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Automorphism group schemes of bielliptic and quasi-bielliptic surfaces

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Abstract. Bielliptic and quasi-bielliptic surfaces form one of the four classes of minimal smooth projective surfaces of Kodaira dimension 0. In this article, we determine the automorphism group schemes of these surfaces over algebraically closed fields of arbitrary characteristic, generalizing work of Bennett and Miranda over the complex numbers; we also find some cases that are missing from the classification of automorphism groups of bielliptic surfaces in characteristic 0.

Keywords. Automorphisms; group schemes; bielliptic surfaces; quasi-bielliptic surfaces; hyperelliptic surfaces; quasi-hyperelliptic surfaces; positive characteristic

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Contents

1. Introduction

We are working over an algebraically closed field k of characteristic $p \geq 0$. Bielliptic and quasi-bielliptic surfaces form one of the four types of minimal smooth projective surfaces of Kodaira dimension 0. Each bielliptic surface X is a quotient $\pi : E \times C \to (E \times C)/G = X$, where E and C are elliptic curves and $G \subseteq E$ is a finite subgroup scheme of E that acts faithfully on C via $\alpha : G \to \mathrm{Aut}_C$. Moreover, the image of α is not entirely contained in the group of translations C. This latter condition guarantees that X is not an Abelian surface. All possible combinations of E, C, G and α have been determined: if $p = 0$ by Bagnera and de Franchis in [BdF10], and if $p \neq 0$ by Bombieri and Mumford in [BM77].

Similarly, quasi-bielliptic surfaces, which exist if and only if $p \in \{2, 3\}$, are obtained by replacing C by a cuspidal plane cubic curve and by imposing on α the condition that the cusp of C is not a fixed point of the group scheme $\alpha(G)$. As in the bielliptic case, it is possible to determine all combinations of E, C, G and α. We refer the reader to [BM76], but note that not all cases listed there actually occur (see Remark 5.12 and Remark 5.13).

Bielliptic and quasi-bielliptic surfaces come with two natural fibrations: one of them is the Albanese map $f_E : X \to E/G =: E'$, which is quasi-elliptic if X is quasi-bielliptic, and elliptic if X is bielliptic. All closed fibers of f_E are isomorphic to C, since this holds after pulling back along the faithfully flat morphism $E \to E/G$. The second fibration $f_C : X \to C/\alpha(G) =: C' \cong \mathbb{P}^1$ is always elliptic, but has multiple fibers.

The purpose of this article is to determine the automorphism group scheme Aut_X of X. If $p = 0$, this has been carried out by Bennett and Miranda in [BM90]. By Proposition 3.1, the actions of the centralizers $C_{\mathrm{Aut}_E}(G)$ and $C_{\mathrm{Aut}_C}(\alpha(G))$ on the first and second factor of $E \times C$, respectively, descend to X and we consider them as subgroup schemes of Aut_X via these actions. Then, the following theorem is the key result of this article.

Theorem 1.1. *Let $X = (E \times C)/G$ be a bielliptic or quasi-bielliptic surface. Then, there is a short exact sequence of group schemes*

$$1 \to (C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G)))/G \xrightarrow{\pi_*} \mathrm{Aut}_X \to M \to 1,$$

where G is embedded via $\mathrm{id} \times \alpha$ and M is a finite and étale group scheme. In particular, Aut_X is of finite type.

We refer the reader to Theorem 4.3 for a refined statement including a description of the group schemes $\mathrm{Aut}_{X/E'}$ and $\mathrm{Aut}_{X/C'}$ of automorphisms of X over E' and over C', respectively. While the part of Aut_X coming from the centralizers is straightforward to calculate and understand, the part M is more elusive. In particular, we note that M can be non-trivial even in characteristic 0, contrary to what is claimed in [BM90, Section 2]. Even though we do not see an a priori reason for this, M always comes from automorphisms of $E \times C$. Then, a posteriori, Theorem 4.3 (6) gives a complete description of M (see Remark 4.5). By Proposition 3.1 and Lemma 4.2 (3), we have the following corollary of our analysis.

Corollary 1.2. *Let $X = (E \times C)/G$ be a bielliptic or quasi-bielliptic surface. Then,*

$$\mathrm{Aut}_X \cong N_{\mathrm{Aut}_E \times \mathrm{Aut}_C}(G)/G,$$

where $N_{\mathrm{Aut}_E \times \mathrm{Aut}_C}(G)$ is the normalizer of G in $\mathrm{Aut}_E \times \mathrm{Aut}_C$.

By Corollary 4.7, we have $E \cong (\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ and $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ is normal in Aut_X, so we can write the quotient Aut_X/E as an extension of M by $(C_{\mathrm{Aut}_E}(G)/E) \times (C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G))$. These group schemes can be calculated explicitly and this will be carried out in Section 5. In the following Tables 1, 2, and 3, the groups S_n, A_n, and D_{2n} are the symmetric, alternating, and dihedral group (of order $2n$), respectively, and M_2 is the p-torsion subscheme of a supersingular elliptic curve. The stars and daggers in Table 1 will be explained in Remark 1.4.

Corollary 1.3. *Let $X = (E \times C)/G$ be a bielliptic or quasi-bielliptic surface. Then, depending on the group scheme G and the j-invariants $j(E)$ and $j(C)$, the group schemes $C_{\mathrm{Aut}_E}(G)/E$, $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G)$ and M are as in Table 1, 2, and 3.*

G	$j(E)$	$C_{\mathrm{Aut}_E}(G)/E$	$j(C)$	$C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G)$	M	p
$\mathbb{Z}/2\mathbb{Z}$	$a)$ any $b)$ 1728^*	$a)$ $\mathbb{Z}/2\mathbb{Z}$ $b)$ $\mathbb{Z}/4\mathbb{Z}$	$i)$ $\neq 0, 1728$ $ii)$ 1728 $iii)$ 0	$i)$ $(\mathbb{Z}/2\mathbb{Z})^2$ $ii)$ D_8 $iii)$ A_4	$\{1\}$	$\neq 2,3$
$(\mathbb{Z}/2\mathbb{Z})^2$	$a)$ any $b)$ $1728^\dagger$	$\mathbb{Z}/2\mathbb{Z}$	$i)$ any $ii)$ 1728^*	$i)$ $\mathbb{Z}/2\mathbb{Z}$ $ii)$ $(\mathbb{Z}/2\mathbb{Z})^2$	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/2\mathbb{Z}$	$\neq 2,3$
$\mathbb{Z}/3\mathbb{Z}$	$a)$ any $b)$ 0^*	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/3\mathbb{Z}$	0	S_3	$\{1\}$	$\neq 2,3$
$(\mathbb{Z}/3\mathbb{Z})^2$	$a)$ any $b)$ $0^\dagger$	$\{1\}$	0	$\{1\}$	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/3\mathbb{Z}$	$\neq 2,3$
$\mathbb{Z}/4\mathbb{Z}$	any	$\{1\}$	1728	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\neq 2$
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	any	$\{1\}$	1728	$\{1\}$	$\{1\}$	$\neq 2$
$\mathbb{Z}/6\mathbb{Z}$	any	$\{1\}$	0	$\{1\}$	$\{1\}$	$\neq 2,3$
$\mathbb{Z}/2\mathbb{Z}$	$a)$ $\neq 0$ $b)$ 0	$a)$ $\mathbb{Z}/2\mathbb{Z}$ $b)$ $\mathbb{Z}/4\mathbb{Z}$	$i)$ $\neq 0$ $ii)$ 0	$i)$ $(\mathbb{Z}/2\mathbb{Z})^2$ $ii)$ $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_3$	$\{1\}$	3
$(\mathbb{Z}/2\mathbb{Z})^2$	$a)$ $\neq 0$ $b)$ 0	$\mathbb{Z}/2\mathbb{Z}$	$i)$ $\neq 0$ $ii)$ 0	$i)$ $\mathbb{Z}/2\mathbb{Z}$ $ii)$ $(\mathbb{Z}/2\mathbb{Z})^2$	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/2\mathbb{Z}$	3
$\mathbb{Z}/3\mathbb{Z}$	$\neq 0$	$\{1\}$	0	$\alpha_3 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3
$\mathbb{Z}/6\mathbb{Z}$	$\neq 0$	$\{1\}$	0	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	3
$\mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$i)$ $\neq 0$ $ii)$ 0	$i)$ $\mu_2 \times \mathbb{Z}/2\mathbb{Z}$ $ii)$ $M_2 \rtimes A_4$	$\{1\}$	2
$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
$\mathbb{Z}/3\mathbb{Z}$	$a)$ $\neq 0$ $b)$ 0	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/3\mathbb{Z}$	0	S_3	$\{1\}$	2
$(\mathbb{Z}/3\mathbb{Z})^2$	$a)$ $\neq 0$ $b)$ 0	$\{1\}$	0	$\{1\}$	$a)$ $\{1\}$ $b)$ $\mathbb{Z}/3\mathbb{Z}$	2
$\mathbb{Z}/4\mathbb{Z}$	$\neq 0$	$\{1\}$	0	α_2	$\mathbb{Z}/2\mathbb{Z}$	2
$\mathbb{Z}/6\mathbb{Z}$	$\neq 0$	$\{1\}$	0	$\{1\}$	$\{1\}$	2

Table 1. Automorphism group schemes of bielliptic surfaces

G	$j(E)$	$C_{\mathrm{Aut}_E}(G)/E$	$C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G)$	M	p
μ_3	$\neq 0$	$\{1\}$	S_3	$\{1\}$	3
$\mu_3 \times \mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\{1\}$	$\{1\}$	$\{1\}$	3
$\mu_3 \times \mathbb{Z}/3\mathbb{Z}$	$\neq 0$	$\{1\}$	$\{1\}$	$\{1\}$	3
α_3	0	$\mathbb{Z}/3\mathbb{Z}$	$\alpha_3 \rtimes \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	3
$\alpha_3 \times \mathbb{Z}/2\mathbb{Z}$	0	$\{1\}$	$\{1\}$	$\mathbb{Z}/4\mathbb{Z}$	3

Table 2. Automorphism group schemes of quasi-bielliptic surfaces in characteristic 3

G	$j(E)$	$C_{\mathrm{Aut}_E}(G)/E$	λ	$C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G)$	M	p
μ_2	$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$i)$ $\neq 0$ $ii)$ 0	$i)$ $(\mathbb{Z}/2\mathbb{Z})^2$ $ii)$ A_4	$i)$ $\{1\}$ $ii)$ $\{1\}$	2
$\mu_2 \times \mathbb{Z}/3\mathbb{Z}$	$\neq 0$	$\{1\}$	–	$\{1\}$	$\{1\}$	2
$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	*any*	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
μ_4	$\neq 0$	$\{1\}$	–	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	2
$\mu_4 \times \mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\{1\}$	–	$\{1\}$	$\{1\}$	2
α_2	0	Q_8	$i)$ 1 $ii)$ 0	$i)$ $\alpha_2^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ $ii)$ $(\alpha_4 \rtimes \alpha_4) \rtimes \mathbb{Z}/3\mathbb{Z}$	$i)$ $\{1\}$ $ii)$ $\mathbb{Z}/3\mathbb{Z}$	2
$\alpha_2 \times \mathbb{Z}/3\mathbb{Z}$	0	$\{1\}$	–	$\{1\}$	$\mathbb{Z}/3\mathbb{Z}$	2
M_2	0	$\mathbb{Z}/2\mathbb{Z}$	$\neq 0$	$\alpha_2 \times \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	2

Table 3. Automorphism group schemes of quasi-bielliptic surfaces in characteristic 2

Remark 1.4. Let us explain the meaning of the stars and daggers in Table 1. We denote by $O \in E$ the neutral element with respect to the group law on E:

- Stars: If $p \neq 2,3$ and $j(E) = 1728$, then every automorphism h_E of (E,O) of order 4 fixes a unique cyclic subgroup of E of order 2. Similarly, if $p \neq 2,3$ and $j(E) = 0$, then every automorphism h_E of (E,O) of order 3 fixes a unique cyclic subgroup of E of order 3. A star after a j-invariant in Table 1 denotes that the translation subgroup of G or $\alpha(G)$ coincides with this cyclic subgroup. By Lemma 5.1, this implies that h_E is in the corresponding centralizer. We note that such special 2 and 3-torsion points do not exist if $p = 2,3$, because (E,O) has more automorphisms in these characteristics.
- Daggers: A dagger after $j(E)$ denotes that the special 2 or 3-torsion points described above maps to a translation in $\alpha(G)$. In these cases, the automorphism (h_E, h_C), where h_E is an automorphism of order 4 or 3 of (E,O) and h_C is translation by a suitable 4 or 3-torsion point, respectively, normalizes the G-action on $E \times C$ and hence descends to X. Since (h_E, h_C) does not centralize the G-action, it induces a non-trivial element of M. See the proof of Proposition 5.5 for a precise description of the automorphism (h_E, h_C) in these cases.

These cases seem to be missing from [BM90], since they were not listed in [BM90, Table 1.1], which is why [BM90, Table 3.2] differs from our Table 1.

Remark 1.5. In the quasi-bielliptic case in characteristic 2, the action of G on $E \times C$ sometimes depends on a parameter $\lambda \in k$ and so does Aut_X. For an explicit description of λ, see Section 5.2.2. The parameter λ should be thought of as a replacement for the j-invariant of the curve C.

Recall that the space $H^0(X, T_X)$ is the tangent space of Aut_X at the identity. Since $E \cong (\mathrm{Aut}_X^\circ)_{\mathrm{red}}$, Aut_X is smooth if and only if $h^0(X, T_X) = 1$. A careful inspection of Tables 1, 2, and 3, and of the orders of the canonical bundle ω_X determined in [BM77] and [BM76] shows the following.

Corollary 1.6. *Let X be a bielliptic or quasi-bielliptic surface. Then, the following hold:*

(1) $h^0(X, T_X) \leq 3$.
(2) *If X is bielliptic or $p \neq 2$, then* $h^0(X, T_X) \leq 2$.
(3) $h^0(X, T_X) = 1$ *if and only if* $\omega_X \not\cong \mathcal{O}_X$ *if and only if* Aut_X *is smooth.*

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2. Notation and generalities on automorphism group schemes

Let $\pi : Y \to X$ be a morphism of proper varieties over an algebraically closed field k. There are several k-group schemes of automorphisms associated to π. We follow the notation of [Bri18, Section 2.4], which we recall for the convenience of the reader. Throughout, T is an arbitrary k-scheme.

- The *automorphism group scheme* Aut_X *of* X is the k-group scheme whose group of T-valued points $\mathrm{Aut}_X(T) := \mathrm{Aut}(X \times_k T)$ is the group of automorphisms of $X_T := X \times_k T$ over T. By [MO68, Theorem (3.7)], Aut_X is a group scheme locally of finite type over k. The identity component of Aut_X is denoted by Aut_X°.
- The *automorphism group scheme* Aut_π *of the morphism* π is the k-group scheme such that $\mathrm{Aut}_\pi(T)$ consists of pairs $(g, h) \in \mathrm{Aut}_Y(T) \times \mathrm{Aut}_X(T)$ making the diagram

$$\begin{array}{ccc} Y_T & \xrightarrow{g} & Y_T \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_T} \\ X_T & \xrightarrow{h} & X_T \end{array}$$

  commutative. In particular, $\mathrm{Aut}_\pi(-)$ is a closed subfunctor of $\mathrm{Aut}_Y(-) \times \mathrm{Aut}_X(-)$, hence Aut_π is representable by a group scheme locally of finite type over k.
- The group scheme Aut_π comes with projections to Aut_Y and Aut_X. If π is faithfully flat, then the first projection $\mathrm{Aut}_\pi \to \mathrm{Aut}_Y$ is a closed immersion and we will use this to consider Aut_π as a subgroup scheme of Aut_Y. We denote the second projection by $\pi_* : \mathrm{Aut}_\pi \to \mathrm{Aut}_X$.
- The *automorphism group scheme* $\mathrm{Aut}_{Y/X}$ *of* Y *over* X is the k-group scheme whose group of T-valued points $\mathrm{Aut}_{Y/X}(T)$ consists of automorphisms $g \in \mathrm{Aut}_Y(T)$ such that $\pi_T \circ g = \pi_T$. By definition, there is an exact sequence realizing $\mathrm{Aut}_{Y/X}$ as a subgroup scheme of Aut_π:

$$1 \to \mathrm{Aut}_{Y/X} \to \mathrm{Aut}_\pi \stackrel{\pi_*}{\to} \mathrm{Aut}_X \tag{1}$$

- Given a closed subgroup scheme $G \subseteq \mathrm{Aut}_Y$, the *normalizer* $N_{\mathrm{Aut}_Y}(G)$ *of* G *in* Aut_Y is the k-group scheme whose group of T-valued points is

$$N_{\mathrm{Aut}_Y}(G)(T) = \{h \in \mathrm{Aut}_Y(T) \mid h_{T'} \circ g \circ (h_{T'})^{-1} \in G(T') \text{ for all } T' \to T \text{ and } g \in G(T')\}.$$

  The *centralizer* $C_{\mathrm{Aut}_Y}(G)$ *of* G *in* Aut_Y is the group scheme whose T-valued points satisfy the stronger condition $h_{T'} \circ g \circ (h_{T'})^{-1} = g$ instead. By [ABD+66, Exposé VIB, Proposition 6.2 (iv)], both $N_{\mathrm{Aut}_Y}(G)$ and $C_{\mathrm{Aut}_Y}(G)$ are closed subgroup schemes of Aut_Y.

Caution 2.1. The notation $\mathrm{Aut}_{Y/X}$ is also a standard notation for the group functor on the category of X-schemes that associates to an X-scheme Z the automorphism group of $Y \times_X Z$ over Z. Since these relative automorphism group functors do not occur in this article, we decided to use the notation introduced above instead of more cumbersome, albeit more precise, notation such as $\mathrm{Aut}_{Y/X/k}$.

3. Automorphism group schemes of quotients

In this section, we study Sequence (1) in the case where $\pi : Y \to X$ is a finite quotient.

Proposition 3.1. *If G is a finite group scheme acting freely on a proper variety Y such that the geometric quotient $\pi : Y \to Y/G =: X$ exists as a scheme, then we have $\mathrm{Aut}_{Y/X} = G$ and $\mathrm{Aut}_\pi = N_{\mathrm{Aut}_Y}(G)$ as subgroup schemes of Aut_Y. In particular, Sequence (1) becomes*

$$1 \to G \to N_{\mathrm{Aut}_Y}(G) \xrightarrow{\pi_*} \mathrm{Aut}_X.$$

Proof. First, we show that $\mathrm{Aut}_{Y/X} = G$. By [Brill, Lemma 4.1], there is a G-equivariant isomorphism $\mathrm{Aut}_{Y/X} \cong \mathcal{H}om(Y,G)$, where the T-valued points of the latter are $\mathrm{Hom}(Y \times T, G)$ and G is embedded as $G = \mathcal{H}om(\operatorname{Spec} k, G)$. Since Y is a proper variety and taking global sections commutes with flat base change, we have $H^0(Y \times T, \mathcal{O}_{Y \times T}) = k \otimes_k H^0(T, \mathcal{O}_T) = H^0(T, \mathcal{O}_T)$ for every affine k-scheme T. As G is affine, this implies $\mathrm{Hom}(Y \times T, G) = \mathrm{Hom}(T, G) = G(T)$, which is what we had to show.

Next, we show $\mathrm{Aut}_\pi = N_{\mathrm{Aut}_Y}(G)$. For this, let $h \in \mathrm{Aut}_Y(T)$ be an automorphism of Y_T. Then, $h \in \mathrm{Aut}_\pi(T)$ if and only if there is $h' \in \mathrm{Aut}_X(T)$ such that the following diagram commutes

$$\begin{array}{ccc} Y_T & \xrightarrow{h} & Y_T \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_T} \\ X_T & \xrightarrow{h'} & X_T. \end{array}$$

Comparing degrees, it is easy to check that the geometric quotient of Y_T by the induced free action of G coincides with π_T, so the morphism $\pi : Y \to X$ is a universal geometric quotient of Y, hence also a universal categorical quotient by [MFK94, Proposition 0.1]. Therefore, the automorphism h' exists if and only if $\pi_T \circ h$ is G-invariant, that is, if and only if for every T-scheme T' we have $\pi_{T'} \circ h_{T'} \circ g = \pi_{T'} \circ h_{T'}$ for all $g \in G(T')$. This is equivalent to $h_{T'} \circ g \circ h_{T'}^{-1} \in \mathrm{Aut}_{Y/X}(T') = G(T')$ for all $g \in G(T')$, which is precisely the condition that $h \in N_{\mathrm{Aut}_Y}(G)$. □

Example 3.2. Contrary to the situation for abstract groups, Proposition 3.1 typically fails if Y is a non-proper variety or the action of G is not free. Indeed, consider any infinitesimal subgroup scheme $G \subseteq \mathrm{PGL}_2$ of length p. The k-linear Frobenius $F : Y := \mathbb{P}^1 \to \mathbb{P}^1 =: X$ is the geometric quotient for the action of G on $\mathbb{P}^1$ and $\mathrm{Aut}_{Y/X} = \mathrm{PGL}_2[F]$ is the kernel of Frobenius on PGL_2. Moreover, we have $\mathrm{Aut}_F = \mathrm{PGL}_2$. Thus, $\mathrm{Aut}_{Y/X}$ and Aut_F are strictly bigger than G and $N_{\mathrm{PGL}_2}(G)$ even though F is a G-torsor over an open subscheme of X.

Even though Example 3.2 shows that Proposition 3.1 fails for non-free actions on curves, we can at least describe the k-rational points in Sequence (1) if the quotient is smooth.

Proposition 3.3. *Let G be a finite group scheme acting faithfully on a proper integral curve D with geometric quotient $\varphi : D \to D' := D/G$. Assume that D' is smooth. Then, we have*

$$\mathrm{Aut}_{D/D'}(k) = G(k) \quad \textit{and} \quad \mathrm{Aut}_\varphi(k) = N_{\mathrm{Aut}(D)}(G(k)).$$

Proof. We can consider the four groups as subgroups of the group $\mathrm{Aut}_k(k(D))$ of k-linear field automorphisms of $k(D)$ via the injective restriction map $\mathrm{Aut}(D) \hookrightarrow \mathrm{Aut}_k(k(D))$. We have a tower of field extensions $k(D') \subseteq k(D)^{G(k)} \subseteq k(D)$, where $k(D') \subseteq k(D)^{G(k)}$ is purely inseparable and $k(D)^{G(k)} \subseteq k(D)$ is a Galois extension with Galois group $G(k)$. An elementary calculation shows that $N_{\mathrm{Aut}_k(k(D))}(G(k))$ is the subgroup of $\mathrm{Aut}_k(k(D))$ of automorphisms preserving $k(D)^{G(k)}$. Since D' is a curve, we have $k(D') = (k(D)^{G(k)})^{p^n}$ for some $n \geq 0$, so an automorphism of $k(D)$ preserves $k(D)^{G(k)}$ if and only if it preserves $k(D')$. Hence, $N_{\mathrm{Aut}_k(k(D))}(G(k))$ is also the group of automorphisms of $k(D)$ preserving $k(D')$. On the other hand, since D' is smooth and proper, $\mathrm{Aut}_\varphi(k)$ consists precisely of those automorphisms of D which, when restricted to $k(D)$, preserve $k(D')$. Hence, we have $N_{\mathrm{Aut}(D)}(G(k)) = N_{\mathrm{Aut}_k(k(D))}(G(k)) \cap \mathrm{Aut}(D) = \mathrm{Aut}_\varphi(k)$, and $\mathrm{Aut}_{D/D'}(k) = \mathrm{Aut}_{k(D')}(k(D)) \cap \mathrm{Aut}(D) = G(k)$, which is what we had to show. □

4. Proof of Theorem 1.1

Throughout this section, E and C are integral curves of arithmetic genus 1, and we assume that E is smooth and C is either smooth or has a single cusp as singularity. We choose a point $O \in E$ and consider E as an elliptic curve with identity element O. We fix a finite subgroup scheme $G \subseteq E$, and a monomorphism $\alpha : G \to \mathrm{Aut}_C$ such that $\alpha(G)$ is not contained in the group of translations of C if C is smooth, and not contained in the stabilizer of the cusp if C is singular. In particular, the actions of G on E (via translations) and C (via α) give rise to a product action of G on $E \times C$ and we set $X := (E \times C)/G$ with quotient map $\pi : E \times C \to X$. We have the following commutative diagram with two cartesian squares:

$$\begin{array}{ccccc}
E \times C & \xrightarrow{\pi_E} & X \times_{E'} E & \longrightarrow & E \\
\downarrow{\scriptstyle \pi_C} & \searrow^{\pi} & \downarrow & \square & \downarrow \\
X \times_{C'} C & \longrightarrow & X & \xrightarrow[f_E]{} & E/G =: E' \\
\downarrow & \square & \downarrow{\scriptstyle f_C} & & \\
C & \longrightarrow & C/\alpha(G) =: C'. & &
\end{array}$$

Since G acts freely on E, the map π_E induces isomorphisms on the fibers of $E \times C \to E$ and $X \times_{E'} E \to E$ and thus, as both maps are flat, the morphism π_E is an isomorphism. The following lemma shows that the automorphism group scheme of X is controlled by the fibrations f_E and f_C.

Lemma 4.1. *There is a unique action of* Aut_X *on* C' *and on* E' *such that both* $f_E : X \to E'$ *and* $f_C : X \to C'$ *are* Aut_X*-equivariant. In particular, there are exact sequences*

$$1 \to \mathrm{Aut}_{X/E'} \to \mathrm{Aut}_X \stackrel{(f_E)_*}{\to} \mathrm{Aut}_{E'}$$

and

$$1 \to \mathrm{Aut}_{X/C'} \to \mathrm{Aut}_X \stackrel{(f_C)_*}{\to} \mathrm{Aut}_{C'}.$$

Proof. The Aut_X°-action on X descends to both E' and C' by Blanchard's Lemma [BSU13, Proposition 4.2.1]. Since f_E and f_C are the only fibrations of X and $E' \not\cong C' \cong \mathbb{P}^1$, it is also clear that the action of the abstract group $\mathrm{Aut}(X)$ descends to E' and C'. By [Bri18, Lemma 2.20 (ii)], this is enough to prove that the whole Aut_X-action descends uniquely to the two curves E' and C'.

With respect to the Aut_X-actions of the previous paragraph, we have $\mathrm{Aut}_X = \mathrm{Aut}_{f_C} = \mathrm{Aut}_{f_E}$, hence the short exact sequences in the statement of the lemma are special cases of Sequence (1). □

The idea for the proof of Theorem 1.1 is to use the isomorphism π_E to lift group scheme actions from X to $E \times C$. By Proposition 3.1, the automorphisms of X that come from $E \times C$ are induced by the normalizer $N_{\mathrm{Aut}_{E\times C}}(G)$. Therefore, before proving Theorem 1.1, we study $N_{\mathrm{Aut}_{E\times C}}(G)$. For the following lemma, note that there is a natural inclusion $\mathrm{Aut}_E \times \mathrm{Aut}_C \hookrightarrow \mathrm{Aut}_{E\times C}$ given by letting Aut_E and Aut_C act on the first and second factor, respectively. In particular, we can consider $C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G))$ and $N_{\mathrm{Aut}_E}(G) \times N_{\mathrm{Aut}_C}(\alpha(G))$ as subgroup schemes of $\mathrm{Aut}_{E\times C}$.

Lemma 4.2. *The normalizer $N_{\mathrm{Aut}_{E\times C}}(G)$ of G in $\mathrm{Aut}_{E\times C}$ satisfies the following properties:*

(1) $N_{\mathrm{Aut}_{E\times C}}(G) \supseteq C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G))$.
(2) $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G) = C^{\circ}_{\mathrm{Aut}_E}(G) \times C^{\circ}_{\mathrm{Aut}_C}(\alpha(G))$.
(3) $N_{\mathrm{Aut}_{E\times C}}(G)(T) = \{(h_E,h_C) \in N_{\mathrm{Aut}_E}(G)(T) \times N_{\mathrm{Aut}_C}(\alpha(G))(T) \mid \alpha_T \circ \mathrm{ad}_{h_E} = \mathrm{ad}_{h_C} \circ \alpha_T\}$, *where* ad_{h_E} *and* ad_{h_C} *denote conjugation by h_E and h_C, respectively.*
(4) The quotient maps $C \to C'$ and $E \to E'$ are $N_{\mathrm{Aut}_{E\times C}}(G)$-equivariant.

Proof. Claim (1) is clear.

For Claim (2), the inclusion $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G) \supseteq C^{\circ}_{\mathrm{Aut}_E}(G) \times C^{\circ}_{\mathrm{Aut}_C}(\alpha(G))$ follows from Claim (1) and we have to show the other inclusion. By [BSU13, Corollary 4.2.7], we have $\mathrm{Aut}^{\circ}_{E\times C} = \mathrm{Aut}^{\circ}_E \times \mathrm{Aut}^{\circ}_C$. In particular, being connected, $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G)$ is contained in $\mathrm{Aut}^{\circ}_E \times \mathrm{Aut}^{\circ}_C$. Hence, it suffices to show that $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G)$ centralizes G on the first factor of $E \times C$. Since $G \subseteq \mathrm{Aut}^{\circ}_E$ is a subgroup scheme of the connected commutative group scheme Aut°_E, we have $\mathrm{Aut}^{\circ}_E \subseteq C^{\circ}_{\mathrm{Aut}_E}(G) \subseteq N^{\circ}_{\mathrm{Aut}_E}(G) \subseteq \mathrm{Aut}^{\circ}_E$, so $N^{\circ}_{\mathrm{Aut}_E}(G)$ centralizes G. Therefore, $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G)$ centralizes G as well.

Claim (3) holds for $N^{\circ}_{\mathrm{Aut}_{E\times C}}(G)$ by Claim (2), so it suffices to prove the statement for $T = \mathrm{Spec}\, k$. Let $h \in N_{\mathrm{Aut}_{E\times C}}(G)(k)$. Since h normalizes G, it descends to X by Proposition 3.1. The induced automorphism of X preserves both f_C and f_E, because they are the only fibrations of X and E' has genus 1, while $C' \cong \mathbb{P}^1$. Since the projections $E \times C \to E$ and $E \times C \to C$ coincide with the Stein factorizations of $f_E \circ \pi$ and $f_C \circ \pi$, respectively, both projections are preserved by h. Hence, $h \in \mathrm{Aut}(E) \times \mathrm{Aut}(C)$. An automorphism of this form normalizes the G-action on $E \times C$ if and only if it normalizes the G-action on both factors and the automorphisms of G induced by the two conjugations are identified via α. This proves Claim (3).

Claim (4) follows from the $N_{\mathrm{Aut}_{E\times C}}(G)$-equivariance of π, f_E, and f_C, since the two projections $E \times C \to E$ and $E \times C \to C$ are faithfully flat. □

Recall that, by Proposition 3.1, the action of $N_{\mathrm{Aut}_{E\times C}}(G)$ on $E \times C$ descends to X and we denote the corresponding homomorphism by $\pi_* : N_{\mathrm{Aut}_{E\times C}}(G) \to \mathrm{Aut}_X$. After these preparations, we are ready to prove the following refined version of Theorem 1.1.

Theorem 4.3 (*cf.* Theorem 1.1). *Let $X = (E \times C)/G$ be a bielliptic or quasi-bielliptic surface. Then:*

(1) $\mathrm{Aut}_{X/C'} = \pi_*(C_{\mathrm{Aut}_E}(G) \times (C_{\mathrm{Aut}_C}(\alpha(G)) \cap \mathrm{Aut}_{C/C'}))$.
(2) If G is étale, then $\mathrm{Aut}_{X/C'} \cong C_{\mathrm{Aut}_E}(G)$.
(3) $\mathrm{Aut}_{X/E'} = \pi_*((C_{\mathrm{Aut}_E}(G) \cap \mathrm{Aut}_{E/E'}) \times C_{\mathrm{Aut}_C}(\alpha(G)))$.
(4) $\mathrm{Aut}_{X/E'} \cong C_{\mathrm{Aut}_C}(\alpha(G))$.
(5) There is a short exact sequence of group schemes

$$1 \to (C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G)))/G \xrightarrow{\pi_*} \mathrm{Aut}_X \to M \to 1,$$

where G is embedded via $\mathrm{id} \times \alpha$, M is finite and étale, and $M(k)$ is a subquotient of the groups $\mathrm{Aut}_{E'}(k)/((f_E)_* C_{\mathrm{Aut}_E}(G)(k))$ *and* $N_{\mathrm{Aut}(C)}(\alpha(G)(k))/(C_{\mathrm{Aut}_C}(\alpha(G))(k))$.
(6) If every element of $M(k)$ can be represented by an automorphism of X that lifts to $E \times C$, then

$$M(k) \cong \frac{\{(h_E,h_C) \in N_{\mathrm{Aut}_E}(G) \times N_{\mathrm{Aut}_C}(\alpha(G)) \mid \alpha \circ \mathrm{ad}_{h_E} = \mathrm{ad}_{h_C} \circ \alpha\}}{C_{\mathrm{Aut}_E}(G)(k) \times C_{\mathrm{Aut}_C}(\alpha(G))(k)}.$$

This always holds if X is bielliptic.

Proof. For Claim (1), we first show that the $\mathrm{Aut}_{X/C'}$-action lifts to $E \times C$. For this, choose a general point $c \in C$ and let $c' \in C'$ be its image in C', so that π restricted to $E \times \{c\}$ yields an identification of E with the fiber F of f_C over c'. Via this identification, the morphism $(f_E)|_F : F \to E'$ is identified with the quotient map

$E \to E/G = E'$. By Lemma 4.1, the action of $\mathrm{Aut}_{X/C'}$ on X descends to an action on E', and we can use the restriction homomorphism $\mathrm{Aut}_{X/C'} \to \mathrm{Aut}_F$ and the identification of F with E to get a compatible action of $\mathrm{Aut}_{X/C'}$ on E. Using the isomorphism $\pi_E : E \times C \to X \times_{E'} E$, we thus obtain an action of $\mathrm{Aut}_{X/C'}$ on $E \times C$ that lifts the action of $\mathrm{Aut}_{X/C'}$ on X. Hence, $\mathrm{Aut}_{X/C'}$ is in the image of π_* and it remains to describe its preimage.

By Lemma 4.2 (4), a subgroup scheme $H \subseteq N_{\mathrm{Aut}_{E\times C}}(G) \subseteq \mathrm{Aut}_E \times \mathrm{Aut}_C$ maps to $\mathrm{Aut}_{X/C'}$ via π_* if and only if it maps to $\mathrm{Aut}_{C/C'}$ under the second projection. To prove Claim (1), we have to show that such an H in fact centralizes G. By Lemma 4.2 (2) this holds for H°, so we have to prove that $H(k)$ centralizes $\alpha(G)$. Observe that $H(k)$ is mapped to $\mathrm{Aut}_{C/C'}(k)$ under the second projection and $\mathrm{Aut}_{C/C'}(k) = \alpha(G)(k)$ by Proposition 3.3. This, and the fact that G is abelian, implies that $H(k)$ centralizes $\alpha(G)$. Now, Lemma 4.2 (3), shows that $H(k)$ centralizes the G-action on $E \times C$.

For Claim (2), it suffices to show that $C_{\mathrm{Aut}_C}(\alpha(G)) \cap \mathrm{Aut}_{C/C'} = \alpha(G)$, since there is an isomorphism $(C_{\mathrm{Aut}_E}(G) \times \alpha(G))/G \cong C_{\mathrm{Aut}_E}(G)$. This holds if G is étale, for then $\mathrm{Aut}_{C/C'}$ is the constant group scheme associated to $\alpha(G)$ by Proposition 3.3.

For Claim (3), we only have to show that the $\mathrm{Aut}_{X/E'}$-action lifts to $E \times C$, because the description of the preimage of $\mathrm{Aut}_{X/E'}$ under π_* works as in the proof of Claim (1). Since $\mathrm{Aut}_{X/E'}$ acts trivially on E', we can use the trivial action of $\mathrm{Aut}_{X/E'}$ on E to define an action of $\mathrm{Aut}_{X/E'}$ on $X \times_{E'} E$ lifting the action of $\mathrm{Aut}_{X/E'}$ on X. Using the isomorphism $\pi_E : E \times C \to X \times_{E'} E$, we thus obtain the desired lifting.

For Claim (4), we use that the G-action on E is free. By Proposition 3.1 this implies that $\mathrm{Aut}_{E/E'} = G$. Hence, Claim (3) shows that $\mathrm{Aut}_{X/E'} = \pi_*(G \times C_{\mathrm{Aut}_C}(\alpha(G)) \cong C_{\mathrm{Aut}_C}(\alpha(G))$.

Next, let us prove Claim (5). By Proposition 3.1, the image of $C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G))$ under π_* is isomorphic to $(C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G)))/G$. By Claim (1) and Claim (3), this image coincides with the subgroup scheme of Aut_X generated by the two normal subgroup schemes $\mathrm{Aut}_{X/C'}$ and $\mathrm{Aut}_{X/E'}$, hence it is itself normal. In particular, the quotient M and the exact sequence in Claim (3) exist. It remains to describe M.

First, consider the exact sequence

$$1 \to C_{\mathrm{Aut}_C}(\alpha(G)) \to \mathrm{Aut}_X \stackrel{(f_E)_*}{\to} \mathrm{Aut}_{E'} \tag{2}$$

from Lemma 4.1, where we used Claim (4) to describe the kernel of $(f_E)_*$. The homomorphism $(f_E)_*$ identifies the group scheme M with a subgroup scheme of $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G))$. We can choose the image $O' \in E'$ of $O \in E$ as the neutral element of a group law on E'. Then, we have $\mathrm{Aut}_{E'} \cong E' \rtimes \mathrm{Aut}_{E',O'}$ for the finite and étale stabilizer $\mathrm{Aut}_{E',O'}$ of O'. Using the translation action, we can consider E as a subgroup scheme of $\mathrm{Aut}_E \times \mathrm{Aut}_C$. In fact, we have $E \subseteq C_{\mathrm{Aut}_E}(G)$, since $G \subseteq E$ and E is commutative. By Lemma 4.2 (4), the induced action on E' coincides with the translation action of E' on itself. Hence, $E' \subseteq (f_E)_* C_{\mathrm{Aut}_E}(G)$. In particular, M is a subquotient of $\mathrm{Aut}_{E',O'}$ and hence it is finite and étale.

Let $H := (f_E)_*^{-1}(\mathrm{Aut}_{E',O'}) \subseteq \mathrm{Aut}_X$ and let F be the fiber of f_E over O'. Then, the restriction of π to $\{O\} \times C$ gives an identification of C with F such that the quotient map $\varphi : C \to C/\alpha(G) = C'$ is identified with $(f_C)|_F : F \to C'$. In the following, we use this identification to write C instead of F and φ instead of $(f_C)|_F$. Since $\mathrm{Aut}_{E',O'}$ fixes O', the action of H on X preserves C and the morphism φ is H-equivariant, since f_C is Aut_X-equivariant by Lemma 4.1. In other words, the H-action on C factors through Aut_φ. By Claim (1), the kernel of this action is contained in $(C_{\mathrm{Aut}_E}(G) \times C_{\mathrm{Aut}_C}(\alpha(G)))/G$, hence M is a subquotient of $\mathrm{Aut}_\varphi / C_{\mathrm{Aut}_C}(\alpha(G))$. Now, it suffices to observe that $\mathrm{Aut}_\varphi(k) = N_{\mathrm{Aut}(C)}(\alpha(G)(k))$, which follows from Proposition 3.3.

Finally, for Claim (6), the description of M follows immediately from Lemma 4.2 (3) and Proposition 3.1. By the previous paragraph, we can lift every element of $M(k)$ to an automorphism $g \in \mathrm{Aut}(X)$ mapping under $(f_C)_*$ to the image of $\mathrm{Aut}_\varphi(k) \to \mathrm{Aut}_{C'}(k)$. In particular, g lifts to an automorphism h' of $X \times_{C'} C$. Since $\pi_C : E \times C \to E \times_{C'} C$ is birational, we obtain a birational automorphism h of $E \times C$. Now, if X is bielliptic, then $E \times C$ is smooth, minimal, and non-ruled hence h extends to a biregular automorphism of $E \times C$ lifting g. □

Remark 4.4. We remark that if G is not étale, then the group $N_{\mathrm{Aut}(C)}(\alpha(G)(k))$ will usually be bigger than $N_{\mathrm{Aut}_C}(\alpha(G))(k)$. Only later it will turn out that M is in fact a subquotient of the smaller group $N_{\mathrm{Aut}_C}(\alpha(G))(k)/C_{\mathrm{Aut}_C}(\alpha(G))(k)$ in every case.

Remark 4.5. In the case-by-case analysis of quasi-bielliptic surfaces in Section 5.2, we will show that the assumptions of Theorem 4.3 (6) are also satisfied for all quasi-bielliptic surfaces, hence the description of M also holds for these surfaces.

Remark 4.6. It will follow from the calculations of Section 5 that Theorem 4.3 (2) holds for all bielliptic surfaces. Indeed, the situation where X is bielliptic and G is not étale only occurs if $p = 2$ and $G = \mu_2 \times \mathbb{Z}/2\mathbb{Z}$ and in this case explicit calculations show that $C_{\mathrm{Aut}_C}(\alpha(G)) \cap \mathrm{Aut}_{C/C'} = \alpha(G)$, hence the existence of an isomorphism $\mathrm{Aut}_{X/C'} \cong C_{\mathrm{Aut}_E}(G)$ follows from Theorem 4.3 (1). In particular, for bielliptic surfaces, we always have $\mathrm{Aut}_{X/C'} \cap \mathrm{Aut}_{X/E'} = \pi_*(G \times \alpha(G)) \cong G$.

If X is quasi-bielliptic, then it is not true in general that $\mathrm{Aut}_{X/C'} \cong C_{\mathrm{Aut}_E}(G)$. Indeed, for example if $p = 3$ and $G = \alpha_3$, then $C \to C'$ is purely inseparable of degree 3, hence $\mathrm{Aut}_{C/C'} = \mathrm{Aut}_C[F]$. Calculations (see Section 5.2.1, Case (d)) show that $C_{\mathrm{Aut}_C}(\alpha_3)^\circ \cong \alpha_3^2$ and $C_{\mathrm{Aut}_E}(G) \cong E \rtimes \mathbb{Z}/3\mathbb{Z}$. Hence, by Theorem 4.3 (1), $\mathrm{Aut}_{X/C'}$ is non-reduced while $C_{\mathrm{Aut}_E}(G)$ is reduced, so they cannot be isomorphic. In particular, for quasi-bielliptic surfaces, $\mathrm{Aut}_{X/C'} \cap \mathrm{Aut}_{X/E'}$ can be larger than G.

We end this section with a description of $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$. We are thankful to the editors for sharing an observation that allowed us to avoid forward references to Section 5 in the proof of the following proposition.

Corollary 4.7. *We have* $E \cong (\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ *and* $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ *is normal in* Aut_X.

Proof. Since X is not birationally ruled, [Pop16, Theorem 1] implies that $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ does not contain a connected linear algebraic group, hence, by [BSU13, Theorem 1.1.1], $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ is an Abelian variety. Then, by [BSU13, Proposition 2.2.1], the stabilizers of the $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$-action on X are finite. Since X is not an Abelian surface, the $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$-action on X cannot be transitive, hence $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ is either trivial or an elliptic curve. Now, by Theorem 4.3, the action of E on the first factor of $E \times C$ descends to a faithful action of E on X. This yields a monomorphism, and hence an isomorphism, of elliptic curves $E \to (\mathrm{Aut}_X^\circ)_{\mathrm{red}}$. To see that $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ is normal in Aut_X, let $(\mathrm{Aut}_X^\circ)_{\mathrm{ant}}$ be the largest anti-affine subgroup scheme of Aut_X° (see [BSU13, Chapter 5]). By [BSU13, Lemma 5.1.1], $(\mathrm{Aut}_X^\circ)_{\mathrm{ant}}$ is smooth and connected, and it contains $(\mathrm{Aut}_X^\circ)_{\mathrm{red}}$, since the latter is anti-affine. Hence, $(\mathrm{Aut}_X^\circ)_{\mathrm{red}} = (\mathrm{Aut}_X^\circ)_{\mathrm{ant}}$. By [BSU13, Theorem 1.2.1, Remark 1.2.2], $(\mathrm{Aut}_X^\circ)_{\mathrm{ant}} = (\mathrm{Aut}_X^\circ)_{\mathrm{red}}$ is normal in Aut_X°, hence also normal in Aut_X. This finishes the proof. □

5. Computing centralizers and normalizers

First, recall that if D is an integral curve of arithmetic genus $p_a(D) = 1$ with smooth locus D^{sm} and with a chosen point $O \in D^{sm}$, then there is a decomposition $\mathrm{Aut}_D = D^{sm} \rtimes \mathrm{Aut}_{D,O}$, where the group scheme $\mathrm{Aut}_{D,O}$ of automorphisms fixing O acts on the group scheme D^{sm} of translations t_s by points $s \in D^{sm}$ via $g \circ t_s \circ g^{-1} = t_{g(s)}$. This is because $\mathrm{Aut}_{D,O}$ acts on D^{sm} via group scheme automorphisms, see [Sil09, Theorem 4.8] and [BM76, Proposition 6]. We use the letter D here, since, using the terminology of Section 4, the following Lemma 5.1 applies to both $D = E$ and $D = C$.

Lemma 5.1. *Let D be an integral curve with $p_a(D) = 1$ and with a chosen point $O \in D^{sm}$. Let $G_1 \subseteq \mathrm{Aut}_{D,O}$ and $G_2 \subseteq D^{sm}$ be subgroup schemes. Then, the following are equivalent:*

(1) G_2 normalizes G_1.

(2) G_1 and G_2 commute.

(3) $G_2 \subseteq D^{G_1}$.

Proof. Note that if T is a k-scheme, $g \in G_1(T)$, and $t_s \in G_2(T)$, then we have

$$t_s \circ g \circ t_{-s} = t_{s-g(s)} \circ g. \tag{3}$$

In particular, if $s = g(s)$ for all $t_s \in G_2(T)$, then G_1 and G_2 commute, hence (3) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) is clear, hence it remains to prove (1) $\Rightarrow$ (3): if G_2 normalizes G_1, then Equation (3) shows that $t_{s-g(s)}(O_{T'}) = O_{T'}$ for all T-schemes T' and $t_s \in G_2(T)$. This is only possible if $s = g(s)$, hence $G_2(T) \subseteq D^{G_1}(T)$. □

5.1. Bielliptic surfaces

We use the notation of Section 4 and Lemma 5.1, but assume that D is smooth. In each of the cases $p \neq 2,3$, $p = 3$ and $p = 2$, we will recall the structure of the subgroup scheme $\mathrm{Aut}_{D,O} \subseteq \mathrm{Aut}_D$. Moreover, for every commutative subgroup $H \subseteq \mathrm{Aut}_{D,O}$, we list the fixed locus D^H and, if $\mathrm{Aut}_{D,O}$ is non-commutative, also the centralizer and normalizer of H in Lemma 5.2, Lemma 5.6, and Lemma 5.9. All of this is well-known and elementary to check, and we refer the reader to [Sil09, Section III.10 and Appendix A] for details. Together with Lemma 5.1, it will be straightforward to calculate the groups $C_{\mathrm{Aut}_E}(G)$ and $C_{\mathrm{Aut}_C}(\alpha(G))$ of Theorem 1.1 and produce Table 1. We will leave the details to the reader, but we will explain how the calculations work in Example 5.3. Using Theorem 4.3 (6), we calculate M in every case. The results of the calculations of this section are summarized in Table 1. To simplify notation, we define

$$N := N_{\mathrm{Aut}(C)}(\alpha(G)(k))/(C_{\mathrm{Aut}_C}(\alpha(G))(k)).$$

5.1.1. Characteristic $p \neq 2,3$.— By Bombieri and Mumford [BM77, p.37], the group schemes G leading to bielliptic surfaces $X = (E \times C)/G$ are the seven groups

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/3\mathbb{Z})^3, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The translation subgroup of $\alpha(G)$ is trivial in the first four of these cases, and isomorphic to the group $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z}$ in the other three cases, respectively.

Lemma 5.2. *The non-trivial commutative subgroup schemes H of $\mathrm{Aut}_{D,O}$ and their fixed loci D^H are as in Table 4.*

$j(D)$	$\mathrm{Aut}_{D,O}$	H	D^H
$\neq 0, 1728$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
1728	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
0	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
		$\mathbb{Z}/6\mathbb{Z}$	$\{1\}$

Table 4. $\mathrm{Aut}_{D,O}$ and its subgroups in characteristic $\neq 2,3$

Example 5.3. We explain how to calculate the centralizers in the case where the group is $G = \mathbb{Z}/2\mathbb{Z}$.

For the calculation of $C_{\mathrm{Aut}_E}(G)$, recall that translations in E always commute with G. Next, by Lemma 5.1, an automorphism $h_E \in \mathrm{Aut}_{E,O}$ commutes with G precisely if $G \subseteq E^{h_E}$. Now, we apply Lemma 5.2: if $j(E) \neq 1728$, or $j(E) = 1728$ and G does not coincide with the fixed locus of an automorphism h_E of order 4 in $\mathrm{Aut}_{E,O}$, then $C_{\mathrm{Aut}_E}(G)/E \cong \mathbb{Z}/2\mathbb{Z}$. This is Case a) in the first row of Table 1. If $j(E) = 1728$ and

$G = E^{h_E}$, then $C_{\mathrm{Aut}_E}(G)/E \cong \mathbb{Z}/4\mathbb{Z}$. This is Case b) in the first row of Table 1 and it seems to be missing from [BM90, Table 3.2], see also Remark 1.4.

For the calculation of $C_{\mathrm{Aut}_C}(\alpha(G))$, we apply Lemma 5.1 to find the subgroup of translations of C that commute with $\alpha(G)$. By Lemma 5.2, this group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Next, by Lemma 5.2, the group $\alpha(G)$ is in the center of $\mathrm{Aut}_{C,O}$, so $C_{\mathrm{Aut}_C}(\alpha(G)) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathrm{Aut}_{C,O}$. Now, if $j(E) \neq 0, 1728$, then $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$, if $j(E) = 1728$, then $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong D_8$, and if $j(E) = 0$, then $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong A_4$. These are the Cases i), ii), and iii) in the first row of Table 1.

Similarly, one can calculate the centralizers of G and $\alpha(G)$ for all seven possibilities of G. They are listed in Table 1. As for the group N, we have the following:

Lemma 5.4. *The group N is as in Table 5.*

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
N	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	S_3	$\mathbb{Z}/2\mathbb{Z}$

Table 5. The group N in characteristic $\neq 2, 3$

Proof. If $\alpha(G)$ does not contain translations, then $N_{\mathrm{Aut}_C}(\alpha(G)) = C_{\mathrm{Aut}_C}(\alpha(G))$ by Lemma 5.1 and because $\mathrm{Aut}_{C,O}$ is abelian. Hence, N is trivial in these cases.

If $G = (\mathbb{Z}/2\mathbb{Z})^2$, then conjugation by $N_{\mathrm{Aut}_C}(\alpha(G))$ fixes the unique non-trivial 2-torsion point c in $\alpha(G)$. By Lemma 5.2 and Lemma 5.1, this implies $|N| \mid 2$. The non-trivial element of N is induced by a 4-torsion point c' of C with $2c' = c$.

If $G = (\mathbb{Z}/3\mathbb{Z})^2$, then conjugation by $N_{\mathrm{Aut}_C}(\alpha(G))$ preserves the subgroup $\langle c \rangle \subseteq \alpha(G)$ generated by a non-trivial 3-torsion point c in $\alpha(G)$. Thus, the action of $N_{\mathrm{Aut}_C}(\alpha(G))$ descends to $C'' := C/\langle c \rangle$. There, it maps to the normalizer in $\mathrm{Aut}_{C''}$ of a subgroup $G'' \subseteq \mathrm{Aut}_{C'',O''}$ of order 3, where O'' is the image of O. By Lemma 5.1 and Table 4, the normalizer of G'' is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/6\mathbb{Z}$, where G'' sits inside the second factor. Thus, N is isomorphic to a subgroup of S_3. One can check that the involution in $\mathrm{Aut}_{C,O}$ and a 3-torsion point not contained in $\langle c \rangle$ induce non-trivial elements of N, hence $N \cong S_3$.

Finally, if $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then, again, conjugation by $N_{\mathrm{Aut}_C}(\alpha(G))$ fixes the unique non-trivial 2-torsion point c in $\alpha(G)$. In this case, however, the involution in $\alpha(G) \cap \mathrm{Aut}_{C,O}$ is the unique element in $\alpha(G)$ which is divisible by 2, hence it is also fixed by $N_{\mathrm{Aut}_C}(\alpha(G))$. Thus, by Lemma 5.1, a translation can be in $N_{\mathrm{Aut}_C}(\alpha(G))$ only if it is a translation by a 2-torsion point. The non-trivial 2-torsion point that commutes with $\alpha(G)$ is already contained in $\alpha(G)$, hence $N \cong \mathbb{Z}/2\mathbb{Z}$ is generated by one of the other two non-trivial 2-torsion points. □

Proposition 5.5. *The cases where M is non-trivial are precisely the following:*

(1) $G = (\mathbb{Z}/2\mathbb{Z})^2$, $j(E) = 1728$, and the fixed points G^{h_E} of the automorphism h_E of order 4 in $\mathrm{Aut}_{E,O}$ act as translations on C. In this case, $M = \mathbb{Z}/2\mathbb{Z}$.

(2) $G = (\mathbb{Z}/3\mathbb{Z})^2$, $j(E) = 0$, and the fixed points G^{h_E} of the automorphism h_E of order 3 in $\mathrm{Aut}_{E,O}$ act as translations on C. In this case, $M = \mathbb{Z}/3\mathbb{Z}$.

Proof. Assume that M is non-trivial. By Theorem 4.3 (5) and Table 5, this can only happen if $G \in \{(\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/3\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$.

Assume $G = (\mathbb{Z}/2\mathbb{Z})^2$. By Theorem 4.3 (5) and Table 5, we have $|M| \mid 2$. If $j(E) \neq 1728$, then $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G))$ has odd order, hence $M = \{1\}$ by Theorem 4.3 (5). If $j(E) = 1728$, we use Theorem 4.3 (6): by our description of the centralizers and normalizers, both $N_{\mathrm{Aut}_E}(G)/C_{\mathrm{Aut}_E}(G)$ and $N_{\mathrm{Aut}_C}(\alpha(G))/C_{\mathrm{Aut}_C}(\alpha(G))$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and every non-trivial element of $M(k)$ can be represented by $h = (h_E, h_C)$, where $h_E \in \mathrm{Aut}_{E,O}$ is of order 4 and h_C is translation by a non-trivial 4-torsion point such that $h_C^2 \in \alpha(G)$. By Lemma 5.1 and Table 4, we have $\alpha \circ \mathrm{ad}_{h_E} = \mathrm{ad}_{h_C} \circ \alpha$ if and only if the fixed point of h_E maps via α to the unique translation in $\alpha(G)$. This is Case (1).

Next, assume $G = (\mathbb{Z}/3\mathbb{Z})^2$. Let $h = (h_E, h_C)$ be an automorphism of $E \times C$ lifting a non-trivial element of $M(k)$. By our description of $C_{\mathrm{Aut}_E}(G)$ and N, we may assume that h_C is either the involution in $\mathrm{Aut}_{C,O}$ or translation by a 3-torsion point $c' \notin \alpha(G)$, and that $h_E \in \mathrm{Aut}_{E,O}$. If h_E is an involution, then ad_{h_E} fixes only the identity in G, while ad_{h_C} has more fixed points on $\alpha(G)$. Hence, by Theorem 4.3 (6), h does not normalize the G-action on $E \times C$ in this case, a contradiction to Proposition 3.1. Thus, we may further assume that $j(E) = 0$ and h_E has order 3. Then, we may assume that h_C is translation by c'. By Lemma 5.1 and Table 4, we have $\alpha \circ \mathrm{ad}_{h_E} = \mathrm{ad}_{h_C} \circ \alpha$ if and only if the fixed points of h_E on E map to translations in $\alpha(G)$. This is Case (2).

Finally, assume $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Assume M is non-trivial and, using Theorem 4.3 (6), let $h = (h_E, h_C)$ be an automorphism mapping to a non-trivial element in $M(k)$. We may assume that $h_E \in \mathrm{Aut}_{E,O}$ is the involution and h_C is a translation by one of the 2-torsion points not contained in $\alpha(G)$. Observe that ad_{h_E} maps elements of order 4 in G to their inverses while ad_{h_C} maps the automorphism σ of order 4 in $\alpha(G) \cap \mathrm{Aut}_{C,O}$ to $\sigma \circ t_c$, where c is the non-trivial 2-torsion point in $\alpha(G)$. Hence, we have $\alpha \circ \mathrm{ad}_{h_E} \neq \mathrm{ad}_{h_C} \circ \alpha$. This contradiction shows that $M = \{1\}$ in this case. □

5.1.2. Characteristic $p = 3$.— By Bombieri and Mumford [BM77, p.37], the groups G leading to bielliptic surfaces $X = (E \times C)/G$ are the six groups

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The translation subgroup of $\alpha(G)$ is trivial in the first four of these cases, and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ in the other two cases.

Lemma 5.6. *The non-trivial commutative subgroup schemes* H *of* $\mathrm{Aut}_{D,O}$, *their fixed loci* D^H, *centralizers* $C_{\mathrm{Aut}_{D,O}}(H)$ *and normalizers* $N_{\mathrm{Aut}_{D,O}}(H)$ *are as in Table 6.*

$j(D)$	$\mathrm{Aut}_{D,O}$	H	D^H	$C_{\mathrm{Aut}_{D,O}}(H)$	$N_{\mathrm{Aut}_{D,O}}(H)$
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
0	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$
		$\mathbb{Z}/3\mathbb{Z}$	α_3	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$
		$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
		$\mathbb{Z}/6\mathbb{Z}$	$\{1\}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$

Table 6. $\mathrm{Aut}_{D,O}$ and its subgroups in characteristic 3

As in characteristic $\neq 2, 3$, it is straightforward to calculate the centralizers of G and $\alpha(G)$ and they are listed in Table 1.

Lemma 5.7. *The group* N *is as in Table 7.*

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
N	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Table 7. The group N in characteristic 3

Proof. If $\alpha(G)$ does not contain translations, then a translation in Aut_C normalizes $\alpha(G)$ if and only if it centralizes $\alpha(G)$ by Lemma 5.1. Thus, in these cases, N can be read off from the last two columns of Table 6. The proof of the two remaining cases is the same as for Lemma 5.4. □

Proposition 5.8. *The cases where M is non-trivial are precisely the following:*

(1) $G = (\mathbb{Z}/2\mathbb{Z})^2$ and $j(E) = 0$. In this case, $M \cong \mathbb{Z}/2\mathbb{Z}$.
(2) $G \in \{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}\}$. In these cases, $M \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. By Theorem 4.3 (5) and Table 7, we may assume $G \in \{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$. For $G \in \{(\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\}$, the proof is essentially the same as in Proposition 5.5. The only difference is that every non-trivial 2-torsion point of E is fixed by some automorphism of order 4 in $\mathrm{Aut}_{E,O}$, so we do not have an extra condition as in Proposition 5.5.

For $G \in \{\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}\}$, it suffices to find a non-trivial element in M. By Lemma 5.6, there is an element $h_C \in N_{\mathrm{Aut}_{C,O}}(\alpha(G))$ of order 4 such that ad_{h_C} swaps the two generators of $\alpha(G)$. The inversion h_E on E induces the same action on G. By Theorem 4.3 (6), this shows $M \cong \mathbb{Z}/2\mathbb{Z}$. □

5.1.3. Characteristic $p = 2$.— By Bombieri and Mumford [BM77, p.37], the group schemes G leading to bielliptic surfaces $X = (E \times C)/G$ are the six group schemes

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mu_2 \times \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^2.$$

The translation subgroup scheme of $\alpha(G)$ is trivial in the first four of these cases, and isomorphic to μ_2 and $\mathbb{Z}/3\mathbb{Z}$, respectively, in the other two cases.

Lemma 5.9. *The non-trivial commutative subgroup schemes H of $\mathrm{Aut}_{D,O}$, their fixed loci D^H, centralizers $C_{\mathrm{Aut}_{D,O}}(H)$ and normalizers $N_{\mathrm{Aut}_{D,O}}(H)$ are as in Table 8.*

$j(D)$	$\mathrm{Aut}_{D,O}$	H	D^H	$C_{\mathrm{Aut}_{D,O}}(H)$	$N_{\mathrm{Aut}_{D,O}}(H)$
$\neq 0$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
0	$Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	M_2	$Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$	$Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$
		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$
		$\mathbb{Z}/4\mathbb{Z}$	α_2	$\mathbb{Z}/4\mathbb{Z}$	Q_8
		$\mathbb{Z}/6\mathbb{Z}$	$\{1\}$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$

Table 8. $\mathrm{Aut}_{D,O}$ and its subgroups in characteristic 2

As before, it is straightforward to calculate the centralizers of G and $\alpha(G)$ and they are listed in Table 1.

Lemma 5.10. *The group N is as in Table 9.*

G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	$\mu_2 \times \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^2$
N	$\{1\}$	$\{1\}$	$\mathbb{Z}/2\mathbb{Z}$	$\{1\}$	$\{1\}$	S_3

Table 9. The group N in characteristic 2

Proof. If $\alpha(G)$ does not contain translations, then a translation in Aut_C normalizes $\alpha(G)$ if and only if it centralizes $\alpha(G)$ by Lemma 5.1. Thus, in these cases, N can be read off from the last two columns of Table 8. For $G = (\mathbb{Z}/3\mathbb{Z})^2$, the proof is the same as for Lemma 5.4. Finally, if $G = \mu_2 \times \mathbb{Z}/2\mathbb{Z}$, then $N_{\mathrm{Aut}(C)}(\alpha(G)(k))$ is generated by $\alpha(G)(k)$ and the unique non-trivial 2-torsion point in $C(k)$ by the same argument as in the proof of Lemma 5.1. Translation by this 2-torsion point commutes with $\alpha(G)$, hence N is trivial. □

Proposition 5.11. *The cases where M is non-trivial are precisely the following:*

(1) $G = (\mathbb{Z}/3\mathbb{Z})^2$ *and* $j(E) = 0$. *In this case,* $M = \mathbb{Z}/3\mathbb{Z}$.

(2) $G = \mathbb{Z}/4\mathbb{Z}$. *In this case,* $M = \mathbb{Z}/2\mathbb{Z}$.

Proof. By Theorem 4.3 (5) and Table 9, we may assume $G \in \{(\mathbb{Z}/3\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z}\}$. The proof for $G = (\mathbb{Z}/3\mathbb{Z})^2$ is the same as in Proposition 5.5 with the only difference that every non-trivial 3-torsion point in E is fixed by some automorphism of order 3, so we do not have an extra condition as in Proposition 5.5.

If $G = \mathbb{Z}/4\mathbb{Z}$, consider the automorphism $h = (h_E, h_C)$ of $E \times C$ where $h_C \in N_{\mathrm{Aut}_{C,O}}(\alpha(G))$ is of order 4 and not contained in $\alpha(G)$ and h_E is the inversion involution on E. By Lemma 4.2 (3), h normalizes the G-action on $E \times C$ and, by Proposition 3.1, induces a non-trivial element of M. Hence, we have $M \cong \mathbb{Z}/2\mathbb{Z}$. □

5.2. Quasi-bielliptic surfaces

In the case of quasi-bielliptic surfaces, E is still smooth, so the group $C_{\mathrm{Aut}_E}(G)/E$ can be calculated using the results of the previous section. We will thus focus on the calculation of $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G)$ and M. We identify the smooth locus of C with $\mathbb{A}^1 = \operatorname{Spec} k[t]$ and use the description of automorphisms of $\mathbb{A}^1$ coming from C given in [BM76, Proposition 6].

5.2.1. Characteristic $p = 3$.— By [BM76, Proposition 6] the T-valued automorphisms of $\mathbb{A}^1$ coming from C are of the form

$$t \mapsto bt + c + dt^3 \tag{4}$$

with $b \in \mathbb{G}_m(T)$, $c, d \in \mathbb{G}_a(T)$ and $d^3 = 0$. By [BM76, p. 214], the subgroup schemes $\alpha(G)$ leading to quasi-bielliptic surfaces are the following:

(a) μ_3: $t \mapsto at + (1-a)t^3$ with $a^3 = 1$
(b) $\mu_3 \times \mathbb{Z}/2\mathbb{Z}$: μ_3 as in (a) and $t \mapsto \pm t$.
(c) $\mu_3 \times \mathbb{Z}/3\mathbb{Z}$: μ_3 as in (a) and $t \mapsto t + i$ with $i^3 = i$
(d) $\alpha_3 : t \mapsto t + at^3$ with $a^3 = 0$
(e) $\alpha_3 \times \mathbb{Z}/2\mathbb{Z}$: α_3 as in (d) and $t \mapsto \pm t$

Remark 5.12. As noted in [Lan79, p.489], Case (f) of [BM76, p. 214] does not exist, because the group scheme given there is isomorphic to α_9 and thus not a subscheme of an elliptic curve.

Now, let us calculate $C_{\mathrm{Aut}_C}(\alpha(G))$ and M for the surfaces in Case (a),...,(e). To this end, we take a k-scheme T and arbitrary elements $g \in \alpha(G)(T)$ as in the above list and $h \in \mathrm{Aut}_C(T)$ as in (4). One can check that the inverse of h is given by

$$t \mapsto b^{-1}t + b^{-4}(c^3 d - b^3 c) - b^{-4} dt^3$$

(a) We calculate

$$h \circ g \circ h^{-1} : t \mapsto at + (1-a)b^{-1}(c^3 - c) + (1-a)(b^2 - b^{-1}d)t^3.$$

Thus, h normalizes $\alpha(G)$ if and only if it centralizes $\alpha(G)$ if and only if $c^3 = c$ and $b^3 = d + b$. Taking the cube of the second equation, we obtain $b^6 = 1$. Thus, the centralizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto bt + i + (b^3 - b)t^3 \text{ with } b^6 = 1 \text{ and } i^3 = i.$$

This group scheme is isomorphic to $\mu_3 \times S_3$. Therefore, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong S_3$.

To calculate M, first note that $|M| \mid 2$, since E and E' are ordinary, M is a subquotient of $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G))$ by Theorem 4.3 (5), and $\mathrm{Aut}^{\circ}_{E'} \subseteq ((f_E)_* C_{\mathrm{Aut}_E}(G))$. If M is non-trivial, then it can be represented by an automorphism $g \in \mathrm{Aut}(X)$ that induces the inversion involution on E'. This involution can be lifted to E, hence g lifts to an automorphism of $E \times C$. However, by the above calculations there is no element of $\mathrm{Aut}(C)$ that acts as an inversion on $\alpha(G)$. So, Theorem 4.3 (6) shows that M is trivial.

(b) Since μ_3 is the identity component of $\mu_3 \times \mathbb{Z}/2\mathbb{Z}$, the normalizer of $\mu_3 \times \mathbb{Z}/2\mathbb{Z}$ in Aut_C is contained in the normalizer of μ_3 in Aut_C. By Case (a), the latter is isomorphic to $\mu_3 \times S_3$. Thus, $N_{\mathrm{Aut}_C}(\alpha(G))$ equals the normalizer of $\mu_3 \times \mathbb{Z}/2\mathbb{Z}$ in $\mu_3 \times S_3$, hence $N_{\mathrm{Aut}_C}(\alpha(G)) = C_{\mathrm{Aut}_C}(\alpha(G)) = \alpha(G)$. By the same argument as in (a), we also have $M = \{1\}$.

(c) Similar to case (b), we obtain that $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) = \{1\}$ and $M = \{1\}$.

(d) We calculate
$$h \circ g \circ h^{-1} : t \mapsto t + ab^{-1}c^3 + ab^2t^3.$$
Thus, h normalizes $\alpha(G)$ if and only if $c^3 = 0$, and it centralizes $\alpha(G)$ if and only if additionally $b^2 = 1$ holds. Thus, $C_{\mathrm{Aut}_C}(\alpha(G))$ is a semi-direct product $\alpha_3^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $N_{\mathrm{Aut}_C}(\alpha(G))$ is a semi-direct product $(\alpha_3)^2 \rtimes \mathbb{G}_m$. In particular, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong \alpha_3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Next, we calculate M. Using Lemma 5.1 and Lemma 5.6, one can check that $C_{\mathrm{Aut}_E}(G)/E \cong \mathbb{Z}/3\mathbb{Z}$. Thus, there is an isomorphism $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G)) \cong \mathrm{Aut}_{E',O}/(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, where we use the structure of $\mathrm{Aut}_{E',O}$ recalled in Lemma 5.6. So, by Theorem 4.3 (5), M is a subquotient of $\mathbb{Z}/4\mathbb{Z}$.

Choose any automorphism $h_E \in \mathrm{Aut}_{E,O}$ of order 4. Since $\alpha_3 \subseteq E$ is the kernel of Frobenius, it is preserved by h_E. Moreover, by Lemma 5.1 and Lemma 5.6, the centralizer of α_3 in $\mathrm{Aut}_{E,O}$ has order 3, so conjugation by h_E induces an automorphism of α_3 of order 4. By the calculations of the first paragraph, we have a surjection $N_{\mathrm{Aut}_C}(\alpha(G)) \to \mathrm{Aut}_{\alpha(G)} \cong \mathbb{G}_m$, hence we can find an $h_C \in N_{\mathrm{Aut}_C}(\alpha(G))(k)$ such that $h = (h_E, h_C) \in N_{\mathrm{Aut}_E \times \mathrm{Aut}_C}(G)(k)$ by Lemma 4.2 (3). By Proposition 3.1, h descends to an automorphism of X that induces an element of order 4 in M. Therefore, we have $M \cong \mathbb{Z}/4\mathbb{Z}$.

(e) Let $g : t \mapsto -t$. Then,
$$h \circ g \circ h^{-1} : t \mapsto -t + b^{-1}c - b^{-4}c^3 d.$$
Since α_3 is the identity component of $\alpha_3 \times \mathbb{Z}/2\mathbb{Z}$, we can use the results of (d) to deduce that h normalizes $\alpha(G)$ if and only if $c = 0$ and it centralizes $\alpha(G)$ if and only if additionally $b^2 = 1$. Thus, we get $C_{\mathrm{Aut}_C}(\alpha(G)) \cong \alpha_3 \times \mathbb{Z}/2\mathbb{Z}$ and the normalizer of $\alpha(G)$ is $N_{\mathrm{Aut}_C}(\alpha(G)) \cong \alpha_3 \rtimes \mathbb{G}_m$. In particular, $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) = \{1\}$.

Since the automorphism g generates the group $\alpha(G)(k)$, the calculation of the previous paragraph also shows that $N_{\mathrm{Aut}(C)}(\alpha(G)(k)) = \mathbb{G}_m(k)$. Thus, M is isomorphic to a subquotient of $\mathbb{G}_m(k)$ by Theorem 4.3 (5) and, in particular, the order of M is prime to 3. By the same theorem, M is also a subquotient of $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G))$, which is isomorphic to $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ since $C_{\mathrm{Aut}_E}(G) \cong E$ in the current case. Hence, M is a subquotient of $\mathbb{Z}/4\mathbb{Z}$. Using the same construction as in (d), one can show that $M \cong \mathbb{Z}/4\mathbb{Z}$.

5.2.2. Characteristic $p = 2$.— By [BM76, Proposition 6] the T-valued automorphisms of $\mathbb{A}^1$ coming from C are of the form

$$t \mapsto bt + c + dt^2 + et^4 \tag{5}$$

with $b \in \mathbb{G}_m(T)$, $c, d, e \in \mathbb{G}_a(T)$ and $d^4 = e^2 = 0$. The subgroup schemes $\alpha(G)$ leading to quasi-bielliptic surfaces are the following, where $\lambda \in k$:

(a) μ_2: $t \mapsto at + \lambda(a+1)t^2 + (a+1)t^4$ with $a^2 = 1$.
(b) $\mu_2 \times \mathbb{Z}/3\mathbb{Z} : \mu_2$ as in (a) with $\lambda = 0$ and $t \mapsto \omega t$, where $\omega^3 = 1$.
(c) $\mu_2 \times \mathbb{Z}/2\mathbb{Z} : \mu_2$ as in (a) and $t \mapsto t + \zeta$, where ζ is a fixed root of $x^3 + \lambda x + 1$.
(d) $\mu_4 : t \mapsto at + (a + a^2)t^2 + (1 + a^2)t^4$ with $a^4 = 1$
(e) $\mu_4 \times \mathbb{Z}/2\mathbb{Z} : \mu_4$ as in (d) and $t \mapsto t + 1$.

(f) $\alpha_2 : t \mapsto t + \lambda a t^2 + a t^4$ with $a^2 = 0$, and with $\lambda \in \{0,1\}$.
(g) $\alpha_2 \times \mathbb{Z}/3\mathbb{Z} : \alpha_2$ as in (f) with $\lambda = 0$ and $\mathbb{Z}/3\mathbb{Z}$ as in (b)
(h) $M_2 : t \mapsto t + a + \lambda a^2 t^2 + a^2 t^4$ with $a^4 = 0$, and with $\lambda \neq 0$.

Remark 5.13. In [BM76, p. 214], Bombieri and Mumford do not give restrictions on the parameter $\lambda \in k$ in Case (f). However, all the α_2-actions with $\lambda \neq 0$ described by them are conjugate, so we may assume $\lambda \in \{0,1\}$. For more details, we refer the reader to the discussion of Case (f) below.

Remark 5.14. To see that the group scheme in Case (h) is indeed M_2, denote the transformation in Case (h) associated to z_i with $z_i^4 = 0$ by t_{z_i}. Observe that $t_{z_1} \circ t_{z_2} = t_{z_1+z_2+\lambda z_1^2 z_2^2}$. So, if $G = \operatorname{Spec} k[z]/z^4$ is the group scheme in Case (h), then its co-multiplication is given by

$$z \mapsto z_1 \otimes 1 + 1 \otimes z_2 + \lambda z_1^2 \otimes z_2^2.$$

Consider the supersingular elliptic curve E with affine Weierstrass equation $y^2 + \lambda y = x^3$ and set $z = x/y, w = 1/y$, so that the equation becomes $z^3 = w + \lambda w^2$. Then, the 2-torsion subscheme M_2 of E is the subscheme given by $z^4 = w^2 = 0$, and thus $w = z^3$. By [Sil09, p.120] the co-multiplication on $k[z]/z^4$ induced by the group structure on E is precisely the one described above. Hence, we have $G = M_2$.

For later use, we note that by [Sil09, Appendix A, Proposition 1.2], the group of automorphisms of E preserving $w = z = 0$ is given by the substitutions $x \mapsto b^2 x + c^2, y \mapsto y + b^2 c x + d$ with $b^3 = 1, c^4 + \lambda c = 0$ and $d^2 + \lambda d + c^6 = 0$. In particular, they act on $k[z]/z^4$ as

$$z \mapsto \frac{b^2 x + c^2}{y + b^2 c x + d} = \frac{b^2 z + c^2 w}{1 + b^2 c z + d w} = (b^2 z + c^2 z^3)(1 + b^2 c z + d z^3)^3 = b^2 z + b c z^2.$$

In particular, if we think of the substitutions in Case (h) above as defining a homomorphism of group schemes $E \supseteq M_2 \xrightarrow{\alpha} \mathrm{Aut}_C$, then precomposing α with ad_{h_E} where $h_E \in \mathrm{Aut}_{E,O}$ is as described in the previous paragraph, then $\alpha \circ \mathrm{ad}_{h_E}$ corresponds to M_2 acting on C as

$$t \mapsto t + (b^2 a + b c a^2) + b \lambda a^2 t^2 + b a^2 t^4.$$

Now, we are prepared to calculate $C_{\mathrm{Aut}_C}(\alpha_C)$ and M in Cases (a),...,(h). As in characteristic 3, we take a k-scheme T and arbitrary elements $g \in \alpha(G)(T)$ as in the above list and $h \in \mathrm{Aut}_C(T)$ as in (5). One can check that the inverse of h is given by

$$t \mapsto b^{-1} t + b^{-7}(b^6 c + b^2 c^4 e + b^4 c^2 d + c^4 d^3) + b^{-3} d t^2 + b^{-7}(d^3 + b^2 e) t^4.$$

(a) We calculate

$$\begin{aligned} h \circ g \circ h^{-1} : \quad t \mapsto \quad & a t + (a+1) b^{-1}(c + \lambda c^2 + c^4) + (a+1)(b^{-1} d + \lambda b) t^2 \\ & + (a+1)(b^{-1} e + \lambda b^{-1} d^2 + b^3) t^4. \end{aligned}$$

Thus, h normalizes $\alpha(G)$ if and only if it centralizes $\alpha(G)$ if and only if

$$c^4 + \lambda c^2 + c = 0,$$
$$d = \lambda(b^2 + b), \text{ and} \tag{6}$$
$$e = b^4 + b + \lambda^3(b^4 + b^2). \tag{7}$$

If $\lambda \neq 0$, the fourth power of (6) yields $b^4 = 1$, while the square of (7) yields $b^6 = 1$, so we have $b^2 = 1$. Hence, in this case $C_{\mathrm{Aut}_C}(\alpha(G))$ is the group scheme of maps

$$t \mapsto b t + c + \lambda(1+b) t^2 + (1+b) t^4 \text{ with } b^2 = 1 \text{ and } c^4 + \lambda c^2 + c = 0,$$

which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2$ since $c^4 + \lambda c^2 + c$ has 4 distinct roots. Therefore, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

If $\lambda = 0$, then $d = 0$, and the square of (7) yields $b^6 = 1$. Thus, the centralizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto bt + c + (b + b^4)t^4 \text{ with } b^6 = 1 \text{ and } c^4 = c,$$

which is isomorphic to $A_4 \times \mu_2$. We deduce that $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong A_4$.

In both cases $\lambda \neq 0$ and $\lambda = 0$, note that $\mathrm{Aut}_{E'} = (f_E)_* C_{\mathrm{Aut}_E}(G)$, so $M = \{1\}$ follows immediately from Theorem 4.3 (5).

(b) Since μ_2 is the identity component of the group scheme $\mathbb{Z}/3\mathbb{Z} \times \mu_2$, it suffices to calculate the normalizer of $\mathbb{Z}/3\mathbb{Z} \times \mu_2$ in $A_4 \times \mu_2$, which is equal to its centralizer and both are equal to $\mathbb{Z}/3\mathbb{Z} \times \mu_2$. In particular, $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) = \{1\}$. To see that $M = \{1\}$, one can use the same arguments as in Case (a) in characteristic 3 to show that the action of M lifts to $E \times C$. Since $N_{\mathrm{Aut}_C}(\alpha(G)) = C_{\mathrm{Aut}_C}(\alpha(G))$, Theorem 4.3 (6) shows that M is trivial.

(c) We take the centralizer of $\mathbb{Z}/2\mathbb{Z} \times \mu_2$ in $(\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2$ if $\lambda \neq 0$ and in $A_4 \times \mu_2$ if $\lambda = 0$. Both are equal to the normalizer and also equal to $(\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2$. Thus, $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong \mathbb{Z}/2\mathbb{Z}$. As in Case (a), we have $M = \{1\}$.

(d) We calculate

$$\begin{aligned} h \circ g \circ h^{-1} : t \quad \longmapsto \quad & at + (a+1)b^{-1}\left(c + ac^2 + (a+1)(b^{-2}c^2 d + b^{-2}c^4 d + c^4)\right) + (a + a^2)(b^{-1}d + b)t^2 \\ & +(a+1)\left(b^{-1}e + ab^{-1}d^2 + (a+1)(b^{-3}d^3 + bd + b^3)\right)t^4. \end{aligned}$$

Thus, h normalizes $\alpha(G)$ if and only if it centralizes $\alpha(G)$. For h to centralize the subgroup scheme where $a^2 = 1$, we obtain the conditions

$$\begin{aligned} c + c^2 &= 0, \\ d &= b^2 + b, \text{ and} \\ e &= b^4 + b^2. \end{aligned}$$

Since $d^4 = 0$, this implies $b^4 = 1$. Plugging these conditions back into the equation for $h \circ g \circ h^{-1}$, it turns out that the subgroup scheme of transformations satisfying these conditions centralizes all of $\alpha(G)$. Therefore, the centralizer $C_{\mathrm{Aut}_C}(\alpha(G))$ is given by the group scheme of maps

$$t \mapsto bt + c + (b + b^2)t^2 + (1 + b^2)t^4 \text{ with } b^4 = 1 \text{ and } c \in \{0, 1\},$$

which is isomorphic to $\mu_4 \times \mathbb{Z}/2\mathbb{Z}$. Therefore, $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong \mathbb{Z}/2\mathbb{Z}$. By the same argument as in Case (b), we have $M = \{1\}$.

(e) Since μ_4 is the identity component of $\mu_4 \times \mathbb{Z}/2\mathbb{Z}$, we can use the computations of (d) to immediately conclude that centralizer and normalizer of $\alpha(G)$ are both equal to $\mu_4 \times \mathbb{Z}/2\mathbb{Z}$ and thus $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) = \{1\}$. Also, $M = \{1\}$ follows by the same argument as in Case (b).

(f) We calculate

$$h \circ g \circ h^{-1} : t \mapsto t + ab^{-1}(\lambda c^2 + c^4) + \lambda abt^2 + a(\lambda b^{-1}d^2 + b^3)t^4.$$

This shows that all the α_2-actions with $\lambda \neq 0$ are conjugate to the one with $\lambda = 1$ by conjugating with the map $t \mapsto \sqrt{\lambda}t$. Hence, we may assume $\lambda \in \{0, 1\}$.

Suppose $\lambda = 1$. Then, h normalizes $\alpha(G)$ if and only if it satisfies the conditions

$$\begin{aligned} c^2 + c^4 &= 0, \text{ and} \\ d^2 &= b^4 + b^2. \end{aligned} \tag{8}$$

Squaring (8), we get $b^4 = 1$. We also note that h centralizes $\alpha(G)$ if and only if additionally $b = 1$. Therefore, the normalizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto bt + c + dt^2 + et^4 \text{ with } b^4 = 1, c^4 = c^2, d^2 = b^4 + b^2, \text{ and } e^2 = 0,$$

which is isomorphic to a semi-direct product $(\alpha_2^3 \rtimes \mathbb{Z}/2\mathbb{Z}) \rtimes \mu_4$ and the centralizer of $\alpha(G)$ is isomorphic to a semi-direct product $\alpha_2^3 \rtimes \mathbb{Z}/2\mathbb{Z}$. Hence, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong \alpha_2^2 \rtimes \mathbb{Z}/2\mathbb{Z}$. To calculate M, note that $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G)) \cong \mathbb{Z}/3\mathbb{Z}$, so $|M| \mid 3$ by Theorem 4.3 (5). Since $E \to E'$ is purely inseparable we can lift the M-action from E' to E and hence to $E \times C$, where it normalizes G. Since $N_{\mathrm{Aut}_C}(\alpha(G))/C_{\mathrm{Aut}_C}(\alpha(G))(k) \cong \mu_4(k)$ is trivial, this shows that $M = \{1\}$.

If $\lambda = 0$, then h normalizes $\alpha(G)$ if and only if $c^4 = 0$ and it centralizes $\alpha(G)$ if and only if additionally $b^3 = 1$. Thus, the normalizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto bt + c + dt^2 + et^4 \text{ with } c^4 = d^4 = e^2 = 0,$$

which is isomorphic to $(\alpha_4 \rtimes A) \rtimes \mathbb{G}_m$ and the centralizer is isomorphic to $(\alpha_4 \rtimes A) \rtimes \mathbb{Z}/3\mathbb{Z}$, where A is a non-split extension of α_4 by $\alpha_2 = \alpha(G)$. Thus, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong (\alpha_4 \rtimes \alpha_4) \rtimes \mathbb{Z}/3\mathbb{Z}$.

Finally, let us explain how to compute M in the case $\lambda = 0$. As in the case $\lambda = 1$, we have $|M| \mid 3$. Choose an element $h_E \in \mathrm{Aut}_{E,O}$ of order 3. Since $\alpha(G) = \alpha_2$ is the kernel of Frobenius on E, it is preserved by h_E. By Lemma 5.1 and Lemma 5.9, conjugation by h_E induces an automorphism of α_2 of order 3. On the other hand, the conjugation action of $N_{\mathrm{Aut}_C}(\alpha(G))$ on α_2 factors through $N_{\mathrm{Aut}_C}(\alpha(G))/C_{\mathrm{Aut}_C}(\alpha(G))$. By the calculations of the previous paragraph and since $\mathrm{Aut}_{\alpha_2} \cong \mathbb{G}_m$, we can find an automorphism $h_C \in N_{\mathrm{Aut}_C}(\alpha(G))$ of order 9 such that $\alpha \circ \mathrm{ad}_{h_E} = \mathrm{ad}_{h_C} \circ \alpha$. By Lemma 4.2 (3), $h = (h_E, h_C)$ normalizes the G-action on $E \times C$. By Proposition 3.1, h descends to X and induces a non-trivial element of M. Hence, we have $M \cong \mathbb{Z}/3\mathbb{Z}$.

(g) Let $g : t \mapsto \omega t$, where $\omega^2 + \omega = 1$. Then,

$$h \circ g \circ h^{-1} : t \mapsto \omega t + \omega^2 b^{-1}(c + b^{-4}c^4 e + \omega^2 b^{-2} c^2 d + b^{-6} c^4 d^3) + b^{-1} d t^2 + b^{-3} d^3 t^4.$$

Thus, h normalizes $\mathbb{Z}/3\mathbb{Z}$ if and only if it centralizes $\mathbb{Z}/3\mathbb{Z}$ if and only if

$$\begin{aligned} d &= 0, \text{ and} \\ c^4 e + b^4 c &= 0. \end{aligned}$$

Putting this together with the conditions obtained in (f), we deduce that the normalizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto bt + et^4 \text{ with } e^2 = 0,$$

which is isomorphic to $\alpha_2 \rtimes \mathbb{G}_m$. Moreover, we see that $C_{\mathrm{Aut}_C}(\alpha(G)) = \alpha(G)$.

Since the automorphism g generates the group $\alpha(G)(k)$, the calculation of the previous paragraph also shows that $N_{\mathrm{Aut}(C)}(\alpha(G)(k)) = \mathbb{G}_m(k)$. Thus, M is a subquotient of $\mathbb{G}_m(k)$ by Theorem 4.3 (5) and, in particular, the order of M is prime to 2. By the same theorem, M is also a subquotient of $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G))$, which is isomorphic to $Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$ since $C_{\mathrm{Aut}_E}(G) \cong E$ in the current case. Hence, M is a subquotient of $\mathbb{Z}/3\mathbb{Z}$. Using the same construction as in (f), one can show that $M = \mathbb{Z}/3\mathbb{Z}$.

(h) We compute

$$h \circ g \circ h^{-1} : t \mapsto t + ab^{-1}(1 + a(\lambda c^2 + c^4 + b^{-2} d)) + \lambda a^2 b t^2 + a^2(\lambda b^{-1} d^2 + b^3) t^4.$$

This means that h normalizes $\alpha(G)$ if and only if it satisfies

$$\begin{aligned} b^3 &= 1, \text{ and} \\ \lambda d^2 &= b + b^2. \end{aligned} \tag{9}$$

In fact, since $d^4 = 0$, we can square (9) to deduce $b = 1$, and since $\lambda \neq 0$, we get $d^2 = 0$. Now, h centralizes $\alpha(G)$ if and only if additionally

$$d = c^4 + \lambda c^2. \tag{10}$$

Squaring (10), we obtain $c^8 + \lambda^2 c^4 = 0$. Hence, the centralizer $C_{\mathrm{Aut}_C}(\alpha(G))$ of $\alpha(G)$ is the group scheme of maps

$$t \mapsto t + c + (c^4 + \lambda c^2)t^2 + et^4 \text{ with } e^2 = 0 \text{ and } c^8 + \lambda^2 c^4 = 0,$$

which is isomorphic to $(M_2 \times \alpha_2) \rtimes \mathbb{Z}/2\mathbb{Z}$, and the normalizer of $\alpha(G)$ is the group scheme of maps

$$t \mapsto t + c + dt^2 + et^4 \text{ with } d^2 = e^2 = 0,$$

which is isomorphic to $\mathbb{G}_a \rtimes \alpha_2^2$. In particular, we have $C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G) \cong \alpha_2 \times \mathbb{Z}/2\mathbb{Z}$.

To calculate M, note first that M is a subquotient of $\mathrm{Aut}_{E'}/((f_E)_* C_{\mathrm{Aut}_E}(G)) \cong A_4$ by Theorem 4.3 (5). Since $E \to E'$ is purely inseparable, we can lift the action of Aut_X to $E \times C$, where it normalizes the G-action. By the previous paragraph, we have $N_{\mathrm{Aut}_C}(\alpha(G))/C_{\mathrm{Aut}_C}(\alpha(G)) \cong \mathbb{G}_a$ and therefore M is isomorphic to a subquotient of $(\mathbb{Z}/2\mathbb{Z})^2$, again by Theorem 4.3 (5). We may assume that E is given by the equation $y^2 + \lambda y = x^3$. Choose $c, d \in k$ such that $c^3 = \lambda$ and $d^2 + \lambda d + \lambda^2 = 0$ and let $h_{E,c,d}$ be the corresponding automorphism of E as in Remark 5.14. Then, by the calculations of the previous paragraph and by Remark 5.14, $\alpha_T \circ \mathrm{ad}_{h_{E,c,d}} = \mathrm{ad}_{h_{C,c'}} \circ \alpha_T$, where $h_{C,c'}$ is a substitution $h_{C,c'} : t \mapsto t + c'$ with $c'^4 + \lambda c'^2 = c$. Therefore the automorphisms $(h_{E,c,d}, h_{C,c'})$ of $E \times C$ descend to X. The three different values of c yield three distinct non-trivial elements of M, so $M \cong (\mathbb{Z}/2\mathbb{Z})^2$.

This finishes the calculation of the groups $C_{\mathrm{Aut}_E}(G)/E, C_{\mathrm{Aut}_C}(\alpha(G))/\alpha(G), M$, and thus also of the full automorphism group schemes for all bielliptic and quasi-bielliptic surfaces in all characteristics. The results are summarized in Table 1, Table 2 and Table 3.

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