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Volume 5 (2021), Article Nr. 20



Factorization of the Abel-Jacobi maps

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Abstract. As an application of the theory of Lawson homology and morphic cohomology, Walker proved that the Abel-Jacobi map factors through another regular homomorphism. In this note, we give a direct proof of the theorem.

Keywords. Abel-Jacobi maps, Regular homomorphisms, Coniveau filtration

2020 Mathematics Subject Classification. 14C25, 14C30, 14K30

[Français]

Factorisation des applications d'Abel-Jacobi

Résumé. Walker a montré que l'application d'Abel-Jacobi se factorise à travers un autre homomorphisme régulier, en utilisant la théorie de l'homologie de Lawson et la cohomologie morphique. Dans cet article, nous donnons une preuve directe de ce théorème.

Received by the Editors on December 11, 2020, and in final form on November 10, 2021. Accepted on November 23, 2021.

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1. Introduction

For a smooth complex projective variety X, the Abel–Jacobi map AJ^p provides a fundamental tool to study codimension p cycles on X. It is a homomorphism of abelian groups

$$AJ^p: CH^p(X)_{\text{hom}} \to J^p(X),$$

where $CH^p(X)_{\text{hom}}$ is the group of codimension p cycles homologous to zero modulo rational equivalence and

$$J^{p}(X) = H^{2p-1}(X,\mathbb{C})/(H^{2p-1}(X,\mathbb{Z}(p)) + F^{p}H^{2p-1}(X,\mathbb{C}))$$

is the p^{th} Griffiths intermediate Jacobian. Although the Abel–Jacobi map AJ^p is transcendental by nature, it is well-known that we get an algebraic theory by restricting it to the subgroup $A^p(X) \subset CH^p(X)_{\text{hom}}$ of cycles algebraically equivalent to zero. More precisely, the image $J^p_a(X) \subset J^p(X)$ of $A^p(X)$ under AJ^p is an abelian variety, and the induced map

$$\psi^p: A^p(X) \to J_a^p(X),$$

which we also call Abel–Jacobi, satisfies the following [Gri68, Lie70]: for any smooth connected projective variety S with a base point s_0 and for any codimension p cycle Γ on $S \times X$, the composition

$$\begin{cases} S \to A^p(X) \to J_a^p(X) \\ s \mapsto \psi^p(\Gamma_s - \Gamma_{s_0}) \end{cases}$$

is a morphism of algebraic varieties.

Generally, for a given abelian variety A, a homomorphism $\phi: A^p(X) \to A$ with the analogous property is called regular (this definition goes back to the work of Samuel [Sam58]). Remarkably, the Abel–Jacobi map ψ^p factors through another regular homomorphism due to a theorem of Walker [Wal07], which was originally proved as an application of the theory of Lawson homology and morphic cohomology. The purpose of this note is to give a direct proof of the theorem.

Theorem 1.1 ([Wal07, Corollary 5.9]). For a smooth projective variety X, the Abel-Jacobi map ψ^p factors as

$$J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right)$$

$$\downarrow^{\pi^{p}}$$

$$\downarrow^{\pi^{p}}$$

$$\downarrow^{\mu^{p}}$$

$$\downarrow^{\mu^{$$

where $J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))$ is the intermediate Jacobian for the pure Hodge structure of weight -1 given by the (p-1)-stage of the coniveau filtration

$$N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))=\operatorname{Ker}\left(H^{2p-1}(X,\mathbb{Z}(p))\to \varinjlim_{Z\in\mathcal{Z}^{p-1}}H^{2p-1}(X-Z,\mathbb{Z}(p))\right),$$

 π^p is a natural isogeny, and $\widetilde{\psi}^p$ is a surjective regular homomorphism.

Remark 1.2. The Walker map $\widetilde{\psi}^p$ is a unique lift of the Abel–Jacobi map ψ^p . This follows from the fact that $A^p(X)$ is divisible [BO74, Lemma 7.10] and $\text{Ker}(\pi^p)$ is finite.

Theorem 1.1 is related to a classical question of Murre (cf. [Mur85, Section 7] and [GMV93, p. 132]), asking whether the Abel–Jacobi map $\psi^p: A^p(X) \to J^p_a(X)$ is universal among all regular homomorphisms $\phi\colon A^p(X) \to A$, that is, whether every such ϕ factors through ψ^p . This is known to hold for p=1 by the theory of the Picard variety, for $p=\dim X$ by the theory of the Albanese variety, and for p=2 as proved by Murre [Mur83a, Mur83b] (see [Kah18] for the correction of a gap in the original proof) using the Merkurjev-Suslin theorem [MS82]. Nevertheless, it was recently observed by Ottem–Suzuki [OS20] that ψ^p is no longer universal in general for $3 \le p \le \dim X - 1$, which settled Murre's question. In fact, they constructed a 4-fold on which the Walker map $\widetilde{\psi}^3$ is universal and the isogeny π^3 has non-zero kernel. We note that the 4-fold was obtained from a certain pencil of Enriques surfaces with non-algebraic integral Hodge classes of non-torsion type.

Theorem 1.1 has several other consequences. It was recently proved by Voisin [Voi21] that the (n-2)-stage of the coniveau and strong coniveau filtrations

$$N^{n-2}H^{2n-3}(X,\mathbb{Z}), \widetilde{N}^{n-2}H^{2n-3}(X,\mathbb{Z})$$

always coincides modulo torsion on a rationally connected n-fold X (see [BO21] for the definition and properties of the strong coniveau filtration). This result follows from a geometric argument involving families of semi-stable maps from curves to X, combined with an analogue of the Roitman theorem for the Walker maps $\widetilde{\psi}^p$ on a smooth projective variety with small Chow groups [Suz20, Theorems 1.1 and A.3]. The Roitman-type theorem also allows us to describe the torsion part of the kernel of the Abel–Jacobi maps ψ^p in terms of the coniveau under the same assumption on the Chow groups.

Our new proof of Theorem 1.1 only depends on the Bloch-Ogus theory [BO74] and the theory of intermediate Jacobians of mixed Hodge structures. This simplifies to a large extent the original argument due to Walker, which relies on the full machinery of Lawson homology and morphic cohomology.

We work over the complex numbers throughout.

Acknowledgments

The author would like to thank Henri Gillet for interesting discussions. He also thanks the anonymous referees for their careful reading and comments.

2. Proof of the main theorem

Before beginning the proof, we review the construction of the Abel–Jacobi maps using mixed Hodge structures [Jan90] (the reader can consult [Del71, Del74] for basic knowledge about mixed Hodge structures). For a mixed Hodge structure $(H, W_{\bullet}, F^{\bullet})$, we define its intermediate Jacobian J(H) as the extension group

$$J(H) = \operatorname{Ext}^1_{\operatorname{MHS}}(\mathbb{Z}(0), H)$$

in the abelian category MHS of mixed Hodge structures¹. If H is pure of weight -1, then J(H) is isomorphic to a complex torus

$$H_{\mathbb{C}}/(H_{\mathbb{Z}}+F^0H_{\mathbb{C}}).$$

¹We denote by $\mathbb{Z}(m)$ the Hodge structure of Tate $(2\pi i)^m \cdot \mathbb{Z}$, which is a pure Hodge structure of weight -2m.

Let X be a smooth projective variety. Then the cohomology group $H^{2p-1}(X,\mathbb{Z}(p))$ has a pure Hodge structure of weight -1, therefore we have

$$J^p(X) = J\left(H^{2p-1}(X, \mathbb{Z}(p))\right).$$

On the other hand, for a codimension p closed subset $Y \subset X$, the long exact sequence for cohomology groups with supports gives a short exact sequence²

$$0 \to H^{2p-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(X - Y, \mathbb{Z}(p)) \to Z_{\mathcal{V}}^p(X)_{\text{hom}} \to 0.$$

This is a short exact sequence of mixed Hodge structures, where $Z_Y^p(X)_{\text{hom}}$ has the trivial Hodge structure. Then the boundary map in the long exact sequence for $\text{Ext}_{\text{MHS}}^i(\mathbb{Z}(0),-)$ determines a map

$$Z_Y^p(X)_{\text{hom}} \to J^p(X).$$

Now we take the direct limit over all codimension p closed subsets of X to obtain a map

$$Z^p(X)_{\text{hom}} \to J^p(X)$$
.

This coincides with the Abel–Jacobi map AJ^p defined by using currents.

2.1. Construction

We will use a variant of the above construction to construct the Walker maps. For a codimension p closed subset $Y \subset X$, the long exact sequence for cohomology groups with supports gives a commutative diagram

$$0 \longrightarrow H^{2p-1}(X, \mathbb{Z}(p)) \longrightarrow H^{2p-1}(X - Y, \mathbb{Z}(p)) \longrightarrow Z_Y^p(X)_{\text{hom}} \longrightarrow 0$$

$$\downarrow f \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \varinjlim_{Z \in \mathbb{Z}^{p-1}} H^{2p-1}(X - Z, \mathbb{Z}(p)) = \varinjlim_{Z \in \mathbb{Z}^{p-1}} H^{2p-1}(X - Y - Z, \mathbb{Z}(p)) \longrightarrow 0 \longrightarrow 0,$$

where \mathbb{Z}^{p-1} is the set of codimension p-1 closed subsets of X. By the snake lemma, we have an exact sequence

$$0 \to N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X-Y, \mathbb{Z}(p)) \to Z_Y^p(X)_{\text{hom}} \xrightarrow{\delta_Y} \operatorname{Coker}(f).$$

We prove that $Ker(\delta_Y) = Z_Y^p(X)_{alg}$. We have a commutative diagram with exact rows and columns

$$\underline{\lim}_{(Y,Z)\in\mathcal{Z}^{p}/\mathcal{Z}^{p-1}} H_{Z-Y}^{2p-1}(X-Y,\mathbb{Z}(p)) = \underline{\lim}_{(Y,Z)\in\mathcal{Z}^{p}/\mathcal{Z}^{p-1}} H_{Z-Y}^{2p-1}(X-Y,\mathbb{Z}(p))
\downarrow \qquad \qquad \downarrow \partial$$

$$H^{2p-1}(X,\mathbb{Z}(p)) \longrightarrow \underline{\lim}_{Y\in\mathcal{Z}^{p}} H^{2p-1}(X-Y,\mathbb{Z}(p)) \longrightarrow \underline{\lim}_{Y\in\mathcal{Z}^{p}} H_{Y}^{2p}(X,\mathbb{Z}(p)) = Z^{p}(X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

where \mathcal{Z}^p is the set of codimension p closed subsets of X and $\mathcal{Z}^p/\mathcal{Z}^{p-1}$ is the set of pairs $(Y,Z) \in \mathcal{Z}^p \times \mathcal{Z}^{p-1}$ such that $Y \subset Z$. Then the result follows from the diagram and the fact that the map ∂ , which can be identified with the differential $E_1^{p-1,p} \to E_1^{p,p}$ of the coniveau spectral sequence, has the image $Z^p(X)_{\text{alg}} \subset Z^p(X)$ [BO74, Theorem 7.3].

The subgroup of cycles are accordingly defined. $Z^p(X)$ the group of codimension p cycles on X and by $Z^p(X)_{\text{rat}}$ (resp. $Z^p(X)_{\text{alg}}$, $Z^p(X)_{\text{hom}}$) the subgroup of cycles rationally equivalent to zero (resp. algebraically equivalent to zero, homologous to zero) on X. For a codimension p closed subset $Y \subset X$, we denote by $Z^p_Y(X)$ the subgroup of cycles supported on Y; the groups $Z^p_Y(X)_{\text{rat}}$, $Z^p_Y(X)_{\text{alg}}$, and $Z^p_Y(X)_{\text{hom}}$ are accordingly defined.

As a consequence, we have a short exact sequence

$$0 \to N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X-Y,\mathbb{Z}(p)) \to Z^p_{\scriptscriptstyle Y}(X)_{\rm alg} \to 0.$$

This is a short exact sequence of mixed Hodge structures, where $N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))$ has a pure Hodge structure of weight -1 and $Z_Y^p(X)_{\mathrm{alg}}$ has the trivial Hodge structure. Then the boundary map in the long exact sequence for $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}(0),-)$ determines a map

$$\widetilde{\psi}_{Y}^{p} \colon Z_{Y}^{p}(X)_{\text{alg}} \to J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))),$$

where $J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))$ is a complex torus. Now we take the direct limit to obtain a map

$$\widetilde{\psi}^p \colon Z^p(X)_{\mathrm{alg}} \to J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right),$$

which we call the Walker map.

2.2. Basic Properties

To finish the proof of Theorem 1.1, we need to establish several basic properties of the Walker map $\widetilde{\psi}^p$.

Lemma 2.1. We have a commutative diagram

$$Z^{p}(X)_{\text{alg}} \xrightarrow{\widetilde{\psi}^{p}} J\left(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))\right)$$

$$\downarrow^{\pi^{p}}$$

$$Z^{p}(X)_{\text{hom}} \xrightarrow{AJ^{p}} J^{p}(X),$$

where π^p is induced by the inclusion $N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\subseteq H^{2p-1}(X,\mathbb{Z}(p))$.

Proof. We have a commutative diagram of short exact sequences of mixed Hodge structures

for any codimension p closed subset $Y \subset X$. The assertion follows by applying $\operatorname{Ext}^i_{\operatorname{MHS}}(\mathbb{Z}(0),-)$ and taking the direct limit.

Lemma 2.2. Let C be a smooth projective curve and Γ be a codimension p cycle on $C \times X$ each of whose components dominates C. Then we have a commutative diagram:

$$Z^{1}(C)_{\text{hom}} \xrightarrow{AJ^{1}} J^{1}(C)$$

$$\downarrow \Gamma_{*} \qquad \qquad \downarrow \Gamma_{*}$$

$$Z^{p}(X)_{\text{alg}} \xrightarrow{\widetilde{\psi}^{p}} J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))).$$

Proof. We freely use the fact that the Betti cohomology and the Borel-Moore homology form a Poincaré duality theory with supports (see [B-V97, BO74] for the axioms). Let $\pi_C \colon C \times X \to C$ (resp. $\pi_X \colon C \times X \to X$) be the projection to C (resp. X). For a codimension one closed subset $Y \subset C$, setting $Y' = \pi_C^{-1}(Y)$, we have a commutative diagram

$$(2.1) 0 \longrightarrow H^{1}(C, \mathbb{Z}(1)) \longrightarrow H^{1}(C - Y, \mathbb{Z}(1)) \longrightarrow Z_{Y}^{1}(C)_{\text{hom}} \longrightarrow 0$$

$$\downarrow^{(\pi_{C})^{*}} \qquad \downarrow^{(\pi_{C})^{*}} \qquad \downarrow^{(\pi_{C})^{*}}$$

$$0 \longrightarrow H^{1}(C \times X, \mathbb{Z}(1)) \longrightarrow H^{1}(C \times X - Y', \mathbb{Z}(1)) \longrightarrow Z_{Y'}^{1}(C \times X)_{\text{hom}} \longrightarrow 0.$$

Similarly, setting $G = \operatorname{Supp}(\Gamma)$ and $Y'' = Y' \cap G$, we have a commutative diagram

where, letting $i: G-Y'' \to C \times X-Y'$ be a closed immersion and denoting by H_*^{BM} the Borel-Moore homology, the middle vertical map $(\cup \Gamma)'$ is the composition

$$H^{1}(C\times X-Y',\mathbb{Z}(1))\xrightarrow{i^{*}}H^{1}(G-Y'',\mathbb{Z}(1))\xrightarrow{\cap(\Gamma|_{G-Y''})}H^{BM}_{2\dim G-1}(G-Y'',\mathbb{Z}(\dim G-1))\xrightarrow{i_{*}}H^{BM}_{2\dim G-1}(C\times X-Y'',\mathbb{Z}(\dim G-1))=H^{2p+1}(C\times X-Y'',\mathbb{Z}(p)).$$

Since the images of the vertical maps are supported on G, we have another commutative diagram

$$(2.2) \qquad 0 \longrightarrow H^{1}(C \times X, \mathbb{Z}(1)) \longrightarrow H^{1}(C \times X - Y', \mathbb{Z}(1)) \longrightarrow Z_{Y'}^{1}(C \times X)_{\text{hom}} \longrightarrow 0$$

$$\downarrow \cup \Gamma \qquad \qquad \downarrow (\cup \Gamma)' \qquad \qquad \downarrow \cup \Gamma$$

$$0 \longrightarrow N^{p}H^{2p+1}(C \times X, \mathbb{Z}(p+1)) \longrightarrow N^{p}H^{2p+1}(C \times X - Y'', \mathbb{Z}(p+1)) \longrightarrow Z_{Y''}^{p+1}(C \times X)_{\text{alg}} \longrightarrow 0.$$

Finally, setting $Y''' = \pi_X(Y'')$ and letting $j : C \times X - \pi_X^{-1}(Y''') \to C \times X - Y''$ be an open immersion, we have a commutative diagram

$$0 \to H^{2p+1}(C \times X, \mathbb{Z}(p+1)) \to H^{2p+1}(C \times X - Y'', \mathbb{Z}(p+1)) \to Z_{Y''}^{2p+1}(C \times X)_{\text{hom}} \to 0$$

$$\downarrow (\pi_X)_* \qquad \qquad \downarrow (\pi_X)_*^* \qquad \qquad \downarrow (\pi_X)_*$$

$$0 \longrightarrow H^{2p-1}(X, \mathbb{Z}(p)) \longrightarrow H^{2p-1}(X - Y''', \mathbb{Z}(p)) \longrightarrow Z_{Y'''}^p(X)_{\text{hom}} \longrightarrow 0,$$

which restricts to

$$(2.3) \qquad 0 \longrightarrow N^{p}H^{2p+1}(C \times X, \mathbb{Z}(p+1)) \longrightarrow N^{p}H^{2p+1}(C \times X - Y'', \mathbb{Z}(p+1)) \longrightarrow Z_{Y''}^{2p+1}(C \times X)_{\text{alg}} \longrightarrow 0$$

$$\downarrow^{(\pi_{X})_{*}} \qquad \qquad \downarrow^{(\pi_{X})_{*}j^{*}} \qquad \qquad \downarrow^{(\pi_{X})_{*}j^{*}}$$

$$0 \longrightarrow N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \longrightarrow N^{p-1}H^{2p-1}(X - Y''', \mathbb{Z}(p)) \longrightarrow Z_{Y'''}^{p}(X)_{\text{alg}} \longrightarrow 0.$$

By the diagrams (2.1), (2.2), and (2.3), we have a commutative diagram

$$\begin{split} 0 & \longrightarrow H^1(C,\mathbb{Z}(1)) & \longrightarrow H^1(C-Y,\mathbb{Z}(1)) & \longrightarrow Z^1_Y(C)_{\mathrm{hom}} & \to 0 \\ & & \downarrow \Gamma_* & & \downarrow (\pi_X)_*j^*(\cup\Gamma)'(\pi_C)^* & & \downarrow \Gamma_* \\ 0 & \to N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) & \to N^{p-1}H^{2p-1}(X-Y''',\mathbb{Z}(p)) & \to Z^p_{Y'''}(X)_{\mathrm{alg}} & \to 0. \end{split}$$

This is a commutative diagram of mixed Hodge structures. The assertion follows by applying $\operatorname{Ext}^i_{\operatorname{MHS}}(\mathbb{Z}(0),-)$ and taking the direct limit.

Corollary 2.3. The Walker map $\widetilde{\psi}^p$ factors through $A^p(X)$. Moreover we have a commutative diagram

$$A^{p}(X) \xrightarrow{\widetilde{\psi}^{p}} J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right)$$

$$\downarrow^{\pi^{p}}$$

$$CH^{p}(X)_{\text{hom}} \xrightarrow{AJ^{p}} J^{p}(X).$$

Proof. By Lemma 2.2, we have a commutative diagram

$$\bigoplus_{\Gamma} Z^{1}(\mathbb{P}^{1})_{\text{hom}} \xrightarrow{(AJ^{1})} \bigoplus_{\Gamma} J^{1}(\mathbb{P}^{1}) = 0$$

$$\downarrow^{(\Gamma_{*})} \qquad \qquad \downarrow^{(\Gamma_{*})}$$

$$Z^{p}(X)_{\text{alg}} \xrightarrow{\widetilde{\psi}^{p}} J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right),$$

where Γ runs through all codimension p cycles on $\mathbb{P}^1 \times X$ with the components dominating \mathbb{P}^1 . Since the image of the left vertical map is the subgroup $Z^p(X)_{\mathrm{rat}} \subset Z^p(X)$, the first assertion follows. The second assertion is immediate by using Lemma 2.1.

The source of the Walker map $\widetilde{\psi}^p$ will be $A^p(X)$ in the following.

Lemma 2.4. The Walker map $\widetilde{\psi}^p$ is functorial for correspondences: we have a commutative diagram

$$A^{q}(X') \xrightarrow{\widetilde{\psi}^{q}} J\left(N^{q-1}H^{2q-1}(X', \mathbb{Z}(q))\right)$$

$$\downarrow^{\Gamma_{*}} \qquad \qquad \downarrow^{\Gamma_{*}}$$

$$A^{p}(X) \xrightarrow{\widetilde{\psi}^{p}} J\left(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))\right)$$

for any smooth projective varieties X, X' and codimension $(p-q+\dim X')$ cycle Γ on $X'\times X$.

Proof. Note that the action of Γ only depends on its rational equivalence class on $X' \times X$, hence we are allowed to use the moving lemma to ensure that Γ , after moving, comes to intersect properly with finitely many chosen cycles. Now the result follows from an argument similar to that of Lemma 2.2, where we may always assume that involved closed subsets have the correct dimensions.

Corollary 2.5. The Walker map $\widetilde{\psi}^p$ is surjective. Moreover $J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))$ is an abelian variety.

Proof. Let $Z \subset X$ be a closed subset of codimension p-1 such that the natural map

$$H_Z^{2p-1}(X,\mathbb{Z}(p)) \to H^{2p-1}(X,\mathbb{Z}(p))$$

induces a surjection

$$H_Z^{2p-1}(X, \mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)).$$

By the right exactness of the intermediate Jacobian functor J(-) [Beĭ83], we have a surjection

$$(2.4) J(H_Z^{2p-1}(X,\mathbb{Z}(p))) \to J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))).$$

Let \widetilde{Z} be a resolution of Z and \widetilde{Z}_i be the components of \widetilde{Z} . An easy computation shows that the natural map

$$\bigoplus_i H^1(\widetilde{Z}_i,\mathbb{Z}(1)) \to H^{2p-1}_Z(X,\mathbb{Z}(p))$$

is an injection with the cokernel having the trivial Hodge structure. This induces a surjection

(2.5)
$$\bigoplus_{i} J^{1}(\widetilde{Z}_{i}) \to J\left(H_{Z}^{2p-1}(X, \mathbb{Z}(p))\right).$$

Then we combine (2.4) and (2.5) to obtain a surjection

$$\bigoplus_{i} J^{1}(\widetilde{Z}_{i}) \to J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right),\,$$

which coincides with the map induced by the graphs Γ_i of $\widetilde{Z}_i \to Z \to X$. By Lemma 2.4, we have a commutative diagram

$$\bigoplus_{i} CH^{1}(\widetilde{Z}_{i})_{\text{hom}} \xrightarrow{AJ^{1}} \bigoplus_{i} J^{1}(\widetilde{Z}_{i})$$

$$\downarrow^{((\Gamma_{i})_{*})} \qquad \qquad \downarrow^{((\Gamma_{i})_{*})}$$

$$A^{p}(X) \xrightarrow{\widetilde{\psi}^{p}} J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))).$$

The results follow.

Corollary 2.6. The Walker map $\widetilde{\psi}^p$ is regular.

Proof. By Lemma 2.4, we have a commutative diagram

$$CH^{\dim S}(S)_{\text{hom}} \xrightarrow{AJ^{\dim S}} J^{\dim S}(S)$$

$$\downarrow^{\Gamma_{*}} \qquad \qquad \downarrow^{\Gamma_{*}}$$

$$A^{p}(X) \xrightarrow{\widetilde{\psi}^{p}} J\left(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))\right)$$

for any smooth projective variety S and codimension p cycle Γ on $S \times X$. Now the result is immediate using the Albanese map on S.

Remark 2.7. The referee points out that one can slightly modify the definition of a regular homomorphism by allowing S to be any smooth connected (not necessarily projective nor proper) variety. In fact, the Walker map $\widetilde{\psi}^p$ is again regular in this sense: the Nagata compactification reduces the assertion to the proper case, for which the same proof as that of Corollary 2.6 works.

Proof of Theorem 1.1. The Walker map $\widetilde{\psi}^p$ constructed in Subsection 2.1 gives a factorization of the Abel–Jacobi map ψ^p with desired properties, as shown by Corollaries 2.3, 2.5, and 2.6. The proof of Theorem 1.1 is now complete.

3. Questions

Question 3.1. Is the Walker map $\widetilde{\psi}^p: A^p(X) \to J\left(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))\right)$ always universal among all regular homomorphisms?

This is related to another question of Murre:

Question 3.2 ([GMV93, p. 132]). Does there always exist a universal regular homomorphism?

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