Examples of surfaces with canonical map of degree 4

Carlos Rito

Abstract. We give two examples of surfaces with canonical map of degree 4 onto a canonical surface.

Keywords. algebraic surface; surface of general type; abelian cover; canonical map

2020 Mathematics Subject Classification. 14J29
1. Introduction

Let $S$ be a smooth minimal surface of general type with geometric genus $p_g \geq 3$. Denote by $\phi : S \to \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following result of Beauville is well-known.

**Theorem 1.1** ([Bea79]). If the canonical image $\Sigma := \phi(S)$ is a surface, then either:

(A) $p_g(\Sigma) = 0$, or

(B) $\Sigma$ is a canonical surface (i.e. it is the canonical image of a surface with birational canonical map), in particular $p_g(\Sigma) = p_g(S)$.

Moreover, in case (A) $d \leq 36$ and in case (B) $d \leq 9$.

The question of which pairs $(d,p_g)$ can actually occur has been object of study for some authors. Several examples were given for case (A), but case (B) is still mysterious. It is known that if $d > 3$, then $p_g \leq 12$, but so far only the case $(d,p_g) = (5,4)$ has been shown to exist (independently by Tan [Tan92] and by Pardini [Par91b]). We refer the recent preprint by Mendes Lopes and Pardini [MLP21] for a more detailed account on the subject. They leave some open problems, this note is motivated by their last question.

**Question.** For what pairs $(d,p_g)$, with $d > 3$, are there examples of surfaces in case (B) of Theorem 1.1?

Here we give examples for the cases $(d,p_g) = (4,5)$ and $(4,7)$, with canonical images a 40-nodal complete intersection surface in $\mathbb{P}^4$ and a 48-nodal complete intersection surface in $\mathbb{P}^6$, respectively (Beauville also paid some attention to such nodal surfaces, see [Bea17]).

The strategy for the construction is the following. If $X$ is a surface with nodes admitting a Galois covering $Y \to X$ ramified over the nodes and with Galois group $G$, a group with a “big” number of subgroups, then we have a “big” number of intermediate coverings of $X$. By computing the geometric genus $p_g$ of all involved surfaces, we may hope to find some $\rho : W \to Z$ with $p_g(W) = p_g(Z)$, hence such that the canonical map of $W$ factors through $\rho$.

We work explicitly with the equations of a 40-nodal surface from [RRS19], all computations are implemented with Magma [BCP97].

**Notation**

As usual the holomorphic Euler characteristic of a surface $S$ is denoted by $\chi(S)$, the geometric genus by $p_g(S)$, the irregularity by $q(S)$, and a canonical divisor by $K_S$. A $(-m)$-curve is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-m$. A node of $S$ is an ordinary double point of $S$. We say that a set of nodes of $S$ is 2-divisible if the sum $\sum A_i$ of the corresponding $(-2)$-curves in the smooth minimal model of $S$ is 2-divisible in the Picard group.
2. \((\mathbb{Z}/2)^r\)-coverings

The following result is taken from [Cat08, Proposition 7.6]. See also [Par91a].

**Proposition 2.1.** A normal finite \(G \cong (\mathbb{Z}/2)^r\)-covering \(\pi : Y \to X\) of a smooth variety \(X\) is completely determined by the datum of

1. reduced effective divisors \(D_\sigma\), for all \(\sigma \in G\), with no common components;
2. divisor linear equivalence classes \(L_{\chi_1}, \ldots, L_{\chi_r}\), for \(\chi_1, \ldots, \chi_r\) a basis of the group of characters \(G^\vee\), such that

\[
2L_{\chi_i} \equiv \sum_{\chi_i(\sigma) = 1} D_\sigma
\]

(with additive notation for the characters).

Conversely, given (1) and (2), one obtains a normal scheme \(Y\) with a finite \(G \cong (\mathbb{Z}/2)^r\)-covering \(Y \to X\), with branch curves the divisors \(D_\sigma\).

The scheme \(Y\) is irreducible if \(\{\sigma \mid D_\sigma > 0\}\) generates \(G\). We have a splitting

\[
\pi_* \mathcal{O}_Y = \bigoplus_{\chi \in G^\vee} L^{-1}_\chi.
\]

From now on, we assume that \(X\) and \(Y\) are surfaces. If each \(D_\sigma\) is smooth and \(\sum D_\sigma\) has simple normal crossings, then \(Y\) is smooth and its invariants are

\[
\chi(\mathcal{O}_Y) = 2^r \chi(\mathcal{O}_X) + \frac{1}{2} \sum_{\chi \in G^\vee} (t^2_\chi + K_X \cdot L_\chi),
\]

\[
p_g(Y) = p_g(X) + \sum_{\chi \in G^\vee} h^0(X, \mathcal{O}_X(K_X + L_\chi)).
\]

Let \(R_\sigma\) be the support of \(\pi^*(D_\sigma)\). The Hurwitz formula gives

\[
K_Y \equiv \pi^*(K_X) + \sum_{\sigma \in G} R_\sigma.
\]

Now assume that the \(D_\sigma\) are disjoint \((-2)\)-curves. Then the \(R_\sigma\) are disjoint \((-1)\)-curves, the canonical map of \(Y\) factors through the covering \(Y \to X\) if and only if \(p_g(Y) = p_g(X)\), and one has a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X'
\end{array}
\]

where \(Y \to Y'\) is the contraction of the \((-1)\)-curves \(R_\sigma\), the surface \(X'\) has nodes corresponding to the \((-2)\)-curves of \(X\), and \(Y' \to X'\) is a \((\mathbb{Z}/2)^r\)-covering ramified on those nodes. In this case Equation (2.1) becomes

\[
(2.2) \quad \chi(\mathcal{O}_Y) = 2^r (\chi(\mathcal{O}_X) - m/8)
\]

where \(m\) is the number of nodes of \(X'\).

3. Construction

Let \(X_{40}\) be the surface in \(\mathbb{P}^4\) given by the equations

\[
5\left(x^2 + y^2 + z^2 + w^2 + t^2\right) - 7\left(x + y + z + w + t\right)^2 = 0
\]

\[
4\left(x^4 + y^4 + z^4 + w^4 + t^4 + h^4\right) - \left(x^2 + y^2 + z^2 + w^2 + t^2 + h^2\right)^2 = 0
\]

(3.1)
where
\[ h := -(x + y + z + w + t). \]

It is the canonical model of a surface with invariants \( p_g = 5, q = 0 \) and \( K^2 = 8 \). The above quartic \( I \) is classically known as the Igusa quartic; its singular set is the union of 15 lines. The quadric meets these lines transversally, and is tangent to \( I \) at 10 smooth points, thus the singular set of \( X_{40} \) is the union of 40 nodes \( N_1, \ldots, N_{40} \) (for more details see [RRS19]).

Let \( \overline{X}_{40} \) be the smooth minimal model of \( X_{40} \) and denote by \( A_i \) the \((-2)\)-curves in \( \overline{X}_{40} \) corresponding to the nodes \( N_i, i = 1, \ldots, 40 \). Let \( a, b, c \) be the canonical generators of the group \((\mathbb{Z}/2)^3\) and, for \( i, j, k \in \mathbb{Z}/2 \), let \( \chi_{ijk} \) denote the character which takes the value \( i, j, k \) on \( a, b, c \), respectively. We show in Section 4.1 that one can write
\[
A_1 + \cdots + A_{40} = D_a + D_b + D_c + D_{abc} + D_{bc} + D_{ac} + D_{ab}
\]
where each of \( D_a, D_b, D_c, D_{abc} \) is a sum of 4 \((-2)\)-curves, each of \( D_{bc}, D_{ac}, D_{ab} \) is a sum of 8 \((-2)\)-curves, and such that there exist divisors \( L_{100}, L_{010}, L_{001} \) satisfying:
\[
\begin{align*}
D_a + D_{abc} + D_{ac} + D_{ab} &\equiv 2L_{100} \\
D_b + D_{abc} + D_{bc} + D_{ab} &\equiv 2L_{010} \\
D_c + D_{abc} + D_{bc} + D_{ac} &\equiv 2L_{001}.
\end{align*}
\]
(3.2)

It follows from Proposition 2.1 that these data define a \((\mathbb{Z}/2)^3\)-covering \( \pi : \overline{Y} \to \overline{X}_{40} \) branched on the \((-2)\)-curves \( A_i \), equivalently a \((\mathbb{Z}/2)^3\)-covering \( \psi : Y \to X_{40} \) branched on the nodes of \( X_{40} \) (the surface \( Y \) is minimal because \( X_{40} \) is minimal and \( \psi \) is étale in codimension 1). In particular there exist divisors \( L_{111}, L_{110}, L_{010}, L_{011} \) such that:
\[
\begin{align*}
D_a + D_b + D_c + D_{abc} &\equiv 2L_{111} \\
D_a + D_b + D_{bc} + D_{ac} &\equiv 2L_{110} \\
D_a + D_c + D_{bc} + D_{ab} &\equiv 2L_{101} \\
D_b + D_c + D_{ac} + D_{ab} &\equiv 2L_{011}.
\end{align*}
\]
(3.3)

One has
\[
2L_{ijk} = \sum_{\chi_{ijk}(\sigma) = 1} D_\sigma.
\]

Since \( \psi \) is ramified only on nodes, we have \( K_Y \equiv \psi^*(K_{X_{40}}) \) and then \( K_Y^2 = 8K_{X_{40}}^2 = 64 \). We show in Section 4.1 that
\[
h^0(\overline{X}_{40}, \mathcal{O}_{\overline{X}_{40}}(K_{\overline{X}_{40}} + L_{111})) = 2
\]
and
\[
h^0(\overline{X}_{40}, \mathcal{O}_{\overline{X}_{40}}(K_{\overline{X}_{40}} + L_{ijk})) = 0 \text{ for } ijk \neq 111,
\]
thus
\[
p_g(Y) = p_g(X_{40}) + 2 + 0 + \cdots + 0 = 7.
\]

We get from (2.2) that \( \chi(Y) = 8(6 - 5) = 8 \), thus \( q(Y) = 0 \).

The covering \( \psi \) factors as
\[
\begin{align*}
Y &\longrightarrow Y_{32} \longrightarrow Y_{48} \\
X_{16} &\longrightarrow X_{32} \longrightarrow X_{40}
\end{align*}
\]
with \( Y_{48} \) and \( X_{16} \) given by the quotients by the groups \( \langle ab, ac \rangle \) and \( \langle c \rangle \), respectively (the subscript \( n \) means a surface with singular set the union of \( n \) nodes). All these surfaces are regular because \( q(Y) = 0 \).

It follows from (2.2) that \( \chi(X_{16}) = 4(6 - 36/8) = 6 \), thus \( p_g(X_{16}) = p_g(X_{40}) = 5 \), and we conclude that the \((\mathbb{Z}/2)^2\)-covering \( X_{16} \to X_{40} \) is the canonical map of \( X_{16} \).
Analogously, \( p_g(Y) = p_g(Y_{48}) = 7 \) and we claim that
\[
\text{the } (\mathbb{Z}/2)^2\text{-covering } Y \to Y_{48} \text{ is the canonical map of } Y.
\]
For this it suffices to show that \( Y_{48} \) is a canonical surface.

Since the canonical system of \( Y_{48} \) contains the pullback of the canonical system of \( X_{40} \) and since \( p_g(Y_{48}) > p_g(X_{40}) \), the canonical map of \( Y_{48} \) must be birational. But we can be more precise. We follow Beauville [Bea17] and show that \( Y_{48} \) can be embedded in \( \mathbb{P}^6 \) as a complete intersection of 4 quadrics in the following way. The linear system \( L \) of quadrics through the branch locus of the covering \( Y_{48} \to X_{40} \) (16 nodes) is of dimension 2. Using computer algebra it is not difficult to show that \( L \) contains quadrics \( B, C, D \) such that the surface \( X_{40} \) is given by \( Q = 0, B^2 - CD = 0 \), where \( Q \) is the quadric from (3.1) (we write the quadrics as general elements of \( L \), thus depending on some parameters; then we obtain a variety on these parameters by imposing that the hypersurfaces \( Q = 0 \) and \( B^2 - CD = 0 \) are tangent at the 24 nodes of \( X_{40} \) which are disjoint from the 16 nodes of \( B^2 - CD = 0 \); finally we compute points in this variety).

Then \( Y_{48} \) is given in \( \mathbb{P}^6(x, y, z, w, t, u, v) \) by equations
\[
u^2 - C = v^2 - D = uv - B = Q = 0.
\]
We give these equations in Section 4.2 and verify that \( Y_{48} \) is as stated.

Let us explain how we find 2-divisible sets of nodes in \( X_{40} \). The surface \( X_{40} \) contains 40 tropes, which are hyperplane sections \( H_i = 2T_i \) with \( T_i \subset X_{40} \) a reduced curve through 12 nodes of \( X_{40} \), and smooth at these points. Thus in \( \tilde{X}_{40} \) the pullback of such a trope can be written as
\[
\tilde{H}_i = 2\tilde{T}_i + \sum_{j \in J} A_j, \text{ with } \#J = 12.
\]
Thus for each pair of tropes the sum of nodes contained in their union and not contained in their intersection is 2-divisible.

Using these 2-divisibilities, the strategy for finding configurations as in (3.2) is simple: we have used a computer algorithm to list and check possibilities.

### 4. Computations

The computations below are implemented with Magma V2.26-5.

#### 4.1. The covering \( Y \to X_{40} \)

We start by defining the surface \( X_{40} \) and its singular set.

```latex
\begin{verbatim}
K:=Rationals();
R<r>:=PolynomialRing(K);
K<r>:=ext<K|r^2 + 15>;
P<x,y,z,w,t>:=ProjectiveSpace(K,4);
h:=-x-y-z-w-t;
Q:=5*(x^2+y^2+z^2+w^2+t^2)-7*(x+y+z+w+t)^2;
I:=4*(x^4+y^4+z^4+w^4+t^4+h^4)-(x^2+y^2+z^2+w^2+t^2+h^2)^2;
X40:=Surface(P,[Q,I]);
SX40:=SingularSubscheme(X40);
\end{verbatim}
```

The partition of the 40 nodes:

```latex
\begin{verbatim}
Da:=\{P![3,3,-2,-2,3],P![4,-r+1,r-5,-r+1,4],
P![-r+1,4,r-5,-r+1,4],P![r+1,r+1,-r-5,4,4]\};
Db:=\{P![2,-3,-3,-3,2],P![4,r+1,r+1,-r-5,4],
P![-r-5,r-5,r-5,-r-5,10],P![-r-5,-r+1,-r+1,4,4]\};
\end{verbatim}
```
Dc:={$P![-3,-3,2,-3,2], P![-r+1,-r+1,r-5,4,4],
P![-5,-r+5,-r+5,-r+5,10], P![-r+1,-r+1,r-5,4,4]$};

Dabc:={$P![-2,3,3,-2,3], P![-r-5,r+1,r+1,4,4],
P![-5,-r+5,-r+5,-r+5,10], P![-r+1,-r+1,r-5,4,4]$};

Dbc:={$P![-2,-2,3,3,3], P![-r-5,r+1,r+1,4,4],
P![-5,-r+5,-r+5,-r+5,10], P![-r+1,-r+1,r-5,4,4]$};

Dac:={$P![-3,2,-3,-3,2], P![-r-5,r+1,r+1,4,4],
P![-5,-r+5,-r+5,-r+5,10], P![-r+1,-r+1,r-5,4,4]$};

Dab:={$P![-3,-3,-3,2,2], P![-2,3,-2,3,3],
P![-r-5,r+1,r+1,4,4], P![-r+1,-r+1,r-5,4,4]$};

Veri

ication that these are in fact the nodes:

&join[Da,Db,Dc,Dabc,Dbc,Dac,Dab] eq SingularPoints(X40);

HasSingularPointsOverExtension(X40) eq false;

Some of the tropes of $X_{40}$:

tropes:=[
6*x + (-r - 9)*y + (r - 9)*z + (r - 9)*w + (-r - 9)*t,
16*x + (-r - 9)*y + 16*z + (3*r + 11)*w + (3*r + 11)*t,
16*x + (r - 9)*y + 16*z + (-3*r + 11)*w + (-3*r + 11)*t,
6*x + (r - 9)*y + (-r - 9)*z + (r - 9)*w + (-r - 9)*t,
16*x + (3*r + 11)*y + 16*z + (3*r + 11)*w + (-r - 9)*t,
16*x + (-3*r + 11)*y + (-3*r + 11)*z + (r - 9)*w + 16*t,
x + y + w,
16*x + (r - 9)*y + (-3*r + 11)*z + (-3*r + 11)*w + 16*t,
x + z + w
];

The reduced subscheme of these tropes:

red:=[ReducedSubscheme(Scheme(X40,q)):q in tropes];
&and[Degree(q) eq 4:q in red];

They are smooth at the nodes of $X_{40}$:

&and[Dimension(SingularSubscheme(q) meet SX40) eq -1:q in red];

Two 2-divisible disjoint sets of 20 nodes, which confirm that the 40 nodes are 2-divisible:

s1:=Points(Scheme(SX40,tropes[1]*tropes[2])) diff
Points(Scheme(SX40,[tropes[1],tropes[2]]));
s2:=Points(Scheme(SX40,tropes[6]*tropes[7])) diff
Points(Scheme(SX40,[tropes[6],tropes[7]]));
&and[#s1 eq 20,#s2 eq 20,##((s1 join s2) eq 40];

We compute three 2-divisible sets of 24 nodes:
Sets:=[[];
for q in [[2,5],[1,4],[3,8]] do
Examples of surfaces with canonical map of degree 4

\[ \text{pts} := \text{Points}(\text{Scheme}(\text{SX}40, \text{tropes}[q[1]] \ast \text{tropes}[q[2]])) \text{ diff Points}(\text{Scheme}(\text{SX}40, [\text{tropes}[q[1]], \text{tropes}[q[2]]])); \]
\text{Append}(`\text{Sets}, \text{SingularPoints}(\text{X}40) \text{ diff pts}); \]
end for;

and use these sets to check the divisibilities in (3.2):
\[ \text{Da join Dabc join Dac join Dab eq Sets[1];} \]
\[ \text{Db join Dabc join Dbc join Dab eq Sets[2];} \]
\[ \text{Dc join Dabc join Dbc join Dac eq Sets[3];} \]

Now we show that
\[ h^0(\overline{X}_{40}, O_{\overline{X}_{40}}(K_{\overline{X}_{40}} + L_{111})) = 2. \]

Let \( N_1, \ldots, N_{16} \) be the nodes in \( D_a + D_b + D_c + D_{abc} \) and \( A_1, \ldots, A_{16} \) be the corresponding \((-2)\)-curves. Let \( H_1, H_2 \) be the tropes whose pullback to \( \overline{X}_{40} \) is
\[ \overline{H}_1 + \overline{H}_2 = 2\overline{T}_1 + 2\overline{T}_2 + \sum_{i=1}^{16} A_i + 2 \sum_{i=17}^{20} A_i, \]
with \( A_{17}, \ldots, A_{20} \in \overline{H}_1 \cap \overline{H}_2 \). Then
\[ \sum_{i=1}^{16} A_i \equiv 2L_{111}, \quad \text{with} \quad K_{\overline{X}_{40}} + L_{111} \equiv 2\overline{H} - \overline{T}_1 - \overline{T}_2 - \sum_{i=17}^{20} A_i. \]

We compute below that the system of quadrics through the curves \( T_1, T_2 \subset \mathbb{P}^4 \) is generated by 2 elements, modulo the quadric \( Q \). For \( i = 17, \ldots, 20 \), the fact \( (2\overline{H} - \overline{T}_1 - \overline{T}_2) \cdot A_i < 0 \) implies that \( A_i \) is contained in the base component of the linear system \([2\overline{H} - \overline{T}_1 - \overline{T}_2]\). This gives \( h^0(\overline{X}_{40}, O_{\overline{X}_{40}}(K_{\overline{X}_{40}} + L_{111})) = 2. \)

T1:=\text{ReducedSubscheme}(\text{Scheme}(\text{X}40, \text{tropes}[2]));
T2:=\text{ReducedSubscheme}(\text{Scheme}(\text{X}40, \text{tropes}[9]));
\text{pts}:=\text{Points}(\text{SX}40 \text{ meet (T1 join T2)) diff Points(SX40 meet T1 meet T2);}
L:=\text{LinearSystem}(\text{LinearSystem}(\text{P}, 2), \text{T1 join T2});
#\text{Sections}(\text{LinearSystemTrace}(L, \text{X}40)) eq 2;

Let us show that
\[ h^0(\overline{X}_{40}, O_{\overline{X}_{40}}(K_{\overline{X}_{40}} + L_{ijk})) = 0 \]
for \( ijk \neq 111 \). Suppose the opposite. Let \( A_1, \ldots, A_{24} \) be the corresponding \((-2)\)-curves. Then there is a curve \( E \in [K_{\overline{X}_{40}} + L_{ijk}] \), and \( E \cdot A_1 = -1 \) implies that the linear system \([K_{\overline{X}_{40}} + L_{ijk} - \sum_{j=1}^{24} A_j] = [K_{\overline{X}_{40}} - L_{ijk}] \) is nonempty. Therefore \( 2K_{\overline{X}_{40}} - \sum_{j=1}^{24} A_j \) is nonempty, which implies that there is at least one quadric in \( \mathbb{P}^4 \) through the corresponding nodes \( N_1, \ldots, N_{24} \) (modulo the quadric \( Q \)). We show below that this does not happen.

Sets:=[
\text{Da join Dabc join Dac join Dab,}
\text{Db join Dabc join Dbc join Dab,}
\text{Dc join Dabc join Dbc join Dac,}
\text{Da join Db join Dbc join Dac,}
\text{Da join Dc join Db join Dab,}
\text{Db join Dc join Dac join Dab,}
];
for q in Sets do
L:=LinearSystem(LinearSystem(P,2),[P!x:x in q]);
#Sections(LinearSystemTrace(L,X40)) eq 0;
end for;

4.2. The surface $Y_{48}$

Here we give the equations of $Y_{48}$ as a complete intersection of 4 quadrics in $\mathbb{P}^6$. We start by defining $\mathbb{P}^6$ over a certain number field.

$K:=\text{Rationals}(); R<x>:=\text{PolynomialRing}(K);$  
$K<r,m>:=\text{ext}<K|x^2 + 15,x^2 - 95/42*x + 2855/2646>;$  
$K<n>:=\text{ext}<K|n^2 + 443889677/206391214080000*r - 46942774543/619173642240000>$;  
$P6<x,y,z,w,t,u,v>:=\text{ProjectiveSpace}(K,6);$  

The three quadrics $B,C,D$:  

$B:=(675/4802*r+334125/33614)*n*x*z+(-389475/67228*r+3266325/33614)*n*x*w+$  
$(34425/9604*r+451575/67228)*n*y*w+(-389475/67228*r+3266325/67228)*n*z*w+$  
$(-62100/16807*r+348300/16807)*n*w^2+(-8100/2401*r+137700/16807)*n*y*t+$  
$(239625/33614*r+1541025/33614)*n*z*t+$  
$(+6075/9604*r+3007125/67228)*n*w*t+(71550/16807*r+319950/16807)*n*t^2;  

$C:=x*y+1/154*(126*m-181)*y^2+1/42*(-42*m+95)*x*z+y*z+(1/1540*(14*m-25)*r+$  
$+1/924*(-798*m+997))*x*w+(1/420*(42*m-65)*r+1/308*(-294*m+767))*y*w+$  
$(1/1540*(-14*m+25)*r+1/924*(-798*m+997))*z*w+(1/385*(-119*m+185)*r+$  
$+1/662*(-168*m+311))*w^2+(1/1540*(-14*m+25)*r+1/924*(-798*m+$  
$+997))*y*t+(1/420*(-42*m+65)*r+1/308*(-294*m+767))*y*w+(1/1540*(-14*m+$  
$+25)*r+1/924*(-798*m+997))*z*t+1/154*(126*m-71)*w*t+(1/385*(119*m-$  
$+185)*r+1/662*(-168*m+311))*t^2;  

$D:=x*y+1/77*(-63*m+52)*y^2+m*x*z+y*z+(1/2310*(-21*m+10)*r+1/154*(133*m+$  
$+32))*x*w+(1/70*(-7*m+5)*r+1/154*(147*m+51))*y*w+(1/2310*(-21*m+$  
$+10)*r+1/154*(133*m+32))*z*w+(1/2310*(714*m-505)*r+1/154*(56*m-$  
$+23))*w^2+(1/2310*(21*m-10)*r+1/154*(133*m+32))*x*t+(1/70*(-7*m+5)*r+$  
$+1/154*(147*m+51))*y*t+(1/2310*(21*m-10)*r+1/154*(133*m+32))*z*t+$  
$+1/77*(-63*m+107)*w*t+(1/2310*(-714*m+505)*r+1/154*(56*m-23))*t^2;  

We obtain alternative equations for $X_{40}$:  

$F:=B-C*D;$  
$Q:=5*(x^2+y^2+z^2+w^2+t^2)*h^2-(x+y+z+w+t)^2;$  
$X:=\text{Scheme}(P6,[F,Q,u,v]);$  
$h:=-x-y-z-w-t;$  
$I:=4*(x^4+y^4+z^4+w^4+t^4-4*x^2*y^2-4*x^2*z^2-4*x^2*w^2-4*y^2*z^2-4*y^2*w^2-4$  
$+4*z^2*w^2-4*t^2*h^2)*h^2;$  
$X40:=\text{Scheme}(P6,[Q,I,u,v]);$  
$X eq X40;$  

And finally the equations of $Y_{48}$ in $\mathbb{P}^6$:  

$Y48:=\text{Surface}(P6,[u^2-C,v^2-D,u*v-B,Q]);$  
$SY48:=\text{SingularSubscheme}(Y48);$  
$\text{Dimension}(SY48) eq 0;$  
$\text{Degree}(SY48) eq 48;$  
$\text{Degree}(\text{ReducedSubscheme}(SY48)) eq 48;
References


