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A conjectural formula for $\mathrm{DR}_g(a,-a)\lambda_g$

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Abstract. We propose a conjectural formula for $\mathrm{DR}_g(a,-a)\lambda_g$ and check all its expected properties. Our formula refines the one point case of a similar conjecture made by the first named author in collaboration with Guéré and Rossi, and we prove that the two conjectures are in fact equivalent, though in a quite non-trivial way.

Keywords. moduli of curves; tautological ring; double ramification cycle

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Contents

1. Introduction

In [Bur15] the first named author defined new Hamiltonian integrable hierarchies, the so-called double ramification hierarchies, associated to cohomological field theories. They are conjectured in [Bur15] to be Miura equivalent to the Dubrovin–Zhang hierarchies constructed in [DZ01, BPS12]. This conjecture is further refined and made more explicit in [BDGR18], and in [BGR19] it is reduced to a system of conjectural relations between some explicitly defined classes in the tautological ring of the moduli space of curves $R^*(\overline{\mathcal{M}}_{g,n})$.

The one point case of the conjecture in [BGR19] gives a surprisingly simple expression for the product of the top Chern class of the Hodge bundle $\lambda_g \in R^g(\overline{\mathcal{M}}_{g,1})$ and the push-forward of the double ramification cycle $\mathrm{DR}_g(a,-a) \in R^g(\overline{\mathcal{M}}_{g,2})$ under the map that forgets the second marked point. For the definition of the double ramification cycle $\mathrm{DR}_g(a,-a)$ we refer, for instance, to [BSSZ15], and for general information on the tautological rings of the moduli spaces of curves to a recent survey of Schmitt [Sch20].

In this paper we propose a refinement of the one point case of the conjecture in [BGR19]. We conjecture a formula for $\mathrm{DR}_g(a,-a)\lambda_g \in R^{2g}(\overline{\mathcal{M}}_{g,2})$ in terms of a very simple linear combination of natural strata equipped with psi classes of the same type as in [BGR19]. We analyze our formula in detail and we prove that it satisfies virtually all properties one might expect from the class $\mathrm{DR}_g(a,-a)\lambda_g$, including the intersections with all natural boundary divisors in $\overline{\mathcal{M}}_{g,2}$ and with the psi classes, and finally using these properties we also show that our conjecture is in fact equivalent to the one point case of the conjecture in [BGR19].

Organization of the paper

In Section 2 we formulate our conjecture, explain its relation to the one point case of the conjecture in [BGR19], and state the expected properties of our formula. In Section 3 we introduce our main tools, a variety of corollaries of the Liu–Pandharipande relations among the tautological classes [LP11], and prove all properties stated before.

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2. Conjectural formula and its properties

2.1. Notation

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford compactification of the moduli space of curves with n marked points. There is a natural action of the symmetric group S_n on $\overline{\mathcal{M}}_{g,n}$ by relabeling the points. In particular, for

$n = 2$ we will use the morphism that permutes the first and second marked points that we denote by $(12)_*\colon R^*(\overline{\mathcal{M}}_{g,2}) \to R^*(\overline{\mathcal{M}}_{g,2})$.

Let $\sigma\colon \overline{\mathcal{M}}_{g_1,2} \times \overline{\mathcal{M}}_{g_2,2} \to \overline{\mathcal{M}}_{g_1+g_2,2}$ glue the second marked point of $\overline{\mathcal{M}}_{g_1,2}$ and the first marked point of $\overline{\mathcal{M}}_{g_2,2}$ into a node and identify the first marked point in $\overline{\mathcal{M}}_{g_1,2}$ (respectively, the second marked point in $\overline{\mathcal{M}}_{g_2,2}$) with the first (respectively, the second) marked point in $\overline{\mathcal{M}}_{g_1+g_2,2}$. Let $c_1 \in R^*(\overline{\mathcal{M}}_{g_1,2})$, $c_2 \in R^*(\overline{\mathcal{M}}_{g_2,2})$. It is convenient for us to denote throughout the text $c_1 \diamond c_2 := \sigma_*(c_1 \otimes c_2)$ and we use $\diamond$ as an associative operation on classes in moduli spaces with two marked points.

With the first two points distinguished, we can extend the notation $\diamond$ to the push-forwards of the morphisms $\sigma\colon \overline{\mathcal{M}}_{g_1,2} \times \overline{\mathcal{M}}_{g_2,2+n} \to \overline{\mathcal{M}}_{g_1+g_2,2+n}$ that glue the second marked point of $\overline{\mathcal{M}}_{g_1,2}$ with the first marked point of $\overline{\mathcal{M}}_{g_2,2+n}$ into a node and identify the first marked point in $\overline{\mathcal{M}}_{g_1,2}$ (respectively, the second marked point in $\overline{\mathcal{M}}_{g_2,2+n}$) with the first (respectively, the second) marked point in $\overline{\mathcal{M}}_{g_1+g_2,2+n}$. We can do the same for the similar morphisms $\sigma\colon \overline{\mathcal{M}}_{g_1,2+n} \times \overline{\mathcal{M}}_{g_2,2} \to \overline{\mathcal{M}}_{g_1+g_2,2+n}$.

2.2. Conjectural formula

For $g_1,\dots,g_k,g \geq 1$ and $d_1,\dots,d_k \geq 0$ such that $\sum g_i = g$, let $\mathsf{c}^{g_1,\dots,g_k}_{d_1,\dots,d_k} \in R^{d_1+\dots+d_k+k-1}(\overline{\mathcal{M}}_{g,2})$ be the class represented by the bamboo

$$1 - (g_1) \overset{\psi^{d_1}}{\text{———}} (g_2) \overset{\psi^{d_2}}{\text{———}} (g_3) \overset{\psi^{d_3}}{\text{———}} - - - (g_k) \overset{\psi^{d_k}}{\text{—}} 2 \;=\; \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \cdots \diamond \psi_2^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}}.$$

Denote

$$\overrightarrow{\mathsf{c}}^{\,g}_{d|k} := \sum_{\substack{g_1,\dots,g_k \\ d_1,\dots,d_k}} \mathsf{c}^{g_1,\dots,g_k}_{d_1,\dots,d_k} \in R^d(\overline{\mathcal{M}}_{g,2}),$$

where the sum is taken over all $g_1 + \dots + g_k = g$ and all $d_1 + \dots + d_k + k - 1 = d$ satisfying the inequalities

$$d_1 + \dots + d_\ell + \ell - 1 \leq 2(g_1 + \dots + g_\ell) - 1, \quad \ell = 1,\dots,k.$$

Note that by the definition

$$\overrightarrow{\mathsf{c}}^{\,g}_{d|k} = 0 \qquad \text{if } k > g \text{ or } d \geq 2g. \tag{2.1}$$

Let

$$\begin{aligned} \mathsf{B}^g :=\ & \psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}} + \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \sum_{k=1}^{g_1} (-1)^k \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1|k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \\ =\ & \sum_{k=1}^{g} (-1)^{k-1} \sum_{\substack{d_1,\dots,d_k \\ g_1,\dots,g_k}} \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \cdots \diamond \psi_2^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}} \in R^{2g}(\overline{\mathcal{M}}_{g,2}), \end{aligned} \tag{2.2}$$

where the last sum is taken over all $g_1 + \dots + g_k = g$, $g_1,\dots,g_k \geq 1$, and $d_1 + \dots + d_k + k - 1 = 2g$, $d_1,\dots,d_k \geq 0$, with the extra condition that for any $1 \leq \ell \leq k-1$ we have $d_1 + \dots + d_\ell + \ell - 1 \leq 2(g_1 + \dots + g_\ell) - 1$.

Conjecture 2.1. *We have* $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g = \mathsf{B}^g$.

Note that the left-hand side of this equation can be expressed in the tautological classes using the formula of Janda–Pandharipande–Pixton–Zvonkine [JPPZ17] or, taking into account the factor λ_g, it is sufficient to use the Hain formula [Hai13] (see an explanation, e.g., in [BR21, Section 2]). However, the resulting expressions are much more complicated than the one we conjecture here. Observe that the right-hand side is independent of a, which is consistent with Hain's formula, which states that the compact-type part of $\mathrm{DR}_g(a,-a)$ is a homogeneous polynomial in a of degree $2g$. Thus, it is enough to prove the conjecture for the case $a = 1$.

2.3. Relation to an earlier conjecture for the push-forwards

Conjecture 2.1 is a refinement of the one point case of a conjecture of the first named author with Guéré and Rossi [BGR19, Conjecture 2.5]. Indeed, recall the definition of the class $B^g_{2g-1} \in R^{2g-1}(\overline{\mathcal{M}}_{g,1})$ in [BGR19]. We have:

$$B^g_{2g-1} := \sum_{k=1}^{g} (-1)^{k-1} \sum_{\substack{g_1,\dots,g_k\\ a_1,\dots,a_k}} \quad \textcircled{g_1}\overset{\psi^{a_1}}{\text{———}}\textcircled{g_2}\overset{\psi^{a_2}}{\text{———}}\textcircled{g_3}\overset{\psi^{a_3}}{\text{- - - -}}\textcircled{g_k}\overset{\psi^{a_k}}{\text{—}}\,1\ ,$$

where the sum is taken over all $g_1 + \cdots + g_k = g$, $g_1,\dots,g_k \geq 1$, and $a_1,\dots,a_k \geq 0$ such that

$$a_1 + \cdots + a_k + k - 1 = 2g - 1 \text{ and } a_1 + \cdots + a_\ell + \ell - 1 \leq 2(g_1 + \cdots + g_\ell) - 2 \text{ for } \ell = 1,\dots,k-1.$$

Let $\pi\colon \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ be the map that forgets the second marked point. In the one point case the conjecture from [BGR19, Conjecture 2.5] is reduced to the identity

$$a^{-2g}\pi_*(\mathrm{DR}_g(a,-a)\lambda_g) = B^g_{2g-1},$$

see [BGR19, Section 4.2]. On the other hand, we have the following statement.

Proposition 2.2. *We have* $\pi_*(12)_*\mathsf{B}^g = B^g_{2g-1}$.

Proof. It follows from the fact that $\pi_*(\psi_1^d) = \psi_1^{d-1}$ for $d \geq 1$ and $\pi_*(\psi_1^0) = 0$. Thus all terms with $d_1 = 0$ in (2.2) vanish under the push-forward, and all other terms are in one-to-one correspondence with $a_1 = d_1 - 1$ and $a_i = d_i$ for $i = 2,\dots,k$, $k = 1,\dots,g$. □

Remark 2.3. Note that an expected property of $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g$ is that it is invariant under $(12)_*$, and indeed we prove below that $(12)_*\mathsf{B}^g = \mathsf{B}^g$, $g \geq 1$, so in fact we can reformulate the statement of Proposition 2.2 as $\pi_*\mathsf{B}^g = B^g_{2g-1}$.

In fact, it is also possible to prove a much stronger statement than Proposition 2.2.

Theorem 2.4. *The two conjectural formulas,* $a^{-2g}\pi_*(\mathrm{DR}_g(a,-a)\lambda_g) = B^g_{2g-1}$ *and* $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g = \mathsf{B}^g$*, are equivalent.*

The first formula follows from the second one by Proposition 2.2. The implication in the other direction is quite non-trivial, and we postpone its proof until Section 3.7.

2.4. Properties

We write down a list of properties of B^g.

Theorem 2.5. *We have:*

(2.3) $$(12)_*\mathsf{B}^g = \mathsf{B}^g;$$

(2.4) $$\mathsf{B}^g \cdot \begin{smallmatrix}1\\2\end{smallmatrix}\!>\!\textcircled{g-1}\!\bigcirc = 0;$$

(2.5) $$\mathsf{B}^g \cdot \begin{smallmatrix}1\\2\end{smallmatrix}\!>\!\textcircled{g_1}\text{—}\textcircled{g_2} = 0, \qquad g_1 + g_2 = g,\ g_2 \geq 1;$$

(2.6) $$\mathsf{B}^g \cdot 1\text{—}\textcircled{g_1}\text{—}\textcircled{g_2}\text{—}2 = \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2}, \qquad g_1 + g_2 = g,\ g_1, g_2 \geq 1;$$

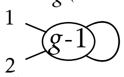
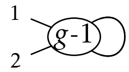
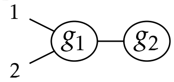
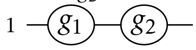
(2.7) $$\pi^*(\mathsf{B}^g)\cdot\psi_1 = \mathsf{B}^g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} \mathsf{B}^{g_1} \diamond \pi^*(\mathsf{B}^{g_2});$$

(2.8) $$\mathsf{B}^g \cdot \psi_1 = \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} \frac{g_2}{g}\mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2},$$

where $\pi\colon \overline{\mathcal{M}}_{g,3}\to\overline{\mathcal{M}}_{g,2}$, $g\geq 1$, *in* (2.7) *is the projection that forgets the third marked point.*

The proof of this theorem is given in Section 3.

Remark 2.6. All these properties are satisfied by $\mathrm{DR}_g(1,-1)\lambda_g=a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g$, namely:

- $\mathrm{DR}_g(1,-1)\lambda_g=\mathrm{DR}_g(-1,1)\lambda_g$ is immediate from Hain's formula [Hai13].
- $\mathrm{DR}_g(1,-1)\lambda_g\cdot$  $=0$, as λ_g restricts to zero on  .
- $\mathrm{DR}_g(1,-1)\lambda_g\cdot$  $=0$, as the classes $\mathrm{DR}_g(1,-1)$ and λ_g respectively restrict to $\mathrm{DR}_{g_1}(1,-1,0)\otimes\mathrm{DR}_{g_2}(0)$ and $\lambda_{g_1}\otimes\lambda_{g_2}$ on $\overline{\mathcal{M}}_{g_1,3}\times\overline{\mathcal{M}}_{g_2,1}$. The vanishing follows after observing $\mathrm{DR}_{g_2}(0)\lambda_{g_2}=(-1)^{g_2}\lambda_{g_2}^2=0$.
- $\mathrm{DR}_g(1,-1)\lambda_g\cdot$  $=\mathrm{DR}_{g_1}(1,-1)\lambda_{g_1}\diamond\mathrm{DR}_{g_2}(1,-1)\lambda_{g_2}$, as $\mathrm{DR}_g(1,-1)$ and λ_g respectively restrict to $\mathrm{DR}_{g_1}(1,-1)\otimes\mathrm{DR}_{g_2}(1,-1)$ and $\lambda_{g_1}\otimes\lambda_{g_2}$ on $\overline{\mathcal{M}}_{g_1,2}\times\overline{\mathcal{M}}_{g_2,2}$.
- The equality

$$\pi^*(\mathrm{DR}_g(1,-1)\lambda_g)\cdot\psi_1=\mathrm{DR}_g(1,-1)\lambda_g\diamond 1|_{\overline{\mathcal{M}}_{0,3}}+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}}\mathrm{DR}_{g_1}\lambda_{g_1}\diamond\pi^*(\mathrm{DR}_{g_2}\lambda_{g_2})$$

  follows from [BSSZ15, Theorem 5]: one should use that $\pi^*\mathrm{DR}_g(1,-1)=\mathrm{DR}_g(1,-1,0)$, apply the formula of [BSSZ15, Theorem 5] with $s=1$ and $n=l=3$, and then multiply the result by λ_g noting that the terms with $p\geq 2$ will vanish after that.
- The identity

$$\mathrm{DR}_g(1,-1)\lambda_g\cdot\psi_1=\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}}\frac{g_2}{g}\mathrm{DR}_{g_1}(1,-1)\lambda_{g_1}\diamond\mathrm{DR}_{g_2}(1,-1)\lambda_{g_2}$$

  follows from the formula of [BSSZ15, Theorem 4] multiplied by λ_g, where one should again note that the terms with $p\geq 2$ vanish after this multiplication.

3. Proofs

3.1. Liu–Pandharipande relations

Fix sets of indices I_1 and I_2 such that $I_1\sqcup I_2=\{1,\dots,n\}$. Let $\Delta_{g_1,g_2}\subset\overline{\mathcal{M}}_{g,n}$ denote the divisor in $\overline{\mathcal{M}}_{g,n}$ whose generic points are represented by two-component curves intersecting at a node, where the two components have genera g_1,g_2 and contain the points with the indices I_1,I_2, respectively. Note that if $g_i=0$, then $|I_i|$ must be at least 2, for the stability condition.

For each Δ_{g_1,g_2} we consider the map $\iota_{g_1,g_2}\colon\overline{\mathcal{M}}_{g_1,|I_1|+1}\times\overline{\mathcal{M}}_{g_2,|I_2|+1}\to\overline{\mathcal{M}}_{g,n}$ that glues the last marked points into a node and whose image is Δ_{g_1,g_2}. Let $\psi_{\circ 1}$ (respectively, $\psi_{\circ 2}$) denote the psi classes at the marked points on the first (respectively, second) component that are glued into the node.

Proposition 3.1 ([LP11, Proposition 1]). *For any* $g\geq 0$, $n\geq 4$, I_1 *and* I_2 *such that* $I_1\sqcup I_2=\{1,\dots,n\}$ *and* $|I_1|,|I_2|\geq 2$, *and an arbitrary* $r\geq 0$ *we have:*

$$\sum_{\substack{g_1,g_2\geq 0\\ g_1+g_2=g}}\ \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=\\ 2g-3+n+r}}(-1)^{a_1}(\iota_{g_1,g_2})_*\psi_{\circ 1}^{a_1}\psi_{\circ 2}^{a_2}=0\in R^{2g-2+n+r}(\overline{\mathcal{M}}_{g,n}).\tag{3.1}$$

This relation has the following corollaries.

Corollary 3.2. *For any $g \geq 1$, $n \geq 1$, $r \geq 0$ we have:*

$$(-1)^{2g+n+r}\psi_2^{2g+n+r} + \sum_{\substack{g_1\geq 0,\, g_2>0\\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2g-1+n+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2+n}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0 \in R^{2g+n+r}(\overline{\mathcal{M}}_{g,n+2}) \tag{3.2}$$

and

$$-\psi_1^{2g+n+r} + \sum_{\substack{g_1>0,\, g_2\geq 0\\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2g-1+n+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2+n}} = 0 \in R^{2g+n+r}(\overline{\mathcal{M}}_{g,n+2}). \tag{3.3}$$

Corollary 3.3 ([LP11, Proposition 2]). *For any $g \geq 1$, $r \geq 0$ we have:*

$$-\psi_1^{2g+r} + (-1)^{2g+r}\psi_2^{2g+r} + \sum_{\substack{g_1,g_2>0\\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2g-1+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0 \in R^{2g+r}(\overline{\mathcal{M}}_{g,2}). \tag{3.4}$$

All corollaries are proved by taking suitable push-forwards of the relations (3.1) under the maps forgetting the marked points, see [LP11, HIS21].

3.2. The symmetry property

Denote $\overleftarrow{\mathsf{c}}^g_{d|k} := (12)_*\overrightarrow{\mathsf{c}}^g_{d|k}$. We will use the following conventions to simplify notation:

$$\overrightarrow{\mathsf{c}}^0_{-1|0} \diamond \psi_2^d|_{\overline{\mathcal{M}}_{g,2}} := \psi_2^d|_{\overline{\mathcal{M}}_{g,2}} \quad \text{and} \quad \psi_1^d|_{\overline{\mathcal{M}}_{g,2}} \diamond \overleftarrow{\mathsf{c}}^0_{-1|0} := \psi_1^d|_{\overline{\mathcal{M}}_{g,2}}.$$

Note that there is the following recursion relation for the classes $\overrightarrow{\mathsf{c}}^g_{d|k}$:

$$\overrightarrow{\mathsf{c}}^g_{d|k+1} = \sum_{\substack{g_1+g_2=g\\ d_1+d_2=d-1}} \overrightarrow{\mathsf{c}}^{g_1}_{d_1|k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}, \quad k\geq 0,\ d\leq 2g-1,$$

where $d_1 = -1$ is allowed in the sum to include the case $k=0$, as explained before. Let us now prove Equation (2.3). Let

$$E := \mathsf{B}^g - (12)_*\mathsf{B}^g = \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_1} (-1)^k \overrightarrow{\mathsf{c}}^{g_1}_{d_1|k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2} (-1)^k \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2|k}.$$

Let E_ℓ denote the terms of E consisting of exactly ℓ components, *i.e.*,

$$E_\ell := \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} (-1)^{\ell-1}\overrightarrow{\mathsf{c}}^{g_1}_{d_1|\ell-1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} (-1)^{\ell-1}\psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2|\ell-1}.$$

Lemma 3.4. *We can write $E_1 + \cdots + E_\ell$ as an expression involving only graphs with $\ell+1$ vertices. In particular:*

$$E_1 + \cdots + E_\ell = (-1)^{\ell+1} \sum_{\substack{r+s=\ell-1\\ g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3}} (-1)^{d_1+d_2}\overrightarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}.$$

Proof. We prove the lemma by induction. The base of induction is the $\ell=1$ case, which follows immediately from (3.4):

$$E_1 = \psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}} - \psi_1^{2g}|_{\overline{\mathcal{M}}_{g,2}} = -\sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} (-1)^{d_1}\overrightarrow{\mathsf{c}}^0_{-1|0} \diamond \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^0_{-1|0}.$$

In order to prove the step of induction, assume the lemma is true for $\ell \geq 1$. Then

$$
\begin{aligned}
E_1 + \cdots + E_{\ell+1} = (-1)^{\ell+1} & \sum_{\substack{r+s=\ell-1 \\ g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-3}} (-1)^{d_1+d_2} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{c}^{g_4}_{d_4|s} \\
& + \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell} \overleftarrow{c}^{g_1}_{d_1|\ell} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell} \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{c}^{g_2}_{d_2|\ell}.
\end{aligned} \tag{3.5}
$$

We can split the first summand into two in the following way:

- $d_1 + d_2 \leq 2(g_1 + g_2) - 2$ and $d_3 + d_4 \geq 2(g_3 + g_4) - 1$;
- $d_1 + d_2 \geq 2(g_1 + g_2) - 1$ and $d_3 + d_4 \leq 2(g_3 + g_4) - 2$.

Thus, the summand with $d_1 + d_2 \leq 2(g_1 + g_2) - 2$ takes the form

$$
\begin{aligned}
(-1)^{\ell+1} & \sum_{\substack{r+s=\ell-1 \\ g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-3 \\ d_1+d_2\leq 2(g_1+g_2)-2}} (-1)^{d_1+d_2} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{c}^{g_4}_{d_4|s} \\
& = (-1)^{\ell} \sum_{\substack{r+s=\ell, r\geq 1 \\ g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-2}} (-1)^{d_1} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{c}^{g_3}_{d_3|s}.
\end{aligned}
$$

Note that the third summand of (3.5) corresponds to the missing terms with $r = 0$ in the last expression. Similarly, for the terms with $d_3 + d_4 \leq 2(g_3 + g_4) - 2$

$$
\begin{aligned}
(-1)^{\ell+1} & \sum_{\substack{r+s=\ell-1 \\ g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-3 \\ d_3+d_4\leq 2(g_3+g_4)-2}} (-1)^{d_1+d_2} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{c}^{g_4}_{d_4|s} \\
& = -(-1)^{\ell} \sum_{\substack{r+s=\ell, s\geq 1 \\ g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-2}} (-1)^{d_1+d_2} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{c}^{g_3}_{d_3|s}.
\end{aligned}
$$

Again, the second summand of (3.5) corresponds to the missing terms with $s = 0$ in the last expression. Putting everything together

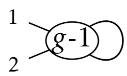
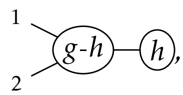
$$
E_1 + \cdots + E_{\ell+1} = (-1)^{\ell} \sum_{\substack{r+s=\ell \\ g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-2}} (-1)^{d_1} \overleftarrow{c}^{g_1}_{d_1|r} \diamond \left(\psi_1^{d_2} - (-1)^{d_2} \psi_2^{d_2}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{c}^{g_3}_{d_3|s}.
$$

Using (2.1) we see that a term in the last sum is equal to zero unless $d_2 \geq 2g_2$. Then the result follows after applying (3.4) to the last expression. $\square$

Applying the lemma above to $E = E_1 + \cdots + E_g$ proves equation (2.3).

3.3. Intersections with divisors of two types

Here we prove Equations (2.4) and (2.5). It is convenient to use the following notations for the classes of the divisors under consideration:

$\delta_g :=$  , $\quad \delta'_g :=$  , if $g \geq h$; 0, if $g < h$, where we fixed $h \geq 1$. So let us prove that

$$\omega_g B^g = 0 \quad \text{if} \quad \omega_g = \delta_g \text{ or } \omega_g = \delta'_g. \tag{3.6}$$

We will use the following property: $\omega_{g_1+g_2}(\alpha \diamond \beta) = \omega_{g_1}\alpha \diamond \beta + \alpha \diamond \omega_{g_2}\beta$, where $\alpha \in R^*(\overline{\mathcal{M}}_{g_1,2})$ and $\beta \in R^*(\overline{\mathcal{M}}_{g_2,2})$. We decompose

$$\omega_g \mathsf{B}^g = \sum_{k\geq 1} E_k, \quad \text{where} \quad E_k := (-1)^{k-1} \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \omega_g \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1|k-1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \right).$$

Lemma 3.5. *We have:*

$$E_1 + \cdots + E_k = (-1)^{k+1} \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ r+s=k-1}} (-1)^{d_1+d_2} \omega_{g_1+g_2} \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \right) \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}. \tag{3.7}$$

Proof. We prove the lemma by induction. The base of induction is the $k = 1$ case, which reads:

$$E_1 = \omega_g \psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}} = - \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} (-1)^{d_1} \omega_{g_1} \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}.$$

If $\omega_i = \delta_i$, then this equation follows from the genus $g-1$ case of (3.2) with $n = 2$ and $r = 0$, after taking the push-forward under the map that glues two marked points. If $\omega_i = \delta'_i$, then the equation follows from the genus $g-h$ case of (3.2) with $n = 1$ and an appropriate r, after taking the product with $1 \in R^0(\overline{\mathcal{M}}_{h,1})$ and then the push-forward under the gluing map $\overline{\mathcal{M}}_{g-h,3} \times \overline{\mathcal{M}}_{h,1} \to \overline{\mathcal{M}}_{g,2}$.

Let us now assume that the lemma holds for $k \geq 1$. Then for $k+1$ we have

$$\begin{aligned} E_1 + \cdots + E_{k+1} =& (-1)^{k+1} \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ r+s=k-1}} (-1)^{d_1+d_2} \omega_{g_1+g_2} \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \right) \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s} \\ &+ (-1)^k \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \omega_g \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1|k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \right). \end{aligned} \tag{3.8}$$

As in the proof of symmetry, we split the first summand in the following way:

- $d_1 + d_2 \leq 2(g_1+g_2) - 2$ and $d_3 + d_4 \geq 2(g_3+g_4) - 1$;
- $d_1 + d_2 \geq 2(g_1+g_2) - 1$ and $d_3 + d_4 \leq 2(g_3+g_4) - 2$.

The terms with $d_1 + d_2 \leq 2(g_1+g_2) - 2$ combine in the following way:

$$(-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}} (-1)^{d_1} \omega_{g_1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3|s}.$$

Note that we do not explicitly impose the condition $r \geq 1$ because we adopt the convention $\omega_0 \overrightarrow{\mathsf{c}}^{0}_{-1|0} := 0$. On the other hand, for the terms with $d_3 + d_4 \leq 2(g_3+g_4) - 2$ we obtain

$$(-1)^{k+1} \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k,\, s\geq 1}} (-1)^{d_1+d_2} \omega_{g_1+g_2} \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \right) \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3|s}.$$

Note that the missing terms with $s=0$ are exactly the ones in the last line of (3.8), *i.e.*, those corresponding to E_{k+1}. Putting everything together

$$E_1+\dots+E_{k+1}=(-1)^k\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}}(-1)^{d_1}\omega_{g_1}\overrightarrow{\mathrm{c}}^{g_1}_{d_1|r}\diamond\left(\psi_1^{d_2}-(-1)^{d_2}\psi_2^{d_2}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\overleftarrow{\mathrm{c}}^{g_3}_{d_3|s}$$
$$+(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}}(-1)^{d_1+d_2}\overrightarrow{\mathrm{c}}^{g_1}_{d_1|r}\diamond\omega_{g_2}\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\overleftarrow{\mathrm{c}}^{g_3}_{d_3|s}.$$

The result follows from applying (3.4) to the first summand and (3.2) to the second one in the expression above. □

Equation (3.6) follows after applying the previous lemma to $E_1+\dots+E_g=\omega_g B^g$ and noting that the right-hand side of (3.7) vanishes in this case because of (2.1).

Remark 3.6. Note that Properties (2.4) and (2.5) can be equivalently stated as

$$\mathrm{gl}_{1*}(\mathrm{gl}_1^*\mathrm{B}^g)=0,\qquad \mathrm{gl}_{2*}(\mathrm{gl}_2^*\mathrm{B}^g)=0,$$

where $\mathrm{gl}_1\colon\overline{\mathcal{M}}_{g-1,4}\to\overline{\mathcal{M}}_{g,2}$ is the gluing map identifying the last two marked points on a curve from $\overline{\mathcal{M}}_{g-1,4}$, and $\mathrm{gl}_2\colon\overline{\mathcal{M}}_{g_1,3}\times\overline{\mathcal{M}}_{g_2,1}\to\overline{\mathcal{M}}_{g,2}$ is the map gluing the third marked point on a curve from $\overline{\mathcal{M}}_{g_1,3}$ with the marked point on a curve from $\overline{\mathcal{M}}_{g_2,1}$. Actually, the arguments from this section can be slightly modified in order to show that $\mathrm{gl}_1^*\mathrm{B}^g=0$ and $\mathrm{gl}_2^*\mathrm{B}^g=0$, which is stronger than what we have proved. The corresponding properties for the DR cycle are clearly true: $\mathrm{gl}_1^*(\mathrm{DR}_g(1,-1)\lambda_g)=0$ and $\mathrm{gl}_2^*(\mathrm{DR}_g(1,-1)\lambda_g)=0$. This observation belongs to the anonymous referee of our paper and we thank him for sharing it with us.

3.4. Intersection with a divisor of curves with marked points on different components

The goal of this section is to prove Equation (2.6). To this end, we need a new notation. Let $g>h>g_1$, $k\ge 1$. Denote

$$\overrightarrow{\mathrm{a}}^{h}_{d|k}:=\sum_{d_1+d_2=d}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\sum_{\substack{i_1,\dots,i_k\\ a_1,\dots,a_k}}\mathrm{c}^{i_1,\dots,i_k}_{a_1,\dots,a_k}+\psi_1\sum_{\substack{j_1,\dots,j_k\\ b_1,\dots,b_k}}\mathrm{c}^{j_1,\dots,j_k}_{b_1,\dots,b_k}\right),$$

where the first sum in the parentheses is taken over all $i_1,\dots,i_k\ge 1$ such that $i_1+\dots+i_k=h-g_1$ and all $a_1,\dots,a_k\ge 0$ such that $a_1+\dots+a_k+k=d_2$ and for any $\ell=1,\dots,k$ we have

$$d_1+a_1+\dots+a_\ell+\ell\le 2(g_1+i_1+\dots+i_\ell).$$

The second sum in the parentheses is taken over all $j_1,\dots,j_k\ge 1$ such that $j_1+\dots+j_k=h-g_1$ and all $b_1,\dots,b_k\ge 0$ such that $b_1+\dots+b_k+k+1=d_2$ and for any $\ell=1,\dots,k$ we have

$$d_1+b_1+\dots+b_\ell+\ell\le 2(g_1+j_1+\dots+j_\ell)-1.$$

In particular,

$$\overrightarrow{\mathrm{a}}^{h}_{d|k}=0\quad\text{if}\quad k>h-g_1\ \text{or}\ d>2h, \tag{3.9}$$

and

$$\overrightarrow{\mathrm{a}}^h_{d|1} = \sum_{d_1+d_2=d}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\psi_2^{d_2-1}+\psi_1\psi_2^{d_2-2}\right)\Big|_{\overline{\mathcal{M}}_{h-g_1,2}}, \qquad d\le 2g,$$

$$\overrightarrow{\mathrm{a}}^h_{d|k+1} = \sum_{\substack{g_1<f<h\\ d_1+d_2=d-1}}\overrightarrow{\mathrm{a}}^f_{d_1|k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{h-f,2}}, \qquad d\le 2g,\ k\ge 1.$$

It is convenient to set $\overrightarrow{\mathrm{a}}^h_{d|k} := 0$ for $h\le g_1$.

Lemma 3.7. *We have:*

$$\mathrm{B}^g\cdot\ 1 - (g_1) - (g_2) - 2 = \mathrm{B}^{g_1}\diamond\mathrm{B}^{g_2} + \sum_{d_1+d_2=2g}\sum_{m=1}^{g_1}(-1)^{m+1}\overrightarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}}$$
$$+\sum_{\substack{g_1<h<g\\ d_1+d_2=2g}}\sum_{k=1}^{g_2-1}(-1)^{k+1}\overrightarrow{\mathrm{a}}^h_{d_1|k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}. \tag{3.10}$$

Proof. This lemma follows directly from the excess intersection formula [GP03, Section A.4]. Let us consider $g=g_1+g_2=f_1+\cdots+f_m$ for some $m\ge 1$. We have:

$$\mathrm{c}^{f_1,\dots,f_m}_{a_1,\dots,a_m}\cdot\ 1 - (g_1) - (g_2) - 2$$
$$=\begin{cases}\mathrm{c}^{f_1,\dots,f_{i-1},f_i',f_i'',f_{i+1},\dots,f_m}_{a_1,\dots,a_{i-1},0,a_i,a_{i+1},\dots,a_m}, & \text{if } g_1=f_1+\cdots+f_{i-1}+f_i' \text{ and } f_i=f_i'+f_i'', \text{ where } f_i',f_i''\ge 1\\[2ex] -\mathrm{c}^{f_1,\dots,f_m}_{a_1,\dots,a_{i-1},a_i+1,a_{i+1},\dots,a_m}-\mathrm{c}^{f_1,\dots,f_i}_{a_1,\dots,a_i}\diamond\psi_1\mathrm{c}^{f_{i+1},\dots,f_m}_{a_{i+1},\dots,a_m}, & \text{if } f_1+\cdots+f_i=g_1.\end{cases} \tag{3.11}$$

Recall that in Formula (2.2) for B^g we have only $\mathrm{c}^{f_1,\dots,f_m}_{a_1,\dots,a_m}$ satisfying the conditions

$$a_1+\cdots+a_i+i-1\le 2(f_1+\cdots+f_i)-1$$

for $i=1,\dots,m-1$ and $a_1+\cdots+a_m+m-1=2g$. We apply Equation (3.11) to all terms of the formula for B^g and we distinguish the following cases:

(1) There exists i such that $f_1+\cdots+f_i=g_1$ and in addition $a_1+\cdots+a_i+i-1=2(f_1+\cdots+f_i)-1$. The first summands in (3.11) applied to these terms form $\mathrm{B}^{g_1}\diamond\mathrm{B}^{g_2}$.
(2) There exists i such that $f_1+\cdots+f_i=g_1$ and in addition $a_1+\cdots+a_i+i-1<2(f_1+\cdots+f_i)-1$. The first summands in (3.11) applied to these terms contribute either to the second (if $i=m-1$) or the third line (if $i<m-1$) of (3.10). More precisely, we can say that in both cases we get terms of the type $\mathrm{c}^{j_1,\dots,j_p}_{t_1,\dots,t_p}$ such that $j_1+\cdots+j_q=g_1$ for some $q<p$, with an extra requirement that $t_q>0$. If $q=p-1$ (respectively, $q<p-1$), these terms land in the second (respectively, third) line of (3.10).
(3) We have $f_1+\cdots+f_{q-1}<g_1<f_1+\cdots+f_q$ for some $1\le q\le m$. Apply (3.11). We get exactly the same terms as in the previous case, but now with an extra requirement that $t_q=0$. This and the previous cases deliver together all terms in the second and the third lines of (3.10) that do not contain ψ_1.
(4) There exists i such that $f_1+\cdots+f_i=g_1$ and $a_1+\cdots+a_i+i-1\le 2(f_1+\cdots+f_i)-1$. The second summands in (3.11) applied to these terms form the summands with ψ_1 in the second and the third lines of (3.10).

□

So our goal is to prove that the sum of the second and the third lines of Equation (3.10) vanishes. To this end, we have a more general statement. Let

$$E := \mathsf{B}^g \cdot \; 1 \,—\, g_1 \,—\, g_2 \,—\, 2 - \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2} = \sum_{\ell \geq 1} E_\ell,$$

where

$$E_\ell := \sum_{d_1+d_2=2g} \sum_{m=1}^{g_1} (-1)^{m+1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1|m} \diamond \left(\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}} \cdot \delta_{\ell,1} + \sum_{\substack{g_1<h<g \\ d_1+d_2=2g}} (-1)^{\ell+1} \overrightarrow{\mathsf{a}}^{h}_{d_1|\ell} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}.$$

Lemma 3.8. *For any* $\ell \geq 1$ *we have:*

$$\begin{aligned} E_1 + \cdots + E_\ell = (-1)^\ell & \sum_{\substack{f_1+f_2=g_2 \\ d_1+d_2+d_3=2g-1}} (-1)^{d_3} \sum_{m=1}^{g_1} (-1)^m \overrightarrow{\mathsf{c}}^{g_1}_{d_1|m} \diamond \left(\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3|\ell} \\ + (-1)^\ell & \sum_{\substack{g_1<h<g \\ f_1+f_2=g-h \\ d_1+d_2+d_3=2g-1}} (-1)^{d_3} \sum_{k=1}^{\ell} \overrightarrow{\mathsf{a}}^{h}_{d_1|k} \diamond \left(\psi_2^{d_2} - (-1)^{d_2} \psi_1^{d_2}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3|\ell-k}. \end{aligned} \tag{3.12}$$

Proof. We prove the lemma by induction. The base of induction is $\ell = 1$, and it is equivalent to the following equation:

$$\begin{aligned} & \sum_{d_1+d_2=2g} \sum_{m=1}^{g_1} (-1)^{m+1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1|m} \diamond \left(\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}} \\ & \qquad = - \sum_{\substack{f_1+f_2=g_2 \\ d_1+d_2+d_3=2g-1}} (-1)^{d_3} \sum_{m=1}^{g_1} (-1)^m \overrightarrow{\mathsf{c}}^{g_1}_{d_1|m} \diamond \left(\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3|1} \\ & \qquad\quad - \sum_{\substack{g_1<h<g \\ d_1+d_2=2g}} (-1)^{d_2} \overrightarrow{\mathsf{a}}^{h}_{d_1|1} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}. \end{aligned} \tag{3.13}$$

We rewrite $\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}$ in the first line of (3.13) as

$$(-1)^{d_2-1} \psi_1^{d_2} + \psi_2^{d_2} - \psi_1 \left((-1)^{d_2-1} \psi_1^{d_2-1} - \psi_2^{d_2-1}\right).$$

Noting that $d_1 \leq 2g_1 - 1$ implies $d_2 - 1 \geq 2g_2$, we apply identity (3.4) twice to obtain

$$\left(\psi_2^{d_2} + \psi_1 \psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}} = \sum_{\substack{f_1+f_2=g_2 \\ a_1+a_2=d_2-1}} (-1)^{a_2} \left(\psi_1 \psi_2^{a_1-1} + \psi_2^{a_1}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{f_2,2}}, \quad d_2 \geq 2g_2 + 1. \tag{3.14}$$

If $a_2 \leq 2f_2 - 1$ (in both summands), then we obtain the second line in (3.13), and if $a_2 \geq 2f_2$, then we obtain the third line in (3.13).

The induction step is equivalent to the following equation:

$$(-1)^{\ell}\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{m=1}^{g_1}(-1)^m\overleftarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell} \tag{3.15}$$

$$+(-1)^{\ell}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell}\overrightarrow{\mathrm{a}}^{h}_{d_1|k}\diamond\left(\psi_2^{d_2}-(-1)^{d_2}\psi_1^{d_2}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell-k} \tag{3.16}$$

$$=(-1)^{\ell+1}\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{m=1}^{g_1}(-1)^m\overleftarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\Big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell+1} \tag{3.17}$$

$$-(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell+1}\overrightarrow{\mathrm{a}}^{h}_{d_1|k}\diamond(-1)^{d_2}\psi_1^{d_2}|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell+1-k} \tag{3.18}$$

$$+(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell+1}\overrightarrow{\mathrm{a}}^{h}_{d_1|k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell+1-k} \tag{3.19}$$

$$+(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ d_1+d_2=2g}}\overrightarrow{\mathrm{a}}^{h}_{d_1|\ell+1}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}, \tag{3.20}$$

where $\ell\geq 1$.

In line (3.15) we have $d_1\leq 2g_1-1$ and $d_3\leq 2f_2-1$, hence $d_2-1\geq 2f_1$ and by (3.14) the expression in line (3.15) is equal to

$$(-1)^{\ell}\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}\sum_{\substack{h_1+h_2=f_1\\ a_1+a_2=d_2-1}}\sum_{m=1}^{g_1}(-1)^m(-1)^{a_2+d_3}\overleftarrow{\mathrm{c}}^{g_1}_{d_1|m}\diamond\left(\psi_2^{a_1}+\psi_1\psi_2^{a_1-1}\right)\Big|_{\overline{\mathcal{M}}_{h_1,2}}\diamond\psi_1^{a_2}|_{\overline{\mathcal{M}}_{h_2,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell}.$$

The part of this sum with $a_2+d_3\leq 2(h_2+f_2)-2$ is equal to the expression in line (3.17) while the part with $a_2+d_3\geq 2(h_2+f_2)-1$ is equal to the $k=1$ term of the expression in line (3.18).

In line (3.16) we have $d_1\leq 2h$ and $d_3\leq 2f_2-1$, hence $d_2\geq 2f_1$ and applying identity (3.4) we get

$$(-1)^{\ell}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}\sum_{\substack{a_1+a_2=d_2-1\\ h_1+h_2=f_1}}\sum_{k=1}^{\ell}(-1)^{a_2+d_3}\overrightarrow{\mathrm{a}}^{h}_{d_1|k}\diamond\psi_2^{a_1}|_{\overline{\mathcal{M}}_{h_1,2}}\diamond\psi_1^{a_2}|_{\overline{\mathcal{M}}_{h_2,2}}\diamond\overleftarrow{\mathrm{c}}^{f_2}_{d_3|\ell-k}.$$

The part of this sum with $a_2+d_3\leq 2(h_2+f_2)-2$ is equal to the part of (3.19) with $k=1,\dots,\ell$ while the part with $a_2+d_3\geq 2(h_2+f_2)-1$ is equal to the part of (3.18) with $k=2,\dots,\ell+1$.

Finally the part of (3.19) with $k=\ell+1$ is equal exactly to (3.20) with the opposite sign. This completes the proof of the induction step and the proof of the lemma. □

Equation (2.6) follows after applying the above lemma to $E=E_1+\cdots+E_{g_2}$ and noting that the right-hand side of (3.12) vanishes for $\ell=g_2$ because of (2.1) and (3.9).

3.5. Evaluation of psi class on a pull-back

To prove (2.7), let us introduce the notation

$$\overleftarrow{\mathsf{d}}^g_{d|k} := \sum_{\ell=1}^{k} \sum_{\substack{g_1,\dots,g_k\\ d_1,\dots,d_k}} \psi_1^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}} \diamond \psi_1^{d_{k-1}}|_{\overline{\mathcal{M}}_{g_{k-1},2}} \diamond \cdots \diamond \psi_1^{d_\ell}|_{\overline{\mathcal{M}}_{g_\ell,3}} \diamond \cdots \diamond \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}, \quad k \geq 1,$$

where the sum is taken over all $g_1,\dots,g_k \geq 1$ and $d_1,\dots,d_k \geq 0$ satisfying $g_1+\cdots+g_k = g$, $d_1+\cdots+d_k+k-1 = d$, and $d_1+\cdots+d_s+s-1 \leq 2(g_1+\cdots+g_s)-1$ for all $1 \leq s \leq k$. Similarly, we define

$$\overleftarrow{\mathsf{e}}^g_{d|k} := \sum_{\ell=1}^{k} \sum_{\substack{g_1,\dots,g_k\\ d_1,\dots,d_k}} \psi_1^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}} \diamond \cdots \diamond \psi_1^{d_{\ell+1}}|_{\overline{\mathcal{M}}_{g_{\ell+1},2}} \diamond 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \psi_1^{d_{\ell-1}}|_{\overline{\mathcal{M}}_{g_{\ell-1},2}} \diamond \cdots \diamond \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}, \quad k \geq 1,$$

where the sum is taken over all $g_1,\dots,g_{\ell-1},g_{\ell+1},\dots,g_k \geq 1$ and $d_1,\dots,d_k \geq 0$ satisfying $g_1+\cdots+g_k = g$, $d_1+\cdots+d_k+k-1 = d$, and $d_1+\cdots+d_s+s-1 \leq 2(g_1+\cdots+g_s)-1$ for all $1 \leq s \leq k$. Note that in particular $d_\ell = g_\ell = 0$. Note also that $\overleftarrow{\mathsf{e}}^g_{d|1} = 0$. We will adopt the convention $\overleftarrow{\mathsf{d}}^g_{d|0} = \overleftarrow{\mathsf{e}}^g_{d|0} := 0$.

Using

$$\pi^*(\psi_1^a|_{\overline{\mathcal{M}}_{g,2}}) = \psi_1^a|_{\overline{\mathcal{M}}_{g,3}} - 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \psi_1^{a-1}|_{\overline{\mathcal{M}}_{g,2}}, \quad a \geq 1,$$

it is straightforward to see that

$$\pi^*(\overleftarrow{\mathsf{c}}^g_{d|k}) = \overleftarrow{\mathsf{d}}^g_{d|k} - \overleftarrow{\mathsf{e}}^g_{d|k+1}. \tag{3.21}$$

As before, let

$$\begin{aligned}
E &:= \pi^*(\mathsf{B}^g)\psi_1 - \mathsf{B}^g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} - \sum_{g_1+g_2=g} \mathsf{B}^{g_1} \diamond \pi^*(\mathsf{B}^{g_2}) \\
&= \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2+1} (-1)^k \left(\psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,3}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2|k} + \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_2}_{d_2|k} + \overleftarrow{\mathsf{e}}^{g_2}_{d_2|k} \right) \right) \\
&\quad - \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_1} (-1)^k \overleftarrow{\mathsf{c}}^{g_1}_{d_1|k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond 1|_{\overline{\mathcal{M}}_{0,3}} &(3.22)\\
&\quad - \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1\\ s\geq 1 \text{ if } g_3=0}} \sum_{r=0}^{g_1} \sum_{s=0}^{g_4} (-1)^{r+s} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s} &(3.23)\\
&\quad - \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1}} \sum_{r=0}^{g_1} \sum_{s=0}^{g_4+1} (-1)^{r+s} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4|s} \right).
\end{aligned}$$

where we have used the already proven symmetry (2.3) and the corresponding mirror formula of (2.2)

$$\mathsf{B}^g = \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2} (-1)^k \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2|k}$$

for the factors on the right-hand side of $\diamond$. Note that (3.22) is exactly the forbidden case $g_3 = 0$ and $s = 0$ in (3.23). Let E_k denote the terms in the expression above that have k components, *i.e.*,

$$E_k :=$$

$$(-1)^{k-1} \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \left(\psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,3}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2|k-1} + \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_2}_{d_2|k-1} + \overleftarrow{\mathsf{e}}^{g_2}_{d_2|k-1}\right)\right) \tag{3.24}$$

$$+ (-1)^{k-1} \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1\\ r+s=k-2}} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s} + \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4|s}\right)\right). \tag{3.25}$$

The following inductive lemma will immediately imply Equation (2.7).

Lemma 3.9. *We can write $E_1 + E_2 + \cdots + E_k$ as an expression involving only graphs with $k+1$ vertices. More precisely, we have:*

$$\begin{aligned} &E_1+E_2 + \cdots + E_k \\ &\quad = (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1}} (-1)^{d_1+d_2} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4|s}\right) + \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}\right). \end{aligned}$$

Proof. We proceed by induction on k. The case $k = 1$ is clear, as

$$\begin{aligned} E_1 = \psi_1^{2g+1}|_{\overline{\mathcal{M}}_{g,3}} &= \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g}} (-1)^{d_1} \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}} \\ &= \sum_{\substack{g_2+g_3=g\\ d_2+d_3=2g}} (-1)^{d_2} \overleftarrow{\mathsf{c}}^{0}_{-1|0} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{0}_{-1|0} \end{aligned}$$

by (3.3).

Assume the lemma holds for $k \geq 1$, then we split $E_1 + \cdots + E_k$ into three kinds of summands, according to the powers of the psi classes:

- $d_1 + d_2 < 2(g_1 + g_2) - 1$ and $d_3 + d_4 > 2(g_3 + g_4) - 1$;
- $d_1 + d_2 > 2(g_1 + g_2) - 1$ and $d_3 + d_4 < 2(g_3 + g_4) - 1$;
- $d_1 + d_2 = 2(g_1 + g_2) - 1$ and $d_3 + d_4 = 2(g_3 + g_4) - 1$.

Note that the summands of the third kind cancel out with (3.25) for E_{k+1}. We rewrite the terms with $d_1 + d_2 < 2(g_1 + g_2) - 1$ as

$$\begin{aligned} &(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1\\ d_1+d_2<2(g_1+g_2)-1}} (-1)^{d_1+d_2} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4|s}\right) + \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}\right) \\ &\quad = (-1)^{k+1} \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k,\, r\geq 1}} (-1)^{d_1} \overleftarrow{\mathsf{c}}^{g_1}_{d_1|r} \diamond \left(\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_3}_{d_3|s} + \overleftarrow{\mathsf{e}}^{g_3}_{d_3|s}\right) + \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3|s}\right). \end{aligned}$$

Note that the expresion in line (3.24) for E_{k+1} consists exactly of those terms with $r=0$. Similarly, for the terms with $d_3+d_4<2(g_3+g_4)-1$:

$$
\begin{aligned}
&(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1\\ d_3+d_4<2(g_3+g_4)-1\\ g_3\geq 1}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\Big(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s}+\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}\Big)\\
&\quad+(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1\\ d_3+d_4<2(g_3+g_4)-1}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\Big(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{e}}^{g_4}_{d_4|s}+\delta_{g_3,0}1|_{\overline{\mathcal{M}}_{0,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4|s}\Big)\\
&\quad=-(-1)^{k+1} \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\Big(\overleftarrow{\mathsf{d}}^{g_3}_{d_3|s}+\overleftarrow{\mathsf{e}}^{g_3}_{d_3|s}\Big).
\end{aligned}
$$

Putting everything together, we get

$$
\begin{aligned}
E_1+\cdots+E_{k+1}=(-1)^{k+1}&\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}} (-1)^{d_1}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\Big(\psi_1^{d_2}-(-1)^{d_2}\psi_2^{d_2}\Big)\Big|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\Big(\overleftarrow{\mathsf{d}}^{g_3}_{d_3|s}+\overleftarrow{\mathsf{e}}^{g_3}_{d_3|s}\Big)\\
+(-1)^{k+1}&\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}} (-1)^{d_1}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}}\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3|s}.
\end{aligned}
$$

We apply (3.4) to the first summand and (3.3) to the second one to obtain

$$
\begin{aligned}
E_1+\cdots+E_{k+1}=(-1)^{k+1}&\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\Big(\overleftarrow{\mathsf{d}}^{g_4}_{d_4|s}+\overleftarrow{\mathsf{e}}^{g_4}_{d_4|s}\Big)\\
+(-1)^{k+1}&\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1|r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4|s},
\end{aligned}
$$

which concludes the proof. □

Equation (2.7) follows from applying the above lemma to $E=E_1+\cdots+E_{g+1}$, again using (2.1).

3.6. Evaluation of psi class

Here we prove Equation (2.8). We derive it from equation (2.7). For $I\subset\{1,\dots,n\}$ with $|I|\geq 2$ denote by $\delta_0^I\in R^1(\overline{\mathcal{M}}_{g,n})$ the class of the closure of the subset of stable curves from $\overline{\mathcal{M}}_{g,n}$ having exactly one node separating a genus 0 component carrying the points marked by I and a genus g component carrying the points marked by $\{1,\dots,n\}\setminus I$. Denote by $\pi^{(g)}\colon\overline{\mathcal{M}}_{g,3}\to\overline{\mathcal{M}}_{g,2}$ the map that forgets the third marked point. Multiplying equation (2.7) by ψ_3 and taking the push-forward by $\pi^{(g)}$, we obtain

$$
\pi^{(g)}_*\Big(\psi_3\psi_1\cdot\pi^{(g)*}\mathsf{B}^g\Big)=\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}}\mathsf{B}^{g_1}\diamond\pi^{(g_2)}_*\Big(\psi_3\cdot\pi^{(g_2)*}\mathsf{B}^{g_2}\Big).
$$

Noting that $\psi_3\psi_1 = \psi_3\left(\pi^{(g)*}\psi_1 + \delta_0^{\{1,3\}}\right) = \psi_3\pi^{(g)*}\psi_1$ and $\pi^{(g)}_*\psi_3 = 2g$, we get

$$2g\psi_1 \cdot \mathsf{B}^g = \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} 2g_2\mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2},$$

as required.

3.7. Equivalence of the conjectural formulas

In this section we prove Theorem 2.4.

Lemma 3.10. *Suppose* $C^g = \mathrm{DR}_g(1,-1)\lambda_g$ *or* $C^g = \mathsf{B}^g$. *Then for* $g \geq 1$ *we have*

$$C^g = \psi_1 \cdot \pi_2^*\pi_{2*}C^g - \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} C^{g_1} \diamond \pi_2^*\pi_{2*}C^{g_2}, \tag{3.26}$$

where $\pi_2 \colon \overline{\mathcal{M}}_{h,2} \to \overline{\mathcal{M}}_{h,1}$ *is the map forgetting the second marked point.*

Proof. The proof is based on Properties (2.3)–(2.8), which are also true for the class $\mathrm{DR}_g(1,-1)\lambda_g$. Consider the following diagram of forgetful maps:

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,3} \arrow[r, "\widetilde{\pi}_2"] \arrow[d, "\pi_3"'] & \overline{\mathcal{M}}_{g,2} \arrow[d, "\widehat{\pi}_2"] \\ \overline{\mathcal{M}}_{g,2} \arrow[r, "\pi_2"] & \overline{\mathcal{M}}_{g,1} \end{tikzcd}$$

where the subindices denote the number of the point that a map forgets. Note that under the map $\widetilde{\pi}_2$ the third marked point on a curve from $\overline{\mathcal{M}}_{g,3}$ becomes the second marked point on the resulting curve from $\overline{\mathcal{M}}_{g,2}$. We then compute

$$\begin{aligned}
\psi_1 \cdot \pi_2^*\pi_{2*}C^g &= \psi_1 \cdot \pi_2^*\widehat{\pi}_{2*}C^g = \psi_1 \cdot \pi_{3*}\widetilde{\pi}_2^*C^g = \pi_{3*}\left(\pi_3^*\psi_1 \cdot \widetilde{\pi}_2^*C^g\right)\\
&= \pi_{3*}\left(\left(\psi_1 - \delta_0^{\{1,3\}}\right)\cdot \widetilde{\pi}_2^*C^g\right) = \pi_{3*}\left(\psi_1 \cdot \widetilde{\pi}_2^*C^g\right) - \pi_{3*}\left(\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*C^g\right)\\
&\overset{(2.7)}{=} \pi_{3*}\left(C^g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} C^{g_1} \diamond \widetilde{\pi}_2^*C^{g_2}\right) - \pi_{3*}\left(\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*C^g\right)\\
&= C^g + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} C^{g_1} \diamond \pi_{3*}\widetilde{\pi}_2^*C^{g_2} - \pi_{3*}\left(\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*C^g\right)\\
&= C^g + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 1}} C^{g_1} \diamond \pi_2^*\widehat{\pi}_{2*}C^{g_2} - \pi_{3*}\left(\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*C^g\right),
\end{aligned}$$

and it is sufficient to check that $\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*C^g = 0$.

Indeed, for $C_g = \mathrm{DR}_g(1,-1)\lambda_g$ we have

$$\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*(\mathrm{DR}_g(1,-1)\lambda_g) = \delta_0^{\{1,3\}} \cdot \mathrm{DR}_g(1,0,-1)\lambda_g = 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \mathrm{DR}_g(0,0)\lambda_g = (-1)^g 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \lambda_g^2 = 0.$$

In the case $C^g = \mathsf{B}^g$, from (3.21) it is easy to see that the class $\widetilde{\pi}_2^*\mathsf{B}^g - \psi_1^{2g}$ is supported on the stratum in $\overline{\mathcal{M}}_{g,3}$ that doesn't intersect the divisor corresponding to $\delta_0^{\{1,3\}}$. Therefore, $\delta_0^{\{1,3\}} \cdot (\widetilde{\pi}_2^*\mathsf{B}^g - \psi_1^{2g}) = 0$, but we also obviously have $\delta_0^{\{1,3\}} \cdot \psi_1^{2g} = 0$, which gives $\delta_0^{\{1,3\}} \cdot \widetilde{\pi}_2^*\mathsf{B}^g = 0$. □

Proof of Theorem 2.4. Assuming $\pi_{2*}\left(\mathrm{DR}_g(1,-1)\lambda_g\right) = \pi_{2*}\mathsf{B}^g$, the equality $\mathrm{DR}_g(1,-1)\lambda_g = \mathsf{B}^g$ immediately follows from Formula (3.26) by the induction on g. This completes the proof of the theorem. □

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