpha} + \widehat{b}_\beta B_\beta + \sum \widehat{c}^+_\bullet C^+_\bullet + \sum \widehat{c}^-_\bullet C^-_\bullet.$$

First the coefficients $\widehat{c}^\pm_\bullet$ are determined from $L^\pm|_{\mathcal{C}^\pm}$ since the cycles $C^\pm_\bullet$ in X_s are also regarded as the ones in $M^\pm_s \subset S^\pm$. Next it follows from Theorem 1.2 (iii) that $(L \cdot A_{\beta\gamma}) = (L \cdot A_{\gamma\alpha}) = 0$, which implies that $\widehat{b}_\alpha = \widehat{b}_\beta = 0$. Moreover, the cycle $A_{\alpha\beta}$ may be regarded as $C^\pm$ in $S^\pm$, which means that $(L \cdot A_{\alpha\beta}) = (L^\pm \cdot C^\pm) = b$ and thus $\widehat{b}_\gamma = b$.

Finally we will determine the coefficients $\widehat{a}_\bullet$. To this end, we put

$$p^\pm := 3p^\pm_0 H^\pm - \sum_{j=1}^{9} p^\pm_j E^\pm_j \in H_2(S^\pm, \mathbb{C}).$$

Then Condition (5.1) shows that $(p^\pm \cdot C^\pm) = \pm\mu$ and the relation (5.3) means that $\sigma|_{\mathcal{C}^\pm} = \pm p^\pm|_{\mathcal{C}^\pm}$. Thus it follows from Equation (5.2) that

$$p^\pm = \left(9p^\pm_0 - \sum_{j=1}^{8} p^\pm_j\right) C^\pm + \left(9p^\pm_0 - \sum_{j=1}^{9} p^\pm_j\right) E^\pm_9 + p^\pm|_{\mathcal{C}^\pm} = (\pm\mu + p^\pm_9) C^\pm \pm \mu E^\pm_9 \pm \sigma|_{\mathcal{C}^\pm}. \tag{5.4}$$

Moreover, by the condition $g_\xi^*(L^-|_{C^-}) = L^+|_{C^+}$, we may assume that $\xi \in \mathbb{C}$ is given by

$$\xi = \frac{1}{b}\left((p^- \cdot L^-) - (p^+ \cdot L^+)\right). \tag{5.5}$$

Thus we have

$$c_9^- = g_\xi(p_9^+) - p_9^- = (p_9^+ + \xi) - p_9^- = \frac{1}{b}\left((p^- \cdot L^-) - (p^+ \cdot L^+)\right) + (p_9^+ - p_9^-). \tag{5.6}$$

Equation (5.4) shows that

$$(\sigma|_{C^\pm} \cdot L^\pm) = \pm(p^\pm \cdot L^\pm) - (\mu \pm p_9^\pm)(C^\pm \cdot L^\pm) - \mu(E_9^\pm \cdot L^\pm) = \pm(p^\pm \cdot L^\pm) \mp b p_9^\pm - b\mu - n_9^\pm \mu, \tag{5.7}$$

and Equations (5.6) and (5.7) show that

$$\begin{aligned} 0 = (\sigma. \cdot L) &= \Big(((2\mu + c_9^-)A_{\alpha\beta} + \mu B_\gamma) \cdot (\widehat{a}_{\alpha\beta}A_{\alpha\beta} + bB_\gamma)\Big) + \Big((xA_{\beta\gamma} + \tau B_\alpha) \cdot \widehat{a}_{\beta\gamma}A_{\beta\gamma}\Big) \\ &\quad + \Big((yA_{\gamma\alpha} + B_\beta) \cdot \widehat{a}_{\gamma\alpha}A_{\gamma\alpha}\Big) + (\sigma|_{C^+} \cdot L^+) + (\sigma|_{C^-} \cdot L^-) \\ &= \Big(\mu\widehat{a}_{\alpha\beta} + \tau\widehat{a}_{\beta\gamma} + \widehat{a}_{\gamma\alpha}\Big) + bc_9^- + (\sigma|_{C^+} \cdot L^+) + (\sigma|_{C^-} \cdot L^-) = \Big(\mu\widehat{a}_{\alpha\beta} + \tau\widehat{a}_{\beta\gamma} + \widehat{a}_{\gamma\alpha}\Big) - (2b + n_9^+ + n_9^-)\mu. \end{aligned}$$

Since $\widehat{a}_{\alpha\beta}, \widehat{a}_{\beta\gamma}, \widehat{a}_{\gamma\alpha}, b, n_9^+, n_9^- \in \mathbb{Z}$ and $(1, \tau, \mu)$ are independent over $\mathbb{Q}$ by the Diophantine condition for the pair (p, q), we have $\widehat{a}_{\alpha\beta} = 2b + n_9^+ + n_9^-$ and $\widehat{a}_{\beta\gamma} = \widehat{a}_{\gamma\alpha} = 0$. □

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